Available at: http://www.ictp.it/~pub_off IC/2005/072

United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

# MATHEMATICAL MODELING AND EXACT SOLUTIONS TO ROTATING FLOWS OF A BURGERS' FLUID

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**Abstract**

The aim of this study is to provide the modeling and exact analytic solutions for hydromagnetic oscillatory rotating flows of an incompressible Burgers' fluid bounded by a plate. The governing time-dependent equation for the Burgers' fluid is different than those from the Navier-Stokes' equation. The entire system is assumed to rotate around an axis normal to the plate. The governing equations for this investigation are solved analytically for two physical problems. The solutions for the three cases, when the two times angular velocity is greater than the frequency of oscillation or it is smaller than the frequency or it is equal to the frequency (resonant case), are discussed in second problem. In Burgers' fluid, it is also found that hydromagnetic solution in the resonant case satisfies the boundary condition at infinity. Moreover, the obtained analytical results reduce to several previously published results as the special cases.

MIRAMARE – TRIESTE
December 2005

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# 1 Introduction

The study of non-Newtonian fluids gained much importance recently in view of its promising applications in engineering and industry. A large class of real fluids does not exhibit the linear relationship between stress and the rate of strain. Due to the non-linear dependence, the analysis of the behavior of fluid motion of the non-Newtonian fluids tends to be much more complicated and subtle in comparison with that of the Newtonian fluids. Flows of fluids with complex microstructure (e.g. molten polymers, polymer solutions, blood, paints, greases, oils, ketchup etc.) cannot be described by a single model of non-Newtonian fluids. Many models exist that are based either on "natural" modifications of established macroscopic theories or molecular considerations [1]. While the theoretical aspects of the Navier-Stokes equation are well advanced, the qualitative behavior of solutions to the equations of motion for viscoelastic fluids have been addressed by several workers [2-11]. The governing equations for viscoelastic fluids are in general of higher order and much more complicated than the Navier-Stokes equation [12-14]. The lack of boundary conditions as well as the non-linearity of the governing equations limit the solutions of the flows involving viscoelastic fluids. One of the viscoelastic fluid models which is most popular recently is called the Burgers's model [15-17]. This model is usually used for modeling asphalt concrete. This model is just an example of how a class of thermodynamically consistent models can be generated to describe the nonlinear behavior of materials such as asphalt concrete. The studies that assume asphalt concrete to be a viscoelastic material have also been carried out and these studies [18-28] use either a spring dashpot analogy in the form of a Burgers's model or some other form of viscoelastic constitutive equation [29-33]. Further, Burgers's model has been used to characterize food products such as cheese [34], soil [35], in the modeling of high temperature viscoelasticity of fine-grained polycrystalline olivine [36,37], in calculating the transient creep properties of the earth's mantle and specifically related to the post-glacial uplift [38-41]. More recently, Ravindran et al. [17] discussed the steady flow of a Burgers' fluid in an orthogonal rheometer.

The rotating flow of an electrically conducting fluid is encountered in cosmical and geophysical fluid dynamics. It is also important in the solar physics involved in the sunspot development, the solar cycle and the structure of rotating magnetic stars [42]. The effect of the Coriolis force due to the earth's rotation is found to be significant as compared to the inertial and viscous forces in the equations of motion. The Coriolis force exerts a strong influence on the hydromagnetic flow in the earth's liquid core which plays an important role in the mean geomagnetic field [43].

Up to now, no investigation on the rotating flow of a Burgers' fluid has been published therefore it is the subject of this investigation. In what follows, a mathematical modeling for an unsteady flow of a Burgers' fluid is given in a rotating frame of reference. Two problems are solved. The first problem describes the flow induced by the general periodic oscillations of a plate. The flow fields, due to certain special plate oscillations, are also derived as special cases of general periodic oscillation. In the second problem, the flow generated is due to the elliptic harmonic plate oscillations. Solving analytically the governing equations the author is able to obtain the velocity fields exactly. The exact analytic solutions for the three cases, is two times of the angular velocity greater than the frequency of oscillations or smaller than the frequency or is equal to the frequency, are given and discussed. If twice of the angular velocity is equal to the frequency of oscillations, the system resonates and in the case of a hydrodynamic fluid the solution does not satisfy the condition at infinity. But, in the hydromagnetic flow, the solution satisfies the condition at infinity which is unlike the hydrodynamic situation.

