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TRANSIENT FLOWS OF A BURGERS' FLUID

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Abstract

An analysis is performed to develop the analytical solutions for some unsteady magneto-hydrodynamic (MHD) flows of a Burgers' fluid between two plates. A uniform magnetic field is applied transversely to the fluid motion. The exact solutions are given for three problems. Results for the velocity fields are discussed and compared with the flows of Oldroyd-B, Maxwell, second grade and Newtonian fluids.

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1 Introduction

An abundance of literature deals with the flows of Newtonian fluids using Navier-Stokes equations. But there are many fluids with complex microstructure that are not well described by Navier-Stokes equations. These fluids are named as non-Newtonian fluids. The importance of the non-Newtonian fluids is well known because of their industrial, biological and technological applications. Interest in these flows has been motivated by the diverse fluids in nature such as polymer solutions, soaps, ketchup, blood, paints, certain oils, greases, pastes and food stuff, to mention just a few fluids. In addition, the motion of non-Newtonian fluids in the absence, as well as in the presence, of a magnetic field has applications in many areas, including the handling of biological fluids and the flow of nuclear fuel slurries, liquid metals and alloys, plasma, mercury amalgams and blood. Undoubtedly, the equations which govern the flow of non-Newtonian fluids are of higher order [1, 2] and much more complicated than the Navier-Stokes equations. Due to diversity of fluids in nature, several constitutive equations have been proposed in the literature.

Amongst the many non-Newtonian fluid models, the viscoelastic fluids have acquired the special status. More recently, Ravindran *et. al.* [3] examined the steady flow of a Burgers' fluid in an orthogonal rheometer. It has been proved that the Burgers' model is a viscoelastic fluid model [4, 5 and several references therein]. This model is just an example of how a class of thermodynamically consistent models can be generated to describe the nonlinear behavior of materials such as asphalt concrete. This model has been used to characterize food products such as cheese [6], soil [7], asphalt and asphalt mixes [8, 9] etc. Some recent studies regarding viscoelastic fluids are given in the references [10 − 15] .

In view of the above motivation, the aim of the present paper is to investigate some unsteady MHD flows of a Burgers' fluid. For this, we developed an equation which governs the unsteady flows and then solved the three problems. An analytical solution for each problem is presented using the Laplace transform technique. The complex variable theory is used for the evaluation of the integrals arising in the inverse Laplace transform. It is seen that the expressions for velocity fields are strongly dependent upon the rheological parameters of the Burgers' fluid. Finally, the graphical results are plotted in order to see the variations of parameters of interest and analyzed in great detail.

2 Constitutive equations

The mathematical modeling of flows is described by the theory of continuum mechanics. The governing equations consist of conservative equations and constitutive equations. The conservative equations are derived from the principle of conservation of mass and the principle of balance of linear momentum.

The balance equations for an incompressible MHD fluid are

$$div\mathbf{V} = 0, \tag{1}$$

$$\rho\frac{d\mathbf{V}}{dt} = div\mathbf{T} + \mathbf{J}\times\mathbf{B}, \tag{2}$$

where ρ is the mass density, $\mathbf{V}$ the velocity, $\mathbf{T}$ the Cauchy stress tensor, $\mathbf{J}$ the current density, $\mathbf{B}$ the total magnetic field so that $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, $\mathbf{b}$ is the induced magnetic field and d/dt denotes the material time derivative. It is assumed that there is no applied electric field and the magnetic Reynolds' number is small so that the induced magnetic field is neglected.

The incompressible Burgers' fluid is characterized by the constitutive equations [4, 5]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{3}$$

$$\mathbf{S} + \lambda\frac{\delta\mathbf{S}}{\delta t} + \beta\frac{\delta^2\mathbf{S}}{\delta t^2} = \mu\left(1 + \lambda_r\frac{\delta}{\delta t}\right)\mathbf{A}_1, \tag{4}$$

where p is the reaction stress due to constraint of incompressibility, $\mathbf{S}$ is the constitutively determined extra stress, $\mathbf{A}_1$ is the first Rivlin-Ericksen tensor, λ and β are the relaxation times, μ is the dynamic viscosity, λ_r $(< \lambda)$ is the retardation time and

$$\frac{\delta\mathbf{S}}{\delta t} = \frac{d\mathbf{S}}{dt} - \mathbf{LS} - \mathbf{SL}^{\mathrm{T}}, \tag{5}$$

is the upper convected time derivative and $\mathbf{L}$ is the velocity gradient.

For the motion under consideration, we assume the velocity field and the extra stress of the form

$$\mathbf{V}(y,t) = \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{S}(y,t) = \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix}. \tag{6}$$

Upon making use of equation (6), the conservation of mass is identically satisfied and equations (2), (3) and (6) give the following scalar equations.

