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Massive Kaluza-Klein Theories and their Spontaneously Broken Symmetries

by

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Massive Kaluza-Klein Theories and their Spontaneously Broken Symmetries

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Abstract

In this thesis we investigate the effective actions for massive Kaluza-Klein states, focusing on the massive modes of spin-3/2 and spin-2 fields. To this end we determine the spontaneously broken gauge symmetries associated to these 'higher-spin' states and construct the unbroken phase of the Kaluza-Klein theory. We show that for the particular background $AdS_3 \times S^3 \times S^3$ a consistent coupling of the first massive spin-3/2 multiplet requires an enhancement of local supersymmetry, which in turn will be partially broken in the Kaluza-Klein vacuum. The corresponding action is constructed as a gauged maximal supergravity in $D = 3$. Subsequently, the symmetries underlying an infinite tower of massive spin-2 states are analyzed in case of a Kaluza-Klein compactification of four-dimensional gravity to $D = 3$. It is shown that the resulting gravity–spin-2 theory is given by a Chern-Simons action of an affine algebra and also allows a geometrical interpretation in terms of 'algebra-valued' differential geometry. The global symmetry group is determined, which contains an affine extension of the Ehlers group. We show that the broken phase can in turn be constructed via gauging a certain subgroup of the global symmetry group. Finally, deformations of the Kaluza-Klein theory on $AdS_3 \times S^3 \times S^3$ and the corresponding symmetry breakings are analyzed as possible applications for the AdS/CFT correspondence.

Zusammenfassung

In dieser Arbeit untersuchen wir die effektiven Wirkungen für massive Kaluza-Klein Zustände, mit besonderer Beachtung von massiven Spin-3/2 und Spin-2-Moden. Zu diesem Zweck bestimmen wir die diesen 'höheren Spin' Feldern assoziierten spontan gebrochenen Symmetrien und konstruieren die ungebrochene Phase der Kaluza-Klein Theorie. Wir zeigen für den speziellen Hintergrund $AdS_3 \times S^3 \times S^3$, dass die konsistente Kopplung des ersten massiven Spin-3/2 Multipletts eine Erweiterung der lokalen Supersymmetrie erfordert, die wiederum im Kaluza-Klein Vakuum partiell gebrochen ist. Die korrespondierende Wirkung wird als eine geeichte maximale Supergravitationstheorie in $D = 3$ konstruiert. Anschliessend werden die Symmetrien analysiert, die im Falle einer Kaluza-Klein Kompaktifizierung von vierdimensionaler Gravitation nach $D = 3$ einem unendlichen Turm massiver Spin-2-Zustände zugrundeliegen. Es wird gezeigt, dass die resultierende Gravitations–Spin-2 Theorie gegeben ist durch die Chern-Simons Wirkung einer affinen Algebra und dass sie weiterhin eine geometrische Interpretation vermittels 'Algebra-wertiger' Differentialgeometrie erlaubt. Die globale Symmetriegruppe, die eine affine Erweiterung der Ehlers-Gruppe enthält, wird bestimmt. Wir zeigen, dass durch Eichung einer bestimmten Untergruppe der globalen Symmetriegruppe die gebrochene Phase konstruiert werden kann. Schlussendlich werden Deformationen der Kaluza-Klein Theorie auf $AdS_3 \times S^3 \times S^3$ und die korrespondierenden Symmetriebrechungen als mögliche Anwendungen der AdS/CFT - Korrespondenz analysiert.

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Chapter 1

Introduction

1.1 String theory as a candidate for quantum gravity

By now it is common wisdom that one of the most important and also most challenging problems of theoretical physics is the reconciliation of gravity, i.e. of general relativity, with quantum theory. Any 'naive' approach to the problem of quantizing gravity along the lines of conventional field theories has failed, in that the resulting theories are non-renormalizable, so that they can be viewed at best only as an effective description.

It is generally appreciated that the conceptually different nature of general relativity lies at the heart of the problem of formulating a consistent theory of quantum gravity. As a so-called background-independent theory it is invariant under the full diffeomorphism group of the underlying manifold. In accordance with that there is no preferred reference frame and no fixed background space-time. Instead, space-time is completely determined dynamically. Since any conventional formulation of quantum field theory relies in contrast on a background space-time (as Minkowski space) with its preferred notion of distance, causality and time, it is not surprising that the quantization of gravity is a complicated problem. Therefore, quantum gravity struggles with a number of conceptually severe problems, like the famous so-called 'problem of time' (for a pedagogical introduction see, e.g., [1].)

Among the approaches to quantize gravity are those which try to maintain the conventional ideas of quantum field theory, like the notion of particles, S-matrix etc., while the others aim to take seriously the lessons which general relativity has taught us by maintaining general covariance also in the quantum theory. For instance, 'loop quantum gravity' belongs to the latter [2, 3, 4]. While these theories are truly background-independent, they suffer from the humbling drawback that it is not clear how to get a semi-classical limit or even how to identify the observables [5]. String theory on the other hand is a perfectly well-defined *perturbative* quantum theory of gravity [6, 7]. Even though it extends the standard concepts of quantum field theory in the sense that it replaces the notion of a point-particle by a one-dimensional string, it still yields a conventional formulation in that it basically predicts certain S-matrix

elements. It provides a framework to compute finite results for, say, graviton-graviton-scattering on a Minkowski background to arbitrary accuracy. Accordingly, string theory in its original formulation is not background-independent, since it quantizes the propagation of strings on a fixed 10-dimensional background space-time (which is usually taken to be Minkowski space). Among the excitations of the string is the massless spin-2 mode, which is interpreted as the graviton, i.e. as the (quantized) fluctuations of the metric around the fixed background space-time.[1] Even though string theory thus provides a consistent theory of quantum gravity, it is unable to answer the conceptually important problems mentioned above, simply because of its intrinsically perturbative formulation. In particular, string theory incorporates the symmetry principles known from classical general relativity at best only in a quite intricate way.

Apart from the massless spin-2 mode, the string spectrum contains massless spin-0 and spin-1 states together with their fermionic superpartners, as well as an infinite tower of massive higher-spin states. The former support the picture of unification, namely that all known interactions and matter fields should combine into one common theory. The latter in turn have been argued to result from a spontaneous symmetry breaking of an infinite-dimensional symmetry [8, 9]. But, what kind of symmetry this might be and how the 'unbroken phase' of string theory might look like has always been unclear. Again we are faced with the problem that the (symmetry) principles underlying string theory remain unknown. It would be clearly desirable to uncover these principles, since they would presumably enrich our understanding in the same sense as the notion of general covariance and the equivalence principle have done for our understanding of gravity [10].

It should be noted that string theory is not unique, instead there are five consistent (super-) string theories in 10 dimensions: Type IIA and type IIB, heterotic string theory with gauge groups $SO(32)$ and $E_8 \times E_8$ and type I superstring theory. In their low-energy description, i.e. in the limit that the strings look effectively like particles, they are described by field theories, which are the associated supergravity theories. Beyond these supergravities related to string theories there exists a unique 11-dimensional supergravity theory, whose origin was unclear from the point of view of 'conventional' string theory.

However, this picture of string theory changed dramatically during the second string revolution in 1995 [11] (for reviews see [12, 13]). Since then various dualities have been argued to exist, which relate the different string theories together with 11-dimensional supergravity. As the result of this change of perspective it is now expected that there should exist a unifying framework for string theory, which is called 'M-theory', that presumably combines all known string theories into one common theory and whose low-energy limit is given by 11-dimensional supergravity. The different string theories will then appear as certain limits of M-theory. This theory is believed to provide a background-independent description of quantum gravity (in which not even the space-time dimension is given a priori), which at the same time unifies all

[1] From a particle physicist's point of view the diffeomorphism symmetry of general relativity is accordingly the gauge symmetry that guarantees consistency of the graviton's self interactions.

known interactions and matter fields. In particular, space-time should appear as an emergent phenomenon [14]. Moreover, such a theory should also be able to answer the conceptual questions about quantum gravity mentioned above. Thus, the question of the 'unbroken phase' of string theory translates in modern parlance into the question of the underlying symmetries of M-theory, whose identification would presumably be the first step in identifying this theory. Even though we are far away from an actual formulation of M-theory, the first glimpses of such a deeper understanding have already shown up, e.g. in the AdS/CFT correspondence or the holographic principle, which will be important in this thesis.

1.2 Holography and Kaluza-Klein theories

It was first proposed by 't Hooft and Susskind that a quantum theory of gravity should pervade a so-called holographic principle [15, 16]. This principle states that quantum gravity ought to have some description in terms of degrees of freedom that are defined in one dimension less. The reason for such an expectation results from the Bekenstein-Hawking formula for the entropy of a black hole, according to which the entropy is given not by the volume, but instead by the area of the black hole's event horizon. This in turn yields a bound for the entropy of a given region of space-time. In fact, if the entropy for such a region would be greater than expected from the area of its boundary one could violate the second law of thermodynamics by throwing matter into this region until a black hole forms. The latter would then have an entropy bounded by its area, thus implying an effective decrease of entropy. Since the entropy counts the number of microstates in that region, the entire information about the degrees of freedom therefore seems to be contained already in the boundary, i.e. effectively in one dimension less.

The first concrete realization of this principle was given by Maldacena's conjecture or the so-called AdS/CFT correspondence [17, 18]. It claims an equivalence between certain string theories on backgrounds containing Anti-de Sitter spaces (AdS) on the one hand and conformal field theories on its boundary on the other hand. The most prominent form of the correspondence is the one between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super-Yang-Mills theory on four-dimensional Minkowski space (which may be viewed as the boundary of AdS_5). Another form of the correspondence, which we will mainly focus on in this thesis, is between type IIB string theory on $AdS_3 \times S^3 \times K$ (where the compact manifold K can be $S^3 \times S^1$, T^4 or $K3$) and certain two-dimensional conformal field theories.

More technically, the AdS/CFT correspondence associates to any field in the supergravity theory on AdS a source ϕ_0 in the CFT.[2] Any of these sources gives rise to a unique solution of the supergravity equations of motion which coincides with the CFT data on the boundary of AdS. The AdS/CFT duality in turn claims a precise relation between the correlation functions in the CFT and the supergravity action.

[2] In the following we will restrict ourselves mainly to the case, where the supergravity approximation of string theory is valid.

Schematically, the correlators can be computed according to the formula

$$\langle \exp \int_{S^d} \phi_0 \mathcal{O} \rangle_{\mathrm{CFT}} = \exp(-S(\phi)) \ , \tag{1.1}$$

where the supergravity action S has to be evaluated on the appropriate solutions of its equations of motion, and $\mathcal{O}$ denotes an operator in the CFT. Thus the knowledge of the full supergravity action allows the computation of all correlation functions, and vice versa. To be more precise, for the evaluation of higher-order correlators, all *non-linear* couplings in the supergravity action have to be known. For instance, if one wants to follow an RG flow in a broken phase of the field theory, one has to analyze arbitrary finite movements of the supergravity scalars in their potential. Therefore the precise shape of the scalar potential has to be known beyond quadratic order.

Concerning the problem of determining the non-linear couplings, it should be stressed that the AdS/CFT correspondence involves the full ten-dimensional string or supergravity theory, and not just the AdS factor. In fact, consistent geometric string backgrounds have to be ten-dimensional and should solve the supergravity equations of motion. Accordingly, the internal manifolds in $AdS_5 \times S^5$ or $AdS_3 \times S^3 \times K$ enter the supergravity action in a crucial way through the appearance of so-called Kaluza-Klein harmonics and their non-linear couplings.

Kaluza-Klein theories were originally introduced as an attempt to unify gravity with Yang-Mills gauge theories through the introduction of higher-dimensional space-times [19]. They are based on the assumption that the ground state of a gravity theory might not be given by a maximally symmetric space (i.e. Minkowski or de Sitter spaces), but instead by a product of a lower-dimensional space with a compact manifold ('spontaneous compactification'). The ground state manifold then may read

$$M_D = \mathbb{R}^{1,d-1} \times K_{D-d} \ , \tag{1.2}$$

where K_{D-d} denotes the compact internal manifold. The fields of the D-dimensional gravity theory can then be expanded in harmonics Y_a, satisfying

$$\left(\Box_{K_{D-d}} + m_a^2\right) Y_a = 0 \ , \tag{1.3}$$

after which one integrates the original action over the internal manifold. The effectively d-dimensional action then looks like a gravity theory composed of a finite number of massless fields (resulting from the zero-modes $m_a = 0$), coupled to an infinite tower of massive modes. Among the massless fields one finds Yang-Mills gauge fields, whose gauge group is given by the isometry group of the internal manifold, thus unifying space-time and internal symmetries. The massive states in turn have a mass scale $1/R$, where R is some characteristic length scale of the internal manifold (as, e.g., the radius of a circle). In most phenomenological considerations one assumes the compact manifold to be very small, such that the higher Kaluza-Klein modes are heavy and can in fact be integrated out. However, the internal manifolds appearing in compactifications on AdS are typically of the same size as the AdS space

itself, i.e. of 'cosmological' scales, in order to be a solution of the supergravity equations of motion.[3] The Kaluza-Klein harmonics have actually masses of order one, in units of the AdS length scale, while in contrast stringy excitations are much heavier. Thus, the inclusion of massive Kaluza-Klein states in the supergravity description is of crucial importance, while so far the focus was mainly on the effective description of the lowest modes. In this thesis we therefore aim to incorporate also Kaluza-Klein states of arbitrary mass. Since the direct construction of effective Kaluza-Klein actions in case of a generic compact manifold is a technically cumbersome problem, we will follow here instead a different strategy of constructing these theories directly in the lower-dimensional spacetime by identifying the underlying symmetries and the allowed couplings.

This strategy has also been followed in the original constructions of the effective supergravity on $AdS_5 \times S^5$ [20, 21, 22] (and also of 11-dimensional supergravity on $AdS_4 \times S^7$ and others [23, 24, 25]). The accomplishment of such a program is based on so-called gauged supergravities, which incorporate non-abelian gauge vectors into locally supersymmetric theories. Since the sphere S^5 leads according to the general Kaluza-Klein recipe to a gauge group $SO(6)$ and preserves moreover all supercharges, the theory can be constructed directly as a gauged maximal supergravity with this particular gauge group. This theory is unique, and it is therefore natural to identify it with a truncation to the 5-dimensional maximal supergravity multiplet of the full Kaluza-Klein theory. Even though this cannot be viewed as an effective description due to the low mass scale of the higher modes, it is believed to provide a consistent truncation.[4] More recently, a tower of massive spin-1 multiplets appearing on $AdS_3 \times S^3$ has been described as a unique gauging of three-dimensional supergravity [26]. (Effective supergravities for massive Kaluza-Klein states have also been considered in different contexts in [27, 28, 29] and more recently in [30, 31].)

As long as only scalar fields and their superpartner are considered, this strategy of constructing the effective supergravities applies. However, these theories require already the introduction of spin-1 fields as gauge fields. Moreover, as supergravities they contain spin-3/2 fields (gravitinos) and a spin-2 field (the metric). These 'higher-spin' fields are associated to local symmetries in the AdS bulk.[5] This in turn is the main obstacle for the direct construction of supergravity theories containing also higher Kaluza-Klein modes. In fact, the construction of interacting higher-spin theories seems to be prohibited in general by various no-go theorems which in principle apply to all fields with $s > 1$. The appearing consistency problems are precisely due to the fact that the local higher-spin symmetries apparently cannot be maintained at the interacting level [32]. Supergravity theories handle these no-go theorems in that they are able to couple the graviton self-consistently (as in general relativity) together with a finite number of gravitinos (due to the existence of local supersym-

[3] This will be shown explicitly in chapter 3 for the $AdS_3 \times S^3 \times S^3 \times S^1$ background.

[4] We will make this more precise in sec. 4.6, and will also try to illuminate why this statement is far from being self-evident. Nevertheless, throughout the thesis we will use the term 'effective' for the resulting Kaluza-Klein actions, even though it is in general not an effective description in the sense of 'integrating out' degrees of freedom in field theory.

[5] These local symmetries do not appear on the CFT side, but correspond to conserved currents.

metry). However, the consistency problems seem to reappear once all Kaluza-Klein modes are taken into account. The higher Kaluza-Klein modes of the metric and the gravitino will in fact show up as infinite towers of massive spin-2 and spin-3/2 fields coupled to gravity. While a finite number of massive spin-3/2 fields can be described within the framework of spontaneously broken supersymmetry (limited by the maximal number of real supercharges ≤ 32), an arbitrary number of them cannot be coupled consistently to gravity – not to speak of the spin-2 fields. On the other hand, we know that the infinite Kaluza-Klein towers of massive spin-2 and spin-3/2 fields have to be consistent, simply because their higher-dimensional ancestors are. Therefore the question appears, how are the no-go results mentioned above circumvented? Moreover, is it possible not only to describe these massive higher-spin states in a consistent way, but to realize them as the spontaneously broken phase of a theory possessing an enhanced local symmetry?

These questions were in part analyzed by Dolan and Duff in [33], where they showed that the higher-dimensional diffeomorphism group shows up as a spontaneously broken infinite-dimensional gauge symmetry in the lower-dimensional Kaluza-Klein theory. More precisely, the diffeomorphism group of the internal manifold will appear as a Yang-Mills-like gauge group, while an infinite-dimensional spin-2 symmetry is supposed to ensure the consistency of the gravity–spin-2 couplings. In subsequent investigations the focus was on the realization of the diffeomorphism algebra as a gauge symmetry [34, 35, 36]. In this thesis we will concentrate instead on the consistency of the infinite-dimensional spin-2 symmetries. Moreover, we will also discuss the similar phenomenon for spin-3/2 symmetries (for the early literature see [37]).

The strategy in this thesis will be to focus on compactifications to three space-time dimensions, and leave possible generalizations to arbitrary dimensions for a second step. Among other things this is motivated by the fact that the existence of so-called Chern-Simons gauge theories allows an investigation of local symmetries, which treats internal and space-time symmetries on an equal footing. For instance, general relativity in three dimensions has an interpretation as a Chern-Simons theory, in which the diffeomorphisms are realized as Yang-Mills gauge transformations [38]. Accordingly, all higher-spin fields – starting with spin-1 vectors – can be described in a similar fashion [39]. Thus Chern-Simons theories provide the natural arena for an analysis of the spin-2 and other local (super-)symmetries, which are expected to appear in Kaluza-Klein theories. Finally, gauged supergravities in three dimensions represent a natural extension of these Chern-Simons theories (which are contained as subsectors) and, correspondingly, are well adapted to an analysis by symmetry arguments. Irrespective of the meaning as a toy-model for higher-dimensional cases, these theories are of significance by themselves, as they afford the necessary background for an investigation of the dual two-dimensional CFT's.

Apart from the relevance for the AdS/CFT correspondence and the conceptual understanding of Kaluza-Klein theories in general, these questions have a striking similarity with the analogous search for the underlying symmetries in string or M-theory mentioned in 1.1, and one may even hope to get some new insights into these subjects.

1.3 Outline of the thesis

This thesis aims to analyze massive Kaluza-Klein theories through their spontaneously broken symmetries, with a focus on the consistency of massive spin-3/2 and spin-2 couplings. The organization is as follows.

In chapter 2 we give a brief review of the consistency problems related to higher-spin couplings in general, and how they are partially resolved in supergravity. Then we give a short introduction into gauged supergravity in three dimensions, which will be applied in forthcoming chapters.

In chapter 3 we turn to the problem of coupling massive spin-3/2 fields to gravity. For this we consider the example of Kaluza-Klein supergravity on $AdS_3 \times S^3 \times S^3 \times S^1$. This background is half-maximally supersymmetric and contains in its Kaluza-Klein spectrum massive supermultiplets with maximal spin 3/2. We construct their respective effective actions as gauged maximal supergravities in $D = 3$, whose supersymmetry is partially broken in the vacuum, thus giving rise to massive spin-3/2 fields via a super-Higgs mechanism.

The analogous problem for spin-2 fields is discussed in chapter 4. We show that an 'unbroken phase' exists, in which the spin-2 fields appear to be massless and therefore possess an enhanced (infinite-dimensional) gauge symmetry, thus circumventing the no-go theorems. It will be shown that a geometrical interpretation exists, which is analogous to the one for general relativity, and consists of a notion of 'algebra-valued' differential geometry developed by Wald. Moreover we will see that the 'broken phase' and the affiliated Higgs mechanism originate from a gauging of certain global symmetries, which is to a certain extent similar to the gauging of supergravity introduced in chapter 2. In particular, the rigid invariance group is enhanced and the gauge fields will show up together with the spin-2 fields in a Chern-Simons form.