## 2 Problem formulation

Let us introduce the Cartesian coordinate system $(x, y, z)$ and consider the motion of a conducting incompressible Burgers' fluid bounded by a plate at $z = 0$. The fluid fills the space $z > 0$. The fluid and the plates are in a state of solid body rotation with constant angular velocity $\mathbf{\Omega} = \Omega\mathbf{k}$ ($\mathbf{k}$ is a unit vector in the $z-$direction). A uniform magnetic flux density $B_0$ fixed relative to the fluid is acting parallel to the $z-$axis. It is assumed that there is no applied voltage which implies the absence of an electrical field. The magnetic Reynolds number is assumed to be very small so that the induced magnetic field and the Hall effect are negligible [44]. In a rotating system, the governing equations can be written as follows:

$$div\mathbf{V} = 0, \tag{1}$$

$$\rho\left(\frac{d\mathbf{V}}{dt} + 2\mathbf{\Omega}\times\mathbf{V}\right) = -\nabla\widehat{p} + div\mathbf{S} - \sigma B_0^2\mathbf{V}, \tag{2}$$

where $\mathbf{V}$ denotes the velocity vector, $t$ the time, $\rho$ the fluid density, $\sigma$ the finite electrical conductivity of the fluid, $d/dt$ the material derivative and the modified pressure $\widehat{p}$ including the centrifugal term is given by

$$\widehat{p} = p - \frac{\rho\Omega^2 r^2}{2}, \tag{3}$$

in which $p$ is the pressure and $r^2 = x^2 + y^2$.

In a Burgers' fluid, the constitutive equation for the extra stress $\mathbf{S}$ is given by [16]

$$\left(1 + \lambda\frac{\delta}{\delta t} + \beta\frac{\delta^2}{\delta t^2}\right)\mathbf{S} = \mu\left(1 + \lambda_r\frac{\delta}{\delta t}\right)\mathbf{A}. \tag{4}$$

In the above equation $\mu$ denotes the dynamic viscosity, $\mathbf{A}$ the first Rivlin-Ericksen tensor, $\lambda$ and $\beta$ the relaxation times, $\lambda_r(< \lambda)$ the retardation time and the upper convective derivative is

$$\frac{\delta\mathbf{S}}{\delta t} = \frac{d\mathbf{S}}{dt} - (grad\mathbf{V})\,\mathbf{S} - \mathbf{S}\,(grad\mathbf{V})^*, \tag{5}$$

where $*$ indicates the matrix transpose.

It should be noted that the Burgers' model reduces to that of an Oldroyd-B fluid for $\beta = 0$. For $\beta = \lambda_r = 0$ and $\beta = \lambda = \lambda_r = 0$, we are left with the Maxwell and classical viscous fluid models, respectively. In some special flows, this model resembles that of second grade fluid model when $\beta = \lambda = 0$.

The extra stress tensor and the velocity field are assumed to be

$$\mathbf{S}(z,t) = \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix}, \quad \mathbf{V}(z,t) = (u, v, 0). \tag{6}$$