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial S_{xy}}{\partial y} - \sigma B_0^2 u, \tag{7}$$

$$0 = -\frac{\partial p}{\partial y} + \frac{\partial S_{yy}}{\partial y}, \tag{8}$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\partial S_{zz}}{\partial y}. \tag{9}$$

Using equations (5) and (6) into (4) and $\mathbf{S}(y,0) = \mathbf{0}$ we have $S_{xz} = S_{yy} = S_{yz} = S_{zz} = 0$ and

$$S_{xx} + \lambda\left(\frac{\partial S_{xx}}{\partial t} - 2S_{xy}\frac{\partial u}{\partial y}\right) + \beta\left[\frac{\partial}{\partial t}\left(\frac{\partial S_{xx}}{\partial t} - 2S_{xy}\frac{\partial u}{\partial y}\right) - 2\frac{\partial S_{xy}}{\partial t}\frac{\partial u}{\partial y}\right] = -2\mu\lambda_r\left(\frac{\partial u}{\partial y}\right)^2, \tag{10}$$

$$\left(1 + \lambda\frac{\partial}{\partial t} + \beta\frac{\partial^2}{\partial t^2}\right)S_{xy} = \mu\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial y}, \tag{11}$$

and equations (8) and (9) thus show that p is independent of y and z and hence p depends upon x and t.

In view of equations (7) and (11) we can write

$$\begin{aligned}\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\frac{\partial u}{\partial t} &= -\frac{1}{\rho}\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\frac{\partial p}{\partial x}+\nu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial y^2}\\ &\quad -\frac{\sigma B_0^2}{\rho}\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)u, \qquad (12)\end{aligned}$$

where $\nu=\mu/\rho$ is the kinematic viscosity of the fluid.

3 Flow due to a jerked plate

We begin with the case of a jerked plate of an incompressible electrically conducting Burgers' fluid between two infinite parallel plates separated by a distance h. Initially, both the fluid and the plates are at rest. At $t=0^+$, the fluid is disturbed by the sudden motion of the bottom plate at $y=0$, in a direction parallel to the $x-$axis, to a constant velocity $U\,(\neq 0)$; i.e. the velocity of the bottom plate is given by $(UH(t),0,0)$, where $H(\cdot)$ denotes the Heaviside unit step function. The upper plate at $y=h$ is kept stationary. In the absence of a pressure gradient, the mathematical model of the flow consists of the following initial-boundary value problem:

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\frac{\partial u}{\partial t}=\nu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0^2}{\rho}\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)u, \qquad (13)$$

$$\begin{aligned}u(0,t) &= UH(t), && t>0,\\ u(h,t) &= 0, && t>0, \qquad (14)\\ \frac{\partial^2 u(y,0)}{\partial t^2} &= \frac{\partial u(y,0)}{\partial t}=u(y,0)=0, && 0\le y\le h.\end{aligned}$$

With the dimensionless quantities

$$u'=\frac{u}{U},\quad y'=\frac{y}{h},\quad t'=\frac{\nu t}{h^2}, \qquad (15)$$

the governing initial-boundary value problem can be rewritten in the dimensionless form

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\frac{\partial u}{\partial t}=\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial y^2}-M^2\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)u, \qquad (16)$$

$$\begin{aligned}u(0,t) &= H(t), && t>0,\\ u(1,t) &= 0, && t>0, \qquad (17)\\ \frac{\partial^2 u(y,0)}{\partial t^2} &= \frac{\partial u(y,0)}{\partial t}=u(y,0)=0, && 0\le y\le 1,\end{aligned}$$

where, for simplicity, the primes of the dimensionless variables have been dropped and the following dimensionless parameters have been introduced

$$\lambda'=\frac{\nu\lambda}{h^2},\quad \lambda_r'=\frac{\nu\lambda_r}{h^2},\quad \beta'=\frac{\nu^2\beta}{h^4},\quad M^2=\frac{\sigma B_0^2}{(\mu/h^2)}. \qquad (18)$$

For the solution of the above initial-boundary value problem, we define

$$\bar{u}(y,s) = \mathcal{L}[u(y,t)] = \int_0^\infty u(y,t)\, e^{-st} dt, \tag{19}$$

as the Laplace transform of $u(y,t)$ (where s is a Laplace transform parameter).

The problem in the transformed s-plane becomes

$$\frac{d^2\bar{u}}{dy^2} - \left\{ \frac{\beta s^3 + \left(\lambda + \beta M^2\right) s^2 + \left(1 + \lambda M^2\right) s + M^2}{1 + \lambda_r s} \right\} \bar{u} = 0, \tag{20}$$

$$\bar{u}(0,s) = \frac{1}{s}; \quad \bar{u}(1,s) = 0. \tag{21}$$

The solution of the problem in the transformed $s-$plane is

$$\bar{u}(y,s) = \frac{\sinh q\,(1-y)}{s \sinh q}, \tag{22}$$

where

$$q = \left\{ \frac{\beta s^3 + \left(\lambda + \beta M^2\right) s^2 + \left(1 + \lambda M^2\right) s + M^2}{1 + \lambda_r s} \right\}^{1/2}.$$

The Laplace inversion of equation (22) yields

$$u(y,t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\sinh q\,(1-y)\, e^{st}}{s \sinh q} ds. \tag{23}$$

Our next job is to solve the integral in equation (23). Note that $s = 0$ is a simple pole and thus the residue at $s = 0$ is

$$\mathrm{Re\,s}\,(0) = \frac{\sinh \mathrm{M}\,(1-\mathrm{y})}{\sinh \mathrm{M}}. \tag{24}$$

