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D-BRANES IN NON-CRITICAL SUPERSTRINGS AND MINIMAL SUPER YANG-MILLS IN VARIOUS DIMENSIONS

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Abstract

We construct and analyze D-branes in superstring theories in even dimensions less than ten. The backgrounds under study are supersymmetric $R^{d-1,1} \times SL(2,R)_k/U(1)$ where the level of the supercoset is tuned such as to provide bona fide string theory backgrounds. We provide exact boundary states for D-branes that are localized at the tip of the cigar $SL(2,R)/U(1)$ supercoset conformal field theory. We analyze the spectra of open strings on these D-branes and show explicitly that they are consistent with supersymmetry in $d = 2, 4$ and 6. The low energy theory on the world-volume of the D-brane in each case is pure Yang-Mills theory with minimal supersymmetry. In the case with four macroscopic flat directions $d = 4$, we realize an $\mathcal{N} = 1$ super Yang-Mills theory, and we interpret the backreaction for the dilaton as the running of the gauge coupling, and study the relation between R-symmetry breaking in the gauge theory and the backreaction on the RR axion.

1. Introduction

1.1. Non-critical Superstrings and Holography

It has been proven useful to study the physics of gauge theories using the geometrical pictures and intuition provided by brane set-ups in string theory (see e.g. [1]). One spectacular outcome of the study of D-branes and their associated geometry [2] has been the impressive list of concrete examples of holography [3], in which gravitational theories are dual to theories without a massless spin two particle.

In this paper, we concentrate on backgrounds of string theory with d flat directions, supplemented with a cigar superconformal field theory $R^{1,d-1}$ times $SL(2,R)/U(1)$ [4,5,6,7]. The background can arise from taking a double scaling limit of string theory near a singularity in a Calabi-Yau manifold, or in the presence of NS5-branes [8]. It can also be thought of as providing a d-dimensional string theory background per se. By dialing the level k the background becomes critical. For even dimension d, the background comes equipped with an $N = 2$ superconformal worldsheet supersymmetry that can be used to GSO project and that provides us with a target-space supersymmetric superstring theory.

These backgrounds and their D-branes are appropriate examples to further study the interplay between holography and D-branes. Indeed, these solutions are of linear dilaton type and can be argued to interpolate between two-dimensional string theories and their ten-dimensional cousins. The dilaton gradient provided by the cigar conformal field theory takes values intermediate between the two-dimensional (strong) gradient and the ten-dimensional (zero) gradient. This mechanism for achieving criticality allows us to interpolate in the dimension of space-time. Since two-dimensional (or generally low-dimensional) examples of holography seem to be under more control than their ten-dimensional counterparts (see e.g. [9] and follow-ups) it may be worthwhile to lay out the playground in between. In the process, we should learn more about linear dilaton holography [10].

1.2. Gauge Theory Physics

Constructing string duals to $\mathcal{N} = 1$ SYM theories has proven to be difficult. Previous approaches [11,12] start from bulk theories with a larger number of supersymmetries, which are then broken through various mechanisms. An unwanted feature in these constructions is the existence of (extra) matter fields (e.g. massive scalars and/or fermions) in the theory. When one goes to the deep infrared (where the extra matter fields are absent), the supergravity backgrounds typically are not under control, either because of strong curvature [11] or strong coupling [12].

Non-critical superstrings seem to be free of most of these problems. The target-space has reduced space-time supersymmetry, and the branes living in them are likewise less supersymmetric.

Thus, we need only carefully construct the bulk, and then the corresponding branes to study gauge theories with less supersymmetry. In this sense, the occurrence of gauge theories with less supersymmetry is natural in the context of lower-dimensional superstring backgrounds. One must keep in mind however that the curvatures in these backgrounds are of string scale; gravity is not a priori a good approximation, and one necessarily has to work with the full sigma model which (after backreaction) involves the difficult problem of dealing with background fluxes.

In this paper, we present a boundary state description of the branes in these backgrounds. Though it *is* a closed string description in principle, the boundary states are in practice more useful to describe open string physics by channel duality, and less so to compute the exact string background; we only compute the linear backreaction to the closed string background. Nevertheless, we consider this an important first step. This approach to $\mathcal{N}=1$ gauge theories through lower-dimensional superstrings may provide us with a new window to gauge theory physics.

1.3. Summary of Results

In this paper, we present the exact conformal field theory description of branes in lower-dimensional superstring theories, and compute the spectrum and low energy theory on these branes for $d=2,4,6$. In the $d=4$ case, we present evidence that our closed string background is dual to a non-gravitational theory which in the IR flows arbitrarily close to $4d\,\mathcal{N}=1$ SYM, and in the UV is completed to a theory which is asymptotically free. In this sense, it is similar to the holographic descriptions in [13] and [12][1].

Furthermore, it is possible to understand instantons and the anomalous breaking of the chiral $U(1)_R$ symmetry to Z_{2N} along the lines of [14] by studying the large distance behavior of the Ramond-Ramond fields: the background value of the RR axion potential spontaneously breaks a $U(1)$ isometry of the solution.

1.4. Organization

In section 2 we briefly review the bulk physics of noncritical superstring theories. The boundary states that describe the D-branes that we concentrate on are presented in section 3. We analyze the spectrum encoded in the one-loop partition function in some detail and argue for the low-energy effective action for these branes in section 4. We follow up by laying bare the physics encoded in the one-point function of the boundary states, in the case of $N=1$ SYM in four dimensions. In the conclusions we summarize our results and indicate possible further developments. Various technicalities, and a generic proof of the vanishing of open string partition functions in Gepner-like (compact or non-compact) models are presented in the appendices.

[1] Indeed the profile of the closed string fields closely resembles that of the solutions of [12].

1.5. Note added in publication

Very recently the interesting paper [15] appeared with some overlap with our paper. In particular, we note the overlap in the construction[2] of the open string spectrum for $d = 4$. However, it is mostly usefully complementary in both subject matter and techniques. In [15] one finds an explicit analysis of the relation to brane set-ups and an analysis of flavor physics in this context. In our work, we focus on pure Yang-Mills, providing many details of the open string theory. We moreover compute properties of the dual closed string background using the boundary states.

2. Superstrings in dimensions less than ten

In this section, we briefly review salient features of the closed string background in which we will embed D-branes in section 3. The closed string background we shall study is the type IIB d dimensional superstring [4] which consists of d-dimensional flat space tensored with a non-trivially curved space: $\mathbb{R}^{d-1,1} \times SL(2,R)/U(1)$. The factor $SL(2)/U(1)$ is a Kazama-Suzuki supersymmetric coset conformal field theory [17], at (supersymmetric) level k. One can write an effective target space action for this coset as:

$$ds^2 = k\left(d\rho^2 + \tanh^2\rho d\theta^2\right), \quad e^{\Phi} = \frac{e^{\Phi_0}}{\cosh\rho} \tag{2.1}$$

This SCFT is known to have a mirror description as the $\mathcal{N}=2$ Liouville theory. The level k of the coset is tuned to make the total central charge $c = 15$. The matter worldsheet theory is tensored with the standard $\mathcal{N} = 2$ superconformal ghosts of central charge $c = -15$, such that the total worldsheet central charge vanishes. From the formula for the flat space and coset central charge, we derive a relation between the dimension d and the level k:

$$c = 15 = \frac{3d}{2} + 3 + \frac{6}{k}. \tag{2.2}$$

For future reference we note that we have the following correspondences:

$$\begin{aligned} &\text{For} \quad d=6 \quad k=2; \qquad &&\text{For} \quad d=4 \quad k=1; \\ &\text{For} \quad d=2 \quad k=\frac{2}{3}; \qquad &&\text{For} \quad d=0 \quad k=\frac{1}{2}. \end{aligned}$$

For $d = 8$ we obtain the familiar superstring in ten dimensional flat space, while for $d = 0$ we obtain a critical two-dimensional black hole background. The supercoset theory asymptotes to a $\mathcal{N} = 2$ linear dilaton with slope $Q = \sqrt{\frac{2}{k}}$. The short calculation above illustrates how the level of the coset (i.e. dilaton gradient) allows us to interpolate in the dimension of space-time.

[2] Following prior work in [16].

We will now discuss briefly the two-dimensional conformal field theories on the worldsheet. In the $d+2$ dimensional theory, the free scalar fields X^μ parameterize the flat space directions. Far from the tip of the cigar, the cigar can be approximated by a cylinder with a dilaton varying linearly along its length. The cylinder directions will be labeled by the fields (ρ,θ).[3] For each worldsheet boson there is a corresponding worldsheet fermion. In the flat directions, we have worldsheet fermions ψ^μ, while the fermions in the cigar directions are named ψ^a_{cig}.

The superconformal field theory on $\mathbb{R}^{d-1,1}$ and the cigar CFT are essentially decoupled: the N=2 worldsheet currents are the sums of the respective currents of the two theories, and states are built in the product state space of the two conformal field theories. The closed string vertex operators are the operators on the cigar $\Phi^j_{m,\overline{m}}$ multiplied with the vertex operators on $\mathbb{R}^{d-1,1}$

$$|\mathcal{V}(k)\rangle = |V_{X,\psi,gh}(k)\rangle \otimes \Phi^j_{m,\overline{m}}|0\rangle_{Cig}\,. \tag{2.3}$$

Here, $(j,m,\overline{m})$ label the primaries of the $SL(2)/U(1)$ supercoset. We review the construction of these states in more detail in appendix B. Here, we note that the quantum number j governs the radial behavior of the wavefunctions, asymptotically $\phi \sim e^{2j\rho}$; the quantum numbers $m,\overline{m}$ are related to the momentum and winding around the cigar. The bulk superstring theories have an interesting physical spectrum that depends strongly on the dimension – this was analyzed in [5,6,7] to which we refer for details. Below we explicitly write down the vertex operators necessary for the analysis of one point functions we carry out in section 5.

2.1. Closed String Vertex Operators for $d=4$

a) The Graviton

In the NSNS sector, we shall first consider states in the $(-1,-1)$ picture with $m=\overline{m}=0$. The worldsheet states which give us the second rank tensor in spacetime are:

$$|\mathcal{V}^{j\,IJ}(k)\rangle = \psi^I_{-\frac{1}{2}}\,\tilde{\psi}^J_{-\frac{1}{2}}\left[|0,k\rangle_{X,\psi,NS}\otimes\,\Phi^j_{00}|0\rangle_{Cig,NS}\otimes\,|0\rangle_{gh}\right]. \tag{2.4}$$

Physical states obey the condition $L_0 - 1 = \frac{1}{2}+\frac{1}{2}+\frac{1}{2}k^\mu k_\mu - \frac{j(j+1)}{k} - 1 = 0$. We will be interested in the graviton modes which propagate in the radial direction of the cigar. These have $k_\mu k^\mu = 0$ and are in the continuous representation on the cigar $j=-\frac{1}{2}+iP$. The on-shell condition becomes $P^2=-1/4$.

[3] The capital letters $I,J=0,1..d-1,\rho,\theta$ label all the $d+2$ dimensions in the theory while the Greek indices $\mu,\nu=0,1..d-1$ run over the flat space directions only.

b) The Tachyon

As mentioned in section 2, the non-trivial part of the closed string background can be thought of as a supercoset or equivalently as an $\mathcal{N}=2$ Liouville theory with winding condensate. It is immediate from the second description (but it is also seen easily from the first) that there is a stable scalar field (called the tachyon) in the spectrum with asymptotic winding number one. This mode of the tachyon has the form in the $(-1,-1)$ picture

$$|\mathcal{V}^j(k)\rangle = |0,k\rangle_{X,\psi,NS} \otimes \Phi^j_{\frac{1}{2}\frac{1}{2}}|0\rangle_{Cig,NS} \otimes |0\rangle_{gh}. \tag{2.5}$$

The mass shell condition for this mode is $L_0 - 1 = \frac{1}{2} + \frac{1}{2}k^\mu k_\mu + \frac{1}{4} - j(j+1) - 1 = 0$. For $k_\mu k^\mu = 0$, and $j = -\frac{1}{2} + iP$, the on-shell state becomes $P^2 = 0$.

c) The Ramond-Ramond Axion

We are interested in the zero mode of the axion field and so we restrict to operators with $m = \overline{m} = 0$. As we will see below, the calculation of the one point function on the disk forces the Ramond sector vertex operator to be in the $(-\frac{3}{2}, -\frac{1}{2})$ picture. The full vertex operator is obtained by tensoring with the ghost contributions and four dimensional spin fields. For propagating states with $m = \overline{m} = 0$ and $k^\mu k_\mu = 0$, the on-shell condition is $L_0 - 1 = \left[\frac{3}{8} + 2 \cdot \frac{1}{8}\right] + \left[P^2 + \frac{1}{4} + \frac{1}{8}\right] - 1 = P^2 = 0$.

The symmetry of the backreaction problem tells us that the components of the one form field strength along the flat directions vanish: $\partial_\mu \chi = 0$. The remaining modes $\partial_{\theta\pm\rho}\chi$ are constructed [7] from the spin fields such that there is zero spin in the flat four directions and the $U(1)$ R charge of the $\mathcal{N}=2$ algebra on the cigar is $Q = \frac{1}{2}$. The same result holds for the right moving sector. The vertex operator in the $(-\frac{1}{2}, -\frac{1}{2})$ picture may now be written as

$$|\mathcal{V}_{-\frac{1}{2}-\frac{1}{2}}(k)\rangle = S^\alpha\, \tilde{S}_\alpha \left[|0,k\rangle_{X,\psi} \otimes \Phi^j_{\frac{1}{2},\frac{1}{2}}|0\rangle_{Cig} \otimes |0\rangle_{gh}\right] \tag{2.6}$$

where the 4-dimensional spinorial index α is contracted to get a scalar.

We note that the behavior of the graviton which had an effective mass in the six dimensions is different from that of the tachyon and axion which are effectively massless. We shall see later that this difference manifests itself in the difference in the falloff rates in the weak coupling region of the backreaction onto these fields.