Upon making use of (6), the continuity equation is identically satisfied and (2), (4) and (5) give

$$\rho\left(\frac{\partial u}{\partial t} - 2\Omega v\right) = -\frac{\partial \widehat{p}}{\partial x} + \frac{\partial S_{xz}}{\partial z} - \sigma B_0^2 u, \tag{7}$$

$$\rho\left(\frac{\partial v}{\partial t} + 2\Omega u\right) = -\frac{\partial \widehat{p}}{\partial y} + \frac{\partial S_{yz}}{\partial z} - \sigma B_0^2 v, \tag{8}$$

$$0 = -\frac{\partial \widehat{p}}{\partial z} + \frac{\partial S_{zz}}{\partial z}, \tag{9}$$

$$\left(1 + \lambda\frac{\partial}{\partial t} + \beta\frac{\partial^2}{\partial t^2}\right) S_{xx} - 2\left[\left(\lambda + \beta\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial z} S_{xz} + \beta\frac{\partial u}{\partial z}\left(\frac{\partial S_{xz}}{\partial t} - \frac{\partial u}{\partial z} S_{zz}\right)\right] = -2\mu\lambda_r\left(\frac{\partial u}{\partial z}\right)^2, \tag{10}$$

$$\left(1 + \lambda\frac{\partial}{\partial t} + \beta\frac{\partial^2}{\partial t^2}\right) S_{xy} - \left[\begin{array}{c} \left(\lambda + \beta\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial z} S_{yz} + \frac{\partial v}{\partial z} S_{xz}\right) \\ +\beta\frac{\partial u}{\partial z}\left(\frac{\partial S_{yz}}{\partial t} - \frac{\partial v}{\partial z} S_{zz}\right) + \beta\frac{\partial v}{\partial z}\left(\frac{\partial S_{xz}}{\partial t} - \frac{\partial u}{\partial z} S_{zz}\right) \end{array}\right] = -2\mu\lambda_r\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial z}\right), \tag{11}$$

$$\left(1 + \lambda\frac{\partial}{\partial t} + \beta\frac{\partial^2}{\partial t^2}\right) S_{xz} - \left[\left(\lambda + \beta\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial z} S_{zz} + \beta\frac{\partial u}{\partial z}\frac{\partial S_{zz}}{\partial t}\right] = \mu\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial z}, \tag{12}$$

$$\left(1 + \lambda\frac{\partial}{\partial t} + \beta\frac{\partial^2}{\partial t^2}\right) S_{yy} - 2\left[\left(\lambda + \beta\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial z} S_{yz} + \beta\frac{\partial v}{\partial z}\left(\frac{\partial S_{yz}}{\partial t} - \frac{\partial v}{\partial z} S_{zz}\right)\right] = -2\mu\lambda_r\left(\frac{\partial v}{\partial z}\right)^2, \tag{13}$$

$$\left(1 + \lambda\frac{\partial}{\partial t} + \beta\frac{\partial^2}{\partial t^2}\right) S_{yz} - \left[\left(\lambda + \beta\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial z} S_{zz} + \beta\frac{\partial v}{\partial z}\frac{\partial S_{zz}}{\partial t}\right] = \mu\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial z}, \tag{14}$$

$$\left(1 + \lambda\frac{\partial}{\partial t} + \beta\frac{\partial^2}{\partial t^2}\right) S_{zz} = 0. \tag{15}$$

Having in mind that the fluid is at rest up to moment $t = 0$ we get $S_{zz} = 0$ and thus (9) indicates that $\widehat{p}$ is independent upon $z$. Moreover, (10) to (14) give

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)S_{xx}-2\left[\left(\lambda+\beta\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial z}S_{xz}+\beta\frac{\partial u}{\partial z}\frac{\partial S_{xz}}{\partial t}\right]=-2\mu\lambda_r\left(\frac{\partial u}{\partial z}\right)^2, \tag{16}$$

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)S_{xy}-\left[\begin{array}{c}\left(\lambda+\beta\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial z}S_{yz}+\frac{\partial v}{\partial z}S_{xz}\right)\\+\beta\frac{\partial u}{\partial z}\frac{\partial S_{yz}}{\partial t}+\beta\frac{\partial v}{\partial z}\frac{\partial S_{xz}}{\partial t}\end{array}\right]=-2\mu\lambda_r\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial z}\right), \tag{17}$$