For other singular points, we set

$$\sinh q = 0. \tag{25}$$

Writing $q = i\delta$ we find

$$\sin \delta = 0. \tag{26}$$

If $\delta_n = n\pi,\ n = 1, 2, \cdots, \infty$ are the zeros of equation (26) then

$$s_{1n} = -\frac{a}{3} - \frac{2^{1/3}\left(-a^2 + 3b\right)}{3D} + \frac{D}{3\ 2^{1/3}}, \tag{27}$$

$$s_{2n} = -\frac{a}{3} + \frac{\left(1 + \sqrt{3}i\right)\left(-a^2 + 3b\right)}{3\ 2^{2/3} D} - \frac{\left(1 - \sqrt{3}i\right) D}{6\ 2^{1/3}}, \tag{28}$$

$$s_{3n} = -\frac{a}{3} + \frac{\left(1 - \sqrt{3}i\right)\left(-a^2 + 3b\right)}{3\ 2^{2/3} D} - \frac{\left(1 + \sqrt{3}i\right) D}{6\ 2^{1/3}}, \tag{29}$$

$n = 1, 2, \cdots, \infty$ are the poles. These are simple poles and the residues at all these poles are

$$\mathrm{Re\,s}\,(\mathrm{s_{1n}}) = \frac{-2\pi\,(-1)^{\mathrm{n}}\,\mathrm{n}\,(1 + \lambda_{\mathrm{r}} \mathrm{s_{1n}})^2\, \mathrm{e}^{\mathrm{s_{1n}t}}}{\mathrm{s_{1n}} \mathrm{D_1}} \sin \mathrm{n}\pi\,(1-\mathrm{y}), \tag{30}$$

$$\mathrm{Re\,s}\left(\mathrm{s_{2n}}\right)=\frac{-2\pi\left(-1\right)^{\mathrm{n}}\mathrm{n}\left(1+\lambda_{\mathrm{r}}\mathrm{s_{2n}}\right)^{2}\mathrm{e^{s_{2n}t}}}{\mathrm{s_{2n}D_2}}\sin\mathrm{n}\pi\left(1-\mathrm{y}\right), \tag{31}$$

$$\mathrm{Re\,s}\left(\mathrm{s_{3n}}\right)=\frac{-2\pi\left(-1\right)^{\mathrm{n}}\mathrm{n}\left(1+\lambda_{\mathrm{r}}\mathrm{s_{3n}}\right)^{2}\mathrm{e^{s_{3n}t}}}{\mathrm{s_{3n}D_3}}\sin\mathrm{n}\pi\left(1-\mathrm{y}\right), \tag{32}$$

where

$$a=\frac{\lambda+\beta M^2}{\beta},\quad b=\frac{1+\lambda M^2+\lambda_r\delta_n^2}{\beta},\quad c=\frac{M^2+\delta_n^2}{\beta},$$

$$D=\left[-2a^3+9ab-27c+3\sqrt{3}\sqrt{-a^2b^2+4b^3+4a^3c-18abc+27c^2}\right]^{1/3},$$

$$D_1=1+M^2\left(\lambda-\lambda_r\right)+2\left(\lambda+\beta M^2\right)s_{1n}+\left(3\beta+\beta\lambda_rM^2+\lambda\lambda_r\right)s_{1n}^2+2\beta\lambda_rs_{1n}^3,$$

$$D_2=1+M^2\left(\lambda-\lambda_r\right)+2\left(\lambda+\beta M^2\right)s_{2n}+\left(3\beta+\beta\lambda_rM^2+\lambda\lambda_r\right)s_{2n}^2+2\beta\lambda_rs_{2n}^3,$$

$$D_3=1+M^2\left(\lambda-\lambda_r\right)+2\left(\lambda+\beta M^2\right)s_{3n}+\left(3\beta+\beta\lambda_rM^2+\lambda\lambda_r\right)s_{3n}^2+2\beta\lambda_rs_{3n}^3.$$

An addition of residues gives the following expression for the complete solution

$$u\left(y,t\right)=\frac{\sinh M\left(1-y\right)}{\sinh M}-2\pi\sum_{n=1}^{\infty}\left(-1\right)^n n\left\{\begin{array}{c}\frac{\left(1+\lambda_rs_{1n}\right)^2e^{s_{1n}t}}{s_{1n}D_1}+\frac{\left(1+\lambda_rs_{2n}\right)^2e^{s_{2n}t}}{s_{2n}D_2}\\+\frac{\left(1+\lambda_rs_{3n}\right)^2e^{s_{3n}t}}{s_{3n}D_3}\end{array}\right\}\sin n\pi\left(1-y\right). \tag{33}$$

4 Flow due to an oscillating plate

Let us now consider the flow of an incompressible electrically conducting Burgers' fluid between two infinite parallel plates. The lower plate at $y=0$ oscillates in its own plane with frequency ω while the upper plate at $y=h$ is held fixed. It is assumed that there is no pressure gradient and the flow is entirely driven due to oscillation of the lower plate. Thus, the governing problem consists of equation (13) and the following initial and boundary conditions

$$\begin{aligned}u\left(0,t\right)&=Ue^{i\omega t}H\left(t\right), && t>0,\\ u\left(h,t\right)&=0, && t>0,\\ \frac{\partial^2u\left(y,0\right)}{\partial t^2}&=\frac{\partial u\left(y,0\right)}{\partial t}=u\left(y,0\right)=0, && 0\leq y\leq h.\end{aligned} \tag{34}$$