In chapter 5 we discuss potential applications of the results from chapter 3 for the AdS/CFT correspondence. We consider so-called marginal deformations, which leave the AdS background intact, but break some supercharges and part of the gauge group spontaneously. More specifically, we discuss a breaking of $\mathcal{N} = (4, 4)$ to $\mathcal{N} = (4, 0)$ and $\mathcal{N} = (3, 3)$, respectively, together with the resulting spectrum and the reorganization into supermultiplets.

Chapter 6 closes with an outlook and discussions, while the appendices contain the required technical background. These include a review on $E_{8(8)}$, an overview of Kac-Moody and Virasoro algebras and the technicalities of an explicit Kaluza-Klein reduction containing all Kaluza-Klein modes (appendices A – C). An extension of the spin-2 theories analyzed in chapter 4 to generic compactification manifolds will be given in appendix D.

Chapter 2

Higher-spin fields and supergravity

To set the stage for later examinations we give in this chapter a short introduction into the interaction problem for higher-spin fields and its partial resolution within supergravity. Moreover, we briefly review gauged supergravity with special emphasis on in its three-dimensional version, since this will be applied in chapter 3 and generalized in chapter 4.

2.1 Consistency problems of higher-spin theories

As mentioned in the introduction the massive Kaluza-Klein states appearing in supergravity are of significance for the AdS/CFT correspondence. Among these massive fields are an infinite tower of spin-3/2 and spin-2 states. The effective Kaluza-Klein supergravity will describe the coupling of these infinite towers to gravity (or more precisely, to the supergravity multiplet). Even more optimistically we would like to couple massless spin-3/2 and spin-2 fields to gravity that would exhibit an enhanced gauge symmetry and then establish a novel version of the Higgs mechanism, such that they can become massive via breaking the symmetry spontaneously. However, there exist various no-go theorems that forbid the existence of interacting higher-spin theories (where higher-spin means $s > 1$), implying in particular that couplings of spin-3/2 and spin-2 fields to gravity – in the massive and even more severely in the massless case – are impossible.[1] In the following we will briefly discuss the consistency problem related to these higher-spin couplings and how they can be partially solved within supergravity.

Let us start with a non-interacting spin-2 field $h_{\mu\nu}$ on a Minkowski background.

[1] There is however a vast literature on the problem of consistent higher-spin theories. (For early papers see [40, 41, 42] and for recent reviews [43, 32].) Even though there was tremendous progress within the last few years, it is probably fair to say, that we are still far away from a conclusive picture.

Its action has been determined by Pauli and Fierz [40] and is given by

$$S_{\rm PF} = \int d^D x \big[\frac{1}{2}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} - \partial_\mu h^{\mu\nu}\partial^\rho h_{\rho\nu} + \partial_\mu h^{\mu\nu}\partial_\nu \hat{h} - \frac{1}{2}\partial_\mu \hat{h}\partial^\mu \hat{h} - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - \hat{h}^2)\big] \,, \tag{2.1}$$

where $\hat{h} = \eta^{\mu\nu}h_{\mu\nu}$ denotes the trace evaluated in the Minkowski metric. The equations of motion derived from this action read

$$\Box h_{\mu\nu} - \partial_\mu\partial^\rho h_{\rho\nu} - \partial^\rho\partial_\nu h_{\rho\nu} + \eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} - \eta_{\mu\nu}\Box\hat{h} + \partial_\mu\partial_\nu\hat{h} + m^2\left(h_{\mu\nu} - \eta_{\mu\nu}\hat{h}\right) = 0 \,. \tag{2.2}$$

If one takes the divergence and the trace of this equation one gets

$$\begin{aligned} m^2\partial^\mu(h_{\mu\nu} - \eta_{\mu\nu}\hat{h}) &= 0 \,, \\ (D-2)\partial^\mu\partial^\nu(h_{\mu\nu} - \eta_{\mu\nu}) - m^2(D-1)\hat{h} &= 0 \,. \end{aligned} \tag{2.3}$$

From this we conclude

$$\partial^\mu h_{\mu\nu} = 0 \,, \qquad \hat{h} = 0 \,. \tag{2.4}$$

Reinserting these relations into the equations of motion, one obtains

$$(\Box + m^2)h_{\mu\nu} = 0 \,. \tag{2.5}$$

The equations (2.4) and (2.5) guarantee that the dynamical content of (2.1) is given by an irreducible representation of the Poincaré group. In four-dimensional language (2.4) states that $h_{\mu\nu}$ has spin 2, while (2.5) ensures that $h_{\mu\nu}$ is an eigenstate of the mass operator P^2. Accordingly, the number of propagating degrees of freedom can be counted as follows. In D dimensions the symmetric $h_{\mu\nu}$ has $\frac{D(D+1)}{2}$ components, and in case that only (2.5) is present, all of them would propagate. In turn one would have to specify $D(D+1)$ Cauchy data on an initial-value surface, namely the $h_{\mu\nu}$ and $\partial_t h_{\mu\nu}$. However, in the present case the conditions (2.4) have to be taken into account, i.e. they have to be imposed as constraints on the initial data. $\hat{h} = 0$ then implies two constraints, namely that $\hat{h}$ and $\partial_t\hat{h}$ are initially zero, while $\partial^\mu h_{\mu\nu} = 0$ yields $2D$ further constraints.[2] In total one ends up with $D^2 - D - 2$ initial conditions. In other words, there are $\frac{(D-1)D}{2} - 1$ propagating degrees of freedom, which match exactly the number of components of a symmetric traceless 2-tensor under the massive little group $SO(D-1)$.

For the massless case $m^2 = 0$ the conditions (2.4) no longer follow from the equations of motion. Instead, the action develops a local symmetry $\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$, which decouples additional degrees of freedom. With a similar analysis as above one finds $\frac{1}{2}D(D-3)$ propagating degrees of freedom.

[2] That $\partial_t\partial^\mu h_{\mu\nu} = 0$ is a constraint, even though it contains a second-order time derivative, can be seen by rewriting it with the help of the Klein-Gordon equation.

What is now the problem of promoting this spin-2 theory to an interacting theory, e.g. via coupling to gravity? First of all, the massless $h_{\mu\nu}$ can of course be elevated to a self-interacting field, namely to the graviton of general relativity, since (2.1) for $m^2 = 0$ describes nothing else than the linearization of the Einstein-Hilbert action. (Correspondingly, the free spin-2 gauge symmetry is the linearization of the diffeomorphism symmetry of general relativity.) In fact, the existence of this theory has a deep geometrical reason, namely the existence of Riemannian geometry. If instead one wants to couple a number of spin-2 fields to gravity, one would have to replace all partial derivatives in (2.1) by covariant ones (with respect to the space-time metric). However, this would violate either the conditions (2.4) in the massive case or the invariance of the massless theory under any obvious covariantisation of the spin-2 transformation. This happens because of the non-commutativity of covariant derivatives, $[\nabla_\mu, \nabla_\nu] \sim R_{\mu\nu}$. In turn, the number of degrees of freedom would be larger than compared to the free case, or in other words, some of the propagating modes would disappear in the free limit, which clearly indicates an inconsistency. (For the no-go theorems in case of spin-2 fields see [44, 45, 46, 47, 48], and also [49] and [50, 51].)

Similar consistency problems appear for all higher-spin fields, in particular for spin-3/2 fields. However, for the latter we know already how to circumvent the aforementioned no-go theorem, namely by introducing local supersymmetry, to which we will turn now.

The supergravity miracle

A free massless spin-3/2 field ψ_μ on a Minkowski background can be described by the Rarita-Schwinger equation, which reads

$$\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho = 0 \ , \tag{2.6}$$

where $\gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$, and we have suppressed the spinor index on ψ_μ. These equations of motion are clearly invariant under the spin-3/2 gauge symmetry $\delta\psi_\mu = \partial_\mu\epsilon$, which in turn guarantees consistency. If one wants to couple this spin-3/2 field to gravity one encounters the same problems as mentioned in the last section. Namely, due to the non-commutativity of covariant derivatives the Rarita-Schwinger equation transforms under the covariantisation of the spin-3/2 symmetry into

$$\delta(\gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho) = \frac{1}{2}\gamma^{\mu\nu\rho}[\nabla_\nu, \nabla_\rho]\epsilon \sim R^\mu{}_\nu\gamma^\nu\epsilon \ . \tag{2.7}$$

Since the Ricci tensor does not vanish in an interacting theory, but is instead determined by the energy-momentum tensor $T_{\mu\nu}(\psi)$ of the spin-3/2 field, the gauge symmetry is explicitly broken and the theory becomes inconsistent. Therefore we find a no-go theorem for spin-3/2 fields which is analogous to the one for gravity/spin-2 couplings mentioned above.

But, as the discovery of supergravity has shown, there is a loophole in the no-go theorem: It was assumed in the variation (2.7) that only the spin-3/2 field transforms

under the symmetry. If one allows instead for a variation also of the metric, which is schematically of the form

$$\delta g_{\mu\nu} = \frac{1}{2}(\bar{\epsilon}\gamma_\mu\psi_\nu + \bar{\epsilon}\gamma_\nu\psi_\mu) \ , \tag{2.8}$$

the Rarita-Schwinger equation transforms into

$$\delta(\gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho) \sim (R_{\mu\nu} - T_{\mu\nu} + \frac{1}{2}g_{\mu\nu}T^\rho_\rho)\gamma^\nu\epsilon \ . \tag{2.9}$$

This means that a non-trivial transformation for the metric can be introduced in such a way that the Rarita-Schwinger equation rotates exactly into the Einstein equation containing the energy-momentum tensor for the spin-3/2 fields. Correspondingly, the Einstein equation transforms into the Rarita-Schwinger equation. The theory is therefore consistent also at the interacting level and propagates only massless spin-2 and spin-3/2 modes ('the supergravity miracle', [44]).

More precisely, $(g_{\mu\nu}, \psi_\mu)$ build a multiplet for a locally realized $\mathcal{N} = 1$ supersymmetry. This means that the introduction of an extended space-time symmetry allowed consistent couplings of 'higher-spin' fields. In fact, the spin-3/2 fields are realized as gauge fields for supersymmetry. If one extends the symmetry further by introducing additional supercharges Q^I, where $I = 1, ..., \mathcal{N}$, consistent couplings of the same number of spin-3/2 fields (gravitinos) are possible. But, the number of spin-3/2 fields that can be coupled in this way is bounded (thus being only of limited use for the required Kaluza-Klein theories). This can be seen by inspecting the representation theory of the superalgebra [52]. Since the supercharges raise and lower the spins in a given multiplet, the maximal number of real supercharges consistent with spin $s \leq 2$ is 32. In other words, the number of gravitinos is bounded by this number, if not at the same time additional higher-spin fields are introduced. However, the latter seems to be still impossible, even if supersymmetry is used. For instance, one may consider the $\mathcal{N} = 1$ multiplet which contains besides the metric not a massless spin-3/2 field, but a massless spin-5/2 field. This would be equally sensible from the point of view of the representation theory of the superalgebra, but still a consistent field theory ('hypergravity') does not exist [53].

Subsequently we will concentrate on three-dimensional supergravities, which are special for the following reasons. First of all, in $D = 3$ an arbitrary number of spin-3/2 fields *can* be coupled to gravity. This is due to the fact, that they are so-called topological fields, and hence they can be added to an action without affecting the number of local degrees of freedom. Even though this flexibility is lost once the spin-3/2 fields are coupled to matter, one might hope that the necessary avoidance of the no-go theorems can be studied more directly. This topological character actually extends to all massless higher-spin fields in $D = 3$, including spin-2 fields. Thus, also for them an analysis in a three-dimensional framework seems to be more promising.

2.2 Gauged supergravity

2.2.1 Generalities

In this section we review the construction of gauged supergravities. To begin with, we recall that pure supergravity in $D = 3$ consists of a metric (described by the vielbein e^a_μ) and Rarita-Schwinger gravitinos ψ^I_μ, $I = 1, ..., \mathcal{N}$, which together build a supermultiplet for $\mathcal{N}$-extended supersymmetry. According to the counting of degrees of freedom in 2.1 the metric possesses no propagating degrees of freedom in $D = 3$. In this sense it is purely topological. Similarly, also the gravitino is topological, as it should be in order to match bosonic and fermionic degrees of freedom. Due to this topological nature consistent field theories exist for arbitrary $\mathcal{N}$. Namely, the action

$$\mathcal{L}_{\text{s.g.}} = -\frac{1}{4}\varepsilon^{\mu\nu\rho}\left(e^a_\mu R_{\nu\rho a}(\omega) + \bar{\psi_\mu}^I \nabla_\nu \psi^I_\rho\right) \tag{2.10}$$

stays invariant under

$$\delta_\epsilon e^a_\mu = \frac{1}{2}\bar{\epsilon}^I \gamma^a \psi^I_\mu \,, \qquad \delta_\epsilon \psi^I_\mu = \nabla_\mu \epsilon^I \,. \tag{2.11}$$

In fact, the no-go theorem excluding couplings of an arbitrary number of spin-3/2 fields mentioned in 2.1 does not apply, since the multiplet structure refers only to propagating degrees of freedom. In view of the fact that the gravitational fields are topological, an arbitrary number of spin-3/2 fields can in turn be coupled.

Chern-Simons theories

Another way of providing invariant supergravity actions for an arbitrary number of topological fields is given by the so-called Chern-Simons supergravities. To introduce them let us first discuss the description of pure gravity theories in the framework of Chern-Simons theories.

The Chern-Simons action for a gauge field $\mathcal{A}$, taking values in the Lie algebra of a certain gauge group G, reads

$$S_{\text{CS}} = \int \text{Tr}\big(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\big) \,. \tag{2.12}$$

Here the trace refers symbolically to an invariant and non-degenerate quadratic form on the Lie algebra. It has been shown in a classic paper by Witten [38] that three-dimensional gravity (with or without cosmological constant) can be interpreted as a Chern-Simons theory with a particular gauge group. If the cosmological constant Λ is positive, negative or zero, the gauge group is given by the de Sitter, anti-de Sitter of Poincaré group, respectively. We will illustrate this for AdS gravity, in which case the isometry group decomposes as

$$SO(2,2) = SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R \,. \tag{2.13}$$

To make contact with the conventional formulation of AdS gravity one rewrites the respective gauge fields corresponding to the left or right $SL(2,\mathbb{R})$ factors according to

$$\mathcal{A}_\mu^{a\ L,R} = \omega_\mu^a \pm \sqrt{-\Lambda}\, e_\mu^a \ . \tag{2.14}$$

Then e_μ^a will be interpreted as the vielbein and ω_μ^a as the spin connection, which is treated as an independent field in the first order Palatini formalism. It turns out that the difference of the two corresponding Chern-Simons terms

$$\mathcal{L}_{\rm CS} = \mathrm{Tr}\big(\mathcal{A}_L \wedge d\mathcal{A}_L + \frac{2}{3}\mathcal{A}_L \wedge \mathcal{A}_L \wedge \mathcal{A}_L\big) - \mathrm{Tr}\big(\mathcal{A}_R \wedge d\mathcal{A}_R + \frac{2}{3}\mathcal{A}_R \wedge \mathcal{A}_R \wedge \mathcal{A}_R\big) \tag{2.15}$$

coincides exactly with the Einstein-Hilbert term with cosmological constant Λ [38]. In this sense pure AdS-gravity can be interpreted as Chern-Simons gauge theory.

Along these lines AdS supergravity theories can in turn be constructed as Chern-Simons theories for appropriate superextensions of the AdS group (2.13). This has first been observed by Achucarro and Townsend [54], which considered the supergroup $OSp(p|2) \times OSp(q|2)$. Here each factor contains the $SL(2,\mathbb{R})$ factor in (2.13) together with supercharges and internal symmetry generators. More precisely, the entire supergroup carries $\mathcal{N} = p + q$ supersymmetry and yields a $SO(p) \times SO(q)$ gauge symmetry. The resulting Chern-Simons action takes the form (2.10) for the corresponding number of supercharges, augmented by additional Chern-Simons terms for the $SO(p) \times SO(q)$ gauge fields together with gravitino 'mass' terms and a cosmological constant. Since these supergroups exist for arbitrary p, q, an arbitrary number of gravitinos can be consistently coupled to gravity.

Matter couplings in supergravity

In the last section we have seen that three-dimensional supergravities are special in the sense that the purely gravitational fields do not carry local degrees of freedom. This in turn allowed the existence of theories with an arbitrary number $\mathcal{N}$ of supercharges. There is also another respect in which three-dimensional theories are special: For massless fields (on Minkowski backgrounds) there is no notion of spin or helicity.[3] This implies that the standard argument, restricting the number of supercharges to be less or equal to 32 in order to arrange for maximal spin 2, does not apply, also if propagating matter fields are taken into account. Thus one might also hope to get matter-coupled supergravities for arbitrary $\mathcal{N}$. However, it turns out that upon coupling to propagating matter the no-go theorem reappears. Adding a globally supersymmetric σ-model action for scalar multiplets (ϕ^i, χ^i) of the form

$$\mathcal{L}_{\rm matter} = \frac{1}{2} g_{ij}(\phi) \left(\partial_\mu \phi^i \partial^\mu \phi^j - \frac{i}{2} \bar{\chi}^i \gamma^\mu D_\mu \chi^j \right) + \mathcal{L}_{\chi^4} \tag{2.16}$$

[3] In the following we will nevertheless refer to spin-s fields, if they are symmetric tensors in s vector indices and carry an associated gauge symmetry (and analogously for spinors).

to the supergravity action (2.10) implies severe conditions on the geometry of the target space. For instance, $\mathcal{N} = 8$ supersymmetry restricts the scalar manifold to be a coset space of the form

$$G/H = \frac{SO(8,n)}{SO(8) \times SO(n)} \ , \tag{2.17}$$

where n indicates the number of scalar multiplets. For $\mathcal{N} > 8$ the target spaces are uniquely determined, and $\mathcal{N} = 16$ implies the exceptional coset space $E_{8(8)}/SO(16)$. Moreover, one finds that there exist no consistent theories with $\mathcal{N} > 16$ [55]. Thus the bound $\mathcal{N} \leq 16$ (implying ≤ 32 real supercharges) is still satisfied.

Since the homogeneous spaces appearing as the target spaces for $\mathcal{N} = 8$ and $\mathcal{N} = 16$ supergravity play an important role in this thesis, we will briefly review their description [56]. For a coset space G/H a convenient parameterization for the scalar fields is given in terms of a G-valued matrix $L(x)$. This matrix is then subject to local H transformations which reduce the number of physical degrees of freedom to the required $\dim G - \dim H$ of the coset space. Moreover, L transforms under global G transformations, i.e. in total

$$L(x) \longrightarrow gL(x)h^{-1}(x) \ , \qquad g \in G, \ h(x) \in H \ . \tag{2.18}$$

The σ-model action can then be defined through the Lie algebra–valued current $L^{-1}\partial_\mu L = \mathcal{Q}_\mu + \mathcal{P}_\mu$, which we have decomposed according to $\mathcal{Q}_\mu \in \mathfrak{h}$ and $\mathcal{P}_\mu \in \mathfrak{g}\backslash\mathfrak{h}$, where $\mathfrak{h}$ and $\mathfrak{g}$ denote the Lie algebras of G and H, respectively. Then the action reads

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \mathrm{Tr}\left(\mathcal{P}_\mu \mathcal{P}_\nu\right) \ . \tag{2.19}$$

For explicit computations one usually takes a gauged-fixed parametrisation for L, e.g. a gauge, where L takes only values in the non-compact part of G. Writing $L = \exp(\phi^i t_i)$, where $i = 1, ..., \dim G - \dim H$ and t_i are the corresponding generators, and then inserting into (2.19) yields a bosonic σ-model action of the form (2.16). The isometry group G as the rigid invariance group of (2.19) is then realized via a left multiplication on L as in (2.18) together with a compensating H transformation in order to restore the chosen gauge for L. In a given parametrisation this results in a non-linear action on the ϕ^i, generated by the corresponding Killing vectors.

Gauging of supergravity

So far we have described ungauged supergravities, since no gauge fields were involved. Let us now turn to the problem of gauging some of the global symmetries, i.e. to the problem of promoting a certain subgroup of the isometry group G to a local symmetry.

