NON-EQUILIBRIUM DYNAM
A QUANTUM PHASE TRAN

Quench in the Quantum Ising Model

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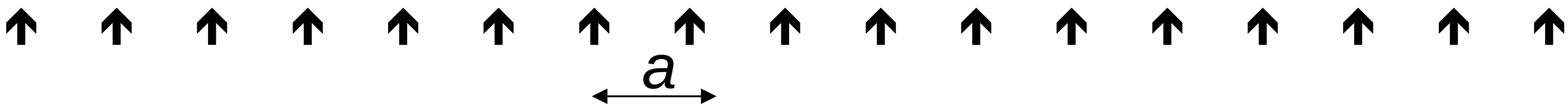
Uwe Dorner, Peter Zoller, WHZ, quant-ph/0503511

QUANTUM ISING MODEL

Lattice of spin 1/2 particles interacting with an external force (e.g., magnetic field along the *x* axis) and with each other (ferromagnetic Ising interaction along the *z* axis):



$$H = -J(t)\sum_{l}\sigma_l^x - W\sum_{l}\sigma_l^z\sigma_{l+1}^z \quad , \quad |\rightarrow\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$$

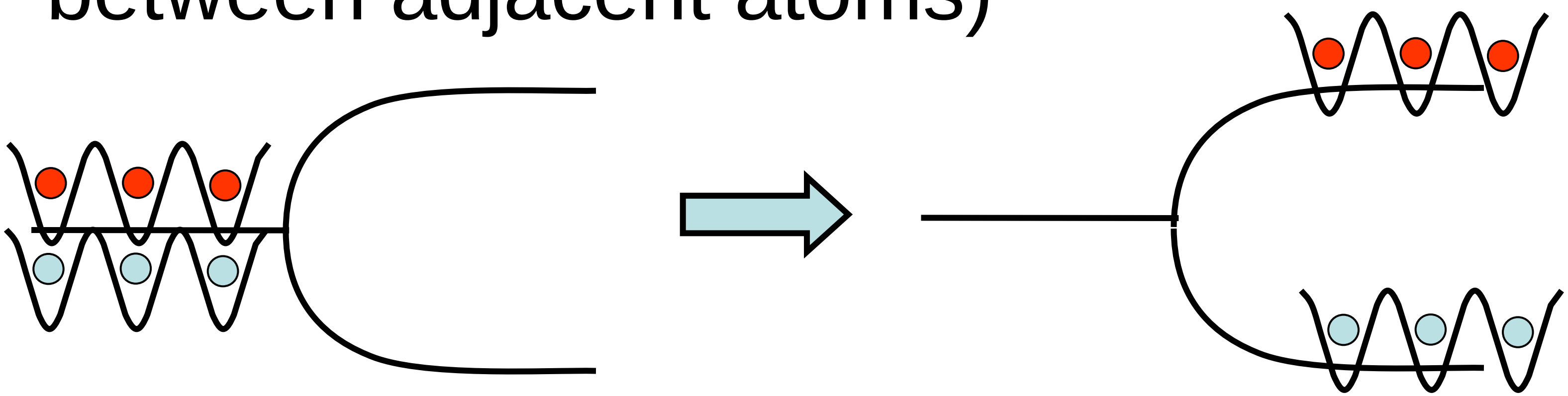
Quantum phase transition occurs as *J(t)* decreases. Then $|\rightarrow\rightarrow\rightarrow\ldots\rangle \propto (|\uparrow\rangle + |\downarrow\rangle)(|\uparrow\rangle + |\downarrow\rangle)(|\uparrow\rangle + |\downarrow\rangle)\cdots$, analogue of the “symmetric vaccum”, is no longer favored energetically: $|\uparrow\uparrow\uparrow\ldots\rangle$ or $|\downarrow\downarrow\downarrow\ldots\rangle$ (or any superposition thereof) are the ground states.

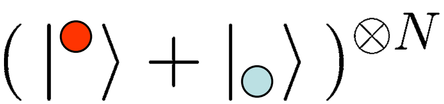
“…one of two canonical models of for quantum phase transitions.

S. Sachdev, *Quantum Phase Transitions, CU*

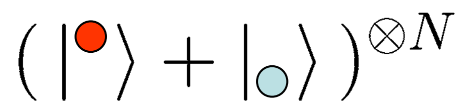
Atoms in 1D lattices

Beam splitter: N x single atom (no interaction between adjacent atoms)