2.2. General remarks about non-critical strings

The string coupling at the tip of the cigar $g_s^{tip} = e^{\Phi_0}$ is a modulus of the theory and is related to the parameter multiplying the Sine-Liouville interaction in the mirror description. Starting from the linear dilaton theory, one can obtain one or the other description by turning on the operator

$\tilde{\mu}\Phi^{-1}_{00}$ or $\mu\Phi^{\frac{k}{2}}_{\frac{1}{2}\frac{1}{2}}$. The parameters are related as $(g_s^{tip})^{-2} = \left(\frac{\mu}{k}\right)^{\frac{2}{k}} = \tilde{\mu}\frac{\Gamma(1/k)}{\Gamma(1-1/k)} \equiv \nu^{-1}$. All the bulk amplitudes of the theory depend on this parameter. This will also be true of the localized branes that we discuss in this paper.

The various theories have a bosonic Poincare symmetry generated by the momenta and Lorentz rotations in flat space The theory with d flat directions has $2^{\frac{d}{2}}$ left moving conserved supercharges. For $\frac{d}{2}$ even, there are two sets of conjugate spinors $\mathcal{S}_\alpha, \tilde{\mathcal{S}}_{\dot{\alpha}}$ and for $\frac{d}{2}$ odd, there are two sets of the same spinor $\mathcal{S}_\alpha, \tilde{\mathcal{S}}_\alpha$. There are an equal number of right moving supercharges, which for the type IIB theory obey exactly the same condition. There is another conserved charge, the $U(1)$ momentum around the cigar, which acts as an R-symmetry in the d-dimensional superalgebra. We have (in the case of even $\frac{d}{2}$, for example):

$$\begin{aligned}
&\{\mathcal{S}_a, \overline{\mathcal{S}}_{\dot{\beta}}\} = 2\gamma^\mu_{\alpha\dot{\beta}} P_\mu, \quad \{\widetilde{\mathcal{S}}_a, \overline{\widetilde{\mathcal{S}}}_{\dot{\beta}}\} = 2\gamma^\mu_{\alpha\dot{\beta}} P_\mu, \quad \{\mathcal{S}, \widetilde{\mathcal{S}}\} = 0. \\
&[P^\theta, \mathcal{S}_\alpha] = \frac{1}{2}\mathcal{S}_\alpha, \quad [P^\theta, \overline{\mathcal{S}}_{\dot{\alpha}}] = -\frac{1}{2}\overline{\mathcal{S}}_{\dot{\alpha}}. \\
&[P^\theta, \widetilde{\mathcal{S}}_\alpha] = \frac{1}{2}\widetilde{\mathcal{S}}_\alpha, \quad [P^\theta, \overline{\widetilde{\mathcal{S}}}_{\dot{\alpha}}] = -\frac{1}{2}\overline{\widetilde{\mathcal{S}}}_{\dot{\alpha}}.
\end{aligned} \tag{2.7}$$

We have set up the discussion of lower-dimensional string theories without referring explicitly to their ten-dimensional origins since we believe they deserve study in their own right, and exhibit physics that are very particular to their precise form (a simple example being the dimension of space-time). However, it is often helpful to realize their roots in ten dimensions. They arise from NS5-branes or singularities inside a Calabi-Yau manifold, in a double scaling limit [8] [18] in which the string coupling is taken to zero while keeping fixed the mass of the relevant non-abelian degrees of freedom (leading to the Higgsed phase of little string theories). From the perspective of the exact description of the near-horizon geometry in terms of coset conformal field theories, one can lose dimensions of space-time by tuning the value of the level of any number of $SU(2)/U(1)$ factors such that they become of central charge zero. (This has its well-known analogue in the Landau-Ginzburg worldsheet description of the strings near singularities in Calabi-Yau manifolds embedded in weighted projective spaces.) The analysis of the exact description of branes in NS5-brane backgrounds [19][16][20] is thus technically close to the analysis that follows in section 3.

3. The boundary state

In this section we review the ingredients that are necessary to construct boundary states in the full lower-dimensional string theory that are consistent with the bulk spectrum and the GSO projection [16]. To that end, we need aspects of boundary states in flat space, as well as boundary states in the supersymmetric cigar conformal field theory [21,22]. We add to this a careful analysis of the Ramond ground state to complete the construction of the full boundary state. We will then use the boundary states assembled in this section to analyze the physics of the spectrum as well as of the one-point function in sections 4, 5 and 6. We assemble some of the details of the set-up in appendix A.

3.1. The terms and factors in the boundary state

We first recall the different ingredients in the boundary state. The branes we focus on are of the form:

$$|\mathbf{B}\rangle = \frac{T}{2}\sum_{\alpha} |B_{X,\psi}\rangle_{\alpha} \otimes |B\rangle_{Cig\,,\alpha} \otimes |B_{gh}\rangle_{\alpha}\,, \tag{3.1}$$

where the $|B_{X,\psi}\rangle$ refers to that part of the boundary state coming from the flat space $\mathbb{R}^{1,d-1}$ directions and $|B\rangle_{Cig}$ refers to a boundary state in the cigar conformal field theory. The part of the boundary state $|B_{gh}\rangle$ in the ghost sector is identical to the one constructed for Dp-branes in ten dimensional superstring theory [23,24]. The sum denoted by α above will run over the periodicity (namely the NS-NS and R-R sectors) and spin structures (which encode how the left and right fermions are glued together for a given periodicity).

The building blocks that constitute the boundary states are the Ishibashi states which satisfy the gluing conditions for a fixed label α. They can be solved for separately in the NS-NS and R-R sectors of the theory. In appendix A, we give a detailed construction of the D-branes that are extended in all of the flat spacetime directions, and that are point-like in the cigar directions.

We briefly recall the solution to the bosonic part of the conditions on the boundary state:

$$|Bp_X\rangle = \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\alpha^{\mu}_{-n}\eta_{\mu\nu}\tilde{\alpha}^{\nu}_{-n}\right]|0,k^{\mu}=0\rangle. \tag{3.2}$$

The ket $|0,k^{\mu}=0\rangle$ denotes the vacuum of the worldsheet bosons. In the fermionic sector, we need to account for the different periodicities and spin structures. The solution to the non-zero modes of the fermionic equations is

$$|Bp_{\psi},\eta\rangle_{NS/R} = \exp\left[i\eta\sum_{r>0}^{\infty}\psi^{\mu}_{-r}\eta_{\mu\nu}\tilde{\psi}^{\nu}_{-r}\right]|0,\eta\rangle_{NS/R}. \tag{3.3}$$

Here $|0,\eta\rangle_{NS/R}$ is the fermionic vacuum. There is a unique NS sector vacuum. However in the R sector, we also need to solve the zero mode fermionic constraints in order to specify the vacuum. Since the total dimension of space-time differs from ten, leading to a different Clifford algebra satisfied by the fermion zeromodes, this calculation differs slightly from the usual one. We give the relevant technical details in the appendix A.

3.2. Assembling the full boundary state

In order to describe the full boundary state, we must be more specific about the sum over periodicities and spin structures in the boundary state (3.1). To construct a GSO invariant boundary state, we need to sum over NS and R sectors, and then insert a projection operator $(1+(-)^F)(1+(-)^{\tilde{F}})$ in the type IIB superstring theory. The sum over the label α is then a sum over four terms, either NS or R and either with or without the insertion of the operator $(-)^F$.[4] This sum is equivalent to a sum over NS and R-sector and the two values of the spin structure η.[5] We index this sum by the label $\alpha = (NS, \widetilde{NS}, R, \widetilde{R})$.

The boundary states of flat space to be tensored with the cigar part are:

$$|Bp_{X,\psi}\rangle_\alpha = |Bp_X\rangle \otimes |Bp_\psi\rangle_\alpha \tag{3.4}$$

where the right-hand side of the equation is given by equations (3.2) and (3.3). Finally, in the cigar sector, we read off from [25,22] (see also [26,27]) the expression for the boundary state corresponding to a point-like brane on the cigar:

$$|D0\rangle_{Cig\,\alpha} = \sum_{j,m,\overline{m}} \Psi^{j,\,\alpha}_{m,\overline{m}}\, \Phi^j_{m,\overline{m}}|0\rangle_\alpha. \tag{3.5}$$

As explained in Appendix B, one can separate the supercoset into a bosonic part and free fermions. The wavefunctions in the different sectors are[6] determined by the purely bosonic part of the CFT:

$$\begin{aligned} \Psi^{j,NS/R}_{m_{bos}\overline{m}_{bos}} &= k^{-\frac{1}{2}}(-1)^{\frac{m_{bos}+\overline{m}_{bos}}{k}}\,\delta_{m_{bos},\overline{m}_{bos}}\,\nu^{1/2+j}\,\frac{\Gamma(-j+m_{bos})\Gamma(-j-m_{bos})}{\Gamma(-2j-1)\Gamma(1-\frac{1+2j}{k})} \\ \Psi^{j,\widetilde{NS}/\widetilde{R}}_{m_{bos}\overline{m}_{bos}} &= i^{\frac{m_{bos}+\overline{m}_{bos}}{k}}\,\Psi^{j,\,NS}_{m_{bos}\overline{m}_{bos}} \end{aligned} \tag{3.6}$$

The full GSO projected boundary state can now be written down using the factors (3.4) and (3.5):

$$|\mathbf{Bp}\rangle = \sum_\alpha |Bp_{X,\psi}\rangle_\alpha \otimes |D0\rangle_{Cig,\alpha} \otimes |B_{gh}\rangle; \qquad \alpha = NS, \widetilde{NS}, R, \widetilde{R}. \tag{3.7}$$

The explicit form of the ghost part of the boundary state will not play a role in the computations that follow and so have not been written out. We have used the notation for the tensor product which is correct only in the NS sector. The GSO projection ties together the R sector vacua of the factors, and the equation should be read as representing the tensor product of the raising operators acting on the total vacuum.

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[4] We note here that the raising operators in the Ishibashi states do not change the relative $(-)^F$ between the left and right movers. In the NS sector, the vacuum and hence all the boundary states that we have been considering have $(-)^F = (-)^{\tilde{F}}$. In the R sector, the vacuum has to be chosen with a certain value of $(-)^{F+\widetilde{F}}$, and all the states then retain that choice. In either case, the sum over eight terms thus reduces to four terms.

[5] In both the NS and R sector, the boundary state satisfies $(-)^F|\eta\rangle = \pm|-\eta\rangle$, and this facilitates the solution of the GSO projected state as $(|\eta=+\rangle \pm |\eta=-\rangle)$. In the boundary state η actually labels whether $(-)^F$ is present or not in the sum.

[6] We choose $u=1$ in the notation of that paper, i.e. the D0-brane on the cigar with only the extended trivial representation in the open string channel.

A Cardy type check can be performed on these D-branes. It consists of a combination of Cardy checks which have already been performed on the individual factors that comprise the full boundary state. The calculation is therefore a combination of the usual Cardy check performed on D-branes in flat space, and the D-branes of the cigar conformal field theory. Note that for the branes that are localized on the cigar, the spectrum in the open string channel is discrete, indeed allowing for a standard Cardy check in terms of the demand that open string degeneracies are positive integers – this is not always the case in non-rational conformal field theories where open string partition functions can depend on continuous quantum numbers, and where volume divergences can spoil this approach tailored on rational conformal field theories. However, for the localized branes on which we concentrated in this paper, no such complication arises. The Cardy check is thus straightforward.

4. The open string theory on the branes

In this section we discuss the physics associated to the D-branes we constructed above. In particular, we study in some detail the low-energy spectrum for the open strings living on the D-branes, and the low-energy effective action that describes their dynamics. The D-branes presented in the previous section break half of the bulk space-time supersymmetry, which will be indicated by the presences of massless fermions (goldstinos). The other half of the bulk supersymmetry is linearly realized on the brane – in appendix C, we exhibit the explicit form of the supercharges preserved by the D-brane in the four-dimensional case $d = 4$, using a worldsheet analysis.
In the following we present for each D-brane, as a function of the dimension d:

1. The spectrum of excitations on the D-brane and information on the low energy limit of the worldvolume theory. In every case, the theory is a pure gauge theory with minimal supersymmetry, and the spectrum consists of gauge bosons and gauginos which are the realization of the goldstinos.
2. The exact form of the full partition function and a proof that it vanishes for $d = 4, 6$, consistent with supersymmetry. The case $d = 2$ is a little subtle, because of the potential existence of unpaired fermion zero modes, on which we shall comment briefly.
3. A considerably more general proof of supersymmetry following arguments used for supersymmetric bulk partition functions [28] is presented in appendix D.