As usual this comprises the introduction of gauge fields and minimal coupling to charged matter. However, in supersymmetric theories one immediately encounters the severe problem that the introduction of additional bosonic degrees of freedom is inconsistent. In generic dimensions one evades this problem via starting from an

ungauged supergravity, which already contains a number of vector multiplets (with abelian gauge fields). The gauging then deforms the theory in such a way that the vectors become gauge fields for a non-abelian gauge group. In contrast, the three-dimensional supergravities mentioned above contain only scalar fields, which in turn is no restriction of generality since abelian vectors can in $D = 3$ always be dualized into scalars. But, in order for the theory to express the maximal rigid symmetry in a coset space structure (as the $E_{8(8)}$ for $\mathcal{N} = 16$) all bosonic degrees of freedom have to reside in scalars. Thus it seems to be impossible to gauge three-dimensional supergravity, while maintaining at the same time the large rigid symmetry. It has been shown in [57, 58] (for a review see [59]) that it is possible, however, to elude this conflict to a certain extent via the introduction of a Chern-Simons term for the gauge vectors instead of a Yang-Mills term. Since the former yields only topological gauge fields, bosonic and fermionic degrees of freedom still match, and accordingly the bosonic degrees of freedom are, also in the gauged theory, all carried by scalars.

Let us now consider the gauging in more detail. To begin with, each partial derivative in the ungauged theory has to be replaced by a covariant one:

$$D_\mu = \partial_\mu + g\Theta_{\mathcal{MN}} t^{\mathcal{M}} A_\mu^{\mathcal{N}} \ . \tag{2.20}$$

Here g denotes the gauge coupling constant which measures the deformation of the ungauged theory into a gauged one. The $A_\mu^{\mathcal{M}}$ are the gauge fields and $\Theta_{\mathcal{MN}}$ is a symmetric tensor, where the indices $\mathcal{M}, \mathcal{N}, ...$ label the adjoint representation of the global symmetry (as, e.g., $E_{8(8)}$ in the $\mathcal{N} = 16$ case). $\Theta_{\mathcal{MN}}$ is the so-called embedding tensor. It describes the embedding of the gauge group G_0 into the rigid symmetry group G in the sense that the Lie algebra $\mathfrak{g}_0$ of G_0 is spanned by the generators $\Theta_{\mathcal{MN}} t^{\mathcal{N}}$, where $t^{\mathcal{M}}$ denote the generators of G. In particular, the rank of Θ is given by the dimension of the gauge group. To be more precise, the embedding tensor is an element in the symmetric tensor product, i.e.

$$\Theta = \Theta_{\mathcal{MN}} t^{\mathcal{M}} \otimes t^{\mathcal{N}} \in \mathrm{Sym}(\mathfrak{g} \otimes \mathfrak{g}) \ . \tag{2.21}$$

The introduction of Θ formally preserves covariance with respect to the full global symmetry group, even though in the gauged theory the latter is broken to the gauge group (since $\Theta_{\mathcal{MN}}$ is constant and does not transform under G). We will see below that this formalism nevertheless substantially simplifies the analysis of gauged supergravities.

Generically, the form of gauged supergravities is highly restricted due to the fact that the minimal substitution (2.20) spoils the invariance under supersymmetry. This has to be compensated by the introduction of additional couplings, and only in special cases this can be done in a consistent way. More specifically, the bosonic terms have to be supplemented by a scalar potential V, such that the matter action for them reads

$$\mathcal{L}_{\text{matter}} \ = \ e\,\mathrm{Tr}\Big\langle [\mathcal{V}^{-1} D_\mu \mathcal{V}]_{\mathfrak{k}}\, [\mathcal{V}^{-1} D^\mu \mathcal{V}]_{\mathfrak{k}} \Big\rangle + e\,V(\mathcal{V}) + \text{fermions} \ , \tag{2.22}$$

where $[\cdot]_{\mathfrak{k}}$ denotes the projection of the Lie algebra $\mathfrak{g}$ associated to G onto its non-compact part. Similarly, Yukawa type couplings between scalars and fermions are

necessary in the gauged theory, and their explicit form will be given in sec. 2.2.2. Finally the Chern-Simons term for the gauge fields has to be added, which can be written as

$$\mathcal{L}_{\text{CS}} = \frac{1}{4}\varepsilon^{\mu\nu\rho}\Theta_{\mathcal{MN}}A^{\mathcal{M}}_{\mu}\big(\partial_{\nu}A^{\mathcal{N}}_{\rho} + \frac{1}{3}f^{\mathcal{NP}}{}_{\mathcal{L}}\Theta_{\mathcal{PK}}A^{\mathcal{K}}_{\nu}A^{\mathcal{L}}_{\rho}\big) \ . \tag{2.23}$$

This coincides with (2.12), where $\Theta_{\mathcal{MN}}$ serves as a non-degenerate quadratic form on the subalgebra that will be gauged. Note that the Chern-Simons term for the compact gauge vectors combines together with the Einstein-Hilbert term and the kinetic terms for the gravitinos into a Chern-Simons theory based on an AdS_3-supergroup, as discussed in sec. 2.2.1. Thus the gauged supergravities are the natural matter-coupled extensions of the topological theories of 2.2.1.

Depending on the amount of required supersymmetry the constructed theory is not automatically supersymmetric, but still several constraints have to be satisfied. However, the advantage of the given formalism based on the embedding tensor consists of the fact that all conditions implied by supersymmetry translate into purely algebraic constraints on the embedding tensor. First of all Θ has to fulfill some purely group-theoretical constraints. In order for the subset G_0 to be a closed algebra, Θ has to be invariant under the gauge group. In terms of the structure constants $f^{\mathcal{MN}}{}_{\mathcal{K}}$ of G this invariance, i.e. the fact that Θ commutes with all gauge group generators $\Theta_{\mathcal{MN}}t^{\mathcal{N}}$, implies the quadratic condition

$$\Theta_{\mathcal{KP}}\Theta_{\mathcal{L}(\mathcal{M}}f^{\mathcal{KL}}{}_{\mathcal{N})} = 0 \ . \tag{2.24}$$

Second, supersymmetry requires an algebraic constraint, whose explicit form we will discuss in the next subsection for $\mathcal{N} = 8$ and $\mathcal{N} = 16$ supergravity, respectively. The full supergravity action is entirely determined by the embedding tensor.

We have seen that consistent gaugings of supergravity are possible in $D = 3$ through the introduction of Chern-Simons gauge fields. On the other hand, we know that gauged supergravities exist which carry Yang-Mills vectors as gauge fields. The latter can, for instance, be constructed by Kaluza-Klein compactification of higher-dimensional supergravities. Therefore one might be tempted to conclude that the gauged supergravities mentioned here are not the most general ones. But this is not the case: All supergravities in $D = 3$ with Yang-Mills type gauging are on-shell dual to a Chern-Simons gauged supergravity with an enlarged number of scalar fields, as has been shown in [60, 61]. In the following we will shortly review this equivalence.

We start from the generic form of a Yang-Mills gauged supergravity

$$\mathcal{L} = -\frac{1}{4}eR - \frac{1}{4}eM_{ab}(\phi)F^{a}_{\mu\nu}F^{\mu\nu b} + \mathcal{L}'(A,\phi) \ , \tag{2.25}$$

where A^{a}_{μ} and $F^{a}_{\mu\nu}$ denote the non-abelian gauge field and field strength for a certain gauge group G. Moreover, $\mathcal{L}'$ indicates some additional matter couplings and fermionic terms. We will assume that they are separately gauge-invariant, which in turn implies that A^{a}_{μ} enters only through a covariant derivative or maybe an additional Chern-Simons term. It is exactly this explicit dependence on A^{a}_{μ} that forbids a

standard dualization into scalars. But we will show that a dualization is still possible in which the gauge fields survive as topological fields, and where the bosonic degrees of freedom are instead carried by new scalars. To see this we have to introduce for each of the former Yang-Mills fields a scalar φ_a and also an additional gauge field $B_{\mu\,a}$. An improved duality relation can then be written as follows

$$\frac{1}{2}e\varepsilon_{\mu\nu\rho}F^{a\,\nu\rho} = M^{ab}(\phi)(D_\mu\varphi_b - B_{\mu\,b}) \ , \tag{2.26}$$

where D_μ denotes the gauge covariant derivative with respect to G. Moreover we have assumed that the scalar field matrix $M_{ab}(\phi)$ is invertible, such that $M^{ac}M_{cb} = \delta^a_b$. The structure of (2.26) suggests that φ_a transforms under a shift symmetry which is gauged by the $B_{\mu\,a}$. More specifically, if we define the transformations

$$\delta B_{\mu\,a} = D_\mu\Lambda_a \ , \qquad \delta\varphi_a = \Lambda_a \ , \tag{2.27}$$

the duality relation (2.26) stays invariant. Equivalently, the right hand side of (2.26) defines the covariant derivative $\tilde{D}_\mu\varphi_a$ for φ_a with respect to the shift gauge symmetry. In the next step we have to give a new Lagrangian that reproduces the duality relation (2.26), and whose equations of motion are equivalent to those of the original Lagrangian (2.25): It can be written as

$$\mathcal{L} = -\frac{1}{4}eR + \frac{1}{2}eM^{ab}(\phi)\tilde{D}_\mu\varphi_a\tilde{D}^\mu\varphi_b + \frac{1}{2}\varepsilon^{\mu\nu\rho}B_{\mu\,a}F^a_{\nu\rho} + \mathcal{L}'(A,\phi) \ . \tag{2.28}$$

Indeed, we observe the appearance of a Chern-Simons-like term $B \wedge F$. Moreover, varying (2.28) with respect to $B_{\mu\,a}$ yields the duality relation (2.26). That (2.28) is on-shell equivalent to (2.25) can be most easily seen by choosing the gauge $\varphi_a = 0$ (i.e. fixing the gauge symmetry, which is absent in (2.25) anyway), such that $\tilde{D}_\mu\varphi_a = -B_{\mu\,a}$, and then integrating out $B_{\mu\,a}$. The latter coincides with the Yang-Mills gauged action (2.25).

In total we have seen that any Yang-Mills gauged supergravity with gauge group G is on-shell equivalent to a Chern-Simons gauged theory, in which the gauge group has been enhanced by $\dim G$ additional shift gauge fields. To be more precise, the gauge algebra is extended by $\dim G$ nilpotent generators (i.e. commuting among themselves), which transform under the adjoint action of G (see eq. (3.39) below). The Chern-Simons term for this extended gauge group coincides indeed with the Chern-Simons term in (2.28). Moreover, additional Chern-Simons terms can be coupled, which will act as gauge invariant mass terms for the vectors. The latter corresponds to a further enhancement of the gauge algebra, see the second line of eq. (3.39) below. These nilpotent symmetries will be broken in the vacuum and give rise to massive vector fields. Later on the massive spin-1 fields will therefore be identified through their Goldstone scalars [62, 63].

2.2.2 Gauged $\mathcal{N} = 8$ and $\mathcal{N} = 16$ supergravity in $D = 3$

In this subsection we will describe the gauged supergravities required for our analysis in more detail.

The $\mathcal{N}=8$ supergravity

In the $\mathcal{N}=8$ supergravity the scalar fields are, in accordance with the coset structure discussed above, described by a $SO(8,n)$ valued matrix L and are subject to local and global transformations (2.18) [64]. The current covariantized with respect to the gauge group reads

$$L^{-1}(\partial_\mu + g\Theta_{\mathcal{MN}}B_\mu{}^{\mathcal{M}}t^{\mathcal{N}})L = \frac{1}{2}Q_\mu^{IJ}X^{IJ} + \frac{1}{2}Q_\mu^{rs}X^{rs} + \mathcal{P}_\mu^{Ir}Y^{Ir}\ , \tag{2.29}$$

where (X^{IJ}, X^{rs}) and Y^{Ir} denote the compact and noncompact generators of $SO(8,n)$, respectively. In this section $I,J=1,...,8$ and $r,s=1,...,n$ are $SO(8)$ and $SO(n)$ vector indices. The gravitinos ψ_μ^A and the matter fermions $\chi^{\dot{A}r}$ carry spinor and conjugate spinor indices under $SO(8)$, respectively.[4] The action reads

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}eR + \frac{1}{2}\varepsilon^{\mu\nu\rho}\overline{\psi}_\mu^A D_\nu\psi_\rho^A + \frac{1}{4}e\mathcal{P}_\mu^{Ir}\mathcal{P}^{\mu\,Ir} - \frac{1}{2}ie\overline{\chi}^{\dot{A}r}\gamma^\mu D_\mu\chi^{\dot{A}r} + \mathcal{L}_{\rm CS}\\ &+\frac{1}{2}geA_1^{AB}\overline{\psi}_\mu^A\gamma^{\mu\nu}\psi_\nu^B + igeA_2^{A\dot{A}r}\bar{\chi}^{\dot{A}r}\gamma^\mu\psi_\mu^A + \frac{1}{2}geA_3^{\dot{A}r\dot{B}s}\bar{\chi}^{\dot{A}r}\chi^{\dot{B}s} + eV\ ,\end{aligned} \tag{2.30}$$

with the Chern-Simons term $\mathcal{L}_{\rm CS}$ defined in (2.23). The explicit expressions for A_1, A_2, A_3 and the potential V in terms of Θ can be found in [64]. The supersymmetry conditions consist basically of some symmetry requirements on Θ [64].

The $\mathcal{N}=16$ supergravity

We have already seen that for $\mathcal{N}=16$ the supergravity multiplet contains 16 Majorana gravitino fields ψ_μ^I, $I=1,...,16$, and we may view I as a vector index of $SO(16)$. The bosonic matter content consists of the 128 scalar fields that parametrize the coset space $E_{8(8)}/SO(16)$. This means they can be represented by an $E_{8(8)}$ valued matrix $\mathcal{V}(x)$, which transforms under global $E_{8(8)}$ and local $SO(16)$ transformations as follows:

$$\mathcal{V}(x) \to g\mathcal{V}(x)h^{-1}(x)\ , \qquad g\in E_{8(8)}\ , \quad h(x)\in SO(16)\ . \tag{2.31}$$

Their fermionic partners are given by 128 Majorana fermions $\chi^{\dot{A}}$, $\dot{A}=1,...,128$, where $\dot{A}$ indicates the conjugate spinor index of $SO(16)$. Specifying (2.22) to the $\mathcal{N}=16$ case, the maximal supergravity Lagrangian up to quartic couplings in the fermions is given by

$$\begin{aligned}\mathcal{L} \;=\; &-\frac{1}{4}eR + \frac{1}{4}e\,\mathcal{P}^{\mu A}\mathcal{P}_\mu^A + \frac{1}{2}\varepsilon^{\mu\nu\rho}\overline{\psi}_\mu^I\mathcal{D}_\nu\psi_\rho^I - \frac{i}{2}e\,\bar{\chi}^{\dot{A}}\gamma^\mu\mathcal{D}_\mu\chi^{\dot{A}} - \tfrac{1}{2}e\bar{\chi}^{\dot{A}}\gamma^\mu\gamma^\nu\psi_\mu^I\Gamma^I_{A\dot{A}}\,\mathcal{P}_\nu^A\\ &+\frac{1}{2}eg\,A_1^{IJ}\,\bar{\psi}_\mu{}^I\gamma^{\mu\nu}\psi_\nu^J + ieg\,A_2^{I\dot{A}}\,\bar{\chi}^{\dot{A}}\gamma^\mu\psi_\mu^I + \frac{1}{2}eg\,A_3^{\dot{A}\dot{B}}\,\bar{\chi}^{\dot{A}}\chi^{\dot{B}}\\ &+\tfrac{1}{4}\,g\,\varepsilon^{\mu\nu\rho}\,A_\mu^{\mathcal{M}}\,\Theta_{\mathcal{MN}}\Big(\partial_\nu A_\rho^{\mathcal{N}} - \tfrac{1}{3}g\,\Theta_{\mathcal{KL}}f^{\mathcal{NL}}{}_{\mathcal{P}}\,A_\nu^{\mathcal{K}}A_\rho^{\mathcal{P}}\Big) - e\,V(\mathcal{V})\ .\end{aligned} \tag{2.32}$$

[4] In all other parts of this thesis these indices will instead refer to $SO(16)$ quantities. Since the $\mathcal{N}=8$ notation enters only in this section, this cannot be a source of confusion.

As discussed above, the theory is entirely encoded in the symmetric constant matrix $\Theta_{\mathcal{MN}}$. The minimal coupling of vector fields to scalars reads in the given case

$$\mathcal{V}^{-1}D_\mu\mathcal{V} \;\equiv\; \mathcal{V}^{-1}\partial_\mu\mathcal{V} + gA_\mu^{\mathcal{M}}\,\Theta_{\mathcal{MN}}\,(\mathcal{V}^{-1}t^{\mathcal{N}}\,\mathcal{V}) \;\equiv\; \tfrac{1}{2}\mathcal{Q}_\mu^{IJ}X^{IJ} + \mathcal{P}_\mu^A Y^A\,, \qquad (2.33)$$

with X^{IJ} and Y^A labeling the 120 compact and 128 noncompact generators of $E_{8(8)}$, respectively.[5] The Yukawa couplings or fermionic mass terms in (2.32) are defined as linear functions of $\Theta_{\mathcal{MN}}$ via

$$\begin{aligned} A_1^{IJ} &= \frac{8}{7}\theta\delta_{IJ} + \frac{1}{7}T_{IK|JK}\,, \quad A_2^{I\dot{A}} = -\frac{1}{7}\Gamma^J_{A\dot{A}}T_{IJ|A}\,, \\ A_3^{\dot{A}\dot{B}} &= 2\theta\delta_{\dot{A}\dot{B}} + \frac{1}{48}\Gamma^{IJKL}_{\dot{A}\dot{B}}T_{IJ|KL}, \end{aligned} \qquad (2.34)$$

with $SO(16)$ gamma matrices $\Gamma^I_{A\dot{B}}$, and the so-called T-tensor

$$T_{\mathcal{M}|\mathcal{N}} \;=\; \mathcal{V}^{\mathcal{K}}{}_{\mathcal{M}}\mathcal{V}^{\mathcal{L}}{}_{\mathcal{N}}\Theta_{\mathcal{KL}}\,. \qquad (2.35)$$

The scalar potential $V(\mathcal{V})$ is given by

$$V = -\frac{g^2}{8}\left(A_1^{KL}A_1^{KL} - \frac{1}{2}A_2^{K\dot{A}}A_2^{K\dot{A}}\right). \qquad (2.36)$$

For later use we give the condition of stationarity of this potential, as it has been shown in [57]

$$\delta V = 0 \qquad \Longleftrightarrow \qquad 3A_1^{IM}A_2^{M\dot{A}} = A_3^{\dot{A}\dot{B}}A_2^{I\dot{B}}\,. \qquad (2.37)$$

The quartic fermionic couplings and the supersymmetry transformations of (2.32) can be found in [57]. For consistency of the theory, the embedding tensor $\Theta_{\mathcal{MN}}$ needs to satisfy two algebraic constraints. First, it has to satisfy the quadratic constraint (2.24). Second, Θ as an element of the symmetric $E_{8(8)}$ tensor product

$$(\mathbf{248}\otimes\mathbf{248})_{\rm sym} \;=\; \mathbf{1}\oplus\mathbf{3875}\oplus\mathbf{27000}\,, \qquad (2.38)$$

is required to only live in the $\mathbf{1}\oplus\mathbf{3875}$ representation, i.e. to satisfy the projection constraint

$$(\mathbb{P}_{\mathbf{27000}})_{\mathcal{MN}}{}^{\mathcal{PQ}}\,\Theta_{\mathcal{PQ}} = 0\,. \qquad (2.39)$$

Explicitly, this constraint takes the form [65]

$$\Theta_{\mathcal{MN}} + \frac{1}{62}\,\eta_{\mathcal{MN}}\,\eta^{\mathcal{KL}}\,\Theta_{\mathcal{KL}} + \frac{1}{12}\,\eta_{\mathcal{PQ}}f^{\mathcal{KP}}{}_{\mathcal{M}}f^{\mathcal{LQ}}{}_{\mathcal{N}}\,\Theta_{\mathcal{KL}} \;=\; 0\,, \qquad (2.40)$$

with the $E_{8(8)}$ structure constants $f^{\mathcal{MN}}{}_{\mathcal{K}}$ and Cartan-Killing form

$$\eta^{\mathcal{MN}} = \frac{1}{60}\,f^{\mathcal{MK}}{}_{\mathcal{L}}\,f^{\mathcal{NL}}{}_{\mathcal{K}}\,. \qquad (2.41)$$

Any solution of (2.40) and (2.24) defines a consistent maximally supersymmetric theory (2.32) in three dimensions.

[5] See appendix A.1 for our $E_{8(8)}$ and $SO(16)$ conventions.

