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THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

OSCILLATING FLOW OF A BURGERS' FLUID IN A PIPE

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Abstract

An analysis is made to see the influences of Hall current on the flow of a Burgers' fluid. The velocity field corresponding to flow in a pipe is determined. The closed form analytical solutions for several Newtonian and non-Newtonian fluid models can be obtained from the present analysis as the limiting cases.

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1 Introduction

The flow problem in a pipe has been drawing attention from various researchers, see for example [1-3]. Recently, in ref.[3] Yin and Zhu examined the oscillating flow in a pipe using fractional Maxwell model. The purpose of this work is twofold. Firstly, to investigate the oscillating flow in a pipe using Burgers' fluid model. Secondly, to see the effects of Hall current on the velocity field. The flow in a pipe is induced due to imposition of an oscillating pressure gradient. An exact analytical solution to the governing problem is given using the Fourier transform technique. The obtained expression for the velocity field shows that there are pronounced effects of Hall and rheological parameters. The considered fluid model is a viscoelastic model and has been used to characterize food products such as cheese [4], soil [5], asphalt and asphalt mixes [6,7] etc.

2 Description of the problem

The constitutive equation for a Burgers' fluid is

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda\frac{\delta\mathbf{S}}{\delta t} + \beta\frac{\delta^2\mathbf{S}}{\delta t^2} = \mu\left(1 + \lambda_r\frac{\delta}{\delta t}\right)\mathbf{A}_1, \tag{1}$$

in which $\mathbf{T}$ is the Cauchy stress tensor, p is the reaction stress due to constraint of incompressibility, $\mathbf{S}$ is the constitutively determined extra stress, $\mathbf{A}_1$ is the first Rivlin-Ericksen tensor, λ and β are the relaxation times, μ is the dynamic viscosity, $\lambda_r\ (< \lambda)$ is the retardation time and the upper convected derivative is

$$\frac{\delta\mathbf{S}}{\delta t} = \frac{d\mathbf{S}}{dt} - \mathbf{LS} - \mathbf{SL}^{\mathrm{T}}, \tag{2}$$

where $\mathbf{L}$ is the velocity gradient.

In the following we consider an axially symmetric and fully developed flow of a Burgers' fluid whose extra stress tensor and velocity field, in a system of cylindrical polar coordinates, are of the form

$$\mathbf{S}(r,t) = \begin{pmatrix} S_{rr} & S_{r\theta} & S_{rz} \\ S_{\theta r} & S_{\theta\theta} & S_{\theta z} \\ S_{zr} & S_{z\theta} & S_{zz} \end{pmatrix}, \qquad \mathbf{V}(r,t) = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}, \tag{3}$$

where the initial condition $\mathbf{S}(r,0) = \mathbf{0}$ i.e. the fluid being at rest up to the moment at $t = 0$ holds and the imposed oscillating pressure gradient is

$$\frac{\partial p}{\partial z} = P_0 e^{i\omega_0 t}, \tag{4}$$

where ω_0 is the oscillating frequency and P_0 is the amplitude. Moreover the $z-$axis acts as the axis of the cylinder and a uniform magnetic field $\mathbf{B}_0$ is applied along the axis of a circular cylinder. The magnetic Reynolds number is assumed to be very small, so that the induced magnetic field is negligible. There is no applied voltage so the electric field $\mathbf{E} = \mathbf{0}$. If the Hall term is retained in generalized Ohm's law, then the following expression holds [9]

$$\mathbf{J} \times \frac{\omega_e\tau_e}{B_0}(\mathbf{J} \times \mathbf{B}) = \sigma\left[\mathbf{E} + \mathbf{V} \times \mathbf{B} \times \frac{1}{en_e}\nabla p_e\right], \tag{5}$$

in which $\mathbf{B}$ is the total magnetic field, ω_e is the cyclotron frequency of electrons, τ_e is the electron collision time, σ is the electrical conductivity, e is the electron charge, n_e is the number density of electron and p_e is the electron pressure. The ion-slip and thermoelectric effects are not included in equation (5). Further it is assumed that $\omega_e\tau_e \sim o(1)$ and $\omega_i\tau_i < 1$ where ω_i and τ_i are cyclotron frequency and collision time for ions respectively.