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)S_{xz}=\mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial z}, \tag{18}$$

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)S_{yy}-2\left[\left(\lambda+\beta\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial z}S_{yz}+\beta\frac{\partial v}{\partial z}\frac{\partial S_{yz}}{\partial t}\right]=-2\mu\lambda_r\left(\frac{\partial v}{\partial z}\right)^2, \tag{19}$$

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)S_{yz}=\mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial z}. \tag{20}$$

Eliminating $\widehat{p}$ from (7) and (8) by differentiating with respect to $z$ and then substituting (18) and (20) in the resulting equations we obtain

$$\rho\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 F}{\partial z\partial t}+2i\Omega\frac{\partial F}{\partial z}\right)+\sigma B_0^2\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\frac{\partial F}{\partial z}=\mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial^3 F}{\partial z^3}, \tag{21}$$

where

$$F = u + iv. \tag{22}$$

## 3 First boundary value problem and its solution

We consider a plate at $z = 0$ which makes periodic oscillations $f(t)$ with period $T_0$. The fluid motion is set up from rest and for some time after the initiation of the motion, the flow field contains transients. The fluid velocity gradually becomes a harmonic function of $t$, with the same frequency as the velocity of the plate and only this state is considered here. Note that the fluid far away from the plate remains at rest. The governing differential equation is (21) and the corresponding boundary conditions are

$$F(0,t) = Uf(t), \qquad F(z,t) \to 0 \quad as \quad z \to \infty, \tag{23}$$

where $U$ is a constant velocity. By Fourier series, one can write

$$f(t) = \sum_{k=-\infty}^{\infty} e^{iknt} \tag{24}$$

in which the Fourier coefficients

$$a_k = \frac{1}{T_0}\int_{T_0} f(t)e^{-iknt}dt$$

with non-zero frequency $n = 2\pi/T_0$.

For the solution, the author employed a similar procedure (Fourier transform technique) as used in references [45,46]. To avoid the details of calculations, the flow field is given by

$$F(z,t) = U\sum_{k=-\infty}^{\infty} a_k e^{-\zeta_k z + i(knt - \eta_k z)}, \tag{25}$$

where $\zeta_k$ and $\eta_k$ are real and positive and are given by

$$\zeta_k = \sqrt{\frac{a_{1k} + \sqrt{a_{1k}^2 + a_{2k}^2}}{2}}, \tag{26}$$

$$\eta_k = \sqrt{\frac{-a_{1k} + \sqrt{a_{1k}^2 + a_{2k}^2}}{2}}, \tag{27}$$

$$a_{1k} = \frac{\sigma B_0^2\left(1 - \beta k^2 n^2 + \lambda\lambda_r k^2 n^2\right) - kn\rho\,(kn + 2\Omega)\left(\lambda - \lambda_r + \beta\lambda_r k^2 n^2\right)}{\mu\,(1 + \lambda_r^2 k^2 n^2)}, \tag{28}$$

$$a_{2k} = \frac{kn\sigma B_0^2\left(\lambda - \lambda_r + \beta\lambda_r k^2 n^2\right) + \rho\,(kn + 2\Omega)\left(1 - \beta k^2 n^2 + \lambda\lambda_r k^2 n^2\right)}{\mu\,(1 + \lambda_r^2 k^2 n^2)}. \tag{29}$$