Chapter 3

Massive spin-3/2-multiplets in supergravity

In this chapter we will construct the effective supergravities for massive supermultiplets on $AdS_3 \times S^3 \times S^3 \times S^1$. First, in sec. 3.1, we review the solution of type IIB supergravity leading to this background and discuss in sec. 3.1.2 the resulting Kaluza-Klein spectrum. In sec. 3.2 we discuss the effective supergravities for the lowest multiplets, i.e. spin-1/2 and spin-1 multiplets, and in sec. 3.3 we finally turn to the first massive spin-3/2 multiplet.

3.1 IIB supergravity on $AdS_3 \times S^3 \times S^3 \times S^1$

3.1.1 The 10-dimensional supergravity solution

First we are going to discuss the required solution of type IIB supergravity, which is the effective low-energy theory for the corresponding string theory. The bosonic spectrum consists of the metric g_{MN}, two 2-forms B_{MN}, C_{MN}, two scalars ϕ (the dilaton) and C_0 and finally a self-dual 4-form C_4. Its action reads [66] (with $\alpha' = 1/(2\pi)^2$)

$$\begin{aligned} S_{\text{IIB}} = &\frac{2\pi}{g_B^2}\int d^{10}x\sqrt{-g}e^{-2\phi}(R + 4(\nabla\phi)^2) - \frac{\pi}{g_B^2}\int e^{-2\phi}H \wedge *H \\ &- \pi\int R_1 \wedge *R_1 - \pi\int R_3 \wedge *R_3 - \frac{\pi}{2}\int R_5 \wedge *R_5 + \pi\int C_4 \wedge H \wedge F_3 \,, \end{aligned} \tag{3.1}$$

where $H_3 = dB_2$, $R_1 = dC_0$, $R_3 = dC_2 - C_0 H_3$, $R_5 = dC_4 - H_3 \wedge C_2$ and $F_3 = dC_2$. This action is actually not a true off-shell formulation of type IIB supergravity, since the self-duality of the 4-form has to be imposed by hand,

$$*R_5 = R_5 \,. \tag{3.2}$$

We are now looking for solutions of the equations of motion which contain an AdS factor, i.e. which possess in its space-time part a maximally symmetric manifold.

Since the type IIB action (3.1) does not allow for a cosmological constant term, AdS spaces of generic dimension without matter sources cannot be a solution of (3.1). Thus we have to give vacuum expectation values to certain matter fields in order to provide for the right source for the gravitational field. For the pure NS solution we are going to construct we set $R_5 = F_3 = \phi = 0$ and assume for the background metric the direct product $AdS_3 \times S^{3+} \times S^{3-} \times S^1$, i.e.

$$ds^2 = ds^2(AdS_3) + R_+^2 ds^2(S_+^3) + R_-^2 ds^2(S_-^3) + L^2(d\theta)^2 \ . \tag{3.3}$$

The source for the metric will be given by a non-trivial vev for the 3-form flux H_3:

$$H = \lambda_0 \omega_0 + \lambda_+ \omega_+ + \lambda_- \omega_- \ , \tag{3.4}$$

while all other fields are set to zero. Here $\omega_0 = (\frac{L_0}{x_2})^3 dt \wedge dx^1 \wedge dx^2$ is the volume form on AdS_3 and $\omega_\pm = \mathrm{vol}(S_\pm^3)$ are the volume forms of the spheres. The Bianchi identity $dH_3 = 0$ is trivially satisfied since all 3-forms are closed on their respective 3-manifolds. The curvature for the AdS part is given in terms of the AdS length scale L_0 by

$$\begin{aligned} R_{\mu\nu\lambda\rho} &= -L_0^{-2}(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}) \ , \\ R_{\mu\nu} &= -2L_0^{-2} g_{\mu\nu} \ , \\ R &= -6L_0^{-2} \ . \end{aligned} \tag{3.5}$$

Similarly the curvature for the two spheres is given in terms of their radii by

$$\begin{aligned} R_{mnpq} &= R_+^{-2}(g_{mp}g_{nq} - g_{mq}g_{np}) \ , \\ R_{mn} &= 2R_+^{-2} g_{mn} \ , \\ R &= 6R_+^{-2} \ , \end{aligned} \tag{3.6}$$

and analogously for S_-^3 (with indices $\bar{m}, \bar{n}, ...$).

Next we have to verify the equations of motion, in particular we have to compare the Ricci tensor with the energy-momentum tensor induced by the 3-form flux. First of all, among the equations of motion for the fields vanishing in the background we also have to check the one for ϕ, since it couples to the non-vanishing H_3. Its equation of motion derived from (3.1)

$$\nabla^M(e^{-2\phi}\partial_M\phi) \sim H_{MNP}H^{MNP} \ , \tag{3.7}$$

implies for $\phi = 0$ the relation $H_{MNP}H^{MNP} = 0$. The latter in turn yield with (3.4)

$$\frac{1}{6} H_{MNP}H^{MNP} = -\lambda_0^2 + \lambda_+^2 + \lambda_-^2 = 0 \ . \tag{3.8}$$

Moreover, it follows from (3.8) that the energy-momentum tensor simplifies and one has $R = 0$. Because of the direct product structure the Einstein equations can in turn be solved, if we set

$$L_0^{-2} = \frac{1}{4}\lambda_0^2 \ , \quad R_+^{-2} = \frac{1}{4}\lambda_+^2 \ , \quad R_-^{-2} = \frac{1}{4}\lambda_-^2 \ , \tag{3.9}$$

which are, however, subject to the constraint (3.8). Therefore the equations of motion for the matter fields – in this case ϕ – imply a relation between the curvature scales of AdS on the one side and the internal manifolds on the other side. Thus the size of the internal manifolds is necessarily of the same order as the AdS space, as mentioned in the introduction.

We have seen that type IIB supergravity admits a spontaneous compactification to a three-dimensional Anti-de Sitter space. In the next section we will discuss the resulting three-dimensional spectrum.

3.1.2 The Kaluza-Klein spectrum

Now we will turn to the Kaluza-Klein spectrum of type IIB supergravity on $AdS_3 \times S^3 \times S^3 \times S^1$. Usually for this one would have to expand all fields in type IIB (or rather of the nine-dimensional theory constructed by reducing on the S^1) into spherical harmonics of $S^3 \times S^3$ and linearize the resulting couplings. Masses and spin[1] of the various fields could then be extracted. This is a technically cumbersome procedure and has been done explicitly, e.g., for the case $AdS_3 \times S^3$ [67]. There is however a more convenient way, based on a group-theoretical analysis due to Salam and Strathdee [68]. The latter is applicable to all coset spaces (like the sphere $S^3 = SO(4)/SO(3)$), and can be explained as follows.

Suppose first for simplicity that we would compactify on a group manifold G. A (scalar) field ϕ would then be expanded according to

$$\phi(x, g) = \sum_n \sum_{p,q} \phi^n_{qp}(x)\ D^n_{pq}(g)\ , \tag{3.10}$$

where $g \in G$. The sum runs over all irreducible representations of G, labeled by n. Due to the Peter-Weyl theorem, in case of a group manifold the harmonics are determined by the matrix elements $D^n_{pq}(g)$ of these irreducible representations.[2] This expansion reflects the fact that the Kaluza-Klein modes can be organized into representations of the isometry group. (In contrast to generic coset spaces G/H, where the isometry group is G, in the special case of a group manifold the isometry group is actually enhanced to $G \times G$.) If one compactifies instead on a coset space G/H, the fields generically transform in some representation of H (which will later be identified with the local Lorentz group). Specifically,

$$\phi^i(hg) = D^i{}_j(h)\phi^j(g)\ . \tag{3.11}$$

This is only consistent with an expansion of the type (3.10), if the irreducible representations $D^n(g)$ fulfill the condition

$$D^n(hg) = D(h)D^n(g)\ , \tag{3.12}$$

[1] To be more precise, one should talk about the representations of the AdS_3 isometry group.

[2] One may think of the illustrative example of a compactification on S^1, i.e. on the group manifold $U(1)$. Here the irreducible representations are labeled by an integer n according to $\theta \to e^{in\theta}$, which are on the other hand the Fourier modes in which one would expand.

where $D(h)$ is the representation under which ϕ transforms. In other words, for a coset space only those representations of the isometry group will appear in the mode expansion that include the given representation $D(h)$ upon restricting to H.

Applied to a compactification on S^3, this gives us a criterion to determine which Kaluza-Klein modes actually appear [69]. Namely, any field transforms in a representation R_3 of the local Lorentz group $SO(3)$ of S^3. In the Kaluza-Klein tower then only those representations R_4 of the isometry group $SO(4)$ will appear, that contain R_3 in a decomposition of R_4 into $SO(3)$ representations. For illustration let us consider a scalar, i.e. a singlet under the Lorentz group. Denoting the representations by spin quantum numbers of $SO(3)$ and $SO(4) \cong SO(3) \times SO(3)$ (which contains the Lorentz group as the diagonal subgroup), we see that the $SO(4)$ representations contained in the Kaluza-Klein tower have to be of the form (j, j), since only these contain a singlet in the decomposition $(j, j) \to 0 \oplus 1 \oplus ... \oplus 2j$. Similarly, the vector-tower contains also states of the form $(j, j+1)$ and $(j+1, j)$, and thus the Kaluza-Klein towers on top of scalars and vectors read

$$0 \longrightarrow \bigoplus_{j\geq 0} (j,j) \ , \qquad 1 \longrightarrow \bigoplus_{j\geq 1/2} (j,j) \ \oplus \ \bigoplus_{j\geq 0} (j,j+1) \ \oplus \ \bigoplus_{j\geq 0} (j+1,j) \ , \quad (3.13)$$

and analogously for all other states.

Next we are going to apply this procedure to the compactification of type IIB supergravity on $AdS_3 \times S^3 \times S^3 \times S^1$. The spectrum in $D = 10$ is organized into representations of the little group $SO(8)$ and reads [6]

$$(\mathbf{8}_V - \mathbf{8}_S)^2 = (\mathbf{8}_V \otimes \mathbf{8}_V \oplus \mathbf{8}_S \otimes \mathbf{8}_S) - (\mathbf{8}_S \otimes \mathbf{8}_V \oplus \mathbf{8}_V \otimes \mathbf{8}_S) \ , \quad (3.14)$$

where the negative sign indicates fermionic states. Since we compactify also on a circle, but take only the zero-modes here, we start effectively from the nine-dimensional theory, i.e. the little group is $SO(7)$. The relevant $SO(8)$ representations decompose according to

$$\mathbf{8}_V \to \mathbf{1} \oplus \mathbf{7}_V, \qquad \mathbf{8}_S \to \mathbf{8} \ , \quad (3.15)$$

where $\mathbf{8}$ denotes the spinor representation of $SO(7)$. The $SO(7)$ gets actually further reduced to the little group of the local Lorentz group of $AdS_3 \times S^{3+} \times S^{3-}$. As the little group of AdS_3 becomes trivial, this means that the $SO(7)$ gets reduced to $SO(3)^+ \times SO(3)^-$, so that the representations decompose as

$$\mathbf{7}_V \to (0,0) \oplus (1,0) \oplus (0,1) \ , \qquad \mathbf{8} \to (\tfrac{1}{2},\tfrac{1}{2}) \oplus (\tfrac{1}{2},\tfrac{1}{2}) \ . \quad (3.16)$$

According to the criterion we have to associate to each of the representations of the Lorentz group $SO(3)^+ \times SO(3)^-$ appearing in the IIB spectrum the tower of those representations of the isometry group $SO(4)^+ \times SO(4)^-$, which contains in the decomposition in representations of $SO(3)^+ \times SO(3)^-$ the given representation of the Lorentz group. For instance, if we start from a field which is a scalar $(0, 0)$, we get the Kaluza-Klein tower

$$(0,0) \ \longrightarrow \ \bigoplus_{j_1\geq 0, j_2\geq 0} (j_1, j_2; j_1, j_2) =: T_{KK} \ , \quad (3.17)$$

which extends (3.13). Here we have denoted the representations of the isometry groups $SO(4)^{\pm} \equiv SO(3)^{\pm}_L \times SO(3)^{\pm}_R$ according to the decomposition

$$G_{\rm c} \;=\; SO(3)^{+}_L \times SO(3)^{-}_L \times SO(3)^{+}_R \times SO(3)^{-}_R \tag{3.18}$$

as $(\ell^+_L, \ell^-_L; \ell^+_R, \ell^-_R)$. Specifically, the $SO(3) \times SO(3)$ content derived from the type IIB spectrum (3.14) by use of (3.16) reads

$$\begin{aligned} T_{\rm IIB} = [&10\,(0,0) \;\oplus\; 9\,(1,0) \;\oplus\; 9\,(0,1) \;\oplus\; 6\,(1,1) \;\oplus\; (2,0) \;\oplus\; (0,2)]_{\rm B} \,\oplus \\ &[16\,(\tfrac{1}{2},\tfrac{1}{2}) \;\oplus\; 4\,(\tfrac{3}{2},\tfrac{1}{2}) \;\oplus\; 4\,(\tfrac{1}{2},\tfrac{3}{2})]_{\rm F}\;, \end{aligned} \tag{3.19}$$

where the first line contains bosonic and the second line fermionic states. The lowest $SO(4)^+ \times SO(4)^-$ states appearing on top of these states are summarized in tab. 3.1. Here h_{mn}, $b_{\mu m}$, etc. denote the fluctuations of the metric, the scalars, the 2-forms and the 4-form, respectively. We have omitted all components which do not give rise to propagating degrees of freedom on AdS_3, and have also suppressed the components of the 4-form in the S^1 direction denoted by 9, in accordance with the self-duality constraint. In addition, the spectrum contains the 2-form C_2, whose Kaluza-Klein tower is identical to the one for B_2 given in tab. 3.1. Note that the fields of the first column in tab. 3.1 represent the bosonic $SO(3)^+ \times SO(3)^-$ states in (3.19).[3]

If one now associates to each of the $SO(3) \times SO(3)$ states in (3.19) a tower of $SO(4) \times SO(4)$ representations as in (3.17), one gets the Kaluza-Klein tower, which can be organized into supermultiplets, as has been shown in [70]. In the present case there exists, however, a more concise way to describe the Kaluza-Klein tower. Instead of reducing each $SO(7)$ representation to $SO(3) \times SO(3)$ and then lifting to a tower of $SO(4) \times SO(4)$ representations as explained above, the $SO(7)$ representation can be lifted to a $SO(8)$ representation,

$$\mathbf{8} \to \mathbf{8}_S\;, \qquad \mathbf{7}_V \to \mathbf{8}_V - \mathbf{1}\;, \tag{3.20}$$

and then directly reduced to $SO(4) \times SO(4)$ according to

$$\mathbf{8}_V \to (\tfrac{1}{2},0;\tfrac{1}{2},0) \oplus (0,\tfrac{1}{2};0,\tfrac{1}{2})\;, \qquad \mathbf{8}_S \to (\tfrac{1}{2},0;0,\tfrac{1}{2}) \oplus (0,\tfrac{1}{2};\tfrac{1}{2},0)\;. \tag{3.21}$$

The claim is that if one associates to each $SO(8)$ representation $R_{SO(8)}$ and the resulting $SO(4)\times SO(4)$ representation $R_{SO(4)\times SO(4)}$ an entire tower of those representations according to

$$R_{SO(8)} \longrightarrow R_{SO(4)\times SO(4)} \otimes T_{KK}\;, \tag{3.22}$$

where T_{KK} denotes the Kaluza-Klein tower corresponding to the singlet defined in (3.17), then this gives the same $SO(4) \times SO(4)$ content for the Kaluza-Klein tower as the procedure introduced in (3.17) above.

[3] For instance, the singlets in tab. 3.1 are given by h^m_m, $h^{\bar m}_{\bar m}$, $h_{\mu 9}$, h_{99}, ϕ, C_0, $b_{\mu 9}$, $c_{\mu 9}$, $a_{\mu mnk}$ and $a_{\mu\bar m\bar n\bar k}$, in accordance with the 10 singlets in (3.19).

	$(0,0;0,0)$	$(\frac{1}{2},0;\frac{1}{2},0)$	$(0,\frac{1}{2};0,\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})$	$(1,0;1,0)$	$(0,1;0,1)$	$(\frac{1}{2},1;\frac{1}{2},1)$	$(1,\frac{1}{2};1,\frac{1}{2})$	$(1,1;1,1)$
h_{mn}					+			+	+
h^m_m	+	+	+	+	+	+	+	+	+
$h_{\bar{m}\bar{n}}$						+	+		+
$h^{\bar{m}}_{\bar{m}}$	+	+	+	+	+	+	+	+	+
$h_{\mu m}$		+		+	+		+	+	+
$h_{\mu\bar{m}}$			+	+		+	+	+	+
$h_{m\bar{n}}$				+			+	+	+
$h_{\mu 9}$	+	+	+	+	+	+	+	+	+
h_{m9}		+		+	+		+	+	+
$h_{\bar{m}9}$			+	+		+	+	+	+
h_{99}	+	+	+	+	+	+	+	+	+
ϕ, C_0	+	+	+	+	+	+	+	+	+
$b_{\mu m}$		+		+	+		+	+	+
$b_{\mu\bar{m}}$			+	+		+	+	+	+
b_{mn}		+		+	+		+	+	+
$b_{\bar{m}\bar{n}}$			+	+		+	+	+	+
$b_{m\bar{n}}$				+			+	+	+
$b_{\mu 9}$	+	+	+	+	+	+	+	+	+
b_{m9}		+		+	+		+	+	+
$b_{\bar{m}9}$			+	+		+	+	+	+
$a_{\mu mnk}$	+	+	+	+	+	+	+	+	+
$a_{\mu\bar{m}\bar{n}\bar{k}}$	+	+	+	+	+	+	+	+	+
$a_{\mu mn\bar{k}}$				+			+	+	+
$a_{\mu m\bar{n}\bar{k}}$				+			+	+	+
$a_{mnk\bar{l}}$			+	+		+	+	+	+
$a_{mn\bar{k}\bar{l}}$				+			+	+	+
$a_{m\bar{n}\bar{k}\bar{l}}$		+		+	+		+	+	+
Σ_{scal}	5	10	10	18	11	11	19	19	20
Σ_{vect}	3	6	6	9	6	6	9	9	9
Σ_{form}	2	3	3	7	3	3	7	7	7

Table 3.1: The lowest KK states on top of the IIB fields.

This is trivially the case for the scalars. To confirm this statement in general one has to check explicitly that both procedures coincide for the $SO(7)$ representations $\mathbf{7}_V$ and $\mathbf{8}$. Let us show this for $\mathbf{7}_V$. Reducing it directly to $SO(3)^+ \times SO(3)^-$ according to (3.16) and then lifting as in (3.17) results in

$$\begin{aligned}
\mathbf{7}_V \;&\longrightarrow\; (0,0) \,\oplus\, (1,0) \,\oplus\, (0,1) \qquad (3.23)\\
&\longrightarrow \bigoplus_{j_1,j_2\leq 0} (j_1,j_2;j_1,j_2) \oplus \bigoplus_{j_1\leq 1/2,j_2\leq 0} (j_1,j_2;j_1,j_2) \oplus \bigoplus_{j_1\leq 0,j_2\leq 1/2} (j_1,j_2;j_1,j_2)\\
&\oplus \bigoplus_{j_1,j_2\leq 0} (j_1+1,j_2;j_1,j_2) \oplus \bigoplus_{j_1,j_2\leq 0} (j_1,j_2;j_1+1,j_2) \oplus \bigoplus_{j_1,j_2\leq 0} (j_1,j_2+1;j_1,j_2)\\
&\oplus \bigoplus_{j_1,j_2\leq 0} (j_1,j_2;j_1,j_2+1)\ .
\end{aligned}$$

For the second procedure we first have to lift to $SO(8)$, then reducing to $SO(4) \times SO(4)$ according to (3.21) and finally tensoring with the T_{KK} in (3.17). This yields

$$\big((\tfrac{1}{2},0;\tfrac{1}{2},0) \,\oplus\, (0,\tfrac{1}{2};0,\tfrac{1}{2}) - (0,0;0,0)\big) \,\otimes \bigoplus_{j_1,j_2\geq 0} (j_1,j_2;j_1,j_2)\ , \qquad (3.24)$$

and one may check explicitly that this tensor product coincides with (3.23).