The no-slip boundary condition for the problem under consideration is

$$u(a,t) = 0, \tag{6}$$

where a is the radius of the cylinder.

By virtue of equation (3), the continuity equation is automatically satisfied and equations (1), (2) and balance of linear momentum gives $S_{rr} = S_{r\theta} = S_{\theta\theta} = S_{\theta z} = \partial p/\partial r = \partial p/\partial\theta = 0$ and

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rS_{rz}) - \frac{\sigma B_0^2}{1-im}u, \tag{7}$$

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)S_{rz} = \mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial r}, \tag{8}$$

$$S_{zz} + \lambda\left(\frac{\partial S_{zz}}{\partial t} - 2S_{rz}\frac{\partial u}{\partial r}\right) + \beta\left[\frac{\partial}{\partial t}\left(\frac{\partial S_{zz}}{\partial t} - 2S_{rz}\frac{\partial u}{\partial r}\right) - 2\frac{\partial S_{rz}}{\partial t}\frac{\partial u}{\partial r}\right] = -2\mu\lambda_r\left(\frac{\partial u}{\partial r}\right)^2, \tag{9}$$

where $m = \omega_e\tau_e$ is the Hall parameter.

Elimination of S_{rz} from equations (7) and (8) yields

$$\begin{aligned}\rho\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\frac{\partial u}{\partial t} &= -\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\frac{\partial p}{\partial z} + \mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right]\\ &\quad -\frac{\sigma B_0^2}{1-im}\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)u.\end{aligned} \tag{10}$$

The above equations can be normalized using the following dimensionless parameters

$$\begin{aligned} u^* &= \frac{u}{U_0},\quad r^* = \frac{r}{a},\quad t^* = \frac{t}{(a^2/\nu)},\quad \omega_0^* = \frac{\omega_0}{(\nu/a^2)},\quad Q_0 = \frac{P_0}{(\mu/a^2)\,U_0},\\ \lambda^* &= \frac{\lambda}{(a^2/\nu)},\quad \lambda_r^* = \frac{\lambda_r}{(a^2/\nu)},\quad \beta^* = \frac{\beta}{(a^4/\nu^2)},\quad M^2 = \frac{\sigma B_0^2}{(\mu/a^2)},\end{aligned} \tag{11}$$

where U_0 is the reference velocity.

Accordingly, equations (6) and (10) after neglecting the dimensionless mark "$*$" for simplicity reduce to

$$\begin{aligned}\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\frac{\partial u}{\partial t} &= -Q_0\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)e^{i\omega_0 t} + \left(1+\lambda_r\frac{\partial}{\partial t}\right)\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right]\\ &\quad -\frac{M^2}{1-im}\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)u,\end{aligned} \tag{12}$$

$$u(1,t) = 0. \tag{13}$$

3 Closed form analytical solution

In order to solve the governing problem, we define the temporal Fourier transform pair as

$$\psi(r,\omega) = \int_{-\infty}^{\infty} u(r,t)\, e^{-i\omega t} dt, \tag{14}$$

$$u(r,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(r,\omega)\, e^{i\omega t} d\omega. \tag{15}$$

Taking Fourier transform to equations (12) and (13) and then solving the resulting problem, we have the following general solution

$$\psi(r,\omega) = \frac{Q_0\left(1-\beta\omega_0^2+i\lambda\omega_0\right)}{\xi^2\left(1+i\lambda_r\omega\right)}\left\{1-\frac{J_0(\xi r)}{J_0(\xi)}\right\}\delta(\omega-\omega_0), \tag{16}$$

where $J_0(\cdot)$ is the zeroth-order Bessel function, $\delta(\cdot)$ is the dirac delta function and

$$\xi^2 = -\left(\frac{M^2}{1-im}+i\omega\right)\left[\frac{1-\beta\omega^2+i\lambda\omega}{1+i\lambda_r\omega}\right].$$