It is worth emphasizing to note that (25) gives the flow field induced by general periodic oscillations of a plate. As the special cases of plate oscillations, one can directly write from it the corresponding flow fields. For example, the flow fields due to the oscillations $\exp(int)$, $\cos nt$, $\sin nt$, $\left[\begin{array}{ll} 1, & |t| < T_1/2 \\ 0, & T_1 < |t| < T_0/2 \end{array}\right]$ and $\sum_{k=-\infty}^{\infty}\delta\,(t - kT_0)$ are respectively given by

$$F_1(z,t) = Ue^{-\zeta_1 z + i(nt - \eta_1 z)}, \tag{30}$$

$$F_2(z,t) = \frac{U}{2}\left[e^{-\zeta_1 z + i(nt - \eta_1 z)} + e^{-\zeta_{-1} z - i(nt + \eta_{-1} z)}\right], \tag{31}$$

$$F_3(z,t) = \frac{U}{2i}\left[e^{-\zeta_1 z + i(nt - \eta_1 z)} - e^{-\zeta_{-1} z - i(nt + \eta_{-1} z)}\right], \tag{32}$$

$$F_4(z,t) = U\sum_{k=-\infty}^{\infty}\frac{\sin knT_1}{k\pi}e^{-\zeta_k z + i(knt - \eta_k z)}, \qquad k \neq 0, \tag{33}$$

$$F_5(z,t) = \frac{U}{T_0}\sum_{k=-\infty}^{\infty} e^{-\zeta_k z + i(knt - \eta_k z)}. \tag{34}$$

## 4 Second boundary value problem and its solution

This problem deals with the hydromagnetic flow generated by the elliptic harmonic oscillations of the plate. The governing problem consists of (21) and the following boundary conditions

$$F(0,t)=U\left(ae^{int}+be^{-int}\right),\qquad F(z,t)\to 0\quad as\quad z\to\infty, \tag{35}$$

in which the constants $a$ and $b$ are complex and it is assumed that the solutions are bounded at infinity.

In order to solve (21) subject to the boundary conditions (35), the author looks for a solution of the form

$$F=aF_1(z)e^{int}+bF_2(z)e^{-int},\quad n>0. \tag{36}$$

Substituting (36) into (21) and boundary conditions (35), the two boundary value problems involving third order ordinary differential equations for $F_1$ and $F_2$ emerge. For $n<2\Omega$, the solutions of these problems give

$$F=U\left(ae^{-m_1z+int}+be^{-m_2z-int}\right), \tag{37}$$

in which

$$m_jz=\Psi_j\left(\alpha_j+i\beta_j\right),\ j=1,2,$$

$$\alpha_j=\frac{1}{\sqrt{2}}\left(S_j+\sqrt{S_j^2+1}\right)^{1/2},$$

$$\beta_j=\frac{1}{\sqrt{2}}\left(-S_j+\sqrt{S_j^2+1}\right)^{1/2},$$

$$\Psi_1=\sqrt{\frac{B_1-n\lambda_rA_1}{\mu\left(1+n^2\lambda_r^2\right)}}z,\qquad \Psi_2=\sqrt{\frac{B_2+n\lambda_rA_2}{\mu\left(1+n^2\lambda_r^2\right)}}z,$$

$$S_1=\frac{A_1+n\lambda_rB_1}{B_1-n\lambda_rA_1},\qquad S_2=\frac{A_2-n\lambda_rB_2}{B_2+n\lambda_rA_2},$$

$$A_1=\sigma B_0^2\left(1-\beta n^2\right)-n\lambda\rho\left(2\Omega+n\right),$$

$$B_1=n\lambda\sigma B_0^2+\rho\left(2\Omega+n\right)\left(1-\beta n^2\right),$$

$$A_2=\sigma B_0^2\left(1-\beta n^2\right)+n\lambda\rho\left(2\Omega-n\right),$$

$$B_2=-n\lambda\sigma B_0^2+\rho\left(2\Omega-n\right)\left(1-\beta n^2\right).$$