Up to now we have defined an algorithm to determine the Kaluza-Klein spectrum in terms of representations of the $SO(4) \times SO(4)$ isometry group. Next we are going to use these results in order to write the entire Kaluza-Klein tower in a very compact form in terms of supermultiplets (see (3.29) below, [70]). However, since this group-theoretical procedure cannot reveal the masses of the various Kaluza-Klein states, we first have to comment on the required uniqueness of the supermultiplet structure. Standard sphere compactifications preserve maximal supersymmetry, so that for them massive long multiplets would contain states with spin up to 4. Therefore the Kaluza-Klein spectra can only be organized into (short) massive BPS multiplets, which are consistent with maximum spin 2. In turn, these BPS multiplets determine the masses of the states completely, such that the entire spectrum of supermultiplets is fixed. Even though the $AdS_3 \times S^3 \times S^3$ background is only half-maximally supersymmetric, we will, following [69, 70], nevertheless assume that the spectrum is organized into short multiplets. This is a reasonable assumption, since it naturally reflects the bound $m^2 = 0$ for the massless fields in the higher-dimensional theory we started with.

Let us first review the structure of supermultiplets on AdS_3. As the background is half-maximally supersymmetric, it preserves $\mathcal{N} = 8$ in $D = 3$, which corresponds to 16 real supercharges. Thus the spectrum is organized under some background (super-) isometry group, which in the given case turns out to be a direct product of two $\mathcal{N} = 4$ supergroups [71]

$$D^1(2,1;\alpha)_L \times D^1(2,1;\alpha)_R\ . \qquad (3.25)$$

This is the required supersymmetrisation of the AdS group (2.13), in which each factor combines a bosonic $SO(3) \times SO(3) \times SL(2,\mathbb{R})$ with eight real supercharges (see [71] for

h			
h_0		(ℓ^+,ℓ^-)	
$h_0+\frac{1}{2}$	$(\ell^+-\frac{1}{2},\ell^--\frac{1}{2})$	$(\ell^++\frac{1}{2},\ell^--\frac{1}{2})$	$(\ell^+-\frac{1}{2},\ell^-+\frac{1}{2})$
h_0+1	(ℓ^+,ℓ^--1)	(ℓ^+-1,ℓ^-)	(ℓ^+,ℓ^-)
$h_0+\frac{3}{2}$		$(\ell^+-\frac{1}{2},\ell^--\frac{1}{2})$	

Table 3.2: The generic short supermultiplet $(\ell^+,\ell^-)_{\rm S}$ of $D^1(2,1;\alpha)$, with $h_0=\frac{1}{1+\alpha}\,\ell^+ + \frac{\alpha}{1+\alpha}\,\ell^-$.

the exact definitions). More precisely, the noncompact factors $SL(2,\mathbb{R})_L\times SL(2,\mathbb{R})_R = SO(2,2)$ join into the isometry group of AdS_3 while the compact factors build the isometry groups $SO(4)^\pm \equiv SO(3)^\pm_L \times SO(3)^\pm_R$ of the two spheres $S^{3+}\times S^{3-}$ in (3.18). Accordingly, this group will show up as the gauge group of the effectively three-dimensional supergravity action. The parameter α of (3.25) corresponds to the ratio of the radii of the two spheres S^3.

The group (3.25) possesses short and long multiplets (analogously to massless and massive supermultiplets for Poincaré supersymmetry). A short $D^1(2,1;\alpha)$ supermultiplet is defined by its highest weight state $(\ell^+,\ell^-)^{h_0}$, where $\ell^\pm$ label spins of $SO(3)^\pm$ and $h_0 = \frac{1}{1+\alpha}\,\ell^+ + \frac{\alpha}{1+\alpha}\,\ell^-$ is the charge under the Cartan subgroup $SO(1,1)\subset SL(2,\mathbb{R})$. The corresponding supermultiplet will be denoted by $(\ell^+,\ell^-)_{\rm S}$. It is generated from the highest weight state by the action of three out of the four supercharges $G^a_{-1/2}$ $(a=1,...,4)$ and carries $8(\ell_+ + \ell_- + 4\ell_+\ell_-)$ degrees of freedom. Its $SO(3)^\pm$ representation content is summarized in table 3.2.

The generic long multiplet $(\ell^+,\ell^-)_{\rm long}$ instead is built from the action of all four supercharges $G^a_{-1/2}$ on the highest weight state and carries $16\,(2\ell_+{+}1)(2\ell_-{+}1)$ degrees of freedom. Its highest weight state satisfies the unitarity bound $h \geq \frac{1}{1+\alpha}\,\ell^+ + \frac{\alpha}{1+\alpha}\,\ell^-$. In case this bound is saturated, the long multiplet decomposes into two short multiplets (table 3.2) according to

$$(\ell^+,\ell^-)_{\rm long} \;=\; (\ell^+,\ell^-)_{\rm S} \,\oplus\, (\ell^+{+}\tfrac{1}{2},\ell^-{+}\tfrac{1}{2})_{\rm S}\,. \tag{3.26}$$

The lowest short supermultiplets $(0,\frac{1}{2})_{\rm S}$, $(0,1)_{\rm S}$, and $(\frac{1}{2},\frac{1}{2})_{\rm S}$ of $D^1(2,1;\alpha)$ are further degenerate and collected in table 3.3, and similar for $\ell^+\leftrightarrow\ell^-$, $\alpha\leftrightarrow 1/\alpha$. The long multiplet $(0,0)_{\rm long}$ is given in table 3.4. Moreover, an unphysical short multiplet $(0,0)_S$ can be defined in such a way that (3.26) applies also to $(0,0)_{\rm long} = (0,0)_{\rm S} \,\oplus\, (\frac{1}{2},\frac{1}{2})_{\rm S}$ (see table 3.4). Here, negative states are understood as constraints that eliminate physical degrees of freedom.

Short representations of the full supergroup (3.25) are constructed as tensor products of the supermultiplets (tab. 3.2) for the left and right factors, and correspondingly will be denoted by $(\ell^+_L,\ell^-_L;\ell^+_R,\ell^-_R)_{\rm S}$. The quantum numbers which denote the representations of the AdS_3 group $SO(2,2)$ are labeled by numbers s and Δ, which encode the AdS analogue of spin and mass, respectively. They are related to the values of h_R and h_L by $s=h_R-h_L, \Delta=h_L+h_R$.

The massive Kaluza-Klein spectrum of nine-dimensional supergravity on $AdS_3\times$

h	(ℓ^+, ℓ^-)
$\frac{\alpha}{2(1+\alpha)}$	$(0, \frac{1}{2})$
$\frac{2\alpha+1}{2(1+\alpha)}$	$(\frac{1}{2}, 0)$

h	(ℓ^+, ℓ^-)
$\frac{\alpha}{1+\alpha}$	$(0, 1)$
$\frac{3\alpha+1}{2(1+\alpha)}$	$(\frac{1}{2}, \frac{1}{2})$
$\frac{2\alpha+1}{1+\alpha}$	$(0, 0)$

h	(ℓ^+, ℓ^-)
$\frac{1}{2}$	$(\frac{1}{2}, \frac{1}{2})$
1	$(0,0) + (0,1) + (1,0)$
$\frac{3}{2}$	$(\frac{1}{2}, \frac{1}{2})$
2	$(0, 0)$

Table 3.3: The lowest short supermultiplets $(0, \frac{1}{2})_{\rm S}$, $(0,1)_{\rm S}$, and $(\frac{1}{2}, \frac{1}{2})_{\rm S}$ of $D^1(2,1;\alpha)$.

$S^3 \times S^3$ can now be written in terms of supermultiplets. For this we use that the IIB spectrum $(\mathbf{8}_V - \mathbf{8}_S)^2$ yields via the reduction in (3.21) a bosonic and fermionic $SO(4) \times SO(4)$ spectrum which coincides exactly with the content of the supermultiplet $(0,0;0,0)_{\rm long}$, which can be easily checked with tab. 3.4. Put differently, to associate to each $SO(8)$ representation of type IIB a Kaluza-Klein tower according to (3.22) is equivalent to take the tensor product of $(0,0;0,0)_{\rm long}$ (to which we will therefore refer as the fundamental multiplet) with the Kaluza-Klein tower in (3.17). Thus the full resulting Kaluza-Klein spectrum is described by

$$(0,0;0,0)_{\rm long} \otimes \bigoplus_{\ell^\pm \geq 0} (\ell^+, \ell^-; \ell^+, \ell^-) \;=\; \bigoplus_{\ell^\pm \geq 0} (\ell^+, \ell^-; \ell^+, \ell^-)_{\rm long} \;, \tag{3.27}$$

where the last equation follows from the fact that the long multiplets are given by tensor products of $(0,0)_{\rm long}$ with the highest-weight state (ℓ^+, ℓ^-),

$$(\ell^+, \ell^-)_{\rm long} = (0,0)_{\rm long} \otimes (\ell^+, \ell^-) \;. \tag{3.28}$$

This can be easily checked by use of (3.26) and table 3.4. Equivalently, as the spectrum is decomposable into short multiplets, it can be rewritten with (3.26) as

$$\begin{aligned} &\bigoplus_{\ell^+ \geq 0, \ell^- \geq 1/2} (\ell^+, \ell^-; \ell^+, \ell^-)_{\rm S} \;\oplus \bigoplus_{\ell^+ \geq 1/2, \ell^- \geq 0} (\ell^+, \ell^-; \ell^+, \ell^-)_{\rm S} \\ &\qquad \oplus \bigoplus_{\ell^+, \ell^- \geq 0} \big((\ell^+, \ell^-; \ell^+ + \tfrac{1}{2}, \ell^- + \tfrac{1}{2})_{\rm S} \oplus (\ell^+ + \tfrac{1}{2}, \ell^- + \tfrac{1}{2}; \ell^+, \ell^-)_{\rm S} \big) \end{aligned} \tag{3.29}$$

where we have omitted the unphysical multiplet $(0,0;0,0)_{\rm S}$ according to tab. 3.4. Note that the multiplets $(\ell^+, \ell^-; \ell^+, \ell^-)_{\rm S}$ generically contain massive fields with spin running from 0 to $\frac{3}{2}$, whereas multiplets of the type $(\ell^+, \ell^-; \ell^+ + \frac{1}{2}, \ell^- + \frac{1}{2})_{\rm S}$ represent massive spin-2 multiplets.

h	(ℓ^+, ℓ^-)
0	$(0,0)$
$\frac{1}{2}$	$(\frac{1}{2}, \frac{1}{2})$
1	$(0,1) + (1,0)$
$\frac{3}{2}$	$(\frac{1}{2}, \frac{1}{2})$
2	$(0,0)$

h	(ℓ^+, ℓ^-)
0	$(0,0)$
$\frac{1}{2}$	0
1	$-(0,0)$

Table 3.4: The long multiplet $(0,0)_{\rm long}$ and the unphysical short multiplet $(0,0)_{\rm S}$

In addition, the tower (3.29) contains $(\frac{1}{2},\frac{1}{2};0,0)_{\rm S}\oplus(0,0;\frac{1}{2},\frac{1}{2})_{\rm S}$, which is the massless supergravity multiplet. It consists of the vielbein, eight gravitinos, transforming as

$$\psi^I_\mu \quad : \quad (\tfrac{1}{2},\tfrac{1}{2};0,0)\oplus(0,0;\tfrac{1}{2},\tfrac{1}{2})\ , \tag{3.30}$$

under (3.18), and topological gauge vectors, corresponding to the $SO(4)_L\times SO(4)_R$ gauge group. In accordance with the counting of table 3.2 it does not contain any physical degrees of freedom.

In total we have seen that generically the short multiplets appearing in the Kaluza-Klein spectrum (3.29) may combine into long multiplets (3.26) [70]. This holds for all multiplets, except for the supergravity multiplet and one of the lowest massive spin-3/2 multiplets $(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})_{\rm S}$, since we have omitted the unphysical short multiplet $(0,0;0,0)_{\rm S}$ in the step from (3.27) to (3.29). This is in contrast to other sphere compactifications, which preserve maximal supersymmetry. The conformal weight of the long representations appearing in (3.29) is not protected by anything and may vary throughout the moduli space. This gives a distinguished role to the supermultiplet $(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})_{\rm S}$ that we shall analyze here.

3.2 Effective supergravities for spin-1/2 and spin-1 multiplets

Next we are going to construct the effective supergravities for the lowest Kaluza-Klein multiplets [72]. As the background preserves half of the supersymmetries, i.e. in total 16 real supercharges, the theories will have $\mathcal{N}=8$ in $D=3$.

The lowest multiplet is the massless supergravity multiplet $(0,0;\frac{1}{2},\frac{1}{2})\oplus(\frac{1}{2},\frac{1}{2};0,0)$. We have already seen in sec. 2.2.1 that it is described by a Chern-Simons theory for the appropriate supergroup.

Furthermore, the lowest massive multiplets in the Kaluza-Klein tower (3.29) are the degenerate multiplets $(0,\frac{1}{2};0,\frac{1}{2})_{\rm S}$ and $(0,1;0,1)_{\rm S}$ (together with $(\frac{1}{2},0;\frac{1}{2},0)_{\rm S}$ and $(1,0;1,0)_{\rm S}$), to which we will refer as the spin-$\frac{1}{2}$ and spin-1 multiplet, respectively, in accordance with their states of maximal spin. Their precise representation content is collected in table 3.5.

To construct their respective gauged supergravities one first of all has to identify the corresponding ungauged theory. Since it should be an $\mathcal{N}=8$ supersymmetric theory, we know already from sec. 2.2.1 the general form of the target spaces in (2.17). It remains to determine the actual dimension of this coset space, i.e. the number n in (2.17). The two spin-$\frac{1}{2}$ multiplets of table 3.5 contain together 16 bosonic degrees of freedom. This suggests that together they are effectively described by a gauging of the $\mathcal{N}=8$ theory with target space $SO(8,2)/(SO(8)\times SO(2))$. To confirm this we have to show that the gauge group $SO(4)\times SO(4)$ can be embedded into the rigid symmetry group $SO(8,2)$ in such a way that the induced spectrum coincides with the representations expected from table 3.5. As reviewed in sec. 2.2.2, the propagating

h_L \ h_R	$\frac{\alpha}{2(1+\alpha)}$	$\frac{1+2\alpha}{2(1+\alpha)}$
$\frac{\alpha}{2(1+\alpha)}$	$(0,\frac{1}{2};0,\frac{1}{2})$	$(0,\frac{1}{2};\frac{1}{2},0)$
$\frac{1+2\alpha}{2(1+\alpha)}$	$(\frac{1}{2},0;0,\frac{1}{2})$	$(\frac{1}{2},0;\frac{1}{2},0)$

h_L \ h_R	$\frac{\alpha}{1+\alpha}$	$\frac{3\alpha+1}{2(1+\alpha)}$	$\frac{2\alpha+1}{1+\alpha}$
$\frac{\alpha}{1+\alpha}$	$(0,1;0,1)$	$(0,1;\frac{1}{2},\frac{1}{2})$	$(0,1;0,0)$
$\frac{3\alpha+1}{2(1+\alpha)}$	$(\frac{1}{2},\frac{1}{2};0,1)$	$(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2};0,0)$
$\frac{2\alpha+1}{1+\alpha}$	$(0,0;0,1)$	$(0,0;\frac{1}{2},\frac{1}{2})$	$(0,0,0,0)$

Table 3.5: The spin-$\frac{1}{2}$ multiplet $(0,\frac{1}{2};0,\frac{1}{2})_{\rm S}$, and the massive spin-1 multiplet $(0,1;0,1)_{\rm S}$.

degrees of freedom of the $\mathcal{N}=8$ theory are carried entirely by the scalars P_μ^{Ir} and the fermions $\chi^{\dot{A}r}$. Thus they transform under $SO(8)$ as $\mathbf{8}_V\oplus\mathbf{8}_C$. And indeed, one verifies that the field content of $(0,\frac{1}{2};0,\frac{1}{2})_{\rm S}$ can be lifted from a representation of the gauge group (3.18) to an $\mathbf{8}_V\oplus\mathbf{8}_C$ with the embedding

$$\mathbf{8}_V\to(0,\tfrac{1}{2};0,\tfrac{1}{2})\ \oplus\ (\tfrac{1}{2},0;\tfrac{1}{2},0)\ ,\qquad \mathbf{8}_C\to(0,\tfrac{1}{2};\tfrac{1}{2},0)\ \oplus\ (\tfrac{1}{2},0;0,\tfrac{1}{2})\ , \tag{3.31}$$

while the supercharges (3.30) lift to the spinor representation $\mathbf{8}_S$ of $SO(8)$. This corresponds to the canonical embedding of the $SO(4)\times SO(4)$ gauge group (see (3.18)) into $SO(8)$ according to $\mathbf{8}_V\to\mathbf{4}_V\oplus\mathbf{4}_V$, etc. Hence, the two spin-1/2 multiplets reproduce the field content $(\mathbf{8}_V\oplus\mathbf{8}_C,\mathbf{2})$ of the ungauged $SO(8,2)/(SO(8)\times SO(2))$ theory. It remains to verify that the embedding (3.31) of the gauge group into $SO(8,2)$ is compatible with the constraints imposed by supersymmetry on the embedding tensor $\Theta_{\mathcal{MN}}$. Along the lines of [64] it can be shown that these requirements determine $\Theta_{\mathcal{MN}}$ completely up to a free parameter corresponding to the ratio α of the two sphere radii. The effective theory is then completely determined. Its scalar potential will be further investigated in chapter 5 (see also [73]) and indeed reproduces the correct scalar masses predicted by table 3.5.

The coupling of the spin-1 multiplets $(0,1;0,1)_{\rm S}\oplus(1,0;1,0)_{\rm S}$ is slightly more involved due to the presence of massive vector fields but can be achieved by a straightforward generalization of the case of a single S^3 compactification [60, 26]. Here, the effective theory for 128 degrees of freedom is a gauging of the $\mathcal{N}=8$ theory with coset space $SO(8,8)/(SO(8)\times SO(8))$. Again, the first thing to verify in this case is that the field content of $(0,1;0,1)_{\rm S}\oplus(1,0;1,0)_{\rm S}$ (table 3.5) can be lifted from a representation of the gauge group (3.18) as above to an $(\mathbf{8}_V\oplus\mathbf{8}_C,\mathbf{8}_V)$ of $SO(8)\times SO(8)$ via the embedding

$$\begin{aligned}&(\mathbf{8}_V,\mathbf{1})\to(0,\tfrac{1}{2};0,\tfrac{1}{2})\oplus(\tfrac{1}{2},0;\tfrac{1}{2},0)\ ,\quad (\mathbf{8}_C,\mathbf{1})\to(0,\tfrac{1}{2};\tfrac{1}{2},0)\oplus(\tfrac{1}{2},0;0,\tfrac{1}{2})\ ,\\ &(\mathbf{1},\mathbf{8}_V)\to(0,\tfrac{1}{2};0,\tfrac{1}{2})\oplus(\tfrac{1}{2},0;\tfrac{1}{2},0)\ ,\quad (\mathbf{1},\mathbf{8}_C)\to(0,\tfrac{1}{2};\tfrac{1}{2},0)\oplus(\tfrac{1}{2},0;0,\tfrac{1}{2})\end{aligned} \tag{3.32}$$

This corresponds to the embedding of groups $SO(8) \times SO(8) \supset SO(8)_D \supset SO(4) \times SO(4)$, where $SO(8)_D$ denotes the diagonal subgroup of the two $SO(8)$ factors. For instance, (3.32) implies that the bosonic part decomposes as

$$\begin{aligned}(\mathbf{8}_V, \mathbf{8}_V) \;&\to\; \Big((0,\tfrac{1}{2};0,\tfrac{1}{2}) \oplus (\tfrac{1}{2},0;\tfrac{1}{2},0)\Big) \otimes \Big((0,\tfrac{1}{2};0,\tfrac{1}{2}) \oplus (\tfrac{1}{2},0;\tfrac{1}{2},0)\Big) \\ &= (0,1;0,1) \oplus (0,1;0,0) \oplus (0,0;0,1) \oplus (1,0;1,0) \oplus (1,0;0,0) \oplus (0,0;1,0) \\ &\quad \oplus 2 \cdot (0,0;0,0) \oplus 2 \cdot (\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}) \,,\end{aligned} \tag{3.33}$$

in agreement with table 3.5 and its conjugate. It is important to note that the massive spin-1 fields show up in this decomposition through their Goldstone scalars, in accordance with the general discussion in 2.2.