The Fourier inversion of equation (16) after using the property of delta function gives

$$u(r,t) = \frac{Q_0\left(1-\beta\omega_0^2+i\lambda\omega_0\right)}{\xi_0^2\left(1+i\lambda_r\omega_0\right)}\left\{1-\frac{J_0(\xi_0 r)}{J_0(\xi_0)}\right\}e^{i\omega_0 t}, \tag{17}$$

where

$$\xi_0 = \xi|_{\omega=\omega_0}.$$

4 Results and discussion

In this section, we present the graphical illustration of the velocity profile for various types of fluids. We interpret these results and verify that they are consistent physically. Special attention has been given to examine the velocity profiles for five different kinds of fluids: a Newtonian fluid for which $\lambda=\lambda_r=\beta=0$, a Maxwell fluid for which $\lambda\neq0,\ \lambda_r=\beta=0$, an Oldroyd-B fluid for which $\lambda\neq0,\ \lambda_r\neq0,\ \beta=0$, a second grade fluid for which $\lambda=0,\ \lambda_r\neq0,\ \beta=0$ and a Burgers' fluid. The effects of various parameters especially the magnetic parameter M, the Hall parameter m and the rheological parameter β of the Burgers' fluid on the velocity profiles for all five types of fluids have been studied.

Figures 1 and 2 are prepared to see the effects of magnetic parameter M on the velocity profiles of different kinds of fluids with and without Hall parameter m, respectively. From figures $1a-1e$, it is noted that, in the presence of magnetic force, an increase of the magnitude in the magnetic parameter M reduces the velocity profiles monotonically due to the effect of the magnetic force against the flow direction. This is in accordance with the fact that the magnetic field is responsible to reduce the velocity.

However, some differences among the classical Newtonian, Maxwell, Oldroyd, second grade and Burgers' fluids can also be observed from figures $1a - 1e$. From these figures, it is observed that the Maxwell fluid has the maximum velocity profiles. Figures $1a$ and $1e$ reveal that the Burgers' fluid has much greater velocity profiles as compared with a Newtonian fluid. Moreover, the Burgers' fluid also has much greater velocity profiles than those for a second grade fluid. We also see that the velocity profiles for an Oldroyd-B fluid are smaller than those for a Burgers' fluid.

Figure 2 is depicted to see the effects of Hall parameter m on the velocity profiles. Figures 1 and 2 show that the increase of Hall parameter m for fixed magnetic parameter M increases the velocity profiles. Moreover, the velocity increases. Further, when the magnetic Reynolds number is very small, the flow pattern with Hall effect is remarkably similar to that for non-conducting flow. Of course, the assumption of very small magnetic Reynolds number will be valid for flow of liquid metals or slightly ionized gas.

Also it appears that the velocity is an increasing function of the rheological parameter β of the Burgers' fluid. But this result can not be generalized for other chosen values of the rheological parameter β since the behaviour of β is non-monotonous. Moreover, the velocity also increases with frequency.

5 Concluding remarks

In this note we have determined the velocity field that corresponds to the flow of a Burgers' fluid in a pipe like domain by means of the Fourier transform. The presented study of Burgers' fluid is of interest not only from an academic point of view, due to the existence of exact solution but also from an industrial point of view, since this kind of flow is frequently found in several applications. It is found from the solution that Burgers' fluid only contributes if there is pressure gradient of the oscillatory nature. For $\omega_0 = 0$, the Burgers' fluid behaves like a Newtonian fluid. It is further noted that for $\omega_0 = B_0 = 0$, there is a trivial solution. Letting $\beta = \lambda_r = \lambda = 0$ in (17), we find the corresponding solution for the Navier-Stokes fluid. The solution tends to the result of an Oldroyd-B fluid for $\beta = 0$ and to Maxwell fluid when $\beta = \lambda_r = 0$. The solution resembles to second grade fluid for $\beta = \lambda = 0$. Moreover, for $\beta = \lambda_r = 0$ and fractional parameters $= 1$, the solution (17) reduces to those obtained in reference [3].