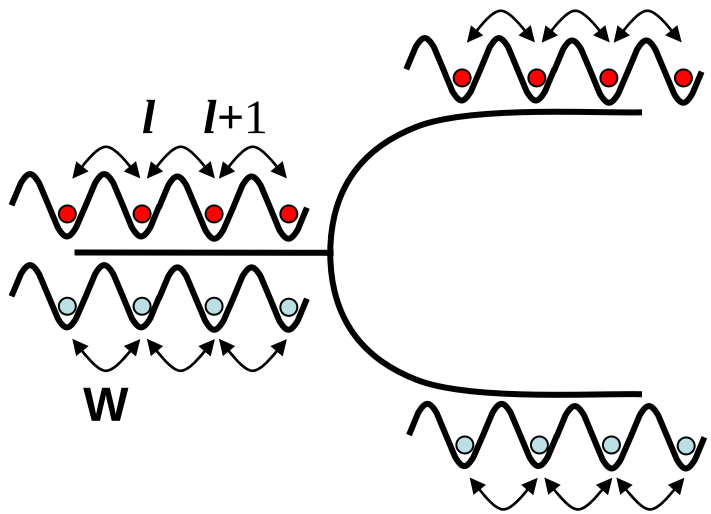


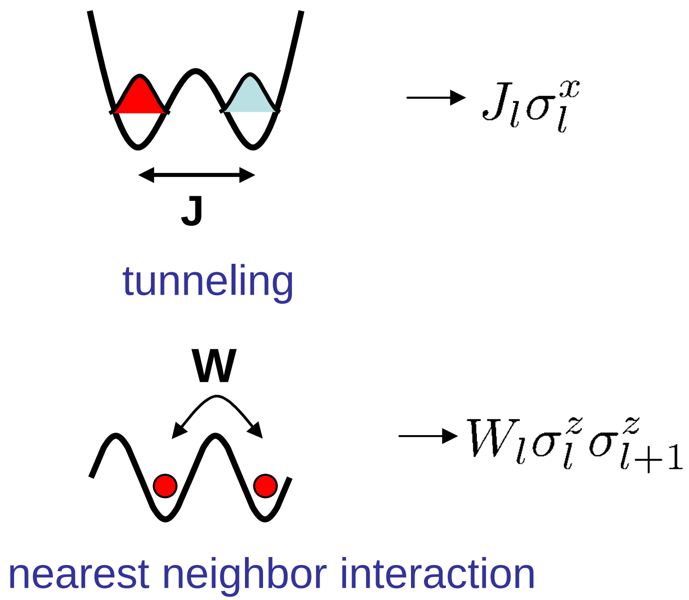
product state

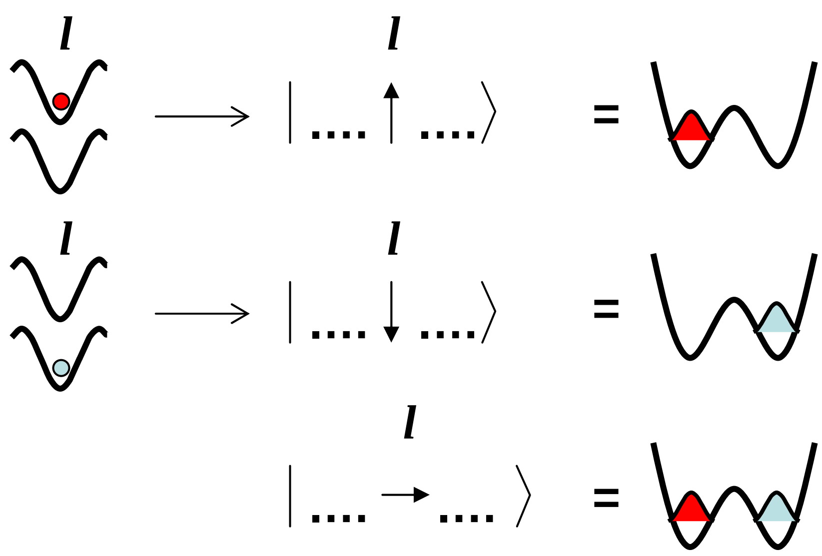


product state

Mapping to Spin Model

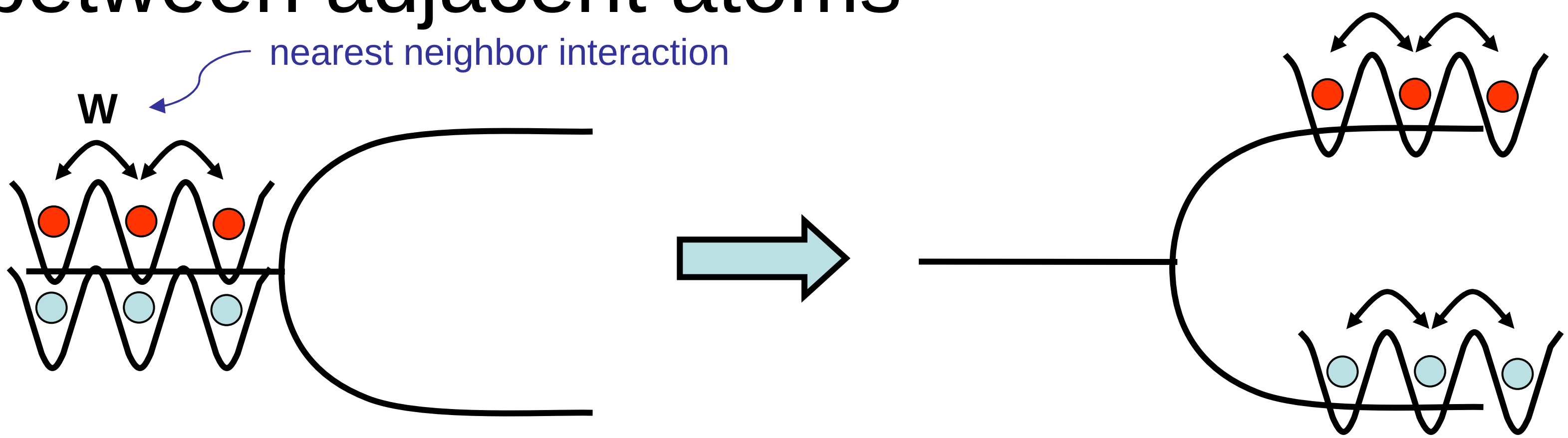


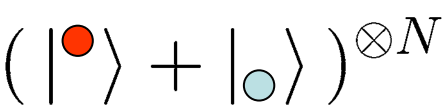




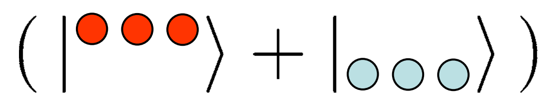
Atoms in 1D lattices

Beam splitter: attractive or repulsive interaction between adjacent atoms

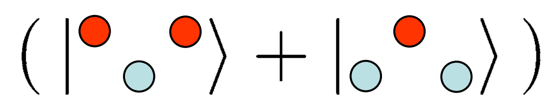




product state

attractive 

entangled state (N-particle GHZ)

repulsive 

entangled state

Nearest neighbor interaction: cold collisions, dipole-dipole (Rydberg atoms)
Jaksch et al. PRL 82, 1975 (1999), Jaksch et al. PRL 85, 2208 (2000)

Symmetry Breaking and Defects

$$H = -J(t)\sum_{l} \sigma_l^x - W\sum_{l} \sigma_l^z \sigma_{l+1}^z$$

Broken symmetry states after the phase transition:

$$\ldots\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\ldots \Rightarrow \ldots\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\ldots$$

$$\ldots\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\ldots \Rightarrow \ldots\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\ldots$$

Also possible “kinks”

$$\ldots\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\ldots \Rightarrow \ldots\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow\uparrow\uparrow\uparrow\ldots$$

Plan

- Introduce the **quantum** Ising model (done)
- Describe dynamics of symmetry breaking in thermodynamic phase transitions (Jim Anglin, Nuno Antunes, Luis Bettencourt, Fernando Cucchietti, Bogdan Damski, Jacek Dziarmaga, Pablo Laguna, Augusto Roncaglia, Augusto Smerzi, Andy Yates…)
- Apply **thermodynamic** approach to **quantum** Ising model & compare with numerical simulations
- Introduce a purely **quantum** approach and compare with thermodynamic approach and with numerical simulations

Spontaneous Symmetry Breaking

$$V_{Ginzburg-Landau}(\varphi) = \varepsilon|\varphi|^2 + |\varphi|^4$$

QuickTime™ and a
Animation decompressor
are needed to see this picture.

During the transition ε changes sign (for instance, “relative temperature” decreases from +1 to -1).

Choice of the phase of φ -- which may be the phase of “the wave function of the condensate” -- is the choice of the broken symmetry state (“vacuum”).

QuickTime™ and a
Animation decompressor
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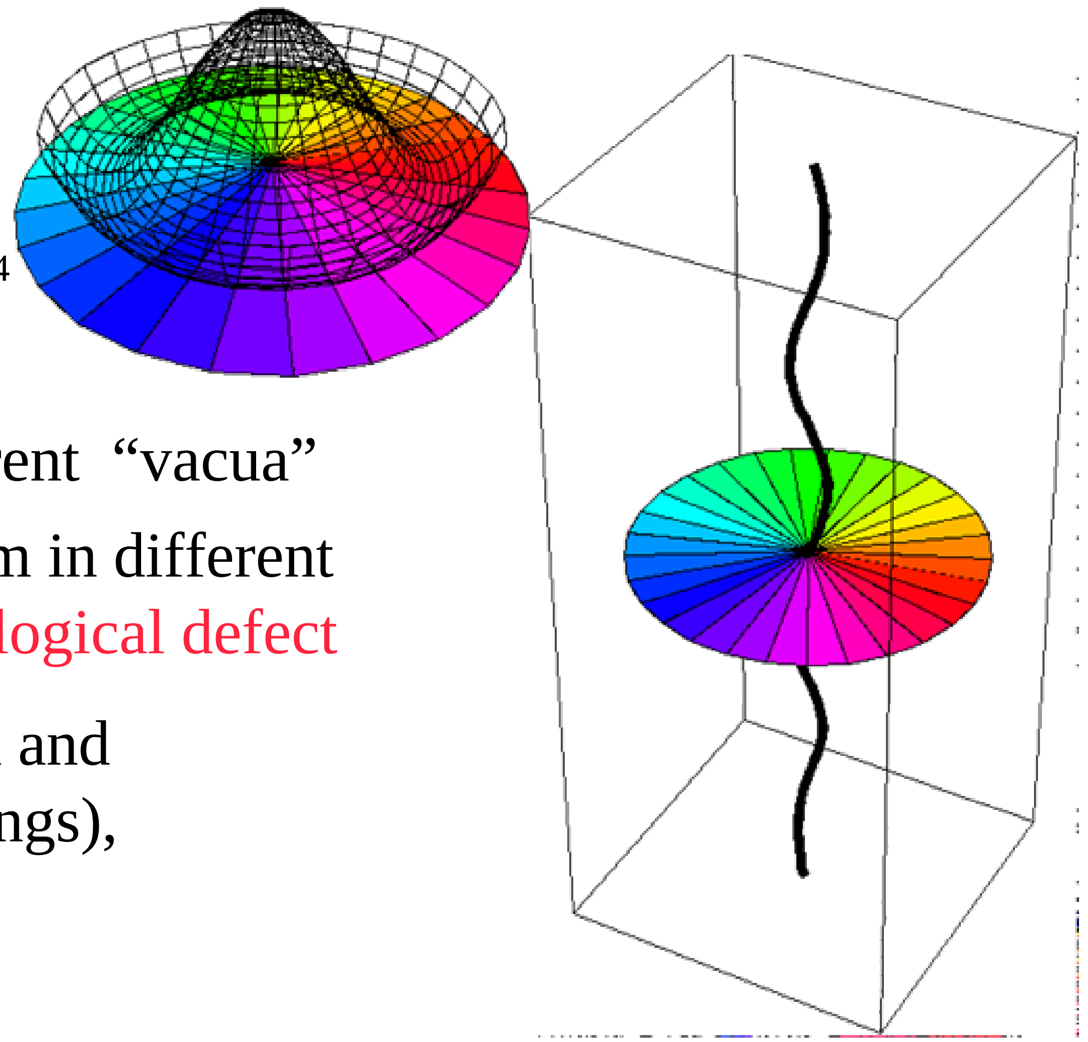
Topological Defects