Making use of (22) into (37) and then separating the real and imaginary parts one obtains

$$u = U\left[\begin{array}{c} e^{-\alpha_1\Psi_1}\left\{a_1\cos\left(\beta_1\Psi_1 - nt\right) + a_2\sin\left(\beta_1\Psi_1 - nt\right)\right\} \\ +e^{-\alpha_2\Psi_2}\left\{b_1\cos\left(\beta_2\Psi_2 + nt\right) + b_2\sin\left(\beta_2\Psi_2 + nt\right)\right\}\end{array}\right], \tag{38}$$

$$v = U\left[\begin{array}{c} e^{-\alpha_1\Psi_1}\left\{a_2\cos\left(\beta_1\Psi_1 - nt\right) - a_1\sin\left(\beta_1\Psi_1 - nt\right)\right\} \\ +e^{-\alpha_2\Psi_2}\left\{b_2\cos\left(\beta_2\Psi_2 + nt\right) - b_1\sin\left(\beta_2\Psi_2 + nt\right)\right\}\end{array}\right], \tag{39}$$

where $a = a_1 + ia_2$ and $b = b_1 + ib_2$.

For $n > 2\Omega$, the expressions for $u$ and $v$ are

$$u = U\left[\begin{array}{c} e^{-\alpha_1\Psi_1}\left\{a_1\cos\left(\beta_1\Psi_1 - nt\right) + a_2\sin\left(\beta_1\Psi_1 - nt\right)\right\} \\ +e^{-\alpha_3\Psi_3}\left\{b_1\cos\left(\beta_3\Psi_3 + nt\right) + b_2\sin\left(\beta_3\Psi_3 + nt\right)\right\}\end{array}\right], \tag{40}$$

$$v = U\left[\begin{array}{c} e^{-\alpha_1\Psi_1}\left\{a_2\cos\left(\beta_1\Psi_1 - nt\right) - a_1\sin\left(\beta_1\Psi_1 - nt\right)\right\} \\ +e^{-\alpha_3\Psi_3}\left\{b_2\cos\left(\beta_3\Psi_3 + nt\right) - b_1\sin\left(\beta_3\Psi_3 + nt\right)\right\}\end{array}\right]. \tag{41}$$

In above equations

$$\alpha_3 = \frac{1}{\sqrt{2}}\left(S_3 + \sqrt{S_3^2 + 1}\right)^{1/2},$$

$$\beta_3 = \frac{1}{\sqrt{2}}\left(-S_3 + \sqrt{S_3^2 + 1}\right)^{1/2},$$

$$\Psi_3 = \sqrt{\frac{n\lambda_r A_3 - B_3}{\mu\left(1 + n^2\lambda_r^2\right)}}z, \qquad S_3 = \frac{A_3 + n\lambda_r B_3}{n\lambda_r A_3 - B_3},$$

$$A_3 = \sigma B_0^2\left(1 - \beta n^2\right) - n\lambda\rho\left(n - 2\Omega\right),$$

$$B_3 = n\lambda\sigma B_0^2 + \rho\left(n - 2\Omega\right)\left(1 - \beta n^2\right).$$

For resonant case $(n = 2\Omega)$, the solution is

$$u = U\left[\begin{array}{c} e^{-\alpha_1\Psi_1}\left\{a_1\cos\left(\beta_1\Psi_1 - nt\right) + a_2\sin\left(\beta_1\Psi_1 - nt\right)\right\} \\ +e^{-\alpha_0\Psi_0}\left\{b_1\cos\left(\beta_0\Psi_0 + nt\right) + b_2\sin\left(\beta_0\Psi_0 + nt\right)\right\}\end{array}\right], \tag{42}$$

$$v = U\left[\begin{array}{c} e^{-\alpha_1\Psi_1}\left\{a_2\cos\left(\beta_1\Psi_1 - nt\right) - a_1\sin\left(\beta_1\Psi_1 - nt\right)\right\} \\ +e^{-\alpha_0\Psi_0}\left\{b_2\cos\left(\beta_0\Psi_0 + nt\right) - b_1\sin\left(\beta_0\Psi_0 + nt\right)\right\}\end{array}\right], \tag{43}$$