We first summarize the parts of the analysis that are common to all space-time dimensions. In all dimensions d, the partition function for the branes filling the flat space $\mathbb{R}^{d-1,1}$ is given by the following sum over sectors labeled by α:

$$Z_{Dp}(t) = \frac{1}{2}\left(Z^{NS}_{Dp}(t) - Z^{\widetilde{NS}}_{Dp}(t) - Z^{R}_{Dp}(t) - Z^{\widetilde{R}}_{Dp}(t)\right), \tag{4.1}$$

where the individual terms are given by the expressions [29,16]:

$$
\begin{aligned}
Z^{NS}_{Dp}(t) &= V_d \int \frac{d^d k}{(2\pi)^d} e^{-2\pi t k^2} \left(\frac{\Theta_{00}(it)}{\eta^3(it)}\right)^{\frac{d-2}{2}} \times \frac{\Theta_{00}(it)}{\eta^3(it)} \sum_{s\in\mathbb{Z}+\frac{1}{2}} \frac{1}{1+q^s}\left(q^{\frac{s^2-s}{k}} - q^{\frac{s^2+s}{k}}\right). \\
Z^{\widetilde{NS}}_{Dp}(t) &= V_d \int \frac{d^d k}{(2\pi)^d} e^{-2\pi t k^2} \left(\frac{\Theta_{01}(it)}{\eta^3(it)}\right)^{\frac{d-2}{2}} \times \frac{\Theta_{01}(it)}{\eta^3(it)} \sum_{s\in\mathbb{Z}+\frac{1}{2}} \frac{(-1)^{s-\frac{1}{2}}}{1-q^s}\left(q^{\frac{s^2-s}{k}} + q^{\frac{s^2+s}{k}}\right). \\
Z^{R}_{Dp}(t) &= V_d \int \frac{d^d k}{(2\pi)^d} e^{-2\pi t k^2} \left(\frac{\Theta_{10}(it)}{\eta^3(it)}\right)^{\frac{d-2}{2}} \times \frac{\Theta_{10}(it)}{\eta^3(it)} \sum_{s\in\mathbb{Z}} \frac{1}{1+q^s}\left(q^{\frac{s^2-s}{k}} - q^{\frac{s^2+s}{k}}\right). \\
Z^{\widetilde{R}}_{Dp}(t) &= V_d \int \frac{d^d k}{(2\pi)^d} e^{-2\pi t k^2} \left(\frac{\Theta_{11}(it)}{\eta^3(it)}\right)^{\frac{d-2}{2}} \times \frac{\Theta_{11}(it)}{\eta^3(it)} \sum_{s\in\mathbb{Z}} \frac{(-1)^{s}}{1-q^s}\left(q^{\frac{s^2-s}{k}} + q^{\frac{s^2+s}{k}}\right).
\end{aligned}
\tag{4.2}
$$

The worldvolume theory on the branes has a bosonic Poincare symmetry in the d flat directions. We show in appendix C that exactly half of the supercharges (2.7) are preserved by the brane. These are of the form $\mathcal{S}_\alpha + \widetilde{\mathcal{S}}_\alpha \equiv \mathcal{S}^{bdry}_\alpha$, $\overline{\mathcal{S}}_{\dot\alpha} + \widetilde{\overline{\mathcal{S}}}_{\dot\alpha} \equiv \overline{\mathcal{S}}^{bdry}_{\dot\alpha}$. In the free theory, the $U(1)_R$ symmetry is preserved. The superalgebra is of the form:

$$
\begin{aligned}
\{\mathcal{S}^{bdry}_\alpha, \overline{\mathcal{S}}^{bdry}_{\dot\beta}\} &= 2\gamma^\mu_{\alpha\dot\beta} P_\mu \\
[P^\theta, \mathcal{S}^{bdry}_\alpha] = \frac{1}{2}\mathcal{S}^{bdry}_\alpha, &\quad [P^\theta, \overline{\mathcal{S}}^{bdry}_{\dot\alpha}] = -\frac{1}{2}\overline{\mathcal{S}}^{bdry}_{\dot\alpha}
\end{aligned}
\tag{4.3}
$$

In all the open string theories, we can write down the following massless states:

$$
\epsilon_\nu \psi^\nu_{-\frac{1}{2}} |k_\mu, NS\rangle, \qquad u^\alpha |k_\mu, \Sigma_\alpha, R\rangle. \tag{4.4}
$$

Here, Σ_α is the spin field on the worldsheet which transforms under $Spin(d)$. These states are BRST invariant on the worldsheet when $k^\mu k_\mu = 0$, and when the polarizations ϵ^μ and u^α obey the conditions $k^\mu \epsilon_\mu = 0$, $\epsilon^\mu \equiv \epsilon^\mu + k^\mu$ and $k^\mu \gamma_\mu u = 0$. The physical interpretation of these modes are clear as a gauge boson and a gaugino in each of the cases. Since the other modes on the brane have masses of order $\alpha'^{-\frac{1}{2}}$, there is a sensible low energy limit in which one can write down a low energy action for these massless modes. In the following, we will turn to the individual cases $d = 4, 6, 2$ and discuss some features of the theories which are typical to the bulk theory in which they are embedded. We start out with the four-dimensional theory.

4.1. $d = 4$ and $\mathcal{N} = 1$ SYM

The low energy theory

Writing the partition function (4.2) in the NS sector as

$$
Z^{NS}(q) = \int \frac{d^4 k}{(2\pi)^4} q^{k^2} A^{NS}(q)\,, \tag{4.5}
$$

the masses of the excitations in sector α are obtained by expanding A^α in powers of $q = e^{-2\pi t}$. The coefficients in the expansion give the degeneracy of states with a given mass. We find:

$$A^{NS}(q) = q^{-\frac{1}{2}} + 2 + 4q^{\frac{1}{2}} + 12q + \dots \tag{4.6}$$

The first state with negative conformal dimension is the NS sector vacuum and will be projected out by the GSO projection. The lowest lying physical states in the NS sector are therefore two massless gauge bosons from the first excited level. In the R-sector, one finds the expansion

$$\frac{1}{2}A^R(q) = \left(\sum_{s \in \mathbb{Z}_+} q^{s^2-s} (1-q^s)^2 \right) \cdot 2 \prod_m \frac{(1+q^m)^4}{(1-q^m)^4} = 2 + 12q + 52q^2 + \dots \tag{4.7}$$

The partition function in the twisted R sector vanishes, and thus, the (GSO projected) Ramond sector gives rise to two physical massless fermionic states in spacetime.

To summarize, we see that the modes on the brane with $k_\mu^2 = 0$ are two physical states each from the NS / R sector. Analyzing (4.4) for this case, we find that the spectrum consists of a massless gauge boson A_μ and a massless gaugino λ_α transforming in the **2** of $Spin(4)$, which are the physical degrees of freedom corresponding to the $\mathcal{N} = 1$ SYM multiplet in four dimensions.

In appendix C, we have shown that the D-brane boundary state (3.7) preserves $\mathcal{N} = 1$ supersymmetry in $d = 4$. We deduce that the low energy ($\alpha' \to 0$) effective action is indeed that of pure super Yang-Mills theory

$$S_{YM} = \frac{1}{g_{YM}^2} \int d^4x \, \mathrm{Tr} \left(\frac{1}{4} F^2 + \bar{\lambda} \partial \lambda \right). \tag{4.8}$$

Full partition function

Using the identities in Appendix E, we can rewrite (4.1) as:

$$\begin{aligned} Z_{D3}(\tau) = \int \frac{d^4k}{(2\pi)^4} e^{2\pi i \tau k^2} \frac{1}{\eta^6(\tau)} \Big[&\Theta_{00}^2(\tau) \Big(-\Theta_{10}(2\tau) + e^{-\frac{i\pi\tau}{2}} \Theta_{00}(2\tau) \Big) \\ - &\Theta_{01}^2(\tau) \Big(\Theta_{10}(2\tau) + e^{-\frac{i\pi\tau}{2}} \Theta_{00}(2\tau) \Big) \\ - &\Theta_{10}^2(\tau) \Big(-\Theta_{00}(2\tau) + e^{-\frac{i\pi\tau}{2}} \Theta_{10}(2\tau) \Big) \Big] \end{aligned} \tag{4.9}$$

Now we use the following identities of theta functions [30,6,31],

$$\begin{aligned} \Theta_{00}^2(\tau) &= \Theta_{00}^2(2\tau) + \Theta_{10}^2(2\tau), \\ \Theta_{01}^2(\tau) &= \Theta_{00}^2(2\tau) - \Theta_{10}^2(2\tau), \\ \Theta_{10}^2(\tau) &= 2\Theta_{00}(2\tau)\Theta_{10}(2\tau), \end{aligned} \tag{4.10}$$

and plugging into (4.9), we find that

$$Z_{D3}(\tau) = 0, \tag{4.11}$$

consistent with supersymmetry.

4.2. $d = 6$, $\mathcal{N} = (0,1)$ SYM

Low energy theory

From (4.2), using manipulations similar to the $d = 4$ case, we find

$$\begin{aligned} A^{NS}(q) &= q^{-\frac{1}{2}} + 4 + 13\,q^{\frac{1}{2}} + 40\,q + 106\,q^{\frac{3}{2}} + 256\,q^2 + \ldots \\ A^{\widetilde{NS}}(q) &= q^{-\frac{1}{2}} - 4 + 13\,q^{\frac{1}{2}} - 40\,q + 106\,q^{\frac{3}{2}} - 256\,q^2 + \ldots \\ \frac{1}{2}A^{R}(q) &= 4 + 40\,q + 256\,q^2 + \ldots \end{aligned} \tag{4.12}$$

The $D5$-branes preserve eight supercharges, and the low energy theory is then determined to be the $(0,1)$ supersymmetric Yang-Mills theory in six dimensions.

Full partition function

As sketched in Appendix E, one can rewrite the partition function as:

$$\begin{aligned} Z^{NS} &= \frac{\Theta_{00}^4(\tau)}{\eta^9(\tau)} \sum_{r=0}^{\infty} (-1)^r \left(q^{-\frac{1}{2}(r-\frac{1}{2})^2} - q^{-\frac{1}{2}(r+\frac{1}{2})^2} \right) \\ Z^{\widetilde{NS}} &= \frac{\Theta_{01}^4(\tau)}{\eta^9(\tau)} \sum_{r=0}^{\infty} (-1)^r \left(q^{-\frac{1}{2}(r-\frac{1}{2})^2} - q^{-\frac{1}{2}(r+\frac{1}{2})^2} \right) \\ Z^{R} &= \frac{\Theta_{10}^4(\tau)}{\eta^9(\tau)} \sum_{r=0}^{\infty} (-1)^r \left(q^{-\frac{1}{2}(r-\frac{1}{2})^2} - q^{-\frac{1}{2}(r+\frac{1}{2})^2} \right) \\ Z^{\widetilde{R}} &= 0. \end{aligned} \tag{4.13}$$

Using the commonly encountered Jacobi identity, we again get the result that the partition function vanishes.

$$Z_{D5}(\tau) = 0. \tag{4.14}$$

4.3. $d = 2$ and $\mathcal{N} = (0,2)$ SYM

Low energy theory

For $d = 2$, we find the q-expansion of the partition functions to be

$$\begin{aligned} A^{NS}(q) &= q^{-\frac{1}{2}} + 0 + q^{\frac{1}{2}} + 2\,q + 5\,q^{\frac{3}{2}} + 6\,q^2 + 8\,q^{\frac{5}{2}} + 14\,q^3 + \ldots \\ A^{\widetilde{NS}}(q) &= q^{-\frac{1}{2}} + 0 + q^{\frac{1}{2}} - 2\,q + 5\,q^{\frac{3}{2}} - 6\,q^2 + 8\,q^{\frac{5}{2}} - 14\,q^3 + \ldots \\ A^{R}(q) &= 2(1 + 2\,q + 6\,q^2 + 14\,q^3 + \ldots.) \\ A^{\widetilde{R}}(q) &= -2. \end{aligned} \tag{4.15}$$

By the now familiar argument of low energy spectrum, and exact supersymmetry, we find that in this case, the low energy theory on the D-strings is $d = 2$, $\mathcal{N} = (0,2)$ SYM. As this case has new features, we present below the full partition function.

Full partition function

We have:

$$
\begin{aligned}
Z^{NS} &= \frac{\Theta_{00}(\tau)}{\eta^3(\tau)} \sum_{s\in\mathbb{Z}+\frac{1}{2}} \frac{1}{1+q^s}\left(q^{\frac{3(s^2-s)}{2}} - q^{\frac{3(s^2+s)}{2}}\right)\\
Z^{\widetilde{NS}} &= \frac{\Theta_{01}(\tau)}{\eta^3(\tau)} \sum_{s\in\mathbb{Z}+\frac{1}{2}} \frac{(-1)^{s-\frac{1}{2}}}{1-q^s}\left(q^{\frac{3(s^2-s)}{2}} + q^{\frac{3(s^2+s)}{2}}\right)\\
Z^{R} &= \frac{\Theta_{10}(\tau)}{\eta^3(\tau)} \sum_{s\in\mathbb{Z}} \frac{1}{1+q^s}\left(q^{\frac{3(s^2-s)}{2}} - q^{\frac{3(s^2+s)}{2}}\right)\\
Z^{\widetilde{R}} &= -2
\end{aligned}
\tag{4.16}
$$

At this point, we would like to make a couple of

Comments on fermion zero modes and vanishing of partition function:

1. Note here that in $d = 2$, we find two fermionic modes with $L_0 = 0$, both having $(-1)^F = -1$. $Tr_R(-1)^F$ does not vanish due to the presence of poles for two terms that come with the same sign. The usual free fermion degeneracy in the R sector is lifted in the coset for very particular states. This is seen as the fact that an operator $G_0^{\pm}$ annihilates a particular state, instead of giving rise to a degenerate state, or as the presence of a pole in the partition function. Whenever the coset is combined with SCFT's with fermion zero modes (like for $d = 4, 6$), we will not notice this subtlety (since the extra excitations re-introduce the degeneracy, and make for the vanishing of the total twisted R-sector partition function).
2. It is still true that the partition function vanishes and two parallel D-strings do not feel any force. However, the fermion zero modes which are projected out need to be understood better, and may lead to interesting physics.

5. Backreaction on the NSNS closed string background

In this section, we concentrate on the case of $d = 4$ and compute the backreaction on the cigar background by N $D3-$branes. To first order, this is given by the appropriately transformed one-point function of closed string fields in the presence of the boundary state (3.7) multiplied by a factor of N. In particular, we shall compute the shift in the background value of the fields in the gravity multiplet, and the tachyon multiplet. We assume that the dilaton (the trace of the second rank tensor) couples to the kinetic term in the gauge theory and interpret the profile of the dilaton in the radial direction of the cigar as the running gauge theory coupling (See also e.g. [32,33,34,20] for similar phenomena in related contexts).

Indeed, earlier work on holography in asymptotically linear dilaton backgrounds leads us to expect that closed string dynamics in the background of N $D3-$branes in the six dimensional theory is dual to an open string theory which flows at low energies ($\rho \to 0$) arbitrarily close to the flow of pure $\mathcal{N}=1$ SYM. Our first interest therefore is the backreaction on the dilaton near the tip of the cigar where one might expect agreement with the familiar logarithmic running of the coupling in the four dimensional supersymmetric gauge theory.

In order to get a "geometric interpretation", we use the following strategy: We use as an input the exact form of the one point functions and reflection amplitudes. As a tool during the computation, we approximate the exact closed string vertex operators with their mini-superspace approximate wavefunction. Even though we work in a strongly curved string background, where the curvature is of order the string length, we can have some faith in our calculations. First of all, the mini-superspace approximation will not be as bad as it looks, since it is known that for $\mathcal{N}=2$ supersymmetric coset models the background metric and dilaton do not receive further curvature corrections [35,36]. Moreover, at appropriate intuitively understandable junctions in the calculation, we replace semi-classical approximations by their known exact counterparts.