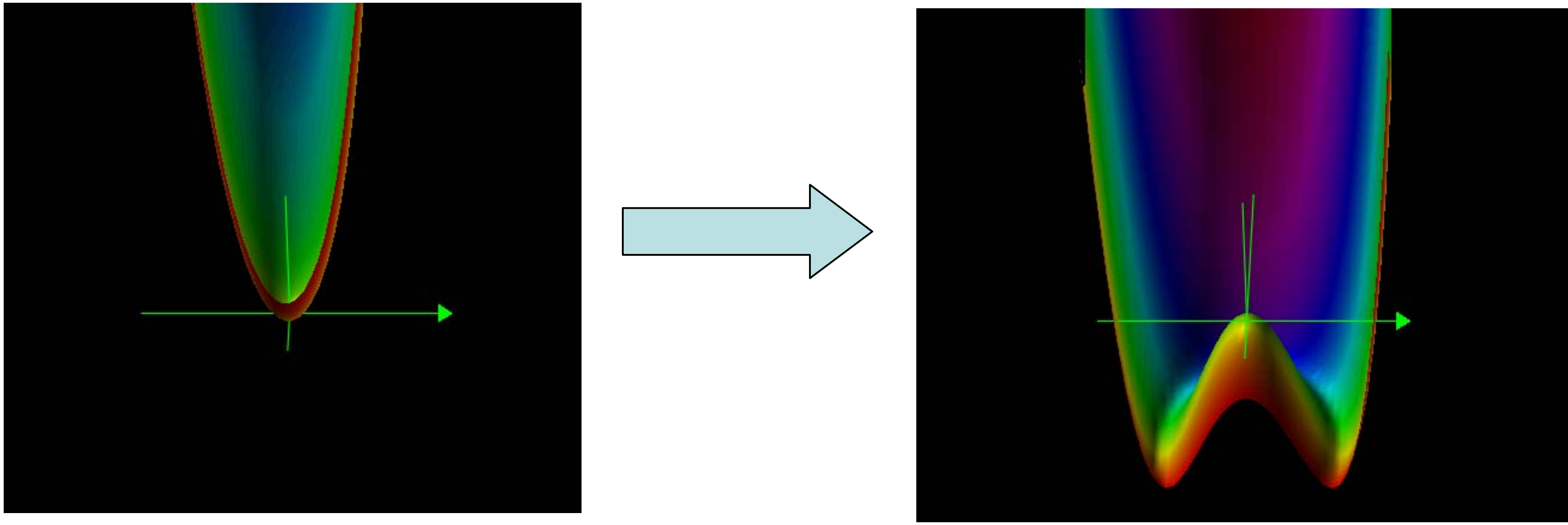
"Mexican hat potential"



$$V_{Ginzburg-Landau}(\varphi) = \varepsilon|\varphi|^2 + |\varphi|^4$$

- Colors – Different "vacua"
- Different vacuum in different directions - topological defect
- Monopoles, flux and vortex lines (strings), domain walls

Local choices may not be globally compatible: topological defects can form during quench!



are needed to see this picture.

$\hat{\xi}$

Density of vortices ("strings"):

$$n = 1/\hat{\xi}^2$$

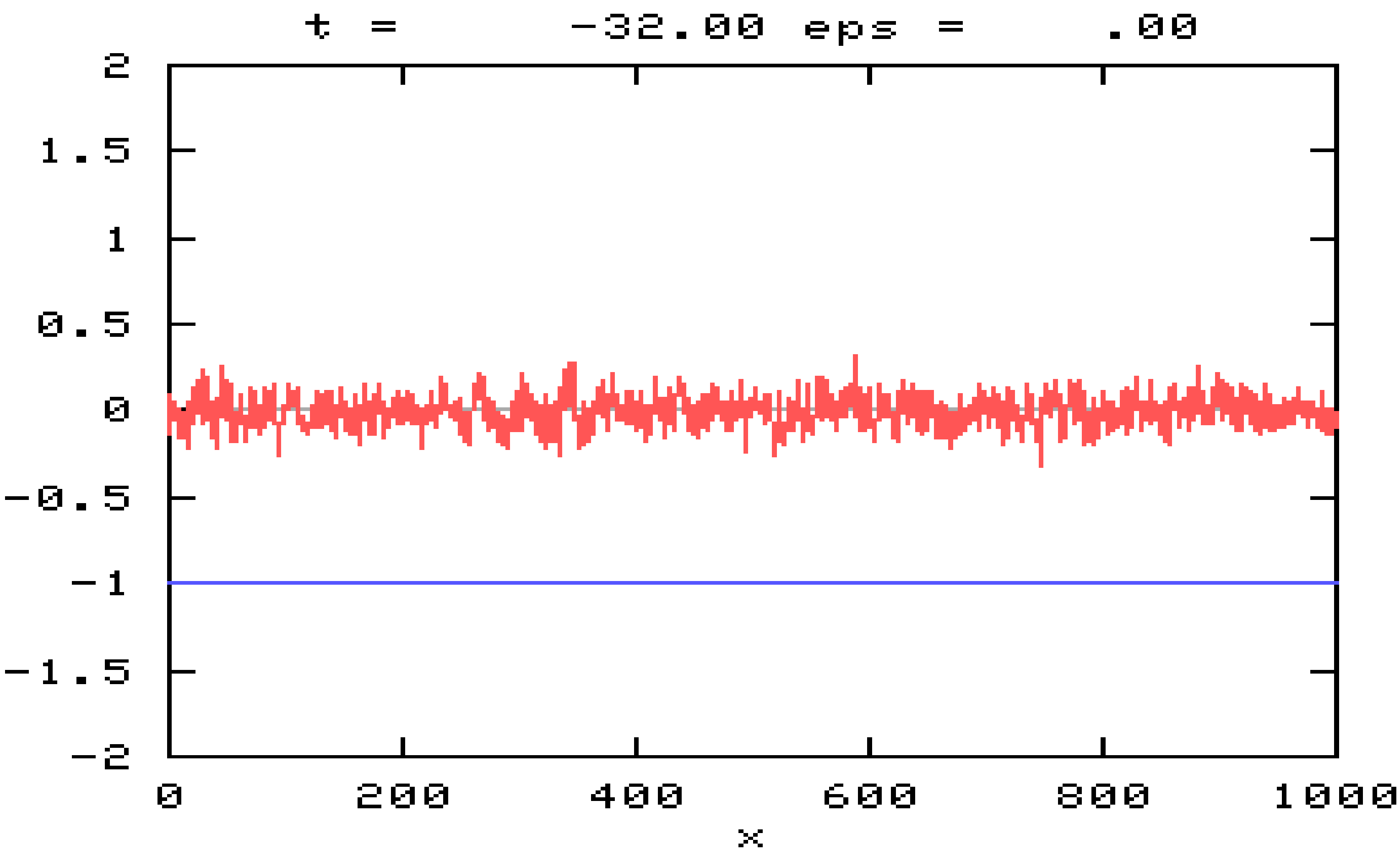
(Kibble, '76)

Formation of kinks in a 1-D Landau-Ginzburg system

$V_{Ginzburg-Landau}(\varphi) = \varepsilon|\varphi|^2 + |\varphi|^4$ with **real** φ driven by white noise.

Overdamped Gross-Pitaevskii evolution with $\varepsilon = t/\tau_Q$ and:

$$\Gamma\dot{\varphi} = -c^{-2}\nabla^2\varphi + \varepsilon\varphi + 2|\varphi|^2\varphi + noise$$



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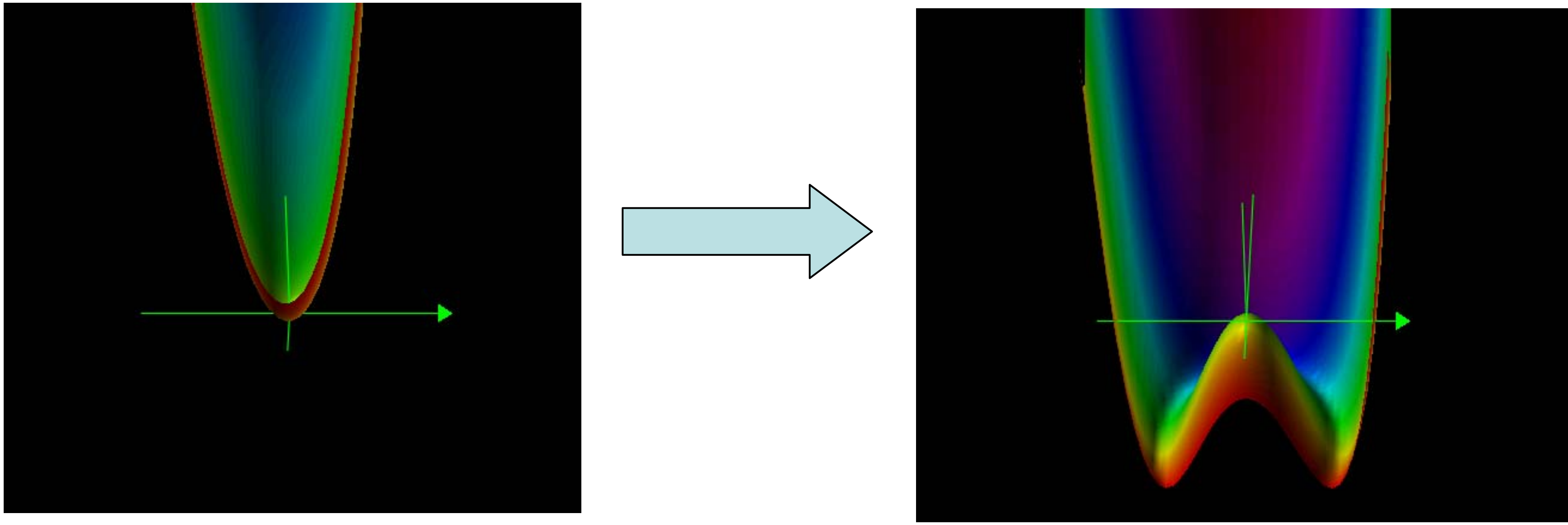