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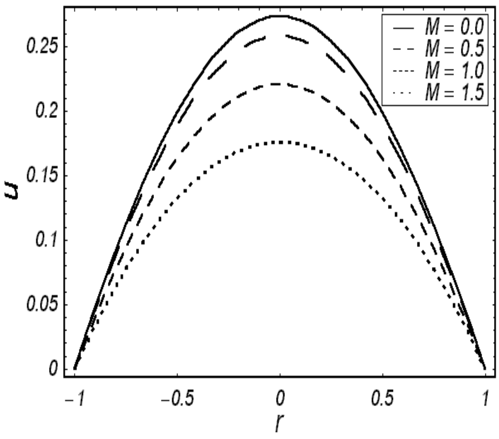
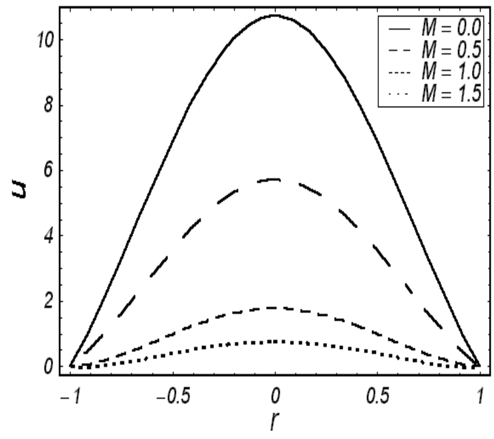
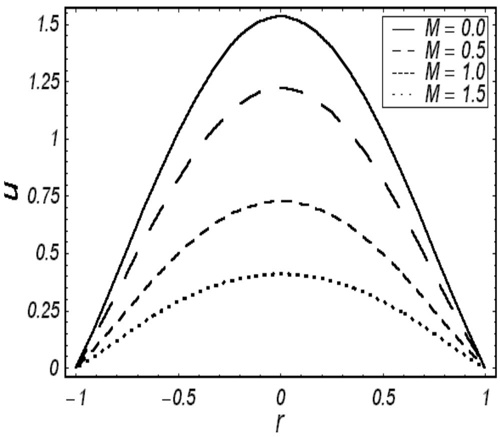


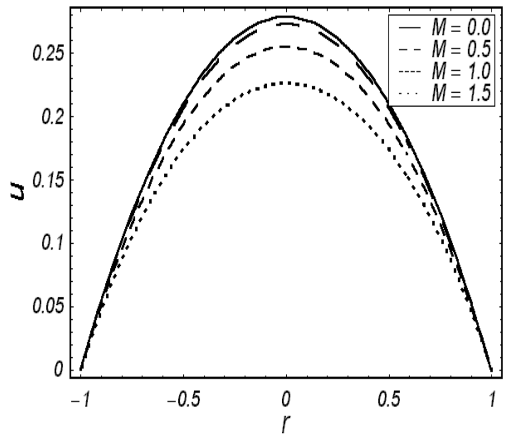
(a) Newtonian fluid ($\lambda = \lambda_r = \beta = 0$)



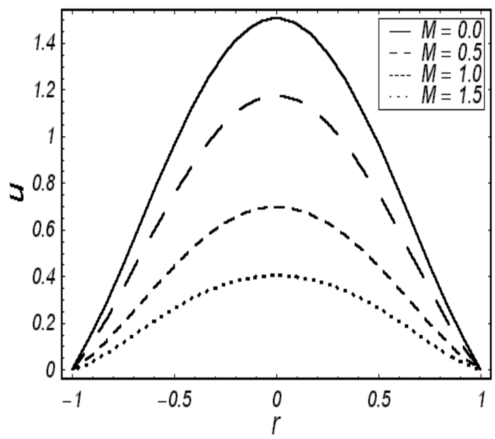
(b) Maxwell fluid ($\lambda = 5, \lambda_r = \beta = 0$)



(c) Oldroyd-B fluid ($\lambda = 5, \lambda_r = 1, \beta = 0$)

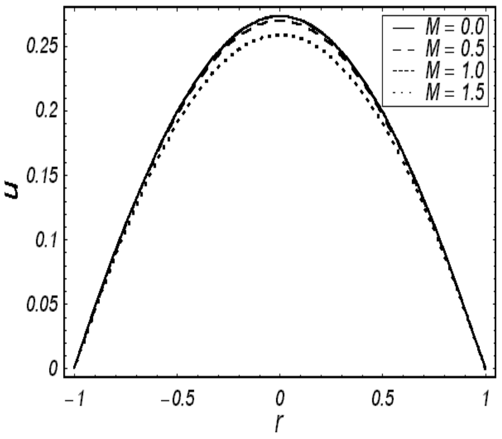


(d) Second grade fluid ($\lambda = 0, \lambda_r = 1, \beta = 0$)