The above conditions are made non-dimensional by scaling the frequency with $\rho h^2/\mu$ and the remaining non-dimensional variables are the same as in the previous section. Thus, the non-dimensional initial-boundary value problem takes the following form

$$\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\frac{\partial u}{\partial t}=\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial^2u}{\partial y^2}-M^2\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)u, \tag{35}$$

$$\begin{aligned}u\left(0,t\right)&=e^{i\omega t}H\left(t\right), && t>0,\\ u\left(1,t\right)&=0, && t>0,\\ \frac{\partial^2u\left(y,0\right)}{\partial t^2}&=\frac{\partial u\left(y,0\right)}{\partial t}=u\left(y,0\right)=0, && 0\leq y\leq 1.\end{aligned} \tag{36}$$

The procedure for determining $u(y,t)$ is similar to that used in the previous section, so here we simply state the solution which is

$$u(y,t)=\frac{\sinh m(1-y)}{\sinh m}e^{i\omega t}-2\pi\sum_{n=1}^{\infty}(-1)^n n\left\{\begin{array}{c}\frac{(1+\lambda_r s_{1n})^2e^{s_{1n}t}}{(s_{1n}-i\omega)D_1}+\frac{(1+\lambda_r s_{2n})^2e^{s_{2n}t}}{(s_{2n}-i\omega)D_2}\\+\frac{(1+\lambda_r s_{3n})^2e^{s_{3n}t}}{(s_{3n}-i\omega)D_3}\end{array}\right\}\sin n\pi(1-y), \tag{37}$$

where

$$m=\left\{\frac{\begin{array}{c}M^2\left(1-\beta\omega^2+\lambda\lambda_r\omega^2\right)-\omega^2\left(\lambda-\lambda_r+\lambda_r\beta\omega^2\right)\\+i\omega\left[1-\beta\omega^2+\lambda\lambda_r\omega^2+M^2\left(\lambda-\lambda_r+\lambda_r\beta\omega^2\right)\right]\end{array}}{1+\lambda_r^2\omega^2}\right\}^{1/2}.$$

5 Plane Poiseuille flow

In this section we consider an incompressible electrically conducting Burgers' fluid between two infinite stationary plates, separated by a distance $2h$. The fluid between them is initially at rest. The motion is caused due to imposition of a periodic pressure gradient in the direction of flow, namely

$$\frac{\partial p}{\partial x}=\rho Q\cos\omega t. \tag{38}$$

The flow is governed by equation (12) along with the following initial and boundary conditions

$$\begin{aligned}u(\pm h,t)&=0, && t>0,\\ \frac{\partial^2u(y,0)}{\partial t^2}&=\frac{\partial u(y,0)}{\partial t}=u(y,0)=0, && -h\le y\le h.\end{aligned} \tag{39}$$

The governing problem in non-dimensional variables becomes

$$\begin{aligned}\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\frac{\partial u}{\partial t}&=Q\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)\cos\omega t+\left(1+\lambda_r\frac{\partial}{\partial t}\right)\frac{\partial^2u}{\partial y^2}\\&\quad-M^2\left(1+\lambda\frac{\partial}{\partial t}+\beta\frac{\partial^2}{\partial t^2}\right)u,\end{aligned} \tag{40}$$

$$\begin{aligned}u(\pm1,t)&=0, && t>0,\\ \frac{\partial^2u(y,0)}{\partial t^2}&=\frac{\partial u(y,0)}{\partial t}=u(y,0)=0, && -1\le y\le 1,\end{aligned} \tag{41}$$

where $Q'=Q/\left(\nu U/h^2\right)$.

The solution of above initial-boundary value problem is given by

$$\begin{aligned}u(y,t)&=Q\,\mathrm{Re}\left[\begin{array}{l}\beta_0^2\left(\frac{\cosh m-\cosh my}{\cosh m}\right)e^{i\omega t}+\beta_1^2\left(\frac{\cosh q_1-\cosh q_1y}{\cosh q_1}\right)e^{s_1t}\\+\beta_2^2\left(\frac{\cosh q_2-\cosh q_2y}{\cosh q_2}\right)e^{s_2t}+\beta_3^2\left(\frac{\cosh q_3-\cosh q_3y}{\cosh q_3}\right)e^{s_3t}\end{array}\right.\\&\quad-\pi\left(1-\beta\omega^2+i\omega\lambda\right)\\&\quad\left.\times\sum_{n=1}^{\infty}(-1)^n(2n+1)\left\{\begin{array}{c}\frac{(1+\lambda_r s_{1n})^2e^{s_{1n}t}}{(s_{1n}-i\omega)D_1^*}+\frac{(1+\lambda_r s_{2n})^2e^{s_{2n}t}}{(s_{2n}-i\omega)D_2^*}\\+\frac{(1+\lambda_r s_{3n})^2e^{s_{3n}t}}{(s_{3n}-i\omega)D_3^*}\end{array}\right\}\cos\left[(2n+1)\frac{\pi}{2}y\right]\right],\end{aligned} \tag{42}$$