Moreover, we have to keep in mind that in order to reproduce the correct coupling for these massive vector fields, the total gauge group $G_0 \subset SO(8,8)$ for the Chern-Simons gauged supergravity should not just be the compact factor G_c (3.18), but is rather extended by some nilpotent generators. In fact, the algebra takes the form of a semi-direct product

$$G_0 \;=\; G_c \ltimes T_{12} \,, \tag{3.34}$$

with the abelian 12-dimensional translation group T_{12} transforming in the adjoint representation of G_c [60]. In the AdS_3 vacuum, these translational symmetries are broken and the corresponding vector fields gain their masses in a Higgs effect. The proper embedding of (3.34) into $SO(8,8)$ is again uniquely fixed by the constraints imposed by supersymmetry on the embedding tensor $\Theta_{\mathcal{MN}}$ up to the free parameter α [26].

Finally, it is straightforward to construct the effective theory that couples both the spin-1/2 and the spin-1 supermultiplets as a gauging of the theory with coset space $SO(8,10)/(SO(8) \times SO(10))$ which obviously embeds the two target spaces described above.

3.3 The spin-3/2 multiplet

Finally we turn to the coupling of the spin-3/2 multiplet $(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})_S$, which is contained twice in the Kaluza-Klein tower (3.29). Its $SO(4) \times SO(4)$ representation content is summarized in table 3.3. In analogy to the aforementioned couplings of the spin-$\frac{1}{2}$ and spin-1 multiplet to $\mathcal{N} = 8$ supergravity, a natural candidate for the effective theory might be an $\mathcal{N} = 8$ gauging of the theory with coset space $SO(8,16)/(SO(8) \times SO(16))$, reproducing the correct number of 128 bosonic degrees of freedom. (The appearance of massive spin-$\frac{3}{2}$ fields would then require some analogue of the dualization taking place in the scalar/vector sector.) Let us check the representation content of table 3.6. It is straightforward to verify that the states of this multiplet may be lifted from a representation of the gauge group (3.18) to a

h_L \ h_R	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$\frac{1}{2}$	$(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2};0,0)$ $(\frac{1}{2},\frac{1}{2};0,1)$ $(\frac{1}{2},\frac{1}{2};1,0)$	$(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2};0,0)$
1	$(1,0;\frac{1}{2},\frac{1}{2})$ $(0,0;\frac{1}{2},\frac{1}{2})$ $(0,1;\frac{1}{2},\frac{1}{2})$	$(0,0;0,0)$ $(0,1;0,0),(0,0;0,1)$ $(1,0;0,0),(0,0;1,0)$ $(0,1;0,1),(1,0;1,0)$ $(0,1;1,0),(1,0;0,1)$	$(0,1;\frac{1}{2},\frac{1}{2})$ $(0,0;\frac{1}{2},\frac{1}{2})$ $(1,0;\frac{1}{2},\frac{1}{2})$	$(0,0;0,0)$ $(0,1;0,0)$ $(1,0;0,0)$
$\frac{3}{2}$	$(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2};0,0)$ $(\frac{1}{2},\frac{1}{2};0,1)$ $(\frac{1}{2},\frac{1}{2};1,0)$	$(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2};0,0)$
2	$(0,0;\frac{1}{2},\frac{1}{2})$	$(0,0;0,0)$ $(0,0;0,1)$ $(0,0;1,0)$	$(0,0;\frac{1}{2},\frac{1}{2})$	$(0,0;0,0)$

Table 3.6: The massive spin-3/2 supermultiplet $(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})_{\rm S}$.

representation $(\mathbf{8}_V\oplus\mathbf{8}_C,\mathbf{8}_V\oplus\mathbf{8}_C)$ of an $SO(8)_L\times SO(8)_R$ according to

$$\begin{aligned}
&(\mathbf{8}_V,\mathbf{1})\to(0,0;0,0)\oplus(0,0;0,0)\oplus(1,0;0,0)\oplus(0,1;0,0)\ ,\\
&(\mathbf{8}_C,\mathbf{1})\to(\tfrac{1}{2},\tfrac{1}{2};0,0)\oplus(\tfrac{1}{2},\tfrac{1}{2};0,0)\ ,\quad (\mathbf{8}_S,\mathbf{1})\to(\tfrac{1}{2},\tfrac{1}{2};0,0)\oplus(\tfrac{1}{2},\tfrac{1}{2};0,0)\ ,\\
&(\mathbf{1},\mathbf{8}_V)\to(0,0;0,0)\oplus(0,0;0,0)\oplus(0,0;1,0)\oplus(0,0;0,1)\ ,\\
&(\mathbf{1},\mathbf{8}_C)\to(0,0;\tfrac{1}{2},\tfrac{1}{2})\oplus(0,0;\tfrac{1}{2},\tfrac{1}{2})\ ,\quad (\mathbf{1},\mathbf{8}_S)\to(0,0;\tfrac{1}{2},\tfrac{1}{2})\oplus(0,0;\tfrac{1}{2},\tfrac{1}{2})
\end{aligned}\tag{3.35}$$

This corresponds to an embedding of groups according to

$$SO(4)_L=\mathrm{diag}[SO(4)\times SO(4)]\subset SO(4)\times SO(4)\subset SO(8)_L\ ,\tag{3.36}$$

and similarly for $SO(4)_R$. In order to be described as a gauging of the $\mathcal{N}=8$ theory, the field content would have to be further lifted to the $(\mathbf{8}_V\oplus\mathbf{8}_C,\mathbf{16})$ of $SO(8)\times SO(16)$. This is only possible, if $SO(8)_R$ is entirely embedded into the $SO(16)$. On the other hand, we know from sec. 2.2.2 that the gravitino of the $\mathcal{N}=8$ theory transform in the $(\mathbf{8}_S,\mathbf{1})$. This in turn implies with the embedding (3.35) that they would decompose as $(\frac{1}{2},\frac{1}{2};0,0)\oplus(\frac{1}{2},\frac{1}{2};0,0)$, in contrast to the gravitinos (3.30) of the Kaluza-Klein spectrum. We conclude that the massive spin-3/2 multiplet cannot be described as a gauging of the $SO(8,16)/(SO(8)\times SO(16))$ theory.

Rather we will find that the effective theory describing this multiplet is a maximally supersymmetric gauging of the $\mathcal{N}=16$ theory in its broken phase. Half of the supersymmetry is broken down to $\mathcal{N}=8$ and correspondingly eight gravitinos

acquire mass via a super-Higgs mechanism. As a first check we observe that the total number of degrees of freedom collected in table 3.6 indeed equals the $16^2 = 256$ of the maximal theory. This is moreover in agreement with our general considerations in sec. 2.1, where we have argued that a consistent coupling of massive spin-3/2 fields requires the existence of spontaneously broken supercharges.

More specifically, we have to check again whether an embedding of the gauge group can be found, that reproduces the right spectrum. In the $\mathcal{N} = 16$ theory we have seen that the fields are described by P_μ^A and $\chi^{\dot{A}}$, i.e. transforming in the spinor and conjugate spinor representation of $SO(16)$. Thus the required spectrum has to be lifted to $\mathbf{128}_S \oplus \mathbf{128}_C$. First we note that according to (3.35), the total spectrum can be lifted to an $(\mathbf{8}_V \oplus \mathbf{8}_S, \mathbf{8}_V \oplus \mathbf{8}_C)$ of $SO(8)_L \times SO(8)_R$ and thus further to $SO(16)$ according to

$$\begin{aligned} &\mathbf{16} \to (\mathbf{8}_C, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}_S) \,, \\ &\mathbf{128}_S \to (\mathbf{8}_V, \mathbf{8}_V) \oplus (\mathbf{8}_S, \mathbf{8}_C) \,, \quad \mathbf{128}_C \to (\mathbf{8}_S, \mathbf{8}_V) \oplus (\mathbf{8}_V, \mathbf{8}_C) \,. \end{aligned} \tag{3.37}$$

This corresponds to the canonical embedding $SO(16) \supset SO(8)_L \times SO(8)_R$ and an additional triality rotation. Finally this lifts the spectrum precisely to the $\mathbf{128}_S \oplus \mathbf{128}_C$ field content of the maximal $\mathcal{N} = 16$ theory with scalar target space $G/H = E_{8(8)}/SO(16)$.

3.3.1 Gauge group and spectrum

In this section, we will identify the full gauge group of the effective three-dimensional theory and determine its embedding into the global $E_{8(8)}$ symmetry group of the ungauged theory. We have already seen in sec. 2.2 that in order to describe a certain number of massive vectors fields, which are on-shell dual to Yang-Mills fields, the gauge algebra has to be enlarged. Its general form has been determined in [60] and will be denoted by

$$G_0 \;=\; G_{\rm c} \ltimes (\hat{T}_p, T_\nu) \,. \tag{3.38}$$

In our case, $G_{\rm c}$ denotes the compact gauge group (3.18) which from the Kaluza-Klein origin of the theory is expected to be realized by propagating vector fields. In the Chern-Simons formulation given above, this compact factor needs to be amended by the nilpotent translation group T_ν whose $\nu = \dim G_{\rm c}$ generators transform in the adjoint representation of $G_{\rm c}$. This allowed an alternative formulation of the theory (2.32) in which part of the scalar sector is redualized into propagating vector fields gauging the group $G_{\rm c}$, which accordingly appear with a conventional Yang-Mills term. The third factor $\hat{T}_p$ in (3.38) is spanned by p nilpotent translations transforming in some representation of $G_{\rm c}$ and closing into T_ν. This part of the gauge group is completely broken in the vacuum and gives rise to p massive vector fields. Specifically, the algebra underlying (3.38) reads

$$\begin{aligned} &[\mathcal{J}^m, \mathcal{J}^n] = f^{mn}{}_k \, \mathcal{J}^k \,, \qquad [\mathcal{J}^m, \mathcal{T}^{\underline{n}}] = f^{mn}{}_k \, \mathcal{T}^{\underline{k}} \,, \qquad [\mathcal{T}^{\underline{m}}, \mathcal{T}^{\underline{n}}] = 0 \,, \\ &[\mathcal{J}^m, \hat{\mathcal{T}}^\alpha] = t^{m\alpha}{}_\beta \, \hat{\mathcal{T}}^\beta \,, \qquad [\hat{\mathcal{T}}^\alpha, \hat{\mathcal{T}}^\beta] = t^{\alpha\beta}{}_m \, \mathcal{T}^{\underline{m}} \,, \qquad [\mathcal{T}^{\underline{m}}, \hat{\mathcal{T}}^\alpha] = 0 \,, \end{aligned} \tag{3.39}$$

with $\mathcal{J}^m$, $\mathcal{T}^{\underline{n}}$, and $\hat{\mathcal{T}}^\alpha$ generating G_c, T_ν, and $\hat{T}_p$, respectively. The $f^{mn}{}_k$ are the structure constants of G_c while the $t^{m\alpha}{}_\beta$ denote the representation matrices for the $\hat{\mathcal{T}}^\alpha$. Indices $m, n, \ldots$ are raised/lowered with the Cartan-Killing form of G_c; raising/lowering of indices α, β requires a symmetric G_c invariant tensor $\kappa^{\alpha\beta}$.

To begin with, we have to reconcile the structure (3.38) with the spectrum collected in table 3.6. With $G_c = SO(4)_L \times SO(4)_R$ from (3.18), T_ν transforms in the adjoint representation $(1,0;0,0)\oplus(0,1;0,0)\oplus(0,0;1,0)\oplus(0,0;0,1)$. Table 3.6 exhibits 34 additional massive vector fields, transforming in the $2\cdot(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})\oplus 2\cdot(0,0;0,0)$ of G_c. In total, we thus expect a gauge group $G_0 = G_c \ltimes (\hat{T}_{34}, T_{12})$ of dimension $\dim G_0 = 12 + 12 + 34 = 58$. Next, we have to identify this group within $E_{8(8)}$. To this end, it proves useful to first consider the embedding of G_c into $E_{8(8)}$ according to the decompositions (see appendix A)

$$\begin{array}{ccccc} E_{8(8)} & \supset & \left\{\begin{array}{ccc} \supset & SO(16) & \supset \\ \\ \supset & SO(8,8) & \supset \end{array}\right\} & \supset & SO(8)_L \times SO(8)_R \\ & \supset & SO(4)_L \times SO(4)_R & & \end{array} \tag{3.40}$$

with the two embeddings of $SO(8)_L \times SO(8)_R$ given by

$$\begin{aligned} SO(16) \;&:\quad \mathbf{16} \to (\mathbf{8}_C, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}_S)\,, \quad \mathbf{128}_S \to (\mathbf{8}_V, \mathbf{8}_V) \oplus (\mathbf{8}_S, \mathbf{8}_C)\,, \\ SO(8,8) \;&:\quad \mathbf{16} \to (\mathbf{8}_V, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}_V)\,, \quad \mathbf{128}_S \to (\mathbf{8}_C, \mathbf{8}_S) \oplus (\mathbf{8}_S, \mathbf{8}_C)\,. \end{aligned} \tag{3.41}$$

Accordingly, the group $E_{8(8)}$ decomposes as

$$\mathbf{248} \;\to\; \Big\{(\mathbf{28},\mathbf{1}) \oplus (\mathbf{1},\mathbf{28}) \oplus (\mathbf{8}_C,\mathbf{8}_S)\Big\} \;\oplus\; \Big\{(\mathbf{8}_V,\mathbf{8}_V) \oplus (\mathbf{8}_S,\mathbf{8}_C)\Big\}\,, \tag{3.42}$$

and further according to (3.35). Here curly brackets indicate the splitting into its compact and noncompact part and $\mathbf{28} = \mathbf{8} \wedge \mathbf{8}$. We have already discussed that with this embedding the noncompact part of $E_{8(8)}$ precisely reproduces the bosonic spectrum of table 3.6.

In order to identify the embedding of the full gauge group $G_0 = G_c \ltimes (\hat{T}_{34}, T_{12})$ we further consider the decomposition of $E_{8(8)}$ according to

$$E_{8(8)} \supset SO(8,8) \supset SO(6,6) \times SO(2,2) \supset SO(6,6) \times SO(1,1) \times SO(1,1)\,, \tag{3.43}$$

and its grading with respect to these two $SO(1,1)$ factors which is explicitly given in table 3.7. From this table we can infer that properly identifying

$$G_c \;\subset\; \mathbf{66}^0_0\,, \quad T_{12} \;=\; \mathbf{12}^{+1}_0\,, \quad \hat{T}_{34} \;\subset\; \mathbf{32}^{+1/2}_{-1/2} \;\oplus\; \overline{\mathbf{32}}^{+1/2}_{+1/2} \;\oplus\; \mathbf{1}^{+1}_{-1} \;\oplus\; \mathbf{1}^{+1}_{+1}\,, \tag{3.44}$$

precisely reproduces the desired algebra structure (3.39). We have thus succeeded in identifying the algebra $\mathfrak{g}_0$ underlying the full gauge group $G_0 = G_c \ltimes (\hat{T}_{34}, T_{12})$, which is entirely embedded in the 'upper light cone' of table 3.7. In the next section, we will explicitly construct the embedding tensor $\Theta_{\mathcal{MN}}$ projecting onto this algebra, and show that it is indeed compatible with the algebraic constraints (2.40), (2.24) imposed by supersymmetry.

$$
\begin{array}{ccccc}
\mathbf{1}^{+1}_{-1} & & \mathbf{12}^{+1}_{0} & & \mathbf{1}^{+1}_{+1} \\
& \mathbf{32}^{+1/2}_{-1/2} & & \overline{\mathbf{32}}^{+1/2}_{+1/2} & \\
\mathbf{12}^{0}_{-1} & & \mathbf{66}^{0}_{0} + \mathbf{1}^{0}_{0} + \mathbf{1}^{0}_{0} & & \mathbf{12}^{0}_{+1} \\
& \overline{\mathbf{32}}^{-1/2}_{-1/2} & & \mathbf{32}^{-1/2}_{+1/2} & \\
\mathbf{1}^{-1}_{-1} & & \mathbf{12}^{-1}_{0} & & \mathbf{1}^{-1}_{+1}
\end{array}
$$

Table 3.7: Grading of $E_{8(8)}$ according to $SO(6,6)\times SO(1,1)\times SO(1,1)$. For later reference, we denote by $SO(1,1)_a$ the factor responsible for the grading from left to right and by $SO(1,1)_b$ the factor responsible for the grading from top to bottom.

3.3.2 The embedding tensor

In this section, we will explicitly construct the embedding tensor $\Theta_{\mathcal{MN}}$ projecting onto the Lie algebra $\mathfrak{g}_0$ of the desired gauge group $G_0 = G_c \ltimes (\hat{T}_{34}, T_{12})$ identified in the previous section. The embedding tensor then uniquely defines the effective action (2.32). We start from the $SO(4)\times SO(4)$ basis of $E_{8(8)}$ defined in appendix A.4. In this basis, the grading of table 3.7 refers to the charges of the generators $X^{0\hat{0}}$ and $X^{\bar{0}\hat{\bar{0}}}$. We further denote the generators of G_c and T_{12} within table 3.7 as

$$
\mathbf{66}^0_0 \;\supset\; \mathbf{3}^+_L \oplus \mathbf{3}^-_L \oplus \mathbf{3}^+_R \oplus \mathbf{3}^-_R\,, \qquad \text{and} \quad \mathbf{12}^{+1}_0 \;=\; \hat{\mathbf{3}}^+_L \oplus \hat{\mathbf{3}}^-_L \oplus \hat{\mathbf{3}}^+_R \oplus \hat{\mathbf{3}}^-_R\,, \tag{3.45}
$$

respectively, with the labels L, R, $\pm$ referring to the four factors of (3.18), i.e. $\mathbf{3}^+_L = (1,0;0,0)$, $\mathbf{3}^-_L = (0,1;0,0)$, etc. Similarly, we will identify the generators of $\hat{T}_{34}$ among

$$
\mathbf{32}^{+1/2}_{-1/2} \equiv \mathbf{16}^{(1)}_- \oplus \mathbf{16}^{(2)}_-\,, \qquad \overline{\mathbf{32}}^{+1/2}_{+1/2} \equiv \mathbf{16}^{(1)}_+ \oplus \mathbf{16}^{(2)}_+\,, \qquad \mathbf{1}^{+1}_{-1} \equiv \mathbf{1}_-\,, \quad \mathbf{1}^{+1}_{+1} \equiv \mathbf{1}_+\,, \tag{3.46}
$$

where $\mathbf{16}$ denotes the $(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})$ of $SO(4)\times SO(4)$, and we use subscripts (1), (2), $\pm$ in order to distinguish the identical representations. The split of the $\mathbf{32}$ representations into two copies of the $\mathbf{16}$ is chosen such that the algebra closes according to

$$
\begin{aligned}
[\mathbf{16}^{(1)}_+, \mathbf{16}^{(1)}_-] \;&\subset\; \hat{\mathbf{3}}^-_L \oplus \hat{\mathbf{3}}^-_R\,, \qquad [\mathbf{16}^{(2)}_+, \mathbf{16}^{(2)}_-] \;\subset\; \hat{\mathbf{3}}^-_L \oplus \hat{\mathbf{3}}^-_R\,, \\
[\mathbf{16}^{(1)}_+, \mathbf{16}^{(2)}_-] \;&\subset\; \hat{\mathbf{3}}^+_L \oplus \hat{\mathbf{3}}^+_R\,, \qquad [\mathbf{16}^{(2)}_+, \mathbf{16}^{(1)}_-] \;\subset\; \hat{\mathbf{3}}^+_L \oplus \hat{\mathbf{3}}^+_R\,.
\end{aligned} \tag{3.47}
$$

The embedding tensor $\Theta_{\mathcal{MN}}$ is an object in the symmetric tensor product of two adjoint representations of $E_{8(8)}$. It projects onto the Lie algebra of the gauge group according to $\mathfrak{g}_0 = \langle X_{\mathcal{M}} \equiv \Theta_{\mathcal{MN}}\, t^{\mathcal{N}} \rangle$. We start from the most general ansatz for $\Theta_{\mathcal{MN}}$ that has entries only on the generators (3.45), (3.46). Since the $\Theta_{\mathcal{MN}}$ relevant for our theory moreover is an $SO(4)\times SO(4)$ invariant tensor, it can only have non-vanishing entries contracting coinciding representations, e.g. $\Theta_{\mathbf{3}^+_L,\hat{\mathbf{3}}^+_L}$, $\Theta_{\mathbf{16}^{(1)}_+,\mathbf{16}^{(2)}_-}$, etc. Using computer algebra (Mathematica), we can then implement the algebraic constraint (2.40). As one of the main results of this chapter, we find that this constraint

determines the embedding tensor Θ with these properties up to five free constants γ, β_1, β_2, β_3, β_4, in terms of which it takes the form[4]

$$\Theta_{\mathbf{3}_L^+,\hat{\mathbf{3}}_L^+} = \beta_1 \ , \quad \Theta_{\mathbf{3}_L^-,\hat{\mathbf{3}}_L^-} = \beta_2 \ , \quad \Theta_{\mathbf{3}_R^+,\hat{\mathbf{3}}_R^+} = \beta_3 \ , \quad \Theta_{\mathbf{3}_R^-,\hat{\mathbf{3}}_R^-} = \beta_4 \ ,$$

$$\Theta_{\mathbf{1}_+,\mathbf{1}_-} = \Theta_{\hat{\mathbf{3}}_R^+,\hat{\mathbf{3}}_R^+} = \Theta_{\hat{\mathbf{3}}_R^-,\hat{\mathbf{3}}_R^-} = -\Theta_{\hat{\mathbf{3}}_L^+,\hat{\mathbf{3}}_L^+} = -\Theta_{\hat{\mathbf{3}}_L^-,\hat{\mathbf{3}}_L^-} = \gamma \ ,$$

$$\begin{aligned} \Theta_{\mathbf{16}_+^{(1)},\mathbf{16}_-^{(1)}} &= -\frac{1}{32\sqrt{2}}(\beta_2+\beta_4) \ , \qquad \Theta_{\mathbf{16}_+^{(1)},\mathbf{16}_-^{(2)}} = -\frac{1}{32\sqrt{2}}(\beta_1+\beta_3) \ , \\ \Theta_{\mathbf{16}_+^{(2)},\mathbf{16}_-^{(1)}} &= \frac{1}{32\sqrt{2}}(\beta_1-\beta_3) \ , \qquad \Theta_{\mathbf{16}_+^{(2)},\mathbf{16}_-^{(2)}} = -\frac{1}{32\sqrt{2}}(\beta_2-\beta_4) \ . \end{aligned} \tag{3.48}$$

A priori, it seems quite surprising that the constraint (2.40) still leaves five free constants in Θ — the **27000** representation of $E_{8(8)}$ gives rise to 1552 different $SO(4) \times SO(4)$ representations that are separately imposed as constraints on our general ansatz for Θ.