5.1. Bulk graviton

The notation is as discussed in Section 2 and the Appendices: the primary vertex operators in the cigar SCFT are denoted $\Phi^{j,n,\overline{n}}_{m_{bos}\overline{m}_{bos}}$. Their overlap with the boundary state is called $\Psi^{j,n,\overline{n}}_{m_{bos}\overline{m}_{bos}}$, as in (3.6). The minisuperspace field configuration for these operators are denoted ϕ^j_m.

The asymptotic behavior of the closed string field perturbation (2.4) in the presence of the D-branes (3.7) is given (in momentum space) by the amplitude with the insertion of a closed string propagator (we are omitting to explicitly show that $n=\overline{n}=0$ to avoid cluttering the equations):

$$\frac{1}{N}\widetilde{h}^{IJ}(k^\mu,P)\equiv\langle\mathcal{V}^{j\,IJ}(k^\mu)|D_{cl}|\mathbf{B3}\rangle=\eta^{IJ}\frac{\delta^4(k^\mu)}{\frac{1}{2}k_\mu k^\mu-j(j+1)}\Psi^{j,NS}_{00}(P)\,, \tag{5.1}$$

where $\mathcal{V}$ is the vertex operator for the graviton introduced in section 2. To get the profile in position space, we fold this with the solution of the Laplacian in the six-dimensional background, which factorizes as $V_{k^\mu,P}(x^\mu,\rho)=e^{ik_\mu X^\mu}\phi_0(\rho,P)$. The delta functions in the flat directions reduce the expression to a one dimensional integral:

$$\frac{1}{N}h^{IJ}(x^\mu,\rho)=\eta^{IJ}\int_0^\infty dP\left(\frac{\phi_0^{NS}(\rho,P)}{P^2+a^2}\right)\Psi^{NS}(P). \tag{5.2}$$

with $a=1/2$. The exact one point function of the operator $\Phi_{00}^{-\frac{1}{2}+iP}$ is (3.6) denoted:

$$\Psi_{00}^{-\frac{1}{2}+iP,NS}\equiv\Psi^{grav}(P)=\nu^{iP}\frac{\Gamma(\frac{1}{2}-iP)^2}{(\Gamma(-2iP)\Gamma(1-2iP))}\,. \tag{5.3}$$

The delta-function normalized minisuperspace field $\phi_0^{-\frac{1}{2}-iP,NS}$ is[7] [21]:

$$\phi_0^{-\frac{1}{2}-iP,NS}(\rho)\equiv\phi_0^{grav}(\rho,P)=-\frac{\Gamma(\frac{1}{2}+iP)^2}{\Gamma(2iP)}F\left(\frac{1}{2}-iP,\frac{1}{2}+iP;1;-\sinh^2\rho\right)\,. \tag{5.4}$$

[7] Note that in (5.2), the field used to implement the "Fourier transform" to get the profile in position space is the complex conjugate of the field whose one point function we compute in (5.3).

where $F(a,b;c;z)$ is the hypergeometric function ${}_2F_1$. The exact reflection amplitude

$$R^{grav}(P) \equiv \frac{\Gamma(2iP)\Gamma(\frac{1}{2}-iP)^2\Gamma(1+2iP)\nu^{iP}}{\Gamma(-2iP)\Gamma(\frac{1}{2}+iP)^2\Gamma(1-2iP)\nu^{-iP}} \tag{5.5}$$

is obtained by putting $m_{bos} = \overline{m}_{bos} = 0$ in the expression for $R(P)$ in Appendix B; it obeys $R^{grav}(P)R^{grav}(-P) = 1$. Using this, the closed string field configuration can be written as:

$$\begin{aligned}\phi_0^{grav}(\rho,P) = \Bigg[&(\sinh\rho)^{2iP-1}F\left(\frac{1}{2}-iP,\frac{1}{2}-iP;1-2iP;-\frac{1}{\sinh^2\rho}\right)\\ &+R^{grav}(-P)(\sinh\rho)^{-2iP-1}F\left(\frac{1}{2}+iP,\frac{1}{2}+iP;1+2iP;-\frac{1}{\sinh^2\rho}\right)\Bigg]\,.\end{aligned} \tag{5.6}$$

In (5.6), we have written the minisuperspace solution in terms of the basis of incoming and outgoing modes propagating on the cigar, which is better suited to the asymptotic description. If we rewrite the solution (5.4) in the above asymptotic basis by using the connection formula (Eg: (15.3.7) of [37]), and replace the semiclassical reflection amplitude by the quantum one, we recover (5.6).

The expression for the linear backreaction on the graviton field is now:

$$\begin{aligned}\frac{1}{N}h^{IJ}(x^\mu,\rho) = \eta^{IJ}&\int_0^\infty dP\frac{1}{P^2+a^2}\times\left[\nu^{iP}\frac{\Gamma(\frac{1}{2}-iP)^2}{(\Gamma(-2iP)\Gamma(1-2iP))}\right.\\ &\left.(\sinh\rho)^{2iP-1}F\left(\frac{1}{2}-iP,\frac{1}{2}-iP;1-2iP;-\frac{1}{\sinh^2\rho}\right)\right]+[P\leftrightarrow -P]\\ = \eta^{IJ}&\int_{-\infty}^\infty dP\frac{1}{P^2+a^2}\nu^{iP}\frac{\Gamma(\frac{1}{2}-iP)^2}{(\Gamma(-2iP)\Gamma(1-2iP))}\times\\ &(\sinh\rho)^{2iP-1}F\left(\frac{1}{2}-iP,\frac{1}{2}-iP;1-2iP;-\frac{1}{\sinh^2\rho}\right)\\ \equiv \eta^{IJ}&I_{grav}\end{aligned} \tag{5.7}$$

In fact the above integral needs a more precise contour prescription for it to be well-defined[8]. First of all, we note that if we had taken only the semi-classical one-point function and reflection amplitude, the above integral (without the factor $\Gamma(1-2iP)$ in the denominator) would allow for closing the contour in the P upper half-plane, leading to the evaluation of the integral as the residue of the pole on the positive imaginary axis[9]. For the above approximation to the exact result, a contour prescription will be more subtle, since the behaviour of the extra Γ function factor does not allow for the naive semi-classical contour. Although this subtlety is important, we believe that a good approximation to the exact result is given by evaluating the above integral at the pole in the propagator on the positive imaginary axis. (One can dictate a corresponding contour prescription.) We will indeed see in the following in various instances that this prescription

[8] We would like to thank Justin David and Edi Gava for emphasizing this important point.

[9] Alternatively, one can compute the integral making use of a Schwinger parametrization of the integral, and a fundamental integral representation of the hypergeometric function.

captures a lot of the expected physics. The caveat that we described will be present in further backreaction computations as well.

The poles of the integrand are only at $P = \pm ia$, and we prescribed to pick up the pole on the positive imaginary axis, $P = +ia$, $a > 0$. We have, with $a = \frac{1}{2}$:

$$
\begin{aligned}
I_{grav} &= \frac{2\pi i}{2ia}\nu^{-a}(\sinh\rho)^{-2a-1}F\left(\frac{1}{2}+a,\frac{1}{2}+a;1+2a;-\frac{1}{\sinh^2\rho}\right) \\
&= 2\pi\nu^{-\frac{1}{2}}(\sinh\rho)^{-2}F\left(1,1;2;-\frac{1}{\sinh^2\rho}\right) \\
&= 2\pi\nu^{-\frac{1}{2}}\log\left(1+\frac{1}{\sinh^2\rho}\right) .
\end{aligned} \tag{5.8}
$$

After multiplying by N, we get the expression for the graviton field:

$$
h^{IJ} = \eta^{IJ}2\pi N\nu^{-\frac{1}{2}}\log\left(1+\frac{1}{\sinh^2\rho}\right) . \tag{5.9}
$$

In the two limits of large and small radial distance, we find that

$$
\begin{aligned}
h^{IJ}(\rho) &\longrightarrow \eta^{IJ}\ \left[2\pi N\nu^{-\frac{1}{2}}e^{-2\rho}\right] \qquad \text{as} \qquad \rho\to\infty \qquad \text{and} \\
&\longrightarrow \eta^{IJ}\ \left[-4\pi N\nu^{-\frac{1}{2}}\log\rho\right] \qquad \text{as} \qquad \rho\to 0\,.
\end{aligned} \tag{5.10}
$$

Comments:

1. We would like to point out here that the "Fourier Transform" we performed in order to convert the momentum space one-point function into the position space profile used only the *continuous series* $j = -\frac{1}{2} + iP$. However, our result (5.8) is simply the profile of an on-shell mode in the *discrete series* with asymptotic behavior $e^{-2\rho}$. The graviton mode with polarization in the cigar directions is precisely the interaction operator in the worldsheet theory which is the first correction from the cylinder towards the cigar.[10]
2. This mode is normalizable at the weak coupling end. In this respect, our result is similar to the ones by [38], in that the the localized branes sources the normalizable mode on the cigar.[11]

5.2. Relation to gauge theory

The following is a short note on understanding the physics of the calculation above from the point of view of the holographic theory. The cigar background had a metric which was asymptotically flat and a dilaton which behaved as $\Phi(\rho) = -\log\cosh\rho$. The one-point function calculation tells us that in the presence of the D-branes, the fields (2.4) whose behaviour in the radial direction

[10] The metric on the cigar (2.1) asymptotically looks like $ds^2 = d\rho^2 + d\theta^2 + e^{-2\rho}d\theta^2$.

[11] Our branes can be thought of as the analog of the ZZ branes of Liouville theory.

is e^{Φ} shift from the background value by (5.7), (5.8). The trace of this second rank tensor gives us the change in the dilaton:

$$\delta(e^{\Phi}) \sim 2\pi N \nu^{-\frac{1}{2}} \log\left(1 + \frac{1}{\sinh^2 \rho}\right). \tag{5.11}$$

To connect with the gauge theory on the D-branes, we use the expansion of this equation near the tip of the cigar where we get:

$$\delta(e^{-\Phi}) \sim 4\pi N \nu^{-\frac{1}{2}} \log \rho. \tag{5.12}$$

This can be understood semiclassically – notice that the region near the tip cigar behaves like flat space[12] with a constant dilaton. In such a region, a pointlike source has a propagator which is logarithmic in two dimensions. If we use a Born-Infeld type action for the world-volume theory on the D-brane using the background closed string fields,

$$S_{D3} = \tau_3 \int d^4x \left(e^{-\Phi} TrF^2 + ...\right) \tag{5.13}$$

and add this as a source to the closed string equations of motion, we recover (5.11). Here τ_3 is a dimensionful quantity entering the tension of the brane.

From the Born-Infeld action (5.13), it is clear that $e^{-\Phi}$ acts as the coupling constant g_{YM}^{-2} of the gauge theory. Also, in a putative holographic duality between $\mathcal{N} = 1$ SYM and the closed string theory in the background of these D-branes, it is reasonable to expect that the closed string field $e^{-\Phi}$ couples to the operator TrF^2 in the action. Putting all of these facts together, we get:

$$\frac{1}{g_{YM}^2} - \frac{1}{g_{YM,0}^2} \sim N \log(\rho/\Lambda) \tag{5.14}$$

Comments;

1. We see here that the radial coordinate on the cigar ρ plays the role of the scale in the gauge theory. The constant Λ, the strong coupling scale in the gauge theory is related to where the RG flow in the IR theory is matched to the full stringy UV complete theory.
2. There is a factor of $\nu^{-\frac{1}{2}}$ present in (5.9). We shall see in the next section that this should be interpreted as the renormalized string coupling at the tip $(g_{s,ren}^{tip})^{-1}$. This renormalization is a redefinition of the zero mode of the dilaton. In the gauge theory, such a redefinition would be a change in the strong coupling scale Λ.

5.3. Bulk tachyon winding mode

This state has $m = \overline{m} = \frac{1}{2}$, and possesses the following reflection amplitude (with $k = 1$):

$$R^{tach}(P) = \nu^{2iP} \frac{\Gamma(2iP)\Gamma(1+2iP)\Gamma(1-iP)\Gamma(-iP)}{\Gamma(-2iP)\Gamma(1-2iP)\Gamma(1+iP)\Gamma(iP)}. \tag{5.15}$$

[12] Note that this is a space of size string scale – there is an overall factor of α' which multiplies the metric.

The (generalized) wavefunction for this mode is

$$
\begin{aligned}
\phi_P^{tach}(\rho) &= \cosh\rho\, e^{\pm i\sqrt{\frac{1}{2}}(\theta-\tilde{\theta})}\, \frac{\Gamma(1+iP)\Gamma(iP)}{\Gamma(1+2iP)\Gamma(2iP)} F(1+iP, 1-iP, 1, -\sinh^2\rho), \\
&= \cosh\rho\, e^{\pm i\sqrt{\frac{1}{2}}(\theta-\tilde{\theta})}\, (\sinh\rho)^{-2iP-2} F\left(1+iP, 1+iP; 1+2iP; -\frac{1}{\sinh^2\rho}\right) + \\
& \qquad R^{tach}(-P)\cosh\rho(\sinh\rho)^{2iP-2} F\left(1-iP, 1-iP; 1-2iP; -\frac{1}{\sinh^2\rho}\right).
\end{aligned}
\tag{5.16}
$$

In the second line, we rewrote the wavefunction in variables suited to the asymptotic region consistent with the above reflection amplitude as in the graviton case. We can now see easily that the asymptotic behavior is $\Phi^{tach}(\rho) \equiv T(\rho) = \mu e^{\pm i\frac{1}{\sqrt{2}}(\theta-\tilde{\theta})-\rho}$. This behavior is on the edge of the Seiberg window of operators non-normalizable at the weak coupling end. In this respect, the $\mathcal{N}=2$ Liouville theory with $k=1$ is similar to the $c=1$ bosonic Liouville theory, and we can extend the understanding gained in that case [39] to this one.