in which

$$\alpha_0 = \frac{1}{\sqrt{2}}\left(S_0 + \sqrt{S_0^2 + 1}\right)^{1/2},$$

$$\beta_0 = \frac{1}{\sqrt{2}}\left(-S_0 + \sqrt{S_0^2 + 1}\right)^{1/2},$$

$$\Psi_0 = \sqrt{\frac{n\sigma B_0^2\left[n\lambda_r\left(1 - \beta n^2\right) - n\lambda\right]}{\mu\left(1 + n^2\lambda_r^2\right)}}z,$$

$$S_0 = \frac{\left(1-\beta n^2\right) + n^2\lambda\lambda_r}{n\lambda_r\left(1-\beta n^2\right) - n\lambda}.$$

Note that for $B_0 = 0$, the second term on the right-hand side of (42) and (43) does not vanish and hence boundary condition at infinity is not satisfied. This means that the velocity components $u$ and $v$ do not vanish for the resonant case.

The expressions (38) to (43) describe physically meaningful hydromagnetic boundary layer flows. The expressions (38) to (41) hold for the non-resonant frequencies ($n \neq 2\Omega$) whereas (42) and (43) hold for the resonant frequency ($n = 2\Omega$). Interestingly, it can be seen that the effects of material parameters of the Burgers' fluid, Coriolis and electromagnetic forces are reflected in the flow fields. Moreover, it is found that the obtained solutions (38) to (43) consist of hydrodynamic and hydromagnetic boundary layers with a double structure, and hence these layers may be treated as the Stokes-Ekman-Hartmann boundary layers. Actually, the imposed elliptical harmonic oscillations of the plate and fluid outside the boundary layers are responsible for the existence of the double structure. The thickness of the boundary layers in each case are as follows:

- For $n < 2\Omega$, the layer thicknesses are $1/\alpha_1\Phi_1$ and $1/\alpha_2\Phi_2$, where

$$\Phi_1 = \sqrt{\frac{B_1 - n\lambda_r A_1}{\mu\left(1+n^2\lambda_r^2\right)}}, \qquad \Phi_2 = \sqrt{\frac{B_2 + n\lambda_r A_2}{\mu\left(1+n^2\lambda_r^2\right)}}.$$

- For $n > 2\Omega$, the layer thicknesses are $1/\alpha_1\Phi_1$ and $1/\alpha_3\Phi_3$, in which

$$\Phi_3 = \sqrt{\frac{n\lambda_r A_3 - B_3}{\mu\left(1+n^2\lambda_r^2\right)}}.$$

- For $n = 2\Omega$, the layer thicknesses are $1/\alpha_1\Phi_1$ and $1/\alpha_0\Phi_0$, where

$$\Phi_0 = \sqrt{\frac{n\sigma B_0^2\left[n\lambda_r\left(1-\beta n^2\right) - n\lambda\right]}{\mu\left(1+n^2\lambda_r^2\right)}}.$$

From these expressions we note that the layer thicknesses are highly dependent upon the nature of the fluid. Close examination of this analysis reveals that the layer thicknesses decrease with the increase in rotation and magnetic field. We have seen here that hydromagnetic diffusive waves are generated in the rotating system.

The presented analysis is more general and the results of several other fluid models (i.e. an Oldroyd-B, Maxwell, second grade and Newtonian) can be taken as the special cases of it. Moreover, the results for hydrodynamic fluid and non-rotating frame can be recovered by putting $B_0 = 0$ and $\Omega = 0$, respectively. This work also paves the way for further investigation of flows of a Burgers' fluid involving the Coriolis and electromagnetic forces. The exact solutions are further very important and rare in the theory of viscoelastic fluids. They provide a standard for checking the accuracies of many approximate methods which may be numerical or empirical. Although computer techniques make the complete integration of the equation of motion of viscoelastic fluids feasible, the accuracy of the results can be established by comparison with an exact solution.

**Acknowledgments.** This work was done within the framework of the Associateship Scheme of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

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