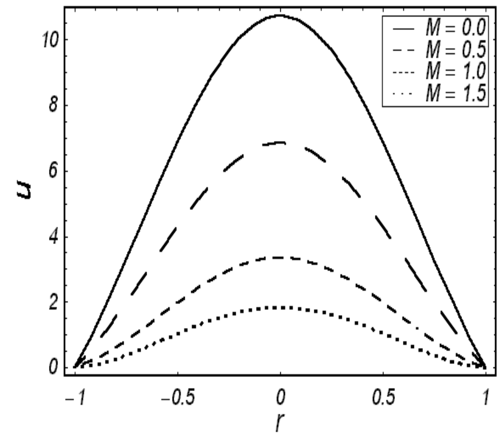


(e) Burgers' fluid ($\lambda = 5, \lambda_r = 1,\ \beta = 1$)

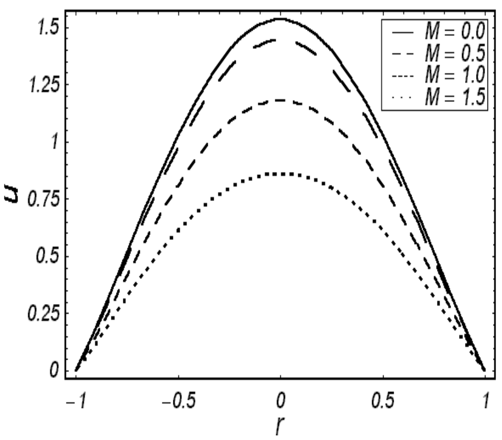
Figure 1: Profiles of dimensionless velocity $u\,(r, t)$ for various values of magnetic parameter M when $t = 1$, $\omega = 1.2$, $Q_0 = -2$ and $m = 0$ are fixed.



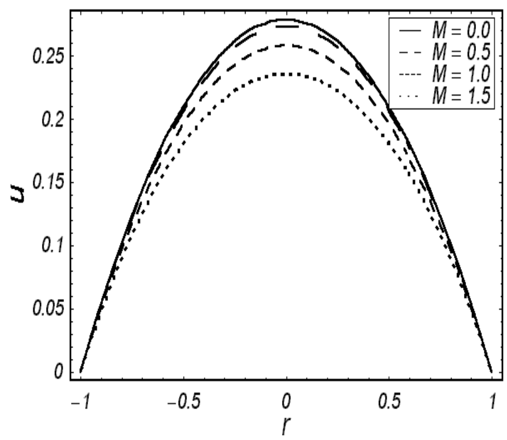
(a) Newtonian fluid ($\lambda = \lambda_r = \beta = 0$)



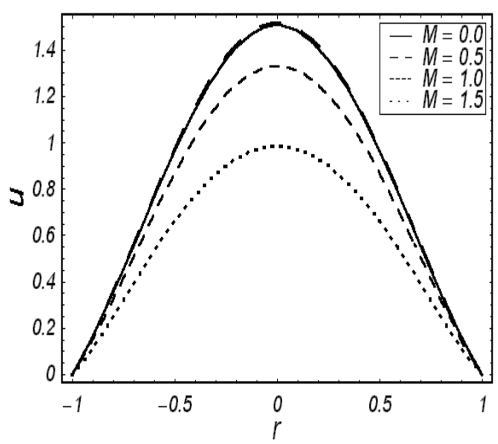
(b) Maxwell fluid ($\lambda = 5, \lambda_r = \beta = 0$)



(c) Oldroyd-B fluid ($\lambda = 5, \lambda_r = 1, \beta = 0$)



(d) Second grade fluid ($\lambda = 0, \lambda_r = 1, \beta = 0$)



(e) Burgers' fluid ($\lambda = 5, \lambda_r = 1, \beta = 1$)

Figure 2: Profiles of dimensionless velocity $u(r,t)$ for various values of magnetic parameter M when $t = 1$, $\omega = 1.2$, $Q_0 = -2$ and $m = 1$ are fixed.