in which

$$
\begin{aligned}
s_1 &= -M^2, \quad s_2 = \frac{-\lambda + \sqrt{\lambda^2 - 4\beta}}{2\beta}, \quad s_3 = \frac{-\lambda - \sqrt{\lambda^2 - 4\beta}}{2\beta}, \\
\beta_0^2 &= \frac{1 - \beta\omega^2 + i\omega\lambda}{M^2\left(1 - \beta\omega^2\right) - \lambda\omega^2 + i\omega\left(1 - \beta\omega^2 + \lambda M^2\right)}, \\
\beta_1^2 &= \frac{1 - \beta\omega^2 + i\omega\lambda}{\left(s_1 - i\omega\right)\left(s_1 - s_2\right)\left(s_1 - s_3\right)}, \\
\beta_2^2 &= \frac{1 - \beta\omega^2 + i\omega\lambda}{\left(s_2 - i\omega\right)\left(s_2 - s_1\right)\left(s_2 - s_3\right)}, \\
\beta_3^2 &= \frac{1 - \beta\omega^2 + i\omega\lambda}{\left(s_3 - i\omega\right)\left(s_3 - s_1\right)\left(s_3 - s_2\right)}, \\
q_1 &= \left.q\right|_{s=s_1}, \quad q_2 = \left.q\right|_{s=s_2}, \quad q_3 = \left.q\right|_{s=s_3}, \\
d_0 &= M^2 + M^4\left(\lambda - \lambda_r\right), \\
d_1 &= 1 + M^2\left(4\lambda - \lambda_r\right) + M^4\left(2\beta + \lambda^2 - \lambda\lambda_r\right), \\
d_2 &= 3\left(\lambda + \left(2\beta + \lambda^2\right) + \lambda\beta M^4\right), \\
d_3 &= 2\beta^2 M^4 + \beta\left(4 + \lambda\lambda_r M^4 + 2M^2\left(4\lambda + \lambda_r\right)\right) + \lambda\left(\lambda_r + 2\lambda + \lambda\lambda_r M^2\right), \\
d_4 &= 5\lambda\beta + 2\lambda_r\beta + \lambda_r\beta^2 M^4 + \lambda^2\lambda_r + \beta\left(5\beta + 4\lambda\lambda_r\right)M^2, \\
d_5 &= 3\beta\left(\beta + \lambda\beta M^2 + \lambda\lambda_r\right), \\
d_6 &= 2\beta^2\lambda_r, \\
D_1^* &= d_0 + d_1 s_{1n} + d_2 s_{1n}^2 + d_3 s_{1n}^3 + d_4 s_{1n}^4 + d_5 s_{1n}^5 + d_6 s_{1n}^6, \\
D_2^* &= d_0 + d_1 s_{2n} + d_2 s_{2n}^2 + d_3 s_{2n}^3 + d_4 s_{2n}^4 + d_5 s_{2n}^5 + d_6 s_{2n}^6, \\
D_3^* &= d_0 + d_1 s_{3n} + d_2 s_{3n}^2 + d_3 s_{3n}^3 + d_4 s_{3n}^4 + d_5 s_{3n}^5 + d_6 s_{3n}^6.
\end{aligned}
$$

6 Results and discussion

In this section, we present various results obtained from the flows analyzed in this investigation. We interpret these results and verify that they are consistent physically. Special emphasis has been given to the comparison among the velocity profiles for four kinds of fluids: a Newtonian fluid for which $\lambda = \lambda_r = \beta = 0$, a Maxwell fluid for which $\lambda \neq 0,\ \lambda_r = \beta = 0$, an Oldroyd-B fluid for which $\lambda \neq 0,\ \lambda_r \neq 0,\ \beta = 0$ and a Burgers' fluid. The effects of various parameters on the velocity profiles especially magnetic parameter M for all four types of fluids have been studied in all three problems through several graphs.

To see the effects of magnetic parameter M on the velocity profiles, we have plotted u against y in figures $1-3$ for flow due to a jerked plate, flow due to an oscillating plate when oscillation is of type $\cos\omega t$ and plane Poiseuille flow, respectively. From all these figures, it is noted that an increase in magnetic parameter M reduces the velocity profiles monotonically due to the effect of the magnetic force against the direction of the flow for all four types of fluids.

This is in accordance to the fact that the magnetic field is responsible to reduce the velocity. However, some differences among Newtonian, Maxwell, Oldroyd-B and Burgers' fluids can also be observed from these figures. Thus, from these figures, it is observed that the velocity profiles for a Burgers' fluid are much greater than those for a Newtonian fluid in all three problems. Moreover, it can be seen that the velocity profiles for a Maxwell fluid are much larger than the velocity profiles for a Burgers' fluid while the velocity profiles for an Oldroyd-B fluid are much smaller than those for a Burgers' fluid.