$\hat{\xi}$

Density of vortices ("strings"):

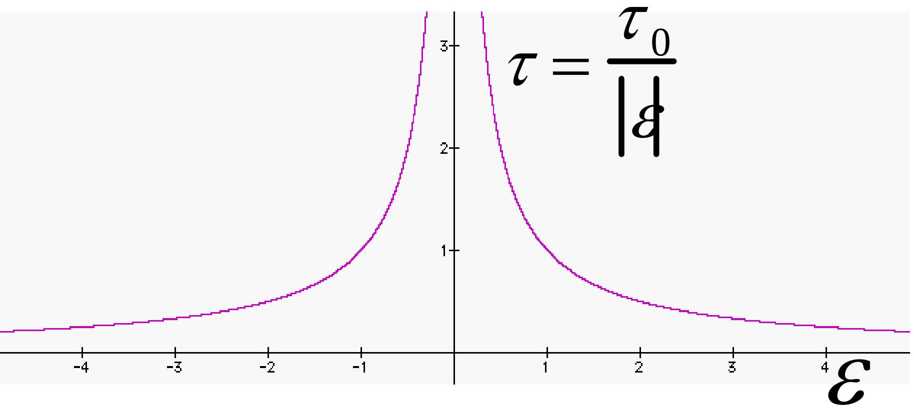
$$n = 1/\hat{\xi}^2$$

(Kibble, '76)

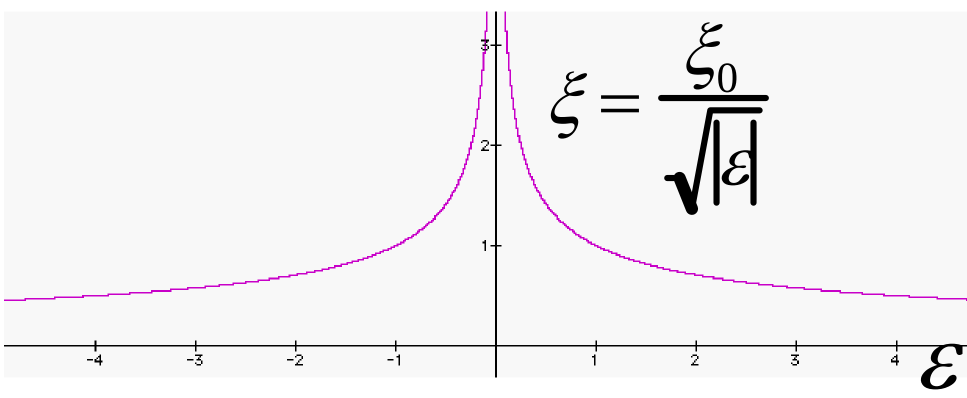
All second order phase transitions fall into "universality classes" characterized by the behavior of quantities such as specific heat, magnetic susceptibility, etc. This is also the case for quantum phase transitions.

For our purpose behavior of the **relaxation time** and of the **healing length** near the critical point will be essential; they determine the density of topological defects formed in the rapid phase transition ("the quench").

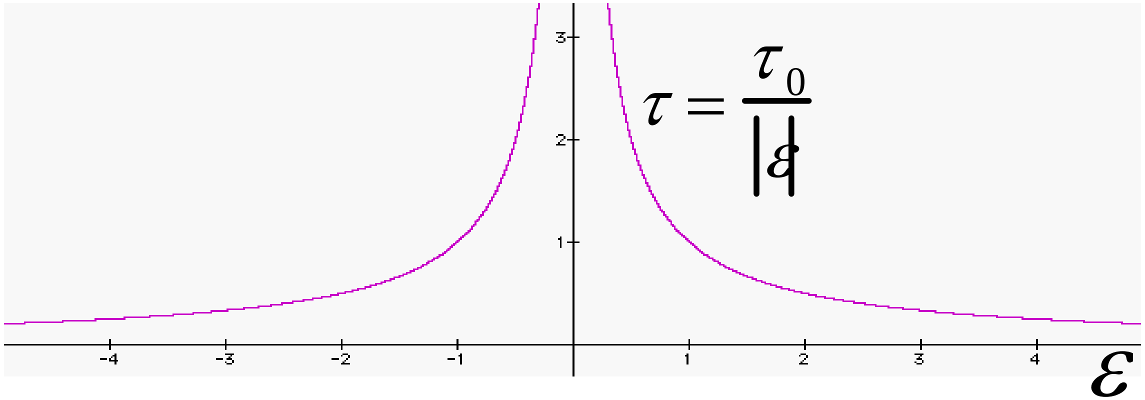
"CRITICAL SLOWING DOWN"



"CRITICAL OPALESCENCE"



Derivation of the "freeze out time"…



Assume:

$$\varepsilon = \frac{time}{"quench\ time"} = \frac{t}{\tau_Q}$$

Relaxation time:

$$\tau = \frac{\tau_0}{|\varepsilon|}$$

determines "reflexes" of the system.

The potential $V_{Ginzburg-Landau}(\varphi) = \varepsilon|\varphi|^2 + |\varphi|^4$ changes at a rate given by:

$$\frac{\varepsilon}{\dot{\varepsilon}} = t$$

Relaxation time is equal to this rate of change when $\tau(\varepsilon(\hat{t})) = \hat{t}$

… and the corresponding "frozen out" healing length $\hat{\xi}$

….. $\tau(\varepsilon(\hat{t})) = \hat{t}$

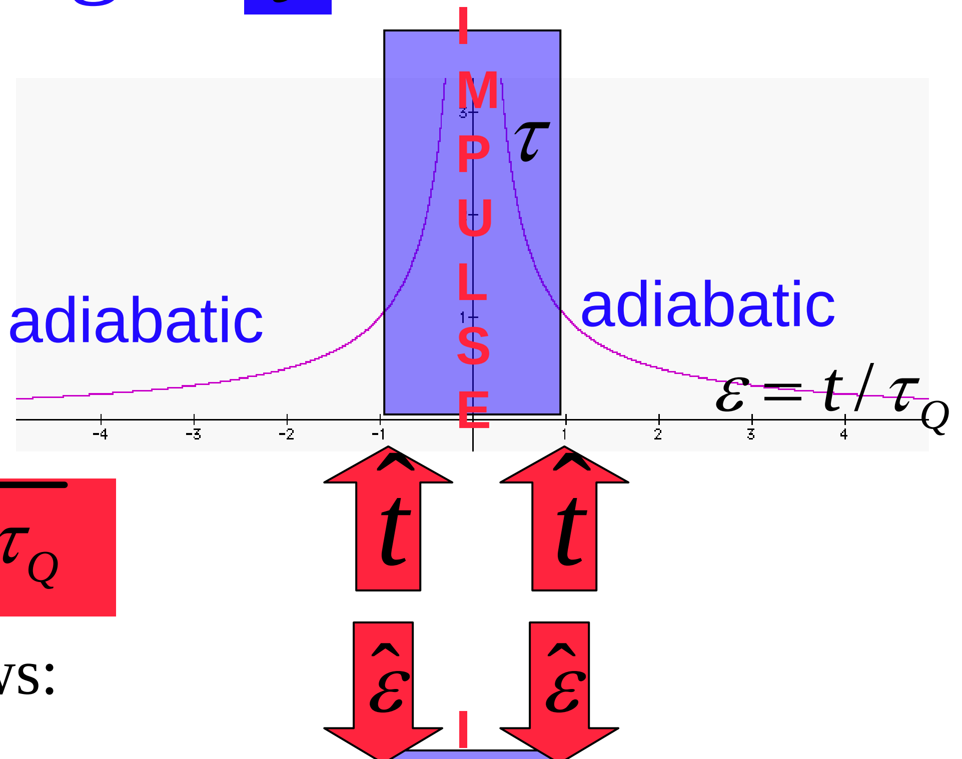
Hence: $\tau_0 / (\hat{t}/\tau_Q) = \hat{t}$