In order to satisfy the full set of consistency constraints it remains to impose the quadratic constraint (2.24) on the embedding tensor $\Theta_{\mathcal{MN}}$. Again using computer algebra, we can compute the form of this constraint for the embedding tensor (3.48) and find that it reduces to a single condition on the parameters:

$$\beta_1^2 + \beta_2^2 = \beta_3^2 + \beta_4^2 \ . \tag{3.49}$$

This suggests a parametrisation as

$$\beta_1 = \kappa \sin\alpha_1 \ , \quad \beta_2 = \kappa\cos\alpha_1 \ , \quad \beta_3 = \kappa\sin\alpha_2 \ , \quad \beta_4 = \kappa\cos\alpha_2 \ . \tag{3.50}$$

Altogether we have shown, that there is a four parameter family of maximally supersymmetric theories, described by the embedding tensor (3.48), which satisfies all the consistency constraints (2.40), (2.24).

For generic values of the parameters, one verifies that the rank of the induced gauge group is indeed 58 as expected.[5] In particular, (3.48), (3.49) imply that on the block of **16** representations one finds

$$\begin{aligned} \Theta_{\mathcal{MN}}\, t^{\mathcal{M}} \otimes t^{\mathcal{N}}\Big|_{\mathbf{16}} &= -\frac{\kappa}{16\sqrt{2}} \Big(\mathbf{16}_+^{(1)} \cos(\tfrac{1}{2}(\alpha_1-\alpha_2)) - \mathbf{16}_+^{(2)} \sin(\tfrac{1}{2}(\alpha_1-\alpha_2))\Big) \otimes \\ &\quad \Big(\mathbf{16}_-^{(1)} \cos(\tfrac{1}{2}(\alpha_1+\alpha_2)) - \mathbf{16}_-^{(2)} \sin(\tfrac{1}{2}(\alpha_1+\alpha_2))\Big) \ . \end{aligned} \tag{3.51}$$

This implies that out of the 64 generators $\mathbf{16}_\pm^{(1)}$, $\mathbf{16}_\pm^{(2)}$, only the 32 combinations

$$\begin{aligned} \mathbf{16}_+ &\equiv \mathbf{16}_+^{(1)} \cos(\tfrac{1}{2}(\alpha_1-\alpha_2)) - \mathbf{16}_+^{(2)} \sin(\tfrac{1}{2}(\alpha_1-\alpha_2)) \ , \\ \mathbf{16}_- &\equiv \mathbf{16}_-^{(1)} \cos(\tfrac{1}{2}(\alpha_1+\alpha_2)) - \mathbf{16}_-^{(2)} \sin(\tfrac{1}{2}(\alpha_1+\alpha_2)) \ , \end{aligned} \tag{3.52}$$

[4] Here we have used a somewhat symbolic notation for Θ, indicating just the multiples of the identity matrix that Θ takes in the various blocks.

[5] Let us note that the degenerate case $\kappa = 0$ induces a theory with 14-dimensional nilpotent abelian gauge group, as can be seen from (3.48). This particular gauge group had already been identified in [74].

form part of the gauge group. These correspond to the $2\cdot(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})$ generators in $\hat{T}_{34}$. The complete gauge algebra spanned by the generators $X_{\mathcal{M}} \equiv \Theta_{\mathcal{MN}}\, t^{\mathcal{N}}$ is precisely of the form anticipated in (3.39).

Let us stress another important property of the embedding tensor (3.48): it is a singlet not only under the $SO(4)\times SO(4)$, but also under the $SO(1,1)_a$ generating the grading from left to right in table 3.7. In other words, the resulting Θ contracts only representations for which these particular charges add up to zero. As a consequence the gauged supergravity (2.32) in addition to the local gauge symmetry $G_0 = G_{\rm c} \ltimes (\hat{T}_{34}, T_{12})$ is invariant under the action of the global symmetry $SO(1,1)_a$. We will discuss the physical consequences of this extra symmetry in section 3.3.4 below.

3.3.3 Ground state and isometries

In the previous section we have found a four-parameter family of solutions $\Theta_{\mathcal{MN}}$ (3.48) to the algebraic constraints (2.40), (2.24) compatible with the gauge algebra $G_0 = G_{\rm c} \ltimes (\hat{T}_{34}, T_{12})$. We will now show that the four free parameters γ, κ, α_1, α_2, can be adjusted such that the theory admits an $\mathcal{N}=(4,4)$ supersymmetric AdS ground state, leaving only two free parameters that correspond to the the radii of the two S^3 spheres. Furthermore, expanding the action (2.32) around this ground state precisely reproduces the spectrum of table 3.6.

In order to show, that the Lagrangian (2.32) admits an AdS ground state, we first have to check the condition (2.37) equivalent to the existence of a stationary point of the scalar potential (2.36). For this in turn we have to compute the tensors A_1, A_2 and A_3 (2.34) from the T-tensor (2.35) evaluated at the ground state $\mathcal{V}=I$. At this point, the T-tensor coincides with the embedding tensor (3.48). The only technical problem is the translation from Θ (3.48) in the $SO(8,8)$ basis of appendix A.2 into the $SO(16)$ basis of appendix A.1, in which the tensors A_1, A_2 and A_3 are defined.

It follows from (3.48) that Θ is traceless, $\theta = 0$, and moreover that all components of Θ, which mix bosonic and spinorial parts, like $\Theta_{ab|\alpha\dot\beta}$, vanish. As a consequence, the tensor A_1 is block-diagonal, and with the index split in appendix A its explicit form turns out to be

$$A_1^{IJ} = \frac{1}{7}\begin{pmatrix} 2\Theta_{\dot\alpha\gamma|\dot\beta\gamma} + \overline{\Gamma}^{\hat a\hat b}_{\dot\alpha\dot\gamma}\overline{\Gamma}^{\hat c\hat d}_{\dot\beta\dot\gamma}\Theta_{\hat a\hat b|\hat c\hat d} & 0 \\ 0 & 2\Theta_{\dot\gamma\alpha|\dot\gamma\beta} + \Gamma^{ab}_{\alpha\gamma}\Gamma^{cd}_{\beta\gamma}\Theta_{ab|cd} \end{pmatrix}, \tag{3.53}$$

with $SO(8)$ Γ-matrices $\Gamma^a_{\alpha\dot\beta}$, see appendix A.3 for details. Similarly, A_2 and A_3 are also block-diagonal and can be written as

$$A_2^{I\dot A} = -\frac{1}{7}\begin{pmatrix} 2\Gamma^a_{\gamma\dot\epsilon}\Theta_{\dot\alpha\gamma|\beta\dot\epsilon} - \Gamma^{\hat b}_{\beta\dot\gamma}\overline{\Gamma}^{\hat c\hat d}_{\dot\alpha\dot\gamma}\Theta_{\hat c\hat d|a\hat b} & 0 \\ 0 & 2\Gamma^{\hat a}_{\delta\dot\gamma}\Theta_{\dot\gamma\alpha|\delta\dot\beta} + \Gamma^{cd}_{\alpha\gamma}\Gamma^{b}_{\gamma\dot\beta}\Theta_{cd|b\hat a} \end{pmatrix}, \tag{3.54}$$

and

$$A_3^{\dot A\dot B} = \begin{pmatrix} \delta^{\alpha\beta}\Theta_{a\hat c|b\hat c} + 2\delta^{ab}\Theta_{\alpha\dot\gamma|\beta\dot\gamma} & 0 \\ 0 & \delta^{\dot\alpha\dot\beta}\Theta_{c\hat a|c\hat b} + 2\delta^{\hat a\hat b}\Theta_{\gamma\dot\alpha|\gamma\dot\beta} \end{pmatrix}. \tag{3.55}$$

Using these tensors one can now check that the ground state condition (2.37) is fulfilled if the parameters of (3.48) satisfy

$$\kappa^2 \;=\; 16\gamma^2 \,. \tag{3.56}$$

Moreover, using (2.36) the value of the scalar potential at the ground state, i.e. the cosmological constant, can be computed and consistently comes out to be negative, $V = -g^2/2$, i.e. the AdS length is given by $L_0 = 1/g$.

In the following, we will absorb κ into the global coupling constant g and set $\gamma = 1/4$ in accordance with (3.56). As a result, there remains a two-parameter family of supergravities admitting an AdS ground state. Let us now determine the number of unbroken supercharges in the ground state. It is derived from the Killing spinor equations

$$\delta\psi^I_\mu = D_\mu\epsilon^I + igA_1^{IJ}\gamma_\mu\epsilon^J \equiv 0 \,, \qquad \delta\chi^{\dot{A}} = gA_2^{I\dot{A}}\epsilon^I \equiv 0 \,. \tag{3.57}$$

As has been shown in [57], the number of solutions to (3.57) and thus the number of preserved supersymmetries is given by the number of eigenvalues α_i of the tensor A_1^{IJ} with $|\alpha_i|gL_0 = 1/2$. Computing these eigenvalues from the explicit form of (3.53) we find that the tensor A_1^{IJ} may be diagonalized as

$$A_1^{IJ} \;=\; \mathrm{diag}\left\{-\tfrac{3}{2},-\tfrac{3}{2},-\tfrac{3}{2},-\tfrac{3}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2}\right\} . \tag{3.58}$$

From this, we infer that the AdS ground state of the theory indeed preserves $\mathcal{N} = (4,4)$ supersymmetries, as expected. The other eight gravitinos become massive through a super-Higgs mechanism [61, 63]. This implies that due to the broken supersymmetries eight of the spin-1/2 fermions

$$\eta^I \;\equiv\; A_2^{I\dot{A}}\chi^{\dot{A}} \,, \tag{3.59}$$

transform by a shift under supersymmetry and act as Goldstone fermions that get eaten by the gravitino fields which in turn become massive propagating spin-3/2 fields. With the relation

$$|\Delta - 1| = |m|\, L_0 \,, \tag{3.60}$$

between the AdS masses m and conformal dimensions Δ of fermions and self-dual massive vectors in three dimensions, (3.58) implies that the massive gravitinos correspond to operators with conformal weights $(\frac{1}{2},2)$ and $(2,\frac{1}{2})$, in precise agreement with the spectrum of table 3.6.

To compute the physical masses for the spin-1/2 fermions, we observe from (2.32) that their mass matrix is given by $gA_3^{\dot{A}\dot{B}}$, except for the eight eigenvalues that correspond to the Goldstone fermions (3.59). From the explicit form (3.55) one computes the spin-1/2 masses and verifies using (3.60) that they coincide with those of table 3.6. Finally, we may check the mass spectrum for the spin-1 fields. Their mass matrix is given by Θ_{AB}, the projection of the embedding tensor onto the non-compact part of

the algebra [61]. From (3.48) one finds by explicit computation for these eigenvalues 46 non-vanishing values in precise accordance with table 3.6.

Altogether, we have shown the existence of a new family of gauged maximally supersymmetric theories in $D = 3$, which are parametrized by the two free parameters α_1 and α_2 and the overall gauge coupling constant g. These theories admit an $\mathcal{N} = (4, 4)$ supersymmetric AdS_3 ground state and linearizing the field equations around this ground state reproduces the correct spectrum of table 3.6. In particular, this spectrum does not depend on the particular values of α_1 and α_2. One may still wonder about the meaning of these two parameters. From the point of view of the Kaluza-Klein reduction the only relevant parameter is the ratio α of the two spheres radii, which enters the superalgebra (3.25). Let us thus compute the background isometry group by expanding the supersymmetry algebra

$$\{\,\delta_{\epsilon_1}, \delta_{\epsilon_2}\,\} \;=\; (\bar\epsilon_1^I \epsilon_2^J)\, \mathcal{V}^{\mathcal{M}}{}_{IJ}\, \Theta_{\mathcal{M}\mathcal{N}}\, t^{\mathcal{N}} + \dots \;, \tag{3.61}$$

around the ground state $\mathcal{V} = I$. The conserved supercharges ϵ^I are the eigenvectors of A_1 from (3.58) to the eigenvalues $\pm 1/2$ where the different signs correspond to the split into left and right supercharges according to (3.25). Correspondingly, the algebra (3.61) splits into two parts, L and R, with anticommutators

$$\{G^i_{-1/2\,L,R}, G^j_{1/2\,L,R}\} = 4\left(\frac{1}{1+\alpha_{L,R}}\,\tau^{+ij}_{kl}\,J^{+kl}_{L,R} + \frac{\alpha_{L,R}}{1+\alpha_{L,R}}\,\tau^{-ij}_{kl}\,J^{-kl}_{L,R}\right) + \dots \;, \tag{3.62}$$

where $\tau^{+ij}_{kl} \equiv \delta^{ij}_{kl} \pm \frac{1}{2}\epsilon_{ijkl}$ denote the projectors onto selfdual and anti-selfdual generators of $SO(4)_{L,R}$ corresponding to the split $SO(4) = SO(3)^+ \times SO(3)^-$. This coincides with the anticommutators of the superalgebra $D^1(2,1;\alpha_{L,R})$ [71]. Specifically, we find the relation

$$\alpha_L = \tan\alpha_1 \;, \qquad \alpha_R = \tan\alpha_2 \;, \tag{3.63}$$

to the parameters (3.50) of the embedding tensor. This shows that the three-parameter family of theories constructed in this section exhibits the background isometry group

$$D^1(2,1;\alpha_L)_L \times D^1(2,1;\alpha_R)_R \;. \tag{3.64}$$

The theories related to the Kaluza-Klein compactification on $AdS_3 \times S^3 \times S^3$ are thus given by further restricting $\alpha_L = \alpha_R \equiv \alpha$ where this parameter corresponds to the ratio of radii of the two spheres.

Putting everything together, we have shown that the effective supergravity action describing the field content of table 3.6 is given by the Lagrangian (2.32) with the following particular form of the embedding tensor $\Theta_{\mathcal{M}\mathcal{N}}$

$$\begin{aligned}
&\Theta_{\mathbf{3}^+_L,\hat{\mathbf{3}}^+_L} = \Theta_{\mathbf{3}^+_R,\hat{\mathbf{3}}^+_R} = \frac{\alpha}{\sqrt{1+\alpha^2}} \;, \qquad \Theta_{\mathbf{3}^-_L,\hat{\mathbf{3}}^-_L} = \Theta_{\mathbf{3}^-_R,\hat{\mathbf{3}}^-_R} = \frac{1}{\sqrt{1+\alpha^2}} \;,\\
&\Theta_{\mathbf{1}_+,\mathbf{1}_-} \;=\; \Theta_{\hat{\mathbf{3}}^+_R,\hat{\mathbf{3}}^+_R} \;=\; \Theta_{\hat{\mathbf{3}}^-_R,\hat{\mathbf{3}}^-_R} \;=\; -\,\Theta_{\hat{\mathbf{3}}^+_L,\hat{\mathbf{3}}^+_L} \;=\; -\,\Theta_{\hat{\mathbf{3}}^-_L,\hat{\mathbf{3}}^-_L} \;=\; \frac{1}{4} \;,\\
&\Theta_{\mathbf{16}^{(1)}_+,\mathbf{16}^{(1)}_-} = -\frac{1}{16\sqrt{2}}\,\frac{1}{\sqrt{1+\alpha^2}} \;, \qquad \Theta_{\mathbf{16}^{(1)}_+,\mathbf{16}^{(2)}_-} = -\frac{1}{16\sqrt{2}}\,\frac{\alpha}{\sqrt{1+\alpha^2}} \;.
\end{aligned} \tag{3.65}$$

We have verified that this tensor indeed represents a solution of the algebraic consistency constraints (2.40), (2.24). The resulting theory admits an $\mathcal{N} = (4,4)$ supersymmetric AdS_3 ground state with background isometry group (3.25) at which half of the 16 supersymmetries are spontaneously broken and the spectrum of table 3.6 is reproduced via a supersymmetric version of the Higgs effect.

3.3.4 The scalar potential for the gauge group singlets

We have identified the gauged supergravity theory, whose broken phase describes the coupling of the massive spin-3/2 multiplet $(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})_{\rm S}$ to the supergravity multiplet. In particular, the scalar potential (2.36) of the effective three-dimensional theory is completely determined in terms of the embedding tensor (3.65). In the holographic context this scalar potential carries essential information about the boundary conformal field theory, in particular about higher point correlation functions and about deformations and renormalization group flows. Explicit computation of the full potential is a highly nontrivial task, as it is a function on the 128-dimensional target space $E_{8(8)}/SO(16)$. For concrete applications it is often sufficient to evaluate this potential on particular subsectors of the scalar manifold.

As an example, let us in this section evaluate the potential on the gauge group singlets. From table 3.6 we read off that there are two scalar fields that are singlets under the $SO(4)_L \times SO(4)_R$ gauge group. Let us denote them by ϕ_1 and ϕ_2. They are dual to a marginal and an irrelevant operator of conformal dimension $(1,1)$ and $(2,2)$, respectively. In particular, the scalar ϕ_1 corresponds to a modulus of the theory. In order to determine the explicit dependence of the scalar potential on these fields, we parametrize the scalar $E_{8(8)}$ matrix $\mathcal{V}$ as

$$\mathcal{V} = \exp\left(\phi_1 X^{0\hat{0}} + \phi_2 X^{\bar{0}\hat{\bar{0}}}\right) , \tag{3.66}$$

where $X^{0\hat{0}}$ and $X^{\bar{0}\hat{\bar{0}}}$ are the generators of the $SO(1,1)_a$ and $SO(1,1)_b$ of table 3.7, respectively. The potential is obtained by computing with this parametrisation the T-tensor from (2.35), (3.65), splitting it into the tensors A_1 and A_2 according to (2.34) and inserting the result into (2.36).

The computation is simplified by first transforming the two singlets into a basis where their adjoint action is diagonal, such that their exponentials can be easily computed and afterwards transforming back to the $SO(16)$ basis of appendix A.1. It becomes now crucial that the embedding tensor Θ is invariant under $SO(1,1)_a$ and thus under the adjoint action of $X^{0\hat{0}}$ as we found in section 3.3.2 above. This implies, that the T-tensor (2.35) is in fact completely independent of ϕ_1. In turn, neither the fermionic mass terms nor the scalar potential carries an explicit dependence on ϕ_1. This scalar thus enters the theory only through its kinetic term and the dual operator is truly marginal.