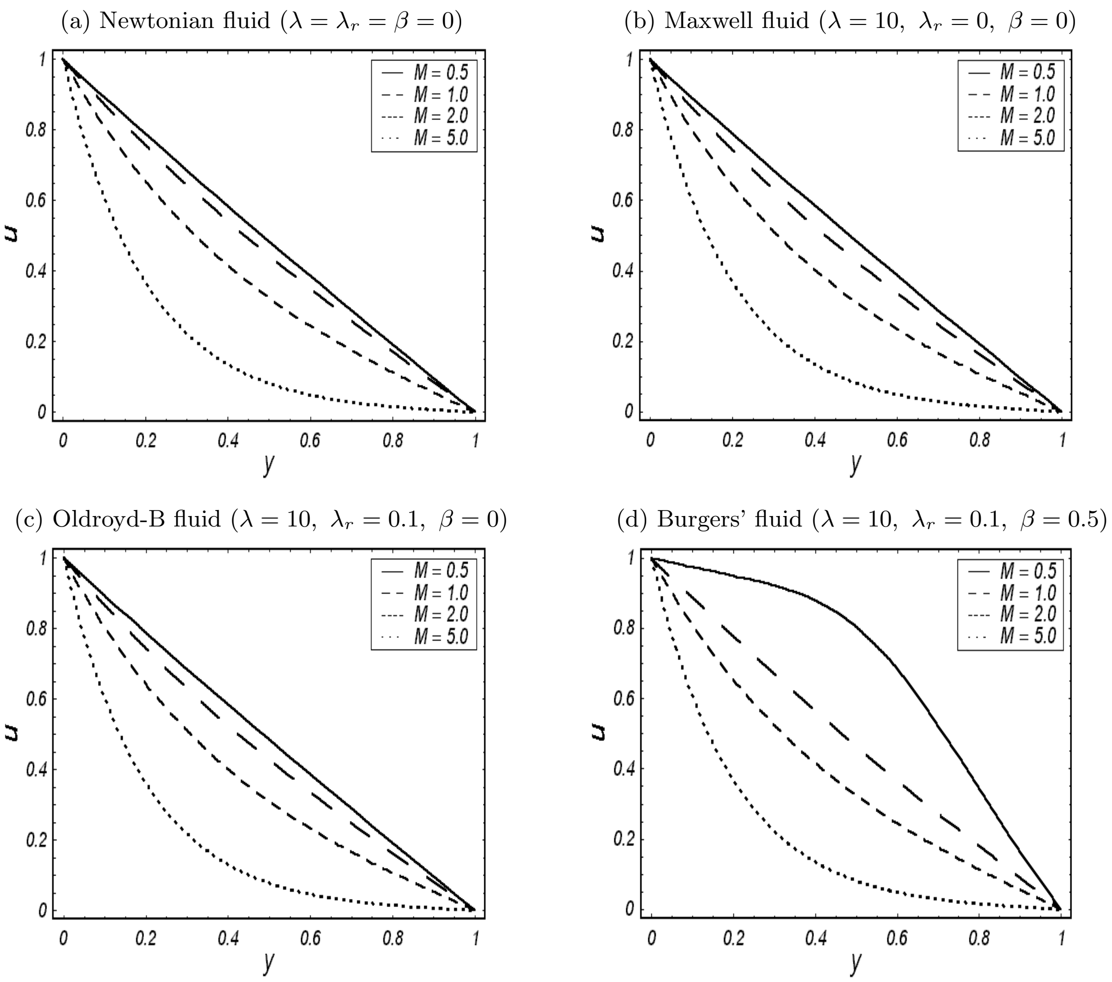

Figure 1: Profiles of the dimensionless velocity $u(y,t)$ (flow due to a jerked plate) for various values of magnetic parameter M when $t = 5.5$ is fixed.

In addition, this analysis also reveals that second grade fluid has smaller velocity profiles when compared with a Burgers' fluid. As expected, it appears that the velocity is an increasing function of the rheological parameter β of the Burgers' fluid. Furthermore, the velocity also increases with frequency.