First, we note that objects in the theory are singular in the limit $k \to 1$, and we regularize as $k = 1+\epsilon$. In order to keep quantities like the two and three point functions in the bulk theory finite, we need to keep $\tilde{\mu}\frac{\Gamma(1/k)}{\Gamma(1-1/k)} \equiv \nu^{-1}$ finite. Using the relation between the mirror parameters, this means that the bare $\mathcal{N}=2$ Liouville interaction diverges. To see what this implies, let us look at the full wavefunction of the tachyon winding mode including the reflected piece. The reflection amplitude for the mode in the action $j = \frac{k}{2}$ has the value $R=-1$. The asymptotic behavior is (keeping only the leading behavior):

$$
\begin{aligned}
\Phi^{tach}(\rho) \equiv T^{phys}(\rho) &= \mu \lim_{\epsilon\to 0} e^{\pm i\sqrt{\frac{1+\epsilon}{2}}(\theta-\tilde{\theta})} \left(e^{-(1+\epsilon)\rho} + R(1-\epsilon)e^{-(1-\epsilon)\rho}\right) \\
&= -(\mu\epsilon)\,\rho\, e^{\pm i\frac{1}{\sqrt{2}}(\theta-\tilde{\theta})} e^{-\rho} \equiv \mu_{ren}\,\rho\, e^{\pm i\frac{1}{\sqrt{2}}(\theta-\tilde{\theta})-\rho}
\end{aligned}
\tag{5.17}
$$

Comments:

1. It is clear that we should keep the quantity μ_{ren} defined above finite and the relations between the various parameters $(g_{s,ren}^{tip})^{-2} = \mu_{ren}^2 = \tilde{\mu}_{ren} \equiv \nu^{-1}$ where all the quantities are now finite and tunable.
2. Keeping track of all the terms in the above computation tells us that the full tachyon winding mode has also a normalizable piece which behaves as $\Phi \sim \mu_{ren}\log\mu_{ren} e^{\pm i\frac{1}{\sqrt{2}}(\theta-\tilde{\theta})-\rho}$. We will see below that this is the mode that is sourced by the brane.

5.4. Backreaction on the tachyon winding mode

The one point function of this mode on our brane is given by:

$$
\Psi^{tach}(P) = -\nu^{iP}\frac{\Gamma(1-iP)\Gamma(-iP)}{\Gamma(1-2iP)\Gamma(-2iP)}
\tag{5.18}
$$

We can collect the above pieces as before to get the expression for the backreaction:

$$
\begin{aligned}
\delta T(x^\mu,\rho) &= e^{\pm i\sqrt{\frac{1}{2}}(\theta-\tilde{\theta})} \int_0^\infty dP \frac{1}{P^2} \times \left[\nu^{iP} \frac{\Gamma(1-iP)\Gamma(-iP)}{\Gamma(1-2iP)\Gamma(-2iP)} \right. \\
&\qquad \left. \cosh\rho\,(\sinh\rho)^{2iP-2} F\left(1-iP,1-iP;1-2iP;-\frac{1}{\sinh^2\rho}\right)\right] + [P\leftrightarrow -P] \\
&= e^{\pm i\sqrt{\frac{1}{2}}(\theta-\tilde{\theta})} \int_{-\infty}^\infty dP \frac{1}{P^2} \nu^{iP} \frac{\Gamma(1-iP)\Gamma(-iP)}{\Gamma(1-2iP)\Gamma(-2iP)} \times \\
&\qquad \cosh\rho\,(\sinh\rho)^{2iP-2} F\left(1-iP,1-iP;1-2iP;-\frac{1}{\sinh^2\rho}\right) \\
&\equiv e^{\pm i\sqrt{\frac{1}{2}}(\theta-\tilde{\theta})}\, I_{tach}
\end{aligned}
\tag{5.19}
$$

We can compute the integral using the principal value prescription[13]:

$$
\begin{aligned}
I_{tach} &= \frac{1}{\epsilon}\cosh\rho\,(\sinh\rho)^{-2} F\left(1,1,1;-\frac{1}{\sinh^2\rho}\right) \\
&= \frac{1}{\epsilon}(\cosh\rho)^{-1}; \\
&\longrightarrow \frac{1}{\epsilon}e^{-\rho},\ \rho\to\infty; \\
&\longrightarrow \frac{1}{\epsilon},\ \rho\to 0;
\end{aligned}
\tag{5.20}
$$

From the discussion of the bulk tachyon field above, it is clear now that the one-point function for the bare tachyon field[14] must diverge, and must be interpreted as $\delta T \sim (\delta\mu)e^{-\rho}$. The physical statement to be inferred from the above is:

$$
\delta T^{phys}(x^\mu,\rho) = \mu_{ren} e^{\pm i\frac{1}{\sqrt{2}}(\theta-\tilde{\theta})}\,(\cosh\rho)^{-1}\ . \tag{5.21}
$$

6. Backreaction on the RR fields

We shall now repeat the analysis for the Ramond-Ramond fields which the Dp-brane source. We shall focus on the case of the $D3$-brane which is charged under the dual of the axion field. The quantum numbers of the RR scalar has been discussed earlier in section 2. The reflection amplitude is given by

$$
R^R(P) = \nu^{2iP}\frac{\Gamma(2iP)}{\Gamma(-2iP)}\frac{\Gamma(1-iP)\Gamma(-iP)}{\Gamma(1+iP)\Gamma(iP)}\frac{\Gamma(1+2iP)}{\Gamma(1-2iP)}\,. \tag{6.1}
$$

We now compute the profile of the RR field in spacetime following the procedure outlined in the NSNS case. We need to compute the overlap of the RR vertex operator with the boundary state in Appendix A in the $(-\frac{3}{2},-\frac{1}{2})$ picture [40,41]:

$$
\frac{1}{N}\mathcal{A}^{\dot{\alpha}\beta}(k,P) = \langle \mathcal{V}^{\dot{\alpha}\beta}_{-\frac{3}{2},-\frac{1}{2}}(k)|D_{cl}|\mathbf{B3}\rangle = \frac{\delta^4(k^\mu)}{P^2}\,\Psi^j_{\frac{1}{2},\frac{1}{2}}\mathcal{N}^{\dot{\alpha}\beta}\,. \tag{6.2}
$$

[13] For the subtleties involved in this procedure: see the discussion following equation (5.7) .

[14] Note that we have done the above backreaction calculation by using the expressions in the theory for general k *without* renormalizing the parameters.

where $j=-\frac{1}{2}+iP$, and $\mathcal{N}^{\dot\alpha\beta}$ is the result of the zero mode overlap between the Ramond sector ground state of the boundary state and the spin fields. In this picture, we actually compute the background value of the gauge potential [41], by taking the trace of (6.2) with the appropriate Γ-matrices:

$$\frac{1}{N}A_{\mu_1\ldots\mu_n}(k)=\mathrm{Tr}(\mathcal{A}(k)\,C\,\Gamma_{\mu_1}\ldots\Gamma_{\mu_n})\,.$$

In position space, we thus get the profile of the gauge field to be

$$\frac{1}{N}A_{\mu_1\ldots\mu_n}(\rho)=\int_0^\infty\frac{dP}{P^2}\,\phi_P^R(\rho)\,\Psi_P^R\,\mathrm{Tr}(\mathcal{N}\,C\,\Gamma_{\mu_1}\ldots\Gamma_{\mu_n})\,. \tag{6.3}$$

where Ψ_P^R is given by (3.6) with $m_{bos}=\overline{m}_{bos}=\frac{1}{2}$

$$\Psi_P^R\equiv\Psi^j_{\frac{1}{2},\frac{1}{2}}=\nu^{iP}\frac{\Gamma(1-iP)\Gamma(-iP)}{\Gamma(-2iP)\Gamma(1-2iP)}\,.$$

The Ramond sector of our boundary state (3.7) tells us that the only non-zero gauge potential is A_{0123}.

It now remains to obtain the solution to the Laplace equation $\phi_P^R(\rho)$ that implements the generalized Fourier transform. For modes with $m=\overline{m}$, this solution should have, in the semiclassical limit, the correct reflection amplitude obtained from the coset algebra. The required wavefunction in the minisuperspace approximation is given by

$$\phi_P^R(\rho)\equiv\phi_P^{R,j=-\frac{1}{2}-iP}=\cosh\rho\;\frac{\Gamma(1+iP)\Gamma(iP)}{\Gamma(2iP)}F(1-iP,1+iP;1;-\sinh^2\rho) \tag{6.4}$$

which has an asymptotic expansion:

$$\begin{aligned}\phi_P^R(\rho)=\cosh\rho\;\Big[&(\sinh\rho)^{-2+2iP}\,F(1-iP,1-iP;1-2iP;-\frac{1}{\sinh^2\rho})\\ &+R^R(-P)\,(\sinh\rho)^{-2-2iP}\,F(1+iP,1+iP;1+2iP,-\frac{1}{\sinh^2\rho})\Big]\,,\end{aligned} \tag{6.5}$$

where the quantum reflection amplitude is given by (6.1). Indeed, as for the NS-NS case, we can use the connection formula for the hypergeometric function to check that it reproduces the classical reflection amplitude in the Ramond-Ramond sector [22] with $k\to\infty$.

We can now repeat the analysis of the NS-NS sector. Substituting the above expressions in (6.3) , we get

$$\begin{aligned}&\frac{1}{N}g_sA_{0123}(\rho)\\ &=\int_{-\infty}^\infty\frac{dP}{P^2}\nu^{iP}\frac{\cosh\rho}{(\sinh\rho)^{2-2iP}}\frac{\Gamma(1-iP)\Gamma(-iP)}{\Gamma(-2iP)\Gamma(1-2iP)}F(1-iP,1-iP;1-2iP;-\frac{1}{\sinh^2\rho})\end{aligned} \tag{6.6}$$

This integral is divergent, with a behavior $\frac{1}{\epsilon}\cosh^{-1}\rho$ with a suitable regulator. To understand this, let us write the vertex operator in the $(-\frac{3}{2},-\frac{1}{2})$ picture [41]. The operator looks like

$$W^{RR}(k)=\mathcal{A}_{\dot\alpha\beta}(k)\mathcal{V}^{\dot\alpha}_{-3/2}(k)\mathcal{V}^{\beta}_{-1/2}(k)+i\mathcal{F}_{\dot\alpha\dot\beta}(k)\mathcal{V}^{\dot\alpha}_{-3/2}(k)\mathcal{V}^{\dot\beta}_{-3/2}(k)\,\overline{\partial}\tilde{c}\,\tilde{\xi}. \tag{6.7}$$

Let us write the vertex operators asymptotically with ρ dependence $e^{(p-1)\rho}$. BRST invariance of the vertex then implies $p\mathcal{F}(p) = 0$ and $p\mathcal{A}(p) = \mathcal{F}(p)$. In position space, these are $g_s d * g_s^{-1}(g_s\mathcal{F}) = 0$ and $g_s d g_s^{-1}\mathcal{A} = g_s\mathcal{F}$.

This shows us the meaning of the divergence – for a constant field strength, we have $\mathcal{A}(p \to 0) = \mathcal{F}(p \to 0)/p$, and in position space, this will translate to $\mathcal{A} = \rho e^{-\rho}$. To keep all our calculations finite, it is also clear what to do: compute the field strength from the beginning:

$$\begin{aligned}
&\frac{1}{N}F_{0123\rho}(\rho) \equiv I_R(\rho) \\
&= \partial_\rho g_s^{-1}\int_{-\infty}^{\infty}\frac{dP}{P^2}\nu^{iP}\frac{\cosh\rho}{(\sinh\rho)^{2-2iP}}\frac{\Gamma(1-iP)\Gamma(-iP)}{\Gamma(-2iP)\Gamma(1-2iP)}F(1-iP,1-iP;1-2iP;-\frac{1}{\sinh^2\rho}) \\
&= \int_{-\infty}^{\infty}\frac{dP}{P^2}\nu^{iP}\partial_\rho\left[\frac{\cosh^2\rho}{(\sinh\rho)^{2-2iP}}\frac{\Gamma(1-iP)\Gamma(-iP)}{\Gamma(-2iP)\Gamma(1-2iP)}F(1-iP,1-iP;1-2iP;-\frac{1}{\sinh^2\rho})\right] \\
&= \int_{-\infty}^{\infty}\frac{dP}{P^2}\nu^{iP}(\sinh\rho)^{2iP}\frac{\Gamma(1-iP)\Gamma(-iP)}{\Gamma(-2iP)\Gamma(1-2iP)} \\
&\qquad\left[\left(2\coth\rho + (-2+iP)\coth^3\rho\right)F(1-iP,1-iP;1-2iP;-\frac{1}{\sinh^2\rho})\right. \\
&\qquad\left. + 2\coth^3\rho\frac{1}{\sinh^2\rho}\frac{(1-iP)^2}{(1-2iP)}F(2-iP,2-iP;2-2iP;-\frac{1}{\sinh^2\rho})\right].
\end{aligned} \tag{6.8}$$

We use again the principal value of the integral and evaluating the pole as $P \to 0$, we get:

$$\begin{aligned}
I_R &= \frac{1}{P}\left[\left(2\coth\rho + (-2+iP)\coth^3\rho\right)\tanh^2\rho + 2\coth^3\rho\frac{1}{\sinh^2\rho}\tanh^4\rho\right] \\
&= \frac{1}{P}iP\coth^3\rho\tanh^2\rho \\
&= \coth\rho.
\end{aligned} \tag{6.9}$$

Dualizing, we get $\chi \sim N\theta$. We shall fix the coefficient in the next section. For now, we note that we can integrate to get an expression for the (non-normalizable) vertex operator for the potential $g_s A_{0123}(\rho) = (\cosh\rho)^{-1}\log\sinh\rho$.

6.1. A short note on instantons and the chiral $U(1)_R$ symmetry breaking.

The $U(1)_R$ symmetry of the Super Yang-Mills theory is realized in the string theory dual as the conserved $U(1)$ momentum around the cigar. Adding the $D3$-branes at the tip sources a constant RR axion field strength, and the axion field hence depends linearly on the angular coordinate with a coefficient proportional to N which can be determined by channel duality of the annulus amplitude. We shall proceed to fix this coefficient using electric-magnetic duality of the action that describes the massless RR fields.