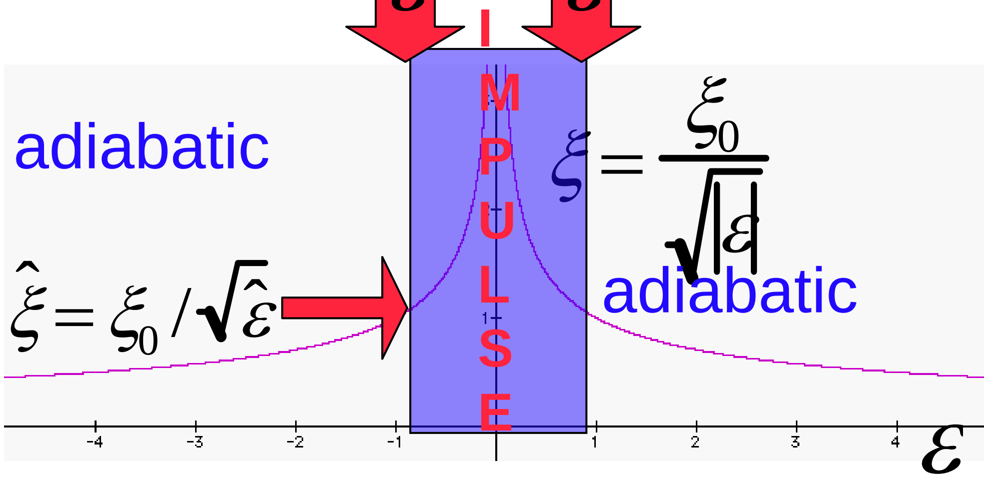
Or:

$$\hat{t} = \sqrt{\tau_0 \tau_Q} \;\&\; \hat{\varepsilon} = \sqrt{\tau_0 / \tau_Q}$$

The corresponding length follows:

$$\hat{\xi} = \xi_0 / \sqrt{\hat{\varepsilon}} = \xi_0 \sqrt[4]{\frac{\tau_Q}{\tau_0}}$$





Formation of kinks in a 1-D Landau-Ginzburg system

$V_{Ginzburg-Landau}(\varphi) = \varepsilon|\varphi|^2 + |\varphi|^4$ with **real** φ driven by white noise.

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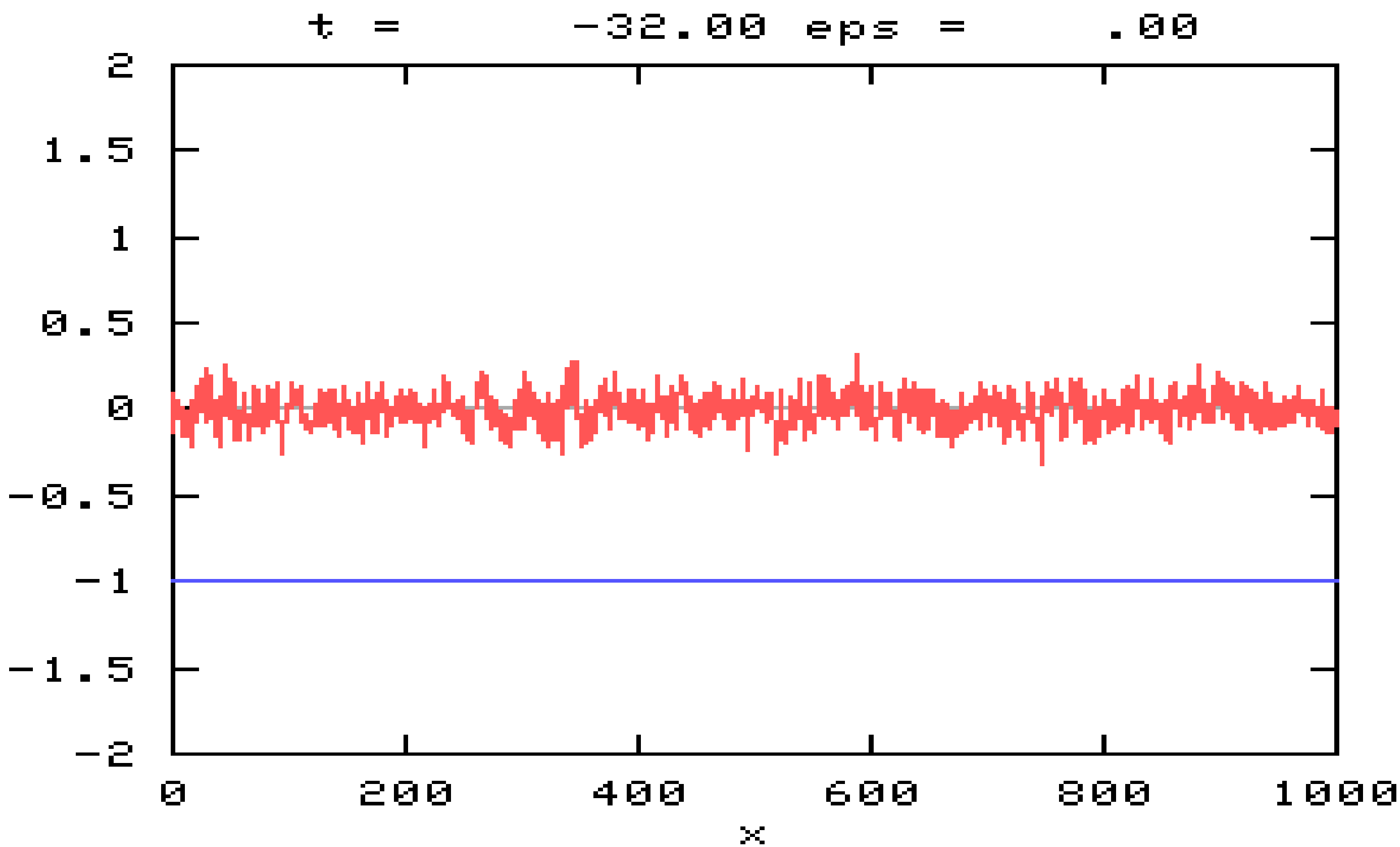
$$\Gamma\dot{\varphi} = -c^{-2}\nabla^2\varphi + \varepsilon\varphi + 2|\varphi|^2\varphi + noise$$



Kinks from a quench

$V_{Ginzburg-Landau}(\varphi) = \varepsilon|\varphi|^2 + |\varphi|^4$ with **real** φ driven by white noise.

Overdamped Gross-Pitaevskii evolution with $\varepsilon = t/\tau_Q$ and:

QuickTime™ and a
Photo - JPEG decompressor
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$$\Gamma\dot{\varphi} = -c^{-2}\nabla^2\varphi + \varepsilon\varphi + 2|\varphi|^2\varphi + noise$$

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$$\tau_0 = \Gamma$$

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QuickDraw decompressor
are needed to see this picture.

$$\hat{\varepsilon} = \sqrt{\Gamma/\tau_Q}$$

$\varphi \uparrow$

$x \rightarrow$

Defect separation:

$$d \approx \hat{\xi} = \xi_0 \sqrt[4]{\tau_Q/\Gamma}$$

Laguna & WHZ, PRL‘97

Kink density vs. quench rate

$$n = 1/(f\hat{\xi}) = (1/f\xi_0)\sqrt[4]{\Gamma/\tau_Q}$$

The observed density of kinks scales with the predicted slope, but with a density corresponding to:

f~ 10-15

Similar values of the factor f multiply $\hat{\xi}$ in 2-D and 3-D numerical experiments.

n

QuickTime™ and a
QuickDraw decompressor
are needed to see this picture.

τ_Q

Vortex line formation in 3-D

(Antunes, Bettencourt, & Zurek, PRL 1999)

QuickTime™ and a
Photo - JPEG decompressor
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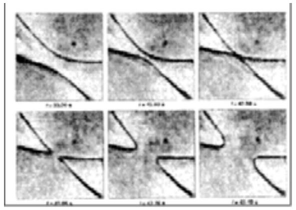
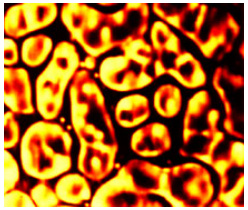
Liquid Crystals

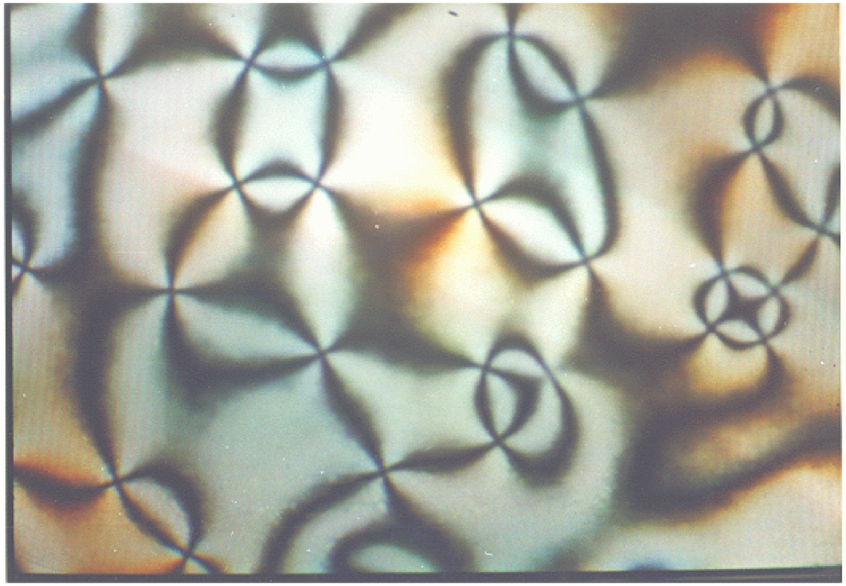
- Chuang et al. (1991): Defect dynamics



- Bowick et al. (1994): Defect formation



- Digal et al. (1999): Defect correlations
  $\nu = 0.26 \pm 0.11$

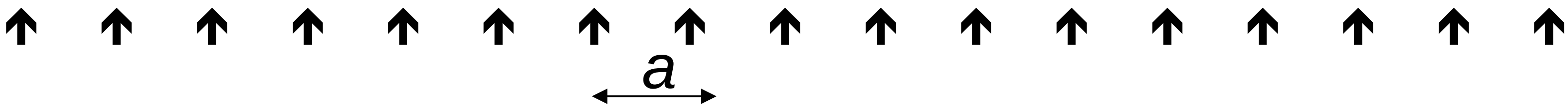


PARTIAL SUMMARY #1:

1. Topological defects as “petrified evidence” of the phase transition dynamics.
2. Universality classes: The mechanism is generally applicable.
3. Initial density of defects after a quench using KZ approach.
4. Numerical simulations.
5. Experiments.