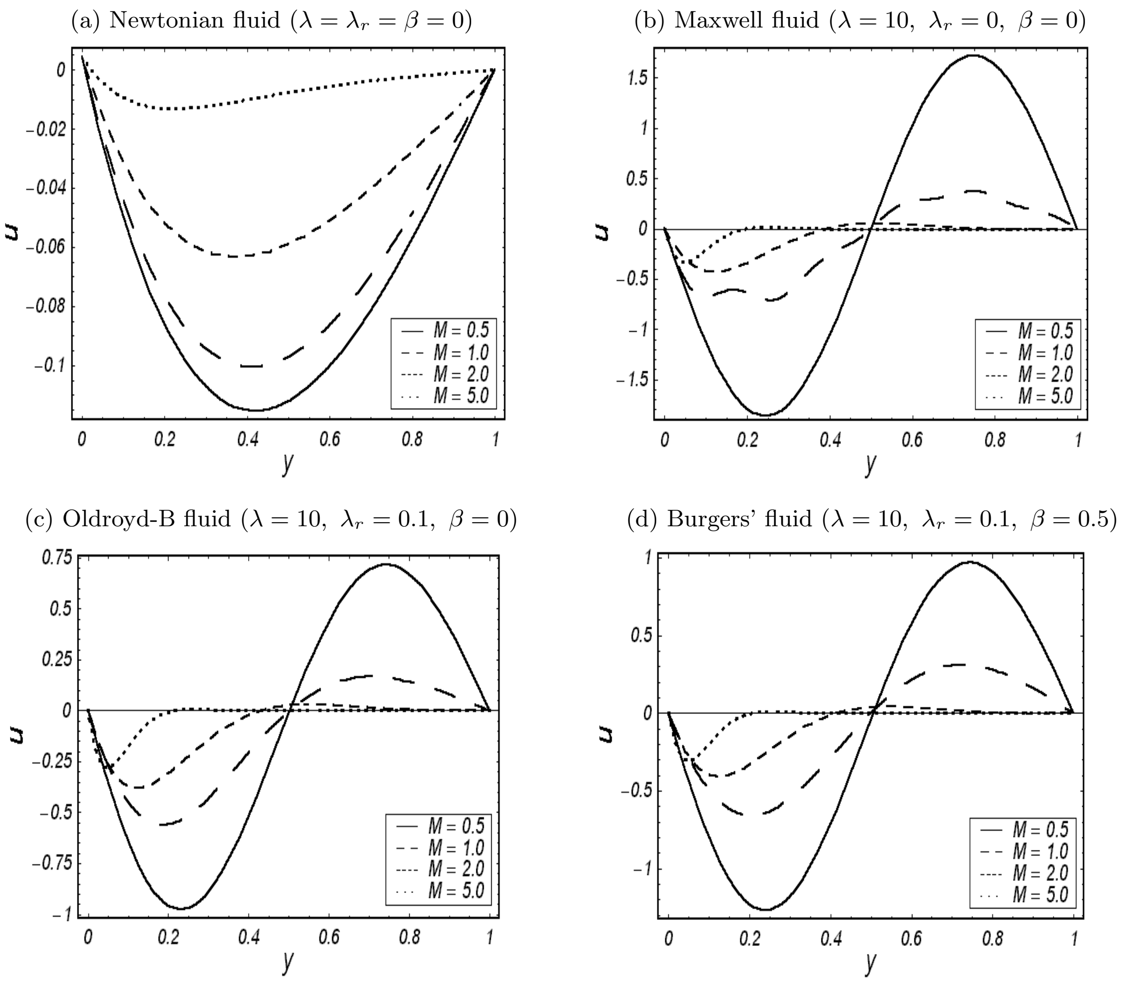

Figure 2: Profiles of the dimensionless velocity $u(y,t)$ (flow due to an oscillating plate) for various values of magnetic parameter M when $\omega = 2$ and $t = 5.5$ are fixed.

7 Concluding remarks

In this communication we have investigated some unidirectional flows of a magnetohydrodynamic (MHD) Burgers' fluid. By using the constitutive equation for a Burgers' fluid, in the literature, the governing time-dependent equation is modeled. Exact analytical solutions are obtained for three flow situations of MHD Burgers' fluid. The analysis for the analytic solutions is carried out using the Laplace transform technique. The presented analysis is more general and the results for several other fluid models (i.e. Oldroyd-B, Maxwell, second-grade and Newtonian) can be taken as special cases of it. Moreover, the result for hydrodynamic fluid can be recovered by putting $B_0 = 0$. This work also paves the way for further investigation of flows of a Burgers' fluid. The present analytical solutions are very important in the theory of viscoelastic fluids because of their exactness. They provide a standard for checking the accuracies of many approximate methods which may be numerical or empirical.