In the ten dimensional superstring, we know that the shift symmetry of the RR axion is non-perturbatively broken. This is also the case in the non-critical superstring theories, where the shift is realized as translation around the angular direction of the cigar θ. To test this, as usual, we consider a D-instanton in the theory which is charged under this axion field. At all orders in perturbation theory, the zero mode of χ is a modulus, but this mode multiplies the action of a D-instanton, and for the string theory path integral to be well-defined even after summing over instanton configurations, we deduce that the zero mode is only defined upto periodic identifications.

The type IIB theory contains odd dimensional D-branes with a Chern-Simons coupling $\mu_p \int C_{p+1}$ for $p = -1, 1, 3, 5$ where the RR potentials C_p are canonically normalized. The action at tree level has a symmetry which exchanges electric and magnetic states under the various gauge potentials. The Dirac quantization condition then implies $\mu_p \mu_{2-p} = 2\pi$.

In the presence of N $D3$-branes, we have $\chi = \mu_3 N \frac{\theta}{2\pi}$. The Chern-Simons coupling of the D-instanton is then $\delta S_{D(-1)} = \mu_{-1}\mu_3 N \frac{\theta}{2\pi} = N\theta$; it follows that the geometric $U(1)$ isometry of the cigar is broken to Z_N. Let us remind ourselves that the normalization of the $U(1)_R$ charge (4.3) was defined such that the fermions in spacetime have a half-integer charge. This normalization is sensible from the geometric point of view as smooth boundary conditions at the tip of the cigar enforce antiperiodicity of the fermions. On the other hand, the normalization of the $U(1)_R$ current in the gauge theory is such that a rotation of 2π gives the gluini a phase of unity. This makes it clear that $\theta_{cig} = 2\theta_{SYM}$.

Putting the above facts together tells us that the modification of the closed string background shows a breaking of the chiral $U(1)_R$ symmetry of the theory to $\mathbf{Z}_{2N}$ as expected. It is not clear from our construction how the chiral symmetry is broken further to Z_2. This is expected to involve the exact form of the axion field in the deep IR which is beyond the scope of this work.

7. Conclusions

We have taken an exact conformal field theory approach towards the construction of interesting gauge theory physics in lower-dimensional superstring theory backgrounds. The construction and explicit analysis of the full open string spectrum was done using known boundary conformal field theory results for the free scalar conformal field theory and the conformal field theory of the cigar $SL(2, R)/U(1)$. We concentrated on the branes that are localized at the tip of the cigar, and analyzed them in different target space dimensions. For $d = 4$, we argued that the low-energy spectrum and effective action are those of $N = 1$ super Yang-Mills theory.

Supersymmetry of the open string spectrum was shown directly in the open string channel, in several complementary ways: technically and precisely, using non-trivial theta-function identities, and conceptually (and very generically), following the techniques applied previously to supersymmetric bulk compactifications. The conceptual proof applies to generic compact and non-compact Gepner models.

We analyzed some of the information encoded in the exact boundary states. We made a first analysis of the resulting backreaction on the closed string background and the physics of the gauge theories that is reflected in it, e.g. the logarithmic running of the coupling constant of $N = 1$ super Yang-Mills theory, and the breaking of the $U(1)_R$ symmetry to $\mathbf{Z}_{2N}$.[15]

Clearly, there is room for further analysis of the closed string backreaction. In particular, one would like to go beyond the linear approximation for the pure $N = 1$ super Yang-Mills theory. Since the compactification is at string scale, an exact conformal field theory approach to this problem would be most convincing – however, supersymmetry may validate a low-energy approach. In particular, it would be desirable to have an analytic solution to the six dimensional non-critical supergravity (along the lines of [42,43]) that corresponds to these D-branes.

It is interesting to include flavors in the construction presented here [42,43,15], to for instance compare the relative normalizations of the running of the gauge coupling. Moreover, it will be interesting to study chiral matter in the $\mathcal{N} = 1$ gauge theory, following the techniques for obtaining chiral matter in brane set-ups (see e.g. [44] and references therein). This should allow for a splitting of the multiplets analyzed in [19,15].

A closer analysis of the gauge theory physics, for every individual even dimension d is equally desirable. We believe that the economical brane construction and tools provided in this paper may serve a further analysis well. Finally, we may hope that the exact conformal field theory treatment of this background allows for a continuous interpolation between the more familiar gauge theory physics and the highly stringy physics at the typical length scale of the cigar (i.e. the string scale). We already saw an example of the convincing simplicity of this interpolation in the analysis of the linear backreaction on the dilaton – one may hope that this gives us a privileged window on little string theory and holography at the string scale.

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[15] It would be desirable to investigate the further breaking to Z_2.

Appendix A. A few details of the construction of the full boundary states

In the following, we will construct D-branes that are extended in all of the flat spacetime directions, and that are point-like in the cigar directions.[16] The notion of being point-like is of course a semiclassical statement. The defining feature of the branes we shall study is that they are BPS (B branes), preserve the momentum around the cigar and localized near the tip.

We will denote the boundary state by $|Bp\rangle$ with $p = d-1$. In the $\mathbb{R}^{1,d-1}$ directions, we have the worldsheet equations

$$\begin{aligned} &\partial_\tau X^\mu(\sigma,0)|Bp_{X,\psi}\rangle = 0, \\ &(\psi^\mu - i\eta\tilde{\psi}^\mu)|Bp_{X,\psi}\rangle = 0, \quad \mu = 0,.,p = d-1. \end{aligned} \tag{A.1}$$

We will follow the conventions that the left-movers are holomorphic and the right movers (indicated by variables with tildes) are anti-holomorphic. In terms of the worldsheet modes, the equations become

$$\begin{aligned} &(\alpha_n^\mu + \tilde{\alpha}_{-n}^\mu)|Bp_{X,\psi}\rangle = 0, \\ &(\psi_r^\mu - i\eta\tilde{\psi}_{-r}^\mu)|Bp_{X,\psi}\rangle = 0, \quad \mu = 0,.,p \quad \forall\ n. \end{aligned} \tag{A.2}$$

The equations (A.1) and (A.2) are written in worldsheet coordinates suited to the closed string channel. Later on we rewrite these conditions in terms of the open string variables to derive the supersymmetries that are left unbroken by the D-brane.

The zero mode of the bosonic oscillators is the momentum $k^\mu = \frac{1}{2}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)$. In the fermionic sector, $r \in \mathbb{Z} + \frac{1}{2}$ in the NS sector and $r \in \mathbb{Z}$ in the R sector. The variable $\eta = \pm$ denotes the spin structure related to the $\mathbb{Z}_2$ automorphism of the gauged $\mathcal{N} = 1$ algebra implemented by the map $G \to \eta G$. In the R sector, there are fermion zero modes, and in that sector η indicates the choice of the eigenvalue of the operator $(-)^F$ acting on the ground state.

In the cigar part, the conditions on the boundary state can be written down in terms of the current algebra [45][22]. The fermionic part involves the conditions like those in equation (A.2) for the fermions $\psi_{cig}^\pm$. The same spin structure η as the flat space part must be imposed on the fermionic modes of the cigar.[17]

There is another $\mathbb{Z}_2$ ambiguity in the choice relating the left and right moving $\mathcal{N} = 2$ currents which tells us whether we have A or B branes. We shall be interested in B-type boundary conditions $J^R = -\overline{J}^R$, $G^\pm = i\eta\tilde{G}^\pm$ which leads to branes of the type (A.1) extended along the flat directions and localized on the cigar. This leads also to Neumann boundary conditions on the angular

[16] We will discuss some of the physics related to other choices of boundary states in due course.

[17] This means that the fermions on the cigar also obey Neumann boundary conditions as shown below in the zero mode equation for the fermions. We thank Angelos Fotopoulos, Vasilis Niarchos and Nikolaos Prezas for correcting us on this point.

direction of the cigar [22] . These imply that the one point functions have a delta function in the momentum n around the cigar.[18] In the conventions introduced earlier, this means that $m = \overline{m}$.

A.1. Ramond ground state

We need to solve for the zero mode factor of the boundary state that solves the conditions in equation (A.2). The R sector ground state of the theory also includes the solution of the zero modes equations of the two fermions of the coset. The fact that the worldsheet fermions are free facilitates this analysis. We can treat this part of the problem as one in flat space $R^{1,d+1}$ with the understanding that the Ramond ground states in the cigar have charges and conformal weights different from those of two dimensional flat space in such a way that the ground states we write down are weight one states of the full string theory. The full set of zero mode equations we wish to solve is then:

$$(\psi_0^I - i\eta\eta^I_J\tilde{\psi}_0^J)|\beta B p\rangle = 0, \quad I = 0, 1..d-1, d = \rho, d+1 = \theta. \tag{A.3}$$

We choose the following representation of the gamma matrices for even dimension d [46] for $\mathbb{R}^{1,d-1}$

$$\begin{aligned}
&\Gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \Gamma^\mu = \gamma^\mu \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\
&\Gamma^{d-2} = I \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \Gamma^{d-1} = I \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mu = 0, 1, ..., d-3.
\end{aligned} \tag{A.4}$$

We further define the parity and charge conjugation matrices:

$$\begin{aligned}
&\Gamma = i^{1-\frac{d}{2}}\Gamma^0\Gamma^1...\Gamma^{d-1} \qquad B_1 = \Gamma^3\Gamma^5...\Gamma^{d-1} \qquad B_2 = \Gamma B_1 \\
&C = B_1\Gamma^0 \quad \text{if} \quad d = 2 \bmod 4 \quad \text{and} \quad C = B_2\Gamma^0 \quad \text{if} \quad d = 0 \bmod 4\,.
\end{aligned}$$

The operator Γ is defined to have eigenvalues ± 1. The gamma matrices obey the relations:

$$\begin{aligned}
&\left\{\Gamma^I, \Gamma^J\right\} = \eta^{IJ} = diag(-1, 1, ...1), \\
&\left\{\Gamma, \Gamma^I\right\} = 0, \quad (\Gamma^I)^t = -C\Gamma^I C^{-1}.
\end{aligned} \tag{A.5}$$

Denoting the vacuum by $|B\rangle_0 = |A\rangle|\widetilde{B}\rangle$ where A and B are the $2^{\frac{d+1}{2}}$ dimensional spinor indices of $Spin(d+2)$, the action of $\psi^I, \tilde{\psi}^I$ are those of the gamma matrices as implied by the fermion

[18] We note that the conventions of [25] are opposite to ours (the BPS localized D-branes in IIB theory are the A branes in [16]) because they consider the theory on the T-dual circle with a condensate of momentum. Indeed, in the exact conformal field theory, the choice of A- or B-type boundary condition is arbitrary, since they are related by an automorphism of the $\mathcal{N} = 2$ superconformal algebra. It is only after we specify a specific semi-classical picture, i.e. give a geometrical interpretation to the $\mathcal{N} = 2$ currents and the branes in the conformal field theory that the distinction between A-type and B-type branes becomes meaningful.

zeromode commutation relations:

$$\begin{aligned}\psi_0^I|A\rangle|\widetilde{B}\rangle &= \frac{1}{\sqrt{2}}(\Gamma^I)^A_C(1)^B_D|C\rangle|\widetilde{D}\rangle\\ \tilde{\psi}_0^I|A\rangle|\widetilde{B}\rangle &= \frac{1}{\sqrt{2}}(\Gamma)^A_C(\Gamma^I)^B_D|C\rangle|\widetilde{D}\rangle.\end{aligned} \tag{A.6}$$

Note that the gamma-matrix Γ in the action of the right movers ensures that the left and right movers anti-commute. If we denote the solution of the zero-mode equation (A.3) by $\mathcal{M}_{AB}|A\rangle|\widetilde{B}\rangle$, we can translate the equation into one for the matrix of coefficients $\mathcal{M}$:

$$(\Gamma^I)^t\mathcal{M} - i\eta\,\eta^I_J\Gamma\mathcal{M}\Gamma^J = 0. \tag{A.7}$$

A solution to the equation is:

$$\mathcal{M} = C\Gamma\frac{(1+i\eta\Gamma)}{(1+i)}. \tag{A.8}$$

It is also useful to decompose the spinors into d dimensional spinors with specific chirality under Γ. Since Γ has eigenvalues ± 1, we can always choose a basis where the top half of the $2^{\frac{d}{2}}$ spinor and the bottom half have eigenvalues ± 1. We can then decompose the matrix $\mathcal{M}_{AB} = \begin{pmatrix} M_{\alpha\beta} & M_{\alpha\dot{\beta}} \\ M_{\dot{\alpha}\beta} & M_{\dot{\alpha}\dot{\beta}}\end{pmatrix}$. The vacuum solution can be written as a superposition:

$$|B,\eta=\pm\rangle_R = |\Omega\rangle_1 + i\eta|\Omega\rangle_2. \tag{A.9}$$

where the two terms are given by:

$$\begin{aligned}|\Omega\rangle_1 &= M_{\alpha\dot{\beta}}|\alpha\rangle|\widetilde{\dot{\beta}}\rangle\\ |\Omega\rangle_2 &= M_{\dot{\alpha}\beta}|\dot{\alpha}\rangle|\tilde{\beta}\rangle.\end{aligned} \tag{A.10}$$

Here, we have chosen the $(-\frac{1}{2},-\frac{3}{2})$ picture for the superghosts (which is a useful to compute one point functions [24][41]). Notice that the two terms have definite fermion number eigenvalue:

$$\begin{aligned}&(-)^F|\Omega\rangle_1 = |\Omega\rangle_1, \quad (-)^{\widetilde{F}}|\Omega\rangle_1 = |\Omega\rangle_1;\\ &(-)^F|\Omega\rangle_2 = -|\Omega\rangle_2, \quad (-)^{\widetilde{F}}|\Omega\rangle_2 = -|\Omega\rangle_2.\end{aligned} \tag{A.11}$$

which implies the relations:

$$(-)^F|B,\eta\rangle_R = (-)^{\widetilde{F}}|B,\eta\rangle_R = |B,-\eta\rangle_R. \tag{A.12}$$

This completes the discussion of the solution to the zero-mode conditions on the boundary state in the R-sector.