QUANTUM ISING MODEL

Lattice of spin 1/2 particles interacting with an external force (e.g., magnetic field along the *x* axis) and with each other (ferromagnetic Ising interaction along the *z* axis):



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Quantum phase transition occurs as *J(t)* decreases. Then $|\rightarrow\rightarrow\rightarrow\ldots\rangle \propto (|\uparrow\rangle + |\downarrow\rangle)(|\uparrow\rangle + |\downarrow\rangle)(|\uparrow\rangle + |\downarrow\rangle)\cdots$, analogue of the "symmetric vaccum", is no longer favored energetically: $|\uparrow\uparrow\uparrow\ldots\rangle$ or $|\downarrow\downarrow\downarrow\ldots\rangle$ (or any superposition thereof) are the ground states.

"…one of two canonical models of for quantum phase transitions.

S. Sachdev, *Quantum Phase Transitions, CU*

CRITICAL REGION OF THE QUANTUM ISING MODEL

The character of the ground state changes when, in the model Hamiltonian;

$$H = -J(t)\sum_l \sigma_l^x - W\sum_l \sigma_l^z \sigma_{l+1}^z$$

the two couplings are equal, that is, when:

$$J(t)/W = 1.$$

In quantum phase transition the parameter ("relative coupling"):

$$\varepsilon = \frac{J(t)}{W} - 1$$

plays the role of the "relative temperature" $(T-T_c)/T_c$: To induce phase transition one can lower the field and, hence, *J(t).*

The **gap** and the critical behavior

The gap (between the ground state and the lowest excited state) plays an essential role. In quantum Ising model it is given by:

$$\Delta = 2\,|W - J(t)| = 2W\;|\varepsilon|$$

This is the energetic "price" of flipping a single spin above J_c or of a pair of kinks in a symmetry broken phase:

Note that the gap is easily related with the "relative coupling".

Relaxation time and healing length in the critical region can be expressed in terms of the gap.

Relaxation time and healing length

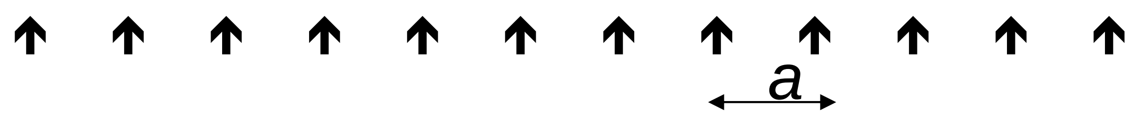
Relaxation time is simply the inverse of the gap:

$$\tau = \frac{\hbar}{\Delta} \propto 1/\varepsilon$$

"critical slowing down"

Once the characteristic velocity is calculated from the coupling *W* and the distance *a* between the spins on the lattice:

$$c = 2Wa/\hbar$$



Healing length is given by:

$$\xi = c \cdot \tau = 2Wa/\Delta \propto 1/\varepsilon$$

"critical opalescence"

This scaling is different than in the mean field case. Still, we have now all of the ingredients of the "K-Z mechanism"……

… and the corresponding "frozen out" healing length $\hat{\xi}$

….. $\tau(\varepsilon(\hat{t})) = \hat{t}$

Hence: $\tau_0 / (\hat{t} / \tau_Q) = \hat{t}$

Or:

$$\hat{t} = \sqrt{\tau_0 \tau_Q} \;\; \& \;\; \hat{\varepsilon} = \sqrt{\tau_0 / \tau_Q}$$

The corresponding length follows:

$$\hat{\xi} = \xi_0 / \sqrt{\hat{\varepsilon}} = \xi_0 \sqrt[4]{\frac{\tau_Q}{\tau_0}}$$