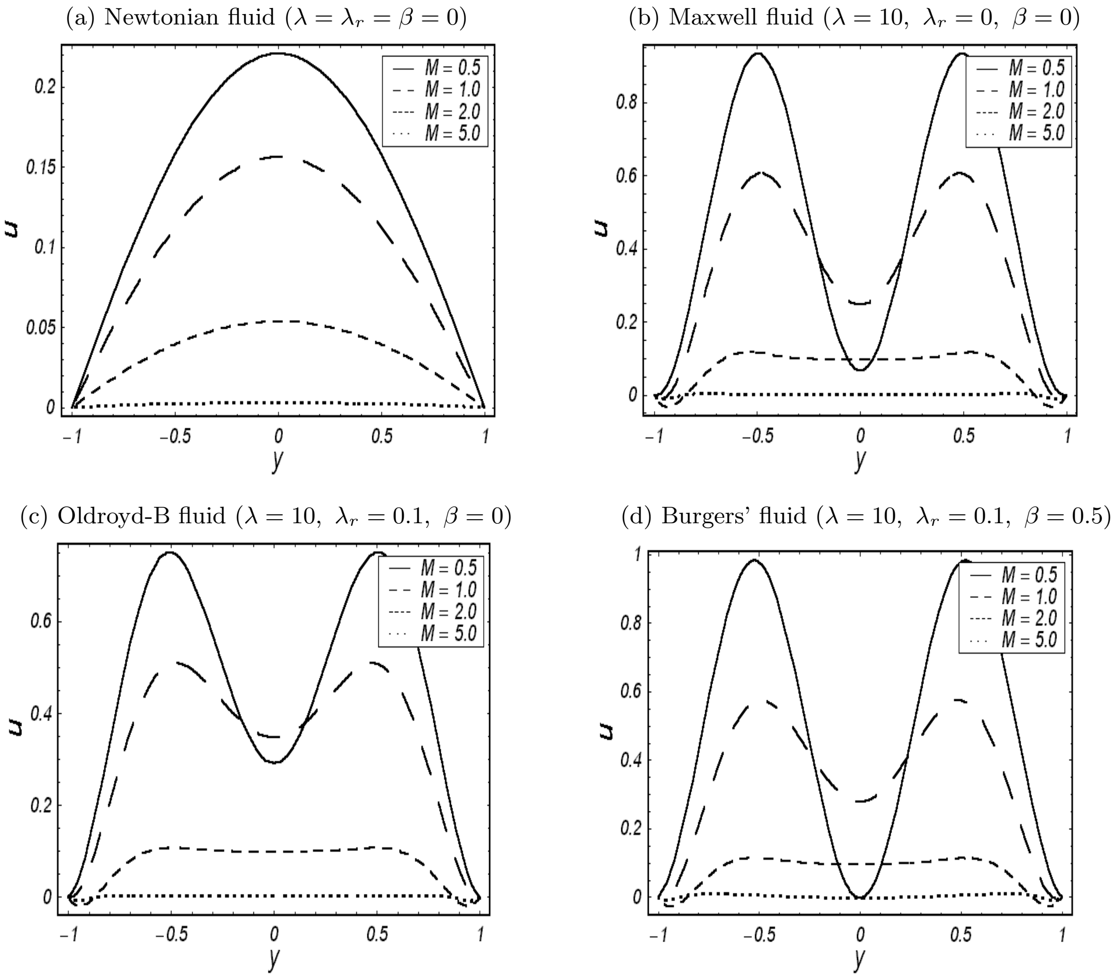

Figure 3: Profiles of the dimensionless velocity $u\,(y,t)$ (plane Poiseuille flow) for various values of magnetic parameter M when $Q=-1,\ \omega=2$ and $t=5.5$ are fixed.

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References

[1] K. R. Rajagopal, On boundary conditions for fluids of the differential type, in A. Sequira (Ed.), Navier-Stokes equations and related non-linear problems, Plenum press, New York, 1995, pp. 273 – 278.

[2] K. R. Rajagopal and P. N. Kaloni, Some remarks on boundary conditions for fluids of the differential type, in G. A. C. Graham, S. K. Malik (Eds.), Continuum mechanics and its applications, Hemisphere, New York, 1989, pp. 935 – 942.

[3] P. Ravindran, J. M. Krishnan and K. R. Rajagopal, A note on the flow of a Burgers' fluid in an orthogonal rheometer, Int. J. Eng. Sci. 42 (2004) 1973 – 1985.

[4] J. M. Krishnan and K. R. Rajagopal, Review of the uses and modeling of bitumen from ancient to modern times, Appl. Mech. Rev. 56 (2003) 149 – 214.

[5] J. M. Krishnan, A. M. Asce and K. R. Rajagopal, Thermodynamic framework for the constitutive modeling of asphalt concrete: theory and application, J. Mater. Civil Eng. 16 (2004) 155 – 166.

[6] C. A. Tovar, C. A. Cerdeirina, L. Romani, B. Prieto and J. Carballo, Viscoelastic behavior of Arzua-Ulloa cheese, J. Texture Stud. 34 (2003) 115 – 129.

[7] M. C. Wang and K. Y. Lee, Creep Behavior of cement stabilized soils, Highway Res. Record 442 (1973) 58 – 69.

[8] K. Majidzadeh and H. E. Schweyer, Viscoelastic response of asphalts in the vicinity of the glass transition point, Assoc. Asphalt Paving Technol. Proc. 36 (1967) 80 – 105.

[9] A. H. Gerritsen, C. A. P. M. van Grup, J. P. J. van der Heide, A. A. A. Molenaar and A. C. Pronk, Prediction and prevention of surface cracking in asphaltic pavements, in: Proceedings of the Sixth International Conference, Structural Design of Asphalt Pavements, vol. I, University of Michigan, 1987.

[10] C. I. Chen, C. K. Chen and Y. T. Yang, Unsteady unidirectional flow of an Oldroyd-B fluid in a circular duct with different given volume flow rate conditions, Heat and Mass Transf. 40 (2004) 203 – 209.

[11] C. I. Chen, C. K. Chen and Y. T. Yang, Unsteady unidirectional flow of second grade fluid between the parallel plates with given volume flow rate conditions, Appl. Math. Comput. 137 (2003) 437 – 450.

[12] W. C. Tan and T. Masuoka, Stokes' first problem for a second grade fluid in a porous half-space with heated boundary, Int. J. Non-Linear Mech. 40 (2005) 515 – 522.

[13] W. C. Tan and T. Masuoka, Stokes' first problem for an Oldroyd-B fluid in a porous half space, Phys. Fluid 17 (2005) 023101 – 023107.

[14] T. Hayat, M. Khan and M. Ayub, Exact solutions of flow problems of an Oldroyd-B fluid, Appl. Math. Comput. 151 (2004) 105 – 119.

[15] S. Asghar, M. Khan and T. Hayat, Magnetohydrodynamic transient flows of a non-Newtonian fluid, Int. J. Non-linear Mech. 40 (2005) 589 – 601.