A.2. Spectral flowed extended characters

The open string partition function for the point-like brane in the cigar supercoset conformal field theory is most easily encoded in characters of the $\mathcal{N}=2$ superconformal algebra that are extended, i.e. summed over spectral flow orbits. Below, the character Ch_t denotes the extended character of the $\mathcal{N}=2$ superconformal algebra associated to the trivial representation of $SL(2,R)$. The trivial representation is a finite, one-dimensional representation. We can associate a spin $j=0$ to this representation (or $u=1$ in the notation of [22]). We recall the trivial unextended characters:

$$\begin{aligned}
ch_t(r;\tau,\nu)\begin{bmatrix}0\\0\end{bmatrix} &= q^{-\frac{1}{4k}+\frac{r^2}{k}}z^{\frac{2r}{k}}\frac{(1-q)}{(1+zq^{\frac{1}{2}+r})(1+z^{-1}q^{\frac{1}{2}-r})}\frac{\theta_3(\tau,\nu)}{\eta(\tau)^3}\\
ch_t(r;\tau,\nu)\begin{bmatrix}0\\1/2\end{bmatrix} &= q^{-\frac{1}{4k}+\frac{r^2}{k}}(-z)^{\frac{2r}{k}}\frac{(1-q)}{(1-zq^{\frac{1}{2}+r})(1-z^{-1}q^{\frac{1}{2}-r})}\frac{\theta_4(\tau,\nu)}{\eta(\tau)^3} \qquad \text{(A.13)}\\
ch_t(r';\tau,\nu)\begin{bmatrix}1/2\\0\end{bmatrix} &= q^{-\frac{1}{4k}+\frac{r'^2}{k}}z^{\frac{2r'}{k}}\frac{(1-q)}{(1+zq^{\frac{1}{2}+r'})(1+z^{-1}q^{\frac{1}{2}-r'})}\frac{\theta_2(\tau,\nu)}{\eta(\tau)^3}
\end{aligned}$$

in the notation of [22] (brought slightly closer to more standard notation). In the R-sector $r'=r+1/2$, i.e. it takes values in a range which is shifted compared to the NS-sector range. To define the extended characters, we perform a sum over spectral flow orbits. Suppose we have a rational level $k=\frac{N}{K}$, where K,N are strictly positive integers (and let's suppose they have greatest common divisor one). We define the extended characters by a sum over spectral flow orbits, determined by the integer N:

$$Ch_t(r;\tau,\nu)\begin{bmatrix}a/2\\b/2\end{bmatrix} = \sum_{n\in NZ} q^{\frac{cn^2}{6}}z^{\frac{cn}{3}}ch_t(r;\tau,\nu+n\tau)\begin{bmatrix}a/2\\b/2\end{bmatrix}. \qquad \text{(A.14)}$$

We compute and find the following characters:

$$\begin{aligned}
Ch_t(r;\tau,\nu)\begin{bmatrix}0\\0\end{bmatrix} &= \sum_{m\in Z} q^{-\frac{K}{4N}}q^{KN(m+\frac{r}{N})^2}z^{2K(m+\frac{r}{N})}\\
&\quad\left(\frac{1}{1+zq^{mN+r+\frac{1}{2}}}-\frac{1}{1+zq^{mN+r-\frac{1}{2}}}\right)\frac{\theta_3(\tau,\nu)}{\eta^3}\\
Ch_t(r;\tau,\nu)\begin{bmatrix}0\\1/2\end{bmatrix} &= \sum_{m\in Z} q^{-\frac{K}{4N}}q^{KN(m+\frac{r}{N})^2}(-z)^{2K(m+\frac{r}{N})}\\
&\quad\left(\frac{1}{1-zq^{mN+r+\frac{1}{2}}}-\frac{1}{1-zq^{mN+r-\frac{1}{2}}}\right)\frac{\theta_4(\tau,\nu)}{\eta^3} \qquad \text{(A.15)}\\
Ch_t(r';\tau,\nu)\begin{bmatrix}1/2\\0\end{bmatrix} &= \sum_{m\in Z} q^{-\frac{K}{4N}}q^{KN(m+\frac{r'}{N})^2}(z)^{2K(m+\frac{r'}{N})}\\
&\quad\left(\frac{1}{1+zq^{mN+r'+\frac{1}{2}}}-\frac{1}{1+zq^{mN+r'-\frac{1}{2}}}\right)\frac{\theta_2(\tau,\nu)}{\eta^3}.
\end{aligned}$$

Again $r'=r+1/2$, i.e. r' takes values in a shifted range compared to the NS-sector where $r\in Z_N$. For the twisted R-sector, generically, we need to be careful. Indeed, although the θ_1 function may be zero, there may appear a pole in the denominator of the other factors, for particular values of z and r'. We therefore note that generically we have:

$$\begin{aligned}
Ch_t(r';\tau,\nu)\begin{bmatrix}1/2\\1/2\end{bmatrix} &= \sum_{m\in Z} q^{-\frac{K}{4N}}q^{KN(m+\frac{r'}{N})^2}(-z)^{2K(m+\frac{r'}{N})}\\
&\quad\left(\frac{1}{1-zq^{mN+r'+\frac{1}{2}}}-\frac{1}{1-zq^{mN+r'-\frac{1}{2}}}\right)\frac{\theta_1(\tau,\nu)}{\eta^3}. \qquad \text{(A.16)}
\end{aligned}$$

The uses of these characters in the paper are as follows. First of all it is important to realize in the light of the proof of the vanishing of the GSO projected open string partition function given in appendix C that the open string partition function can indeed be written as a trivial, extended $N = 2$ character, and in particular that it corresponds to a sum over spectral flow orbits. This is manifest from the papers [25][47][22]. Secondly, it is important to have the twisted R-sector partition function at generic values of z, since it allows for the evaluation of possible singular terms in the limit $z \to 1$ (as in the case of $d = 2$ in the bulk of the paper). Moreover, as observed in [25][47][22] , the powerful formalism allows for easier generalization (to orbifolds, other levels, other backgrounds, etc). The above character formulas can easily be evaluated at the levels $k = 2, 1, 2/3, 1/2$ and yield (the coset factors in) the open string partition functions recorded in the bulk of the paper.

Appendix B. Closed String Vertex Operators on the Cigar

In this section, we construct the primaries of the supersymmetric coset $SL(2, \mathbb{R})/U(1)$ at level k. We follow the conventions of [48,22] and references therein. This theory has an $\mathcal{N} = 2$ superconformal symmetry. The parent supersymmetric $SL(2, \mathbb{R})_k$ theory has currents J^a and ψ^a which have coupled OPEs

$$\begin{aligned} J^a(z)J^b(0) &\sim \frac{\frac{k}{2}g^{ab}}{z^2} + f^{ab}_c J^c z \\ J^a(z)\psi^b(0) &\sim \frac{f^{ab}_c \psi^c}{z} \end{aligned} \tag{B.1}$$

with the fermions satisfying the usual OPEs. This theory is a product of a bosonic $SL(2, \mathbb{R})$ at level $k + 2$, generated by the currents

$$j^a = J^a - \hat{J}^a = J^a - \frac{i}{2} f^a_{bc} \psi^b \psi^c \,,$$

and three free fermions. The $U(1)$ symmetry to be gauged is generated by J^3, ψ^3_{cig}. The currents that make up the $\mathcal{N} = 2$ chiral algebra on the coset are:

$$\begin{aligned} T &= T_{SL_{2,\mathbb{R}}} - T_{U(1)} \\ G^{\pm} &= (\psi^1_{cig} \pm i\psi^2_{cig})(j^1 \mp ij^2) \\ J &= j^3 - i\psi^1_{cig}\,\psi^2_{cig} \equiv iQ\partial\theta + i\partial H_{cig} \equiv i\partial\phi \,. \end{aligned} \tag{B.2}$$

Now, the primaries of the bosonic $SL(2, \mathbb{R})$ (at level $k+2$) are denoted $V^j_{m_{bos}\overline{m}_{bos}}$, where m_{bos} is the charge under the purely bosonic j^3 current

$$j^3(z)\Phi^j_{m_{bos}\overline{m}_{bos}}(0) \sim \frac{m_{bos}\Phi^j_{m_{bos}\overline{m}_{bos}}}{z} \,.$$

They have (left and right) conformal dimensions

$$\Delta(V_{j,m_{bos},\overline{m}_{bos}}) = -\frac{j(j+1)}{k}\,.$$

As these fields are independent of the free fermions, they are also primary fields of the superconformal $SL(2,\mathbb{R})$ at level k. In order to obtain the primaries of the coset, it is useful to bosonize the various currents we have as follows:

$$\begin{aligned} \partial H_{cig} = i\psi^-_{cig}\psi^+_{cig} \qquad & J^3 = -\sqrt{\frac{k}{2}}\partial X_3 \\ J^R = i\sqrt{\frac{c}{3}}X_R \qquad & j^3 = -\sqrt{\frac{k+2}{2}}\partial x_3\,, \end{aligned} \tag{B.3}$$

where the normalizations ensure that the scalars have canonical OPEs. These scalars are not all independent and using the definition of the bosonic currents and the $\mathcal{N}=2$ (B.2), we can rewrite all scalars in terms of X_3 and X_R:

$$\sqrt{\frac{k}{2}}x_3 = iX_R + \sqrt{\frac{k+2}{2}}X_3 \qquad \text{and} \qquad iH_{cig} = \sqrt{\frac{2}{k}}X_3 + i\sqrt{\frac{k+2}{k}}X_R\,.$$

Given these expressions, and knowing that the currents that are gauged in the coset are J^3 (in the bosonic case), we can decompose

$$V^j_{m_{bos}} = \Phi^j_{m_{bos}} e^{m\sqrt{\frac{2}{k+2}}x_3} \equiv \Phi^j_{m_{bos}} e^{2m_{bos}\sqrt{\frac{k+2}{k}}X_R}\, e^{m_{bos}\sqrt{\frac{2}{k}}X_3}\,, \tag{B.4}$$

where $\Phi^j_{m_{bos}}$ is a primary of the bosonic Euclidean coset CFT (at level $k+2$). One infers that

$$\Delta(\Phi^j_{m_{bos}}) = -\frac{j(j+1)}{k} + \frac{m^2_{bos}}{k+2}$$

In the supersymmetric coset, we also gauge the fermionic current ψ^3 and the primary we start with in the parent theory is of the form $V^j_{m_{bos}}e^{inH}$. The coset primaries are found by using (B.4) and

$$e^{inH} = e^{n(\sqrt{\frac{2}{k}}X_3 + i\sqrt{\frac{k+2}{k}}X_R)}\,.$$

These two equations lead to the decomposition of the primary in the parent theory of the form

$$V^j_{m_{bos}}\, e^{inH_{cig}} = \Phi^j_{m_{bos}}\, e^{i(\frac{2m_{bos}}{k+2}+n)\sqrt{\frac{k+2}{k}}X_R}\, e^{\sqrt{\frac{2}{k}}(m_{bos}+n)X_3}\,, \tag{B.5}$$

which allows one to infer that the superconformal coset primary is given by

$$\Phi^{j,n}_{m_{bos}} = \Phi^j_{m_{bos}} e^{i(\frac{2m_{bos}}{k+2}+n)\sqrt{\frac{k+2}{k}}X_R}\,.$$

It is clear from equation (B.5) that the J^3 eigenvalue of the operator is given by

$$m = m_{bos} + n\,.$$

In terms of m and n, the conformal dimension is read off to be

$$\Delta(\Phi^{j,n}_{m_{bos}}) = -\frac{j(j+1)}{k} + \frac{(m_{bos}+n)^2}{k} + \frac{n^2}{2} = -\frac{j(j+1)}{k} + \frac{m^2}{k} + \frac{n^2}{2} \,,$$

while the R-charge is given by

$$Q(\Phi^{j,n}_{m_{bos}}) = \frac{k+2}{k}\left(\frac{2m_{bos}}{k+2} + n\right) = \frac{2m_{bos}}{k} + \frac{n(k+2)}{k} = \frac{2m}{k} + n \,.$$

In the NSNS sector, we have $n \in \mathbb{Z}$, while in the RR sector, we have $n \in \mathbb{Z} + \frac{1}{2}$. The full closed string field is obtained by putting together the left and right moving pieces and yields $\Phi^{j,n,\overline{n}}_{m_{bos},\overline{m}_{bos}}$. The axial gauging of the coset is done such that the J^3 and $\overline{J}^3$ eigenvalues m and $\overline{m}$ are are related to the asymptotic momentum and winding of the circle direction of the cylinder at infinity[19]

$$m = \frac{n+kw}{2} \qquad \overline{m} = -\frac{n-kw}{2} \,.$$

In the full theory, we tensor together these operators with the flat space operators as in (2.3). The physical operators are obtained by imposing BRST invariance and a GSO projection which tie together the two Hilbert spaces.

B.1. Reflection Amplitude

For completeness, we also write down the reflection amplitude for the bulk fields. It is defined by the two point function of the fields

$$\langle \Phi^{j,n,\overline{n}}_{m_{bos},\overline{m}_{bos}}(1)\Phi^{j',n'\overline{n}'}_{m'_{bos},\overline{m}'_{bos}}(0)\rangle = [\delta(j+j'+1) + R(j,m_{bos},\overline{m}_{bos})\delta(j-j')]\,\delta_{m_{bos}+m'_{bos}}\,\delta_{\overline{m}_{bos}+\overline{m}'_{bos}}$$

where

$$R(j,m_{bos},\overline{m}_{bos}) = \nu^{2j+1}\frac{\Gamma(2j+1)\Gamma(-j+m_{bos})\Gamma(-j-\overline{m}_{bos})\Gamma(1+\frac{2j+1}{k})}{\Gamma(-2j-1)\Gamma(j+1+m_{bos})\Gamma(j+1-\overline{m}_{bos})\Gamma(1-\frac{2j+1}{k})} \,. \tag{B.6}$$

Note that irrespective of whether the operators are in the NSNS or RR sector, the reflection amplitude is determined by the quantum numbers $m_{bos}, \overline{m}_{bos}$ from the $k+2$-level bosonic current algebra.