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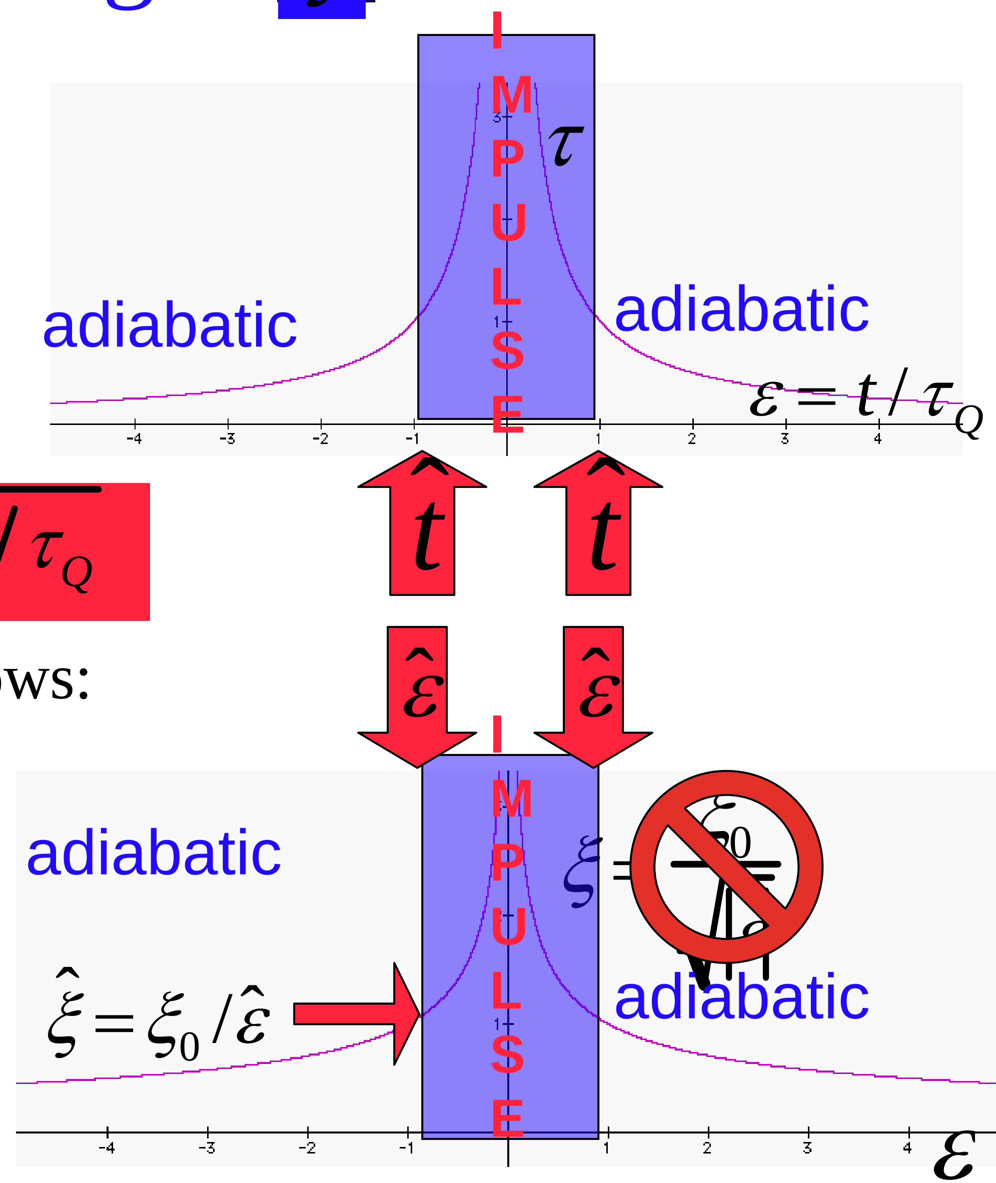
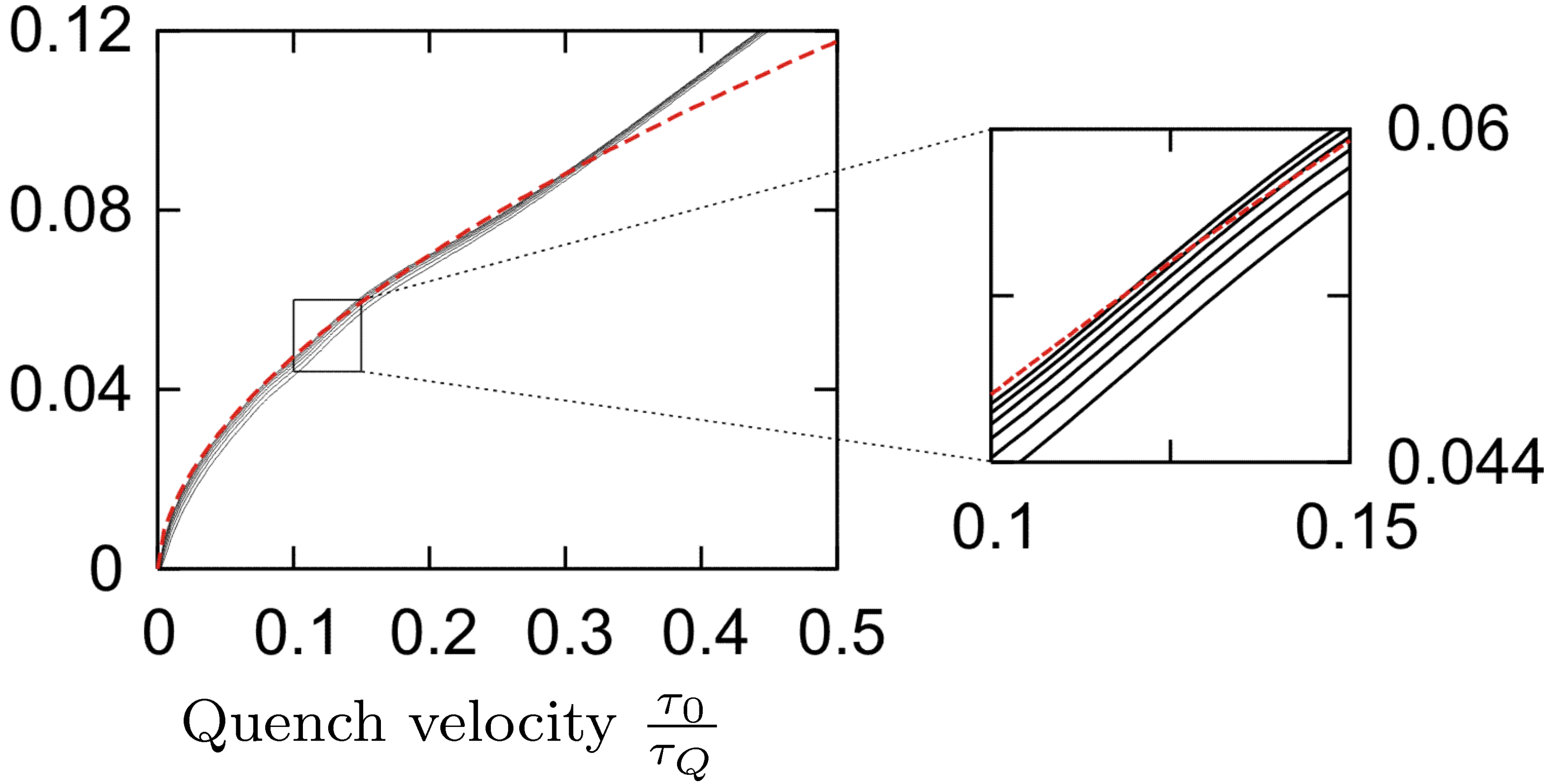
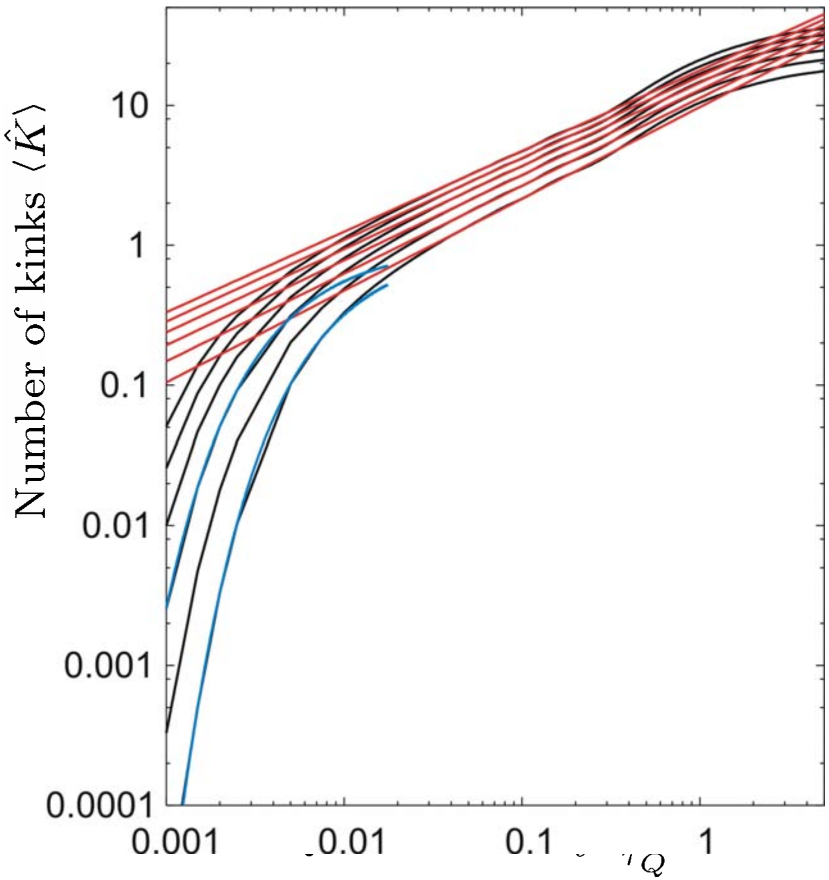


Density of kinks (# of kinks per spin in the Ising chain) as a function of quench rate



(Dorner, Zoller, & WHZ, quant-ph/503511)

Density of Kinks in the Quantum Ising Model



“Kink”-Operator:

(counts number of domain walls)

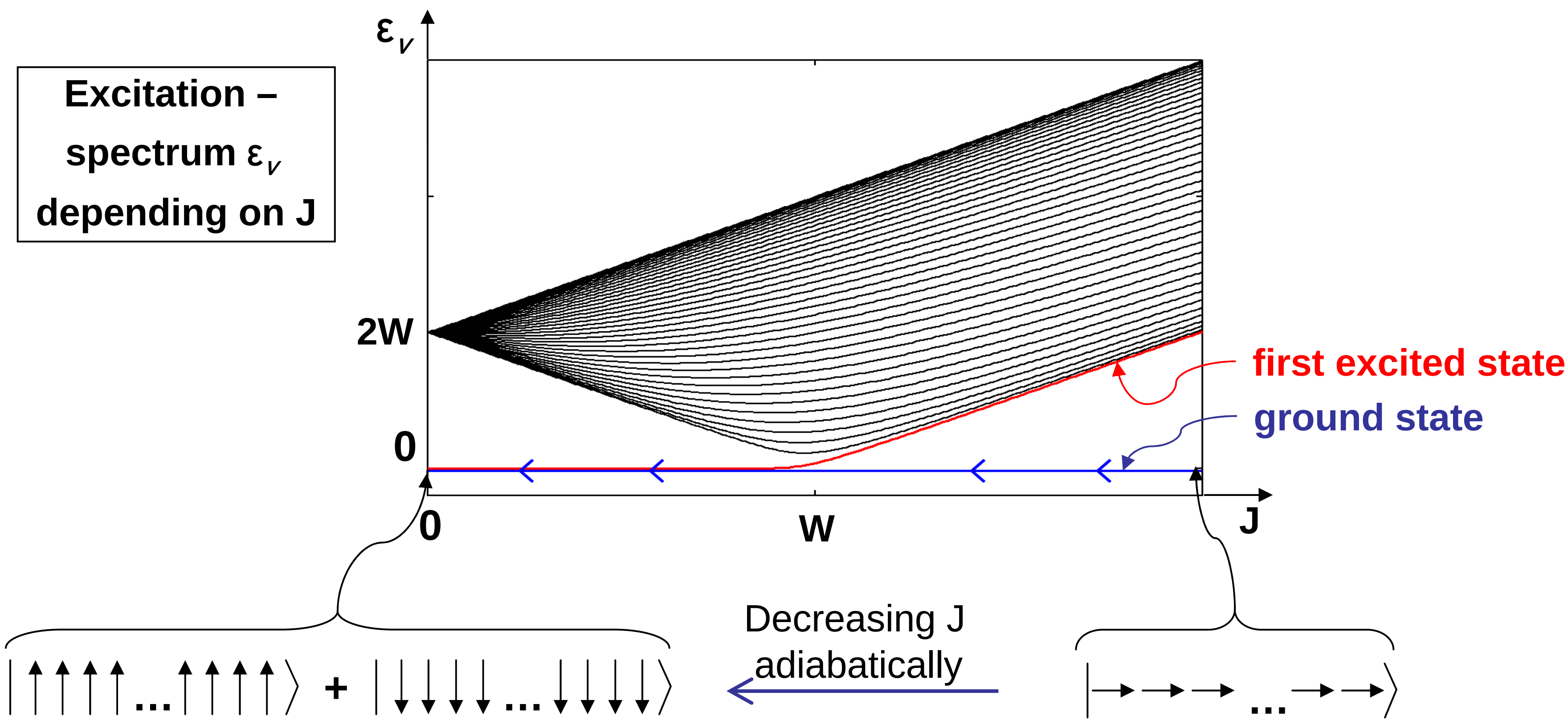
$$\hat{K} = \frac{1}{2}\left(N - 1 - \sum_{l=1}^{N} \sigma_l^z \sigma_{l+1}^z\right)$$