Appendix C. The construction of the conserved space-time supercharges

In this appendix, we show that a combination of the supercurrents $S_\alpha(z)$ and $\tilde{S}_\alpha(\overline{z})$ is conserved on the open string worldsheet; in other words, the supercharges $\int dz\; S_\alpha(z) + \int d\overline{z}\,\tilde{S}_\alpha(\overline{z})$ annihilate

[19] The negative sign follows from the axial gauging of the coset.

the boundary state. This construction is subtle at the quantum mechanical level [49,23,24]. We shall prove it at the classical level by studying the boundary conditions on the worldsheet fields.

As already discussed in Appendix B, the supersymmetric coset $SL_2/U(1)$ is described as a product of the bosonic $SL(2,\mathbb{R})_{k+2}$ and the two free fermions $\psi^{\pm}_{cig}$ which have OPE's which do not mix [17] . Also, the fermions ψ^{a}_{cig} have conformal weight $\frac{1}{2}$ and can be bosonized as

$$(\psi^1_{cig} + i\psi^2_{cig}) \equiv \Psi_{cig} = e^{iH_{cig}}\,. \tag{C.1}$$

We have already exhibited the $\mathcal{N}=2$ algebra of the coset in (B.2). The $\mathcal{N}=2$ supersymmetry in the $\mathbb{R}^{1,3}$ factor can be seen by forming the linear combinations

$$\begin{aligned} &Z^0 = X^0 + X^1 \qquad \overline{Z}^0 = -X^0 + X^1 \qquad Z^1 = X^2 + iX^3 \qquad \overline{Z}^1 = X^2 - iX^3 \ldots \\ &\Psi^0 = \psi^0 + \psi^1 \qquad \overline{\Psi}^0 = -\psi^0 + \psi^1 \qquad \Psi^1 = \psi^2 + i\psi^3 \qquad \overline{\Psi}^1 = \psi^2 - i\psi^3 \ldots \end{aligned} \tag{C.2}$$

in terms of which

$$\begin{aligned} &T_{mat} = -\frac{1}{2}\partial\overline{Z}^a\partial Z_a - \frac{1}{2}(\overline{\Psi}^a\partial\Psi_a + \Psi^a\partial\overline{\Psi}_a) \qquad J_{mat} = -\overline{\Psi}^a\Psi_a \\ &G^+_{mat} = i\Psi^a\partial\overline{Z}_a \qquad G^-_{mat} = i\overline{\Psi}^a\partial Z_a \qquad a = (0,\ldots,\frac{d-2}{2}). \end{aligned} \tag{C.3}$$

We bosonize the fermions and denote them

$$\Psi^a = e^{iH_a} \quad \overline{\Psi}^a = e^{-iH_a}\,. \tag{C.4}$$

Denoting the bosonized superghost by φ, the dimension one operator which is the spacetime supercharge is [4]

$$S = e^{-\frac{\varphi}{2}}\, e^{\frac{i}{2}(H_1 + \ldots + H_{cig} + Q\theta)}\,. \tag{C.5}$$

In order to verify that the boundary state we have constructed is supersymmetric (at least at the classical level), we will begin with the boundary conditions (A.3)

$$\left(\psi^I - i\eta\,\eta^I_J\,\tilde{\psi}^J\right)|_{\partial\Sigma} = 0, \tag{C.6}$$

and construct the linear combination of (C.5) and its right-moving counterpart that is preserved by the D-brane. Rewriting this in the bosonized variables, we get the boundary conditions:

$$\left(H_a = \tilde{H}_a + \frac{\pi\eta}{2}\right)|_{\partial\Sigma}, \quad a = 0,\ldots,\frac{d-2}{2}, cig. \tag{C.7}$$

The superconformal ghosts obey the boundary condition [23]

$$\left(\varphi = \tilde{\varphi} + \frac{i\pi\eta}{2}\right)|_{\partial\Sigma}. \tag{C.8}$$

We need to supplement these standard boundary conditions with boundary conditions for the chiral boson θ which preserve the Neumann condition on the boson:

$$\left(\theta = \tilde{\theta} + \frac{\pi\eta Q}{2}\right)|_{\partial\Sigma}. \tag{C.9}$$

We note as a check that for the case $d = 6$, when the chiral boson is at the free fermion radius, the boundary condition (C.9) is the same as the one for the other fermions (C.7).

Using (C.7), (C.9) and (C.8), one can check that the combination of supercharges that preserve the boundary condition is $\left(S + \tilde{S}\right)$. Other supercharges, which are local with respect to this one are obtained by flipping the signs in front of two of the three bosonized fields H_a. One can show that for all of these supercharges, the same left-right combination preserves the boundary condition.

Appendix D. A generic proof of the vanishing of supersymmetric open string partition functions

We sketch a generic proof of the vanishing of the GSO projected open string partition function. Our aim in this section is to clarify conceptually the mechanism underlying the delicate cancellation of the bosonic and fermionic contributions to the partition functions. To lay bare the generic mechanism, we export some important lessons from the construction of bulk supersymmetric partition functions to our context (see especially [28] for a nice summary of the relevant techniques in the bulk). The basic differences with the bulk analysis is that for the annulus amplitude, we needn't worry about modular invariance, and on the other hand that we wish to prove the supersymmetry of the *open* string spectrum – we therefore apply the relevant part of the strategy of [28] in the open string channel directly.

Let's assume then, following [28] that we are in light-cone gauge – which most straightforwardly encodes the physical spectrum. The (open string channel) $N = 2$ superconformal field theory has central charge $c = 12$. The (total transverse) $U(1)_R$ current can then be written as $J = i2\partial\phi$, where ϕ is a canonically normalized scalar. We define an associated operator (in the open string channel):

$$O_{1/2} = e^{i\phi}\,,$$

which is the operator that implements spectral flow by half a unit. The proof of the vanishing of the partition functions is based on two assumptions. Firstly, we assume that the partition function in the open string channel consists of (supersymmetric) characters which are of the following form:

$$\chi_{susy}(\tau) = \sum_{n\in Z}(-1)^n O_{1/2}^n \chi_n(\tau)\,,$$

namely, it is an alternating sum (over bosons and fermions) of contributions that are related by half a unit of spectral flow, implemented by the operator $O_{1/2}$.

We observe that the degrees of freedom associated to the $U(1)_R$ scalar ϕ may be fermionized using one (chiral) complex fermion. The complex fermion will be in the R-sector, provided that all total light-cone $U(1)_R$ charges are odd. That is the second assumption. (The assumption is realized by performing a GSO projection.)

We expect that the operators $O_{1/2} = e^{i\phi}$ and $O_{1/2}^\dagger = e^{-i\phi}$, leave the total partition function invariant (up to a minus sign). For the first operator, this is manifest, while for the second it follows from the bijective character of spectral flow (see also [28] for a slightly different argument).

The vanishing of the partition function can now be argued for as follows. We concentrate on the fermionized $U(1)_R$ scalar. Decompose the $c = 12$ theory into a $c = 1$ theory of a complex fermion in the R-sector and a theory with central charge $c = 11$. The action of the operators $O_{1/2}, O_{1/2}^\dagger$ is on the first theory only, and it generates all states, starting from a ground state (typically denoted $|s = -\frac{1}{2}\rangle$). The partition function in the open string sector will therefore be of the form:

$$Z = \sum_i \chi^i_{susy} = \sum_i (\frac{\theta_{1,2}}{\eta} - \frac{\theta_{-1,2}}{\eta}) Z^i_{c=11} = \sum_i \frac{\theta_{11}}{\eta} Z^i_{c=11}.$$

The first term in the third expression corresponds to R-charges that are one modulo four, while the second term corresponds to R-charges that are three modulo four. The relative sign is necessary for the partition function to change sign under the action of $O_{1/2}$. The partition function is zero – it has a factor that is zero, corresponding to a twisted R-sector partition function for a chiral complex fermion.

In summary, the proof of the vanishing of the open string partition function is generic, provided we can show that it is of the form we assumed above. It is more straightforward then its closed string counterpart because we needn't worry about modular invariance. The adaptation from the proof in [28] is conceptual, in that we thought of it as applying to the open string channel. Note that the above proof provides the rationale for the theta-function identities proven in the bulk of the paper and a lot of other identities that may be difficult to prove otherwise, and that it explains the generic logic for the implementation of the GSO projection in the open string channel. We note for instance that some identities for which numerical evidence was provided in [50] (referred to in [25] in a context very close to that of our paper) can be proven in the above fashion. Note that the proof equally well applies to the case of D-branes in compact Gepner models [51][52] . In all these cases, the conditions on the proof are met.

Appendix E. Details of the partition functions of the various open string theories

E.1. d=4

For the $D3$-brane in $d=4$, the sum over s in (4.5) can be simplified as follows

$$\begin{aligned}\sum_{s\in\frac{1}{2}+\mathbb{Z}}\frac{1}{(1+q^s)}\left(q^{s^2-s}-q^{s^2+s}\right) &= \left[\sum_{s\in-\frac{1}{2}-\mathbb{Z}_+}+\sum_{s\in\frac{1}{2}+\mathbb{Z}_+}\right]\frac{1}{(1+q^s)}\left(q^{s^2-s}-q^{s^2+s}\right)\\ &=\sum_{s\in\frac{1}{2}+\mathbb{Z}_+}\left(q^{s^2+s}-q^{s^2-s}\right)\left(\frac{1}{(1+q^{-s})}-\frac{1}{(1+q^s)}\right)\\ &=\sum_{s\in\frac{1}{2}+\mathbb{Z}_+}q^{s^2-s}\left(1-q^s\right)^2\\ &=q^{-\frac{1}{4}}-2q^{\frac{1}{4}}+2q^{\frac{3}{4}}+\dots\end{aligned}$$

Using the standard product representation for the Θ-functions, we get

$$\left(\frac{\Theta_{00}(it)}{\eta^3(it)}\right)^2=q^{-\frac{1}{4}}\prod_1^\infty\frac{(1+q^{m-\frac{1}{2}})^4}{(1-q^m)^4}=q^{-\frac{1}{4}}+4q^{\frac{1}{4}}+10q^{\frac{3}{4}}+\dots$$

Multiplying the two contributions leads to[20] formula (4.6).

To prove the vanishing of the exact partition function, we use the identity (where the sum is a formal sum over any set),

$$\begin{aligned}\sum_s\frac{q^{s^2-s}-q^{s^2+s}}{1\pm q^s}&=\sum_s q^{s^2-s}\left(\frac{1-q^{2s}}{1\pm q^s}\right)=\sum_s q^{s^2-s}(1\mp q^s)\\ &=\sum_s q^{s^2-s}\mp\sum_s q^{s^2}=q^{-\frac{1}{4}}\sum_s q^{(s-\frac{1}{2})^2}\mp\sum_s q^{s^2}\end{aligned}\tag{E.1}$$

We can then re-express

$$\begin{aligned}A^{NS}&=\left(\frac{\Theta_{00}(\tau)}{\eta^3(\tau)}\right)^2\left(-\Theta_{10}(2\tau)+e^{-\frac{i\pi\tau}{2}}\Theta_{00}(2\tau)\right)\\ A^{\widetilde{NS}}&=\left(\frac{\Theta_{01}(\tau)}{\eta^3(\tau)}\right)^2\left(\Theta_{10}(2\tau)+e^{-\frac{i\pi\tau}{2}}\Theta_{00}(2\tau)\right)\\ A^{R}&=\left(\frac{\Theta_{10}(\tau)}{\eta^3(\tau)}\right)^2\left(-\Theta_{00}(2\tau)+e^{-\frac{i\pi\tau}{2}}\Theta_{10}(2\tau)\right)\\ A^{\widetilde{R}}&=0.\end{aligned}\tag{E.2}$$

[20] We also plugged the modular functions into a symbolic manipulation program and got the expansion to high order.

E.2. d=6

We use a formal power series expansion to re-express the first term Z^{NS} as[21]:

$$
\begin{aligned}
Z^{NS} &= \frac{\Theta_{00}^3(\tau)}{\eta^9(\tau)} \sum_{s\in\mathbb{Z}+\frac{1}{2}} \sum_{r=0}^{\infty} (-1)^r q^{rs} q^{-\frac{1}{8}} \left(q^{\frac{1}{2}(s-\frac{1}{2})^2} - q^{\frac{1}{2}(s+\frac{1}{2})^2} \right) \\
&= \frac{\Theta_{00}^3(\tau)}{\eta^9(\tau)} \sum_{r=0}^{\infty} (-1)^r q^{\frac{r}{2}-\frac{1}{8}} \sum_{s\in\mathbb{Z}+\frac{1}{2}} q^{r(s-\frac{1}{2})} \left(q^{\frac{1}{2}(s-\frac{1}{2})^2} - q^{\frac{1}{2}(s+\frac{1}{2})^2} \right) \\
&= \frac{\Theta_{00}^3(\tau)}{\eta^9(\tau)} \sum_{r=0}^{\infty} (-1)^r q^{\frac{r}{2}-\frac{1}{8}} \sum_{n\in\mathbb{Z}} \left(q^{\frac{1}{2}n^2+nr} - q^{\frac{1}{2}(n+1)^2+nr} \right) \\
&= \frac{\Theta_{00}^3(\tau)}{\eta^9(\tau)} \sum_{r=0}^{\infty} (-1)^r q^{\frac{r}{2}-\frac{1}{8}} \left(\sum_{m\in\mathbb{Z}} q^{\frac{1}{2}m^2} \right) \left(q^{-\frac{r^2}{2}} - q^{-\frac{r^2}{2}-r} \right) \\
&= \frac{\Theta_{00}^4(\tau)}{\eta^9(\tau)} \sum_{r=0}^{\infty} (-1)^r \left(q^{-\frac{1}{2}(r-\frac{1}{2})^2} - q^{-\frac{1}{2}(r+\frac{1}{2})^2} \right)
\end{aligned}
\tag{E.3}
$$

Using similar manipulations, we get the other the equations in (4.13).

[21] It is dangerous to use the formal power series expansion for $s < 0$. In a more careful analysis, one splits the sum over s into two sums, both containing positive powers of q as we did previously.

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