$$\text{e.g. } \hat{K}|\uparrow\uparrow\downarrow\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow\rangle = 2 \cdot |\uparrow\uparrow\downarrow\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow\rangle$$

Fit results: $\sim \left(\frac{1}{\tau_Q}\right)^{0.58..0.66}$

Creating entanglement

$$H = -W\sum_{l=1}^{N-1}\sigma_l^z\sigma_{l+1}^z - J\sum_{l=1}^{N}\sigma_l^x \rightarrow$$ **Fermionizing** $$\rightarrow \quad H = \sum_{\nu=1}^{N}\varepsilon_\nu\left(a_\nu^\dagger a_\nu - \frac{1}{2}\right)$$



W > 0 (ferromagnetic case)

QUANTUM (LANDAU-ZENER) APPROACH

In an **avoided level crossing**, the probability of transition that "preserves the character of the state but changes the energy level" when the external parameter is used to continuously vary the Hamiltonian is given by:

$$p \approx \exp\left(-\frac{\pi\Delta^2}{2\hbar\,|v\,|}\right)$$

Above:

$$\Delta = E_1 - E_2, \; v = \dot{\Delta} = d(E_1 - E_2)/dt$$

In the adiabatic limit ($v \approx 0$) Landau-Zener formula predicts that the system will remain in the same energy eigenstate. Transitions are induced when the change is sufficiently fast.

THE SIZE OF THE MINIMUM GAP (TO THE LOWEST ACCESSIBLE STATE ABOVE THE GROUND STATE) FOR *N* SPINS DESCRIBED BY ISING MODEL HAMILTONIAN:

$$H = -J(t)\sum_l \sigma_l^x - W\sum_l \sigma_l^z \sigma_{l+1}^z$$

IS:

$$\Delta_{\min} \approx \frac{3\pi W}{N}$$

THEREFORE, THE GROUND STATE IS PRESERVED WITH FIDELITY ***p*** WHEN THE QUENCH IS NO FASTER THAN:

$$\hbar\dot{\Delta} = \pi\Delta_{\min}^2 \big/ |\ln p|$$

BUT $\dot{\Delta} = 2\dot{J}(t) = 2\theta$. CONSEQUENTLY….:

….CONSEQUENTLY, THE CONDITION FOR THE RATE OF QUENCH SUFFICIENTLY SLOW FOR THE SPIN CHAIN TO LIKELY REMAIN IN THE GROUND STATE:

$$\hbar \cdot 2\theta \leq \frac{\pi}{f} \frac{(3\pi W)^2}{N^2}$$

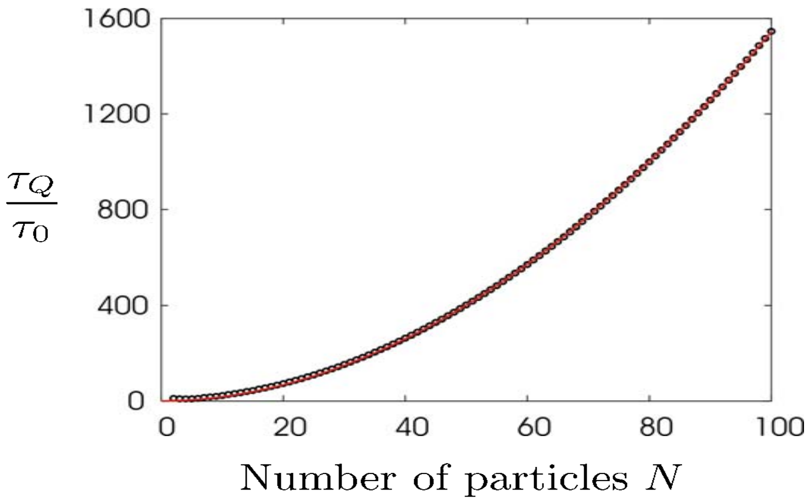
CAN BE TRANSLATED INTO A CONDITION FOR *N*, THE NUMBER OF SPINS IN A CHAIN THAT -- GIVEN FIXED QUENCH RATE -- WILL REMAIN IN THE GROUND STATE:

$$\hat{N} \leq 3\pi W \sqrt{\frac{\pi}{2\hbar\theta |\ln p|}}$$

FOR COMPARISON, DOMAIN SIZE OBTAINED BEFORE:

$$\hat{N}_{KZ} = W \sqrt{\frac{1}{2\hbar\theta}}$$

Dynamics: Landau-Zener



Landau - Zener prediction:

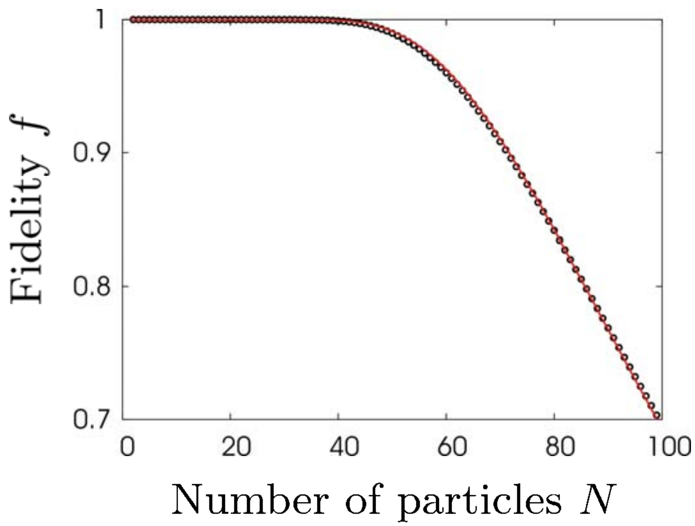
$$\tau_Q \sim N^2$$

Fit result for f = 99% :

$$\tau_Q \sim N^{1.94}$$

W = 10 MHz → τ_Q ~ 15 μs (N=40)

Dynamics: Landau-Zener



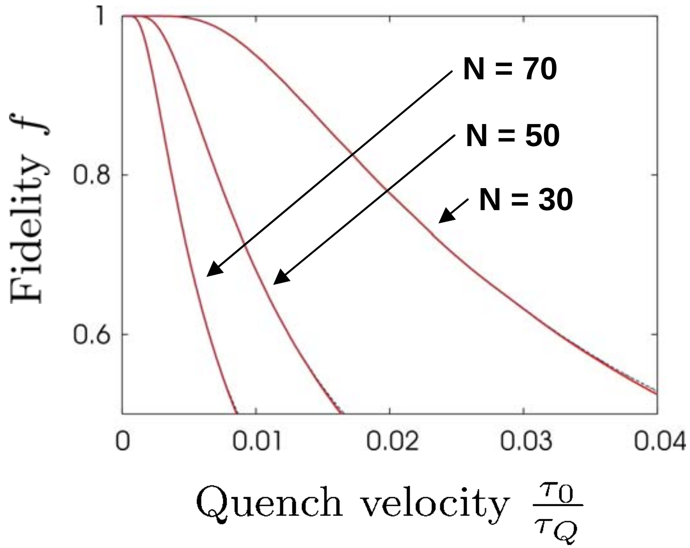
(f = probability of staying in ground state)

Landau - Zener prediction:

$$f = 1 - e^{-\alpha \frac{\tau_Q}{N^2}}$$

Fit result:

$$\alpha = 59\, W/\hbar$$



Fit results:

$$\alpha = 54,\, 57,\, 59\, W/\hbar$$

~ 10-16% deviation in the constant;

perfect fit to the form of dpependence

SUMMARY:

1. Phase transition in the quantum Ising model.
2. Initial density of defects after a quench in a “normal” second order phase transition.
3. Analogous estimates for the quantum Ising model.
4. Quantum calculation.