The scalar potential (2.36) evaluated on the gauge group singlets is finally given

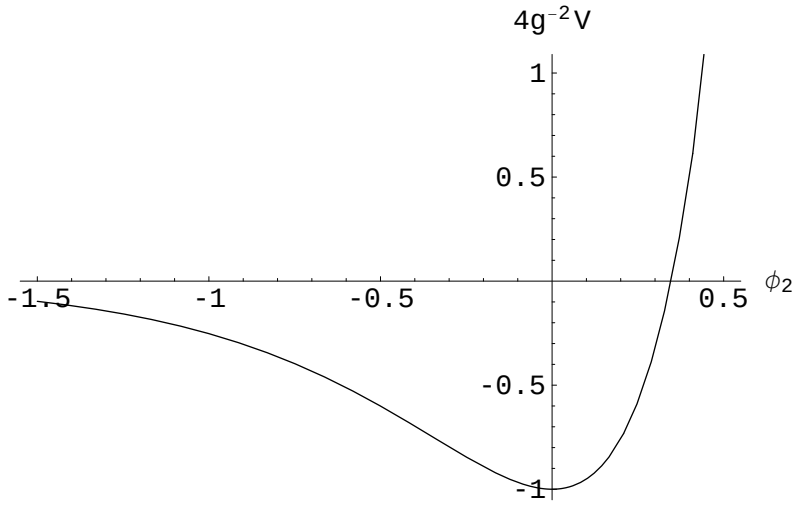


Figure 3.1: The scalar potential for the gauge group singlets

as a function of ϕ_2 as

$$V(\phi_1, \phi_2) = \frac{g^2}{4} e^{2\phi_2} \big(-2 + e^{2\phi_2} \big) \, . \tag{3.67}$$

The profile is plotted in Figure 3.1. Its explicit form shows that the theory has no other ground state which preserves the full $SO(4) \times SO(4)$ symmetry.

Chapter 4

Massive spin-2 fields and their infinite-dimensional symmetries

4.1 Is there a spin-2 Higgs effect?

We have seen in the last chapter that the massive spin-3/2 states appearing in Kaluza-Klein supergravities have to be described within the framework of spontaneously broken supersymmetry. Specifically we have seen that for Kaluza-Klein supergravity on $AdS^3 \times S^3 \times S^3$ the inclusion of the lowest spin-3/2 multiplet requires already an enhancement of supersymmetry from $\mathcal{N} = 8$ to $\mathcal{N} = 16$, which in turn gets spontaneously broken in the Kaluza-Klein vacuum.

Let us now turn to the similar problem of finding the effective action for spin-2 fields. As we have discussed in sec. 2.1 their naive coupling to gravity is inconsistent, both in the massless and in the massive case. In comparison, the analogous inconsistencies appearing for spin-3/2 fields were avoided by the introduction of supersymmetry, which linked the spin-3/2 fields with the metric in such a way that the entire theory becomes consistent. In consideration of the fact that the spin-2 couplings appearing in Kaluza-Klein theories have to be consistent, one might expect to identify a similar phenomenon for spin-2 fields. Indeed we will show the appearance of an infinite-dimensional spin-2 symmetry, that guarantees consistent couplings in the same sense as supersymmetry does for spin-3/2 fields. Even more, we will see that it is possible to parallel the main steps for the construction of gauged supergravities, where the spin-3/2 fields get somehow replaced by the spin-2 fields. For this we will again focus on compactifications to $D = 3$, since here – as we have seen – the gauged supergravities have a more coherent form due to the topological nature of all 'higher-spin' fields (starting with $s = 1$). Thus we expect similar simplifications for spin-2 theories.

In order to accomplish such a program we have to show the existence of an unbroken or ungauged phase, where the spin-2 fields appear to be massless. Furthermore, we expect the bosonic degrees of freedom to be carried entirely by scalars, i.e. we have to show that even in the presence of an infinite number of spin-2 fields all appearing

vector fields (including the Kaluza-Klein vectors) can be dualized into scalars. After dualization we expect this scalar sector to exhibit an enhanced rigid symmetry, in analogy to supergravities, in which, e.g., the $\mathcal{N} = 16$ theory carries the exceptional symmetry group $E_{8(8)}$. These enlarged global symmetries will presumably restrict the possible couplings severely. In addition, there will be a local spin-2 symmetry for infinitely many spin-2 fields in much the same way as one has local supersymmetry in an ungauged supergravity theory. The broken phase will then be constructed by gauging a certain subgroup of the global symmetries, or in other words by switching on a gauge coupling constant. This gauge coupling will later turn out to be given by the mass scale M characterizing the inverse radius of the internal manifold (such that the unbroken phase corresponds to the decompactification limit). The gauging in turn will modify the spin-2 symmetries by M-dependent terms and will induce a spin-2 mass term. The latter enables a novel Higgs effect for the spin-2 fields, which takes place in the same way as the super-Higgs effect in supergravity.

In practice, we have to address the following questions, which also determine the organization of this chapter:

(i) How can we identify the unbroken phase, and how is the spin-2 symmetry realized in this limit? In particular, how does this theory fit into the no-go results discussed in the literature before?

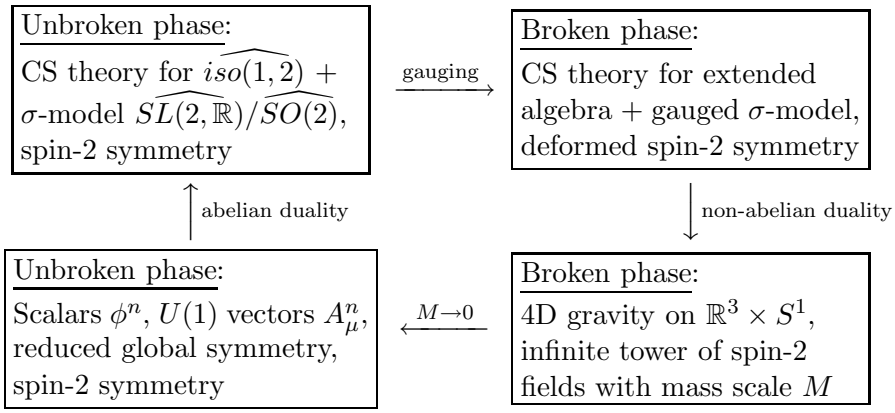
(ii) Which global symmetry is realized on the scalar fields in this phase?

(iii) Which subgroup of the global symmetries has to be gauged in order to get the full Kaluza-Klein theory? Does a formulation exist also for the gauged phase, where all vector and spin-2 fields appear to be topological?

(iv) How does the spin-2 symmetry get modified due to the gauging?

For answering question (i) we have to identify the ungauged theory with its symmetries. The symmetries appearing in Kaluza-Klein theories have in part been analyzed by Dolan and Duff in [33], where they showed that in the simplest case of an S^1 compactification including all massive modes a local Virasoro algebra $\hat{v}$ corresponding to the diffeomorphisms on S^1 as well as the affine extension $\widehat{iso}(1,2)$ of the Poincaré algebra appear. This Kac-Moody algebra describes the infinite-dimensional spin-2 symmetry, which in the Kaluza-Klein vacuum will be broken to the lower-dimensional diffeomorphism group. Answering question (ii) we will see that the scalar fields span a generalization of a non-linear σ-model (which we are going to make precise later) with target space $\widehat{SL(2,\mathbb{R})}/\widehat{SO(2)}$. Thus the global symmetry group contains an enhancement of the Ehlers group $SL(2,\mathbb{R})$ to its affine extension, but moreover it will also contain the Virasoro algebra. By gauging a certain subgroup of the global symmetries, i.e. after answering question (iii), we will see that a formulation is still possible in which all degrees of freedom are carried by scalars. The topological Kaluza-Klein vectors will in turn combine with the spin-2 fields into a Chern-Simons theory for an extended algebra. This algebra structure will also enlighten the deformations of the

spin-2 transformations due to the gauging, thus answering in part question (iv). The strategy underlying this approach can be illustrated schematically as given below.



In order to get the full Kaluza-Klein theory (given in the lower right corner) we show how to construct the ungauged theory with its enlarged symmetry group (given in the upper left corner). Then we argue that via gauging part of the global symmetries we obtain a theory (given in the upper right corner) which is on-shell equivalent to the original theory.[1]

4.2 Kac-Moody symmetries in Kaluza-Klein theories

It has been shown by Dolan and Duff [33] that Kaluza-Klein compactification can be analyzed from the following point of view. The infinite tower of massive modes in the lower-dimensional Kaluza-Klein spectrum can be viewed as resulting from a spontaneous symmetry breaking of an infinite-dimensional Kac-Moody-like algebra down to the Poincaré group times the isometry group of the internal manifold. This infinite dimensional symmetry group is a remnant of the higher dimensional diffeomorphism group.

To be more specific let us review Dolan and Duff's analysis applied to the case of a Kaluza-Klein reduction on $\mathbb{R}^3 \times S^1$. We start from pure Einstein gravity in $D = 4$ and split the vielbein E^A_M in $D = 4$ as follows[2]:

$$E^A_M = \begin{pmatrix} \phi^{-1/2} e^a_\mu & \phi^{1/2} A_\mu \\ 0 & \phi^{1/2} \end{pmatrix} . \tag{4.1}$$

Here we have chosen a triangular gauge and also performed a Weyl rescaling. The fields are now expanded in spherical harmonics of the compact manifold, which for

[1] This chapter is based on [75].

[2] $M, N, ... = 0, 1, 2, 5$ denote $D = 4$ space-time indices, $A, B, ...$ are flat $D = 4$ indices and the coordinates are called $x^M = (x^\mu, \theta/M)$, where M is a mass scale characterizing the inverse radius of the compact dimension. Our metric convention is $(+, -, -)$ for $D = 3$ and similar for $D = 4$.

S^1 simply reads

$$\begin{aligned} e^a_\mu(x,\theta) &= \sum_{n=-\infty}^{\infty} e^{a(n)}_\mu(x) e^{in\theta} \,, \qquad A_\mu(x,\theta) = \sum_{n=-\infty}^{\infty} A^n_\mu(x) e^{in\theta} \,, \\ \phi(x,\theta) &= \sum_{n=-\infty}^{\infty} \phi^n(x) e^{in\theta}, \end{aligned} \tag{4.2}$$

where we have to impose the reality constraint $(\phi^*)^n = \phi^{-n}$ and similarly for the other fields. Truncating to the zero-modes, the effective Lagrangian is given by

$$\mathcal{L} = -eR^{(3)} + \frac{1}{2} e g^{\mu\nu} \phi^{-2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} e \phi^2 g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \,, \tag{4.3}$$

where as usual $F_{\mu\nu}$ denotes the $U(1)$ field strength for A_μ. This action is invariant under three-dimensional diffeomorphisms and $U(1)$ gauge transformations.

Let us next analyze how the four-dimensional symmetries are present in the Kaluza-Klein theory without any truncation. For this we notice that the diffeomorphisms in $D = 4$, which are locally generated by a vector field ξ^M, are restricted by the topology of the assumed ground state $\mathbb{R}^3 \times S^1$ to be periodic in θ. Therefore we have to expand similarly

$$\xi^\mu(x,\theta) = \sum_{n=-\infty}^{\infty} \xi^{\mu(n)}(x) e^{in\theta} \,, \qquad \xi^5(x,\theta) = \sum_{n=-\infty}^{\infty} \xi^{5(n)}(x) e^{in\theta} \,. \tag{4.4}$$

The four-dimensional diffeomorphisms and local Lorentz transformations act on the vielbein as

$$\delta_\xi E^A_M = \xi^N \partial_N E^A_M + \partial_M \xi^N E^A_N \,, \qquad \delta_\tau E^A_M = \tau^A{}_B E^B_M \,. \tag{4.5}$$

By applying a four-dimensional diffeomorphism to (4.1) we get

$$\begin{aligned} \delta_\xi \phi &= \xi^\rho \partial_\rho \phi + \xi^5 \partial_5 \phi + 2\phi \partial_5 \xi^\rho A_\rho + 2\phi \partial_5 \xi^5 \,, \\ \delta_\xi A_\mu &= \xi^\rho \partial_\rho A_\mu + \xi^5 \partial_5 A_\mu + \partial_\mu \xi^\rho A_\rho + \partial_\mu \xi^5 - A_\mu \partial_5 \xi^\rho A_\rho - A_\mu \partial_5 \xi^5 \,, \\ \delta_\xi e^a_\mu &= \xi^\rho \partial_\rho e^a_\mu + \xi^5 \partial_5 e^a_\mu + \partial_\mu \xi^\rho e^a_\rho + \partial_5 \xi^5 e^a_\mu + \partial_5 \xi^\rho A_\rho e^a_\mu \,. \end{aligned} \tag{4.6}$$

Moreover we have to add a compensating Lorentz transformation with parameter $\tau^a_5 = -\phi^{-1} \partial_5 \xi^\rho e^a_\rho$ to restore the triangular gauge,

$$\delta_\tau \phi = 0 \,, \quad \delta_\tau e^a_\mu = -A_\mu \partial_5 \xi^\rho e^a_\rho \,, \quad \delta_\tau A_\mu = -\phi^{-2} \partial_5 \xi^\rho g_{\rho\mu} \,, \tag{4.7}$$

where as usual we have written $g_{\mu\nu} = e^a_\mu e_{\nu a}$, but now with θ-dependent vielbein. Using the mode expansion for the fields in (4.2) and the transformation parameter in (4.4), one gets an infinite-dimensional Kaluza-Klein symmetry acting on the fields

as[3]

$$\begin{aligned}
\delta\phi^n &= \xi^\rho_k \partial_\rho \phi^{n-k} + iM\sum_k (n+k)\xi^5_k \phi^{n-k} + 2iM\sum_{k,l} k\xi^\rho_k \phi^{n-k-l} A^l_\rho \,, \\
\delta A^n_\mu &= \partial_\mu \xi^5_n + iM\sum_k (n-2k)\xi^5_k A^{n-k}_\mu + \xi^\rho_k \partial_\rho A^{n-k}_\mu + \partial_\mu \xi^\rho_k A^{n-k}_\rho \\
&\quad - iM\sum_{k,l} k\xi^\rho_k (\phi^{-2})^{n-k-l} g^l_{\rho\mu} - iM\sum_{k,l} k\xi^\rho_k A^{n-k-l}_\mu A^l_\rho \,, \\
\delta e^{a(n)}_\mu &= \xi^\rho_k \partial_\rho e^{a(n-k)}_\mu + \partial_\mu \xi^\rho_k e^{a(n-k)}_\rho \\
&\quad + iMn\xi^5_k e^{a(n-k)}_\mu + iM\sum_{k,l} k\xi^\rho_k \left(e^{a(n-k-l)}_\mu A^l_\rho - e^{a(n-k-l)}_\rho A^l_\mu\right) \,.
\end{aligned} \tag{4.8}$$

Here $(\phi^{-2})^n$ is implicitly defined by $\phi^{-2} = \sum_{n=-\infty}^{\infty} (\phi^{-2})^n e^{in\theta}$.

We see now that the standard Kaluza-Klein vacuum given by the vacuum expectation values

$$\langle g_{\mu\nu}\rangle = \eta_{\mu\nu} \,, \qquad \langle A_\mu\rangle = 0 \,, \qquad \langle\phi\rangle = 1 \,, \tag{4.9}$$

is only invariant under rigid $k = 0$ transformations, or in other words the infinite-dimensional symmetry is spontaneously broken to the symmetry of the zero-modes, i.e. to three-dimensional diffeomorphisms and a $U(1)$ gauge symmetry.

To explore the group structure which is realized on the whole tower of Kaluza-Klein modes including its spontaneously broken part, Dolan and Duff proceeded as follows. Expanding the generators of the $D = 3$ Poincaré algebra as well as those for the diffeomorphisms on S^1 into Fourier modes, one gets

$$P^n_a = e^{in\theta}\partial_a \,, \quad J^n_{ab} = e^{in\theta}(x_b\partial_a - x_a\partial_b) \,, \quad Q^n = -Me^{in\theta}\partial_\theta \,, \tag{4.10}$$

where we have used flat Minkowski indices. This implies after introducing $J^a = \frac{1}{2}\varepsilon^{abc}J_{bc}$ the following symmetry algebra

$$\begin{aligned}
&[P^m_a, P^n_b] = 0 \,, \qquad [J^m_a, J^n_b] = \varepsilon_{abc}J^{c(m+n)} \,, \qquad [J^m_a, P^n_b] = \varepsilon_{abc}P^{c(m+n)} \,, \\
&[Q^m, Q^n] = iM(m-n)Q^{m+n} \,, \\
&[Q^m, P^n_a] = -iMnP^{m+n}_a \,, \qquad [Q^m, J^n_a] = -iMnJ^{m+n}_a \,,
\end{aligned} \tag{4.11}$$

i.e. one gets the Kac-Moody algebra associated to the Poincaré group as well as the Virasoro algebra, both without a central extension. (For definitions see appendix B.) More precisely, we have a semi-direct product of the Virasoro algebra $\hat{v}$ with the affine Poincaré algebra $\widehat{iso(1,2)}$ in the standard fashion known from the Sugawara construction [76]. This algebra should be realized as a local symmetry. However, this latter statement is a little bit contrived since even the diffeomorphisms (i.e. the $k = 0$ transformations) are known not to be realized in general as gauge transformations. In

[3] We will adopt the Einstein convention also for double indices $m, n = -\infty, ..., \infty$, but indicate summations explicitly, if the considered indices appear more than twice.

fact, even though the idea that general relativity should have some interpretations as a gauge theory has always been around [77, 78], it is known that the diffeomorphism in general do not allow an interpretation as Yang-Mills transformations for a certain Lie algebra, e.g. as gauge transformations for the P_a. One aim of the present chapter is to clarify this question in the case of a Kaluza-Klein reduction to $D = 3$, and we will see how a modification of (4.11) appears as a proper gauge symmetry.

In summary, from this infinite-dimensional symmetry algebra only $iso(1,2) \times u(1)$ remains unbroken in the Kaluza-Klein vacuum. This in turn implies that the fields A^n_μ and ϕ^n for $n \neq 0$, which correspond to the spontaneously broken generators ξ^μ_n and ξ^5_n, can be identified with the Goldstone bosons. They get eaten by the spin-2 fields $e^{a(n)}_\mu$, such that the latter become massive. A massless spin-2 field carries no local degrees of freedom in $D = 3$, while a massless vector as well as a real scalar each carry one degree of freedom in $D = 3$, such that in total the massive spin-2 fields each carry two degrees of freedom, as expected.

4.3 Unbroken phase of the Kaluza-Klein theory

4.3.1 Infinite-dimensional spin-2 theory

To construct the theory containing infinitely many massless spin-2 fields coupled to gravity, we remember that according to (4.11) it should have an interpretation as a gauge theory of the Kac-Moody algebra $\widehat{iso(1,2)}$. However, as we already indicated in the discussion at the end of the previous section, in general dimensions this is not a helpful statement, because not even pure gravity has an honest interpretation as a Yang-Mills-like gauge theory. Fortunately, as we reviewed in 2.2.1, gravity in $D = 3$ can in contrast be viewed as a gauge theory. In case of Poincaré gravity the required gauge theory is a Chern-Simons theory with the Poincaré group $ISO(1,2)$ as (non-compact) gauge group [38]. Furthermore, the symmetries of general relativity, i.e. the diffeomorphisms, are on-shell realized as the non-abelian gauge transformations. We are going to show that correspondingly the Chern-Simons theory of the affine $\widehat{iso(1,2)}$ describes a consistent coupling of infinitely many spin-2 fields to gravity.

To start with, we recall the Chern-Simons theory for a gauge connection $\mathcal{A}$, which is given by

$$S_{\text{CS}} = \int \text{Tr}\big(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\big) \ . \tag{4.12}$$

The invariance of the quadratic form $\langle \, , \rangle$ then implies that under an arbitrary variation one has

$$\delta S_{\text{CS}} = \int \langle \delta\mathcal{A}_\mu, \mathcal{F}_{\nu\rho}\rangle dx^\mu \wedge dx^\nu \wedge dx^\rho \ , \tag{4.13}$$

where $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]$ denotes the field strength. In particular, under a gauge transformation $\delta\mathcal{A}_\mu = D_\mu u$, where D_μ denotes the gauge covariant derivative

$$D_\mu u = \partial_\mu u + [\mathcal{A}_\mu, u] \tag{4.14}$$

of an infinitesimal transformation parameter u, the action is invariant due to the Bianchi identity. In addition, the non-degeneracy of the quadratic form implies the equations of motion $\mathcal{F}_{\mu\nu} = 0$.

Therefore, to construct the Chern-Simons theory for $\widehat{iso(1,2)}$, we have to find such a quadratic form. In contrast to the AdS algebra (2.13), which is the direct product of two semi-simple Lie algebras, the existence of a non-degenerate quadratic form is not self-evident. However, it turns out that

$$\langle P_a^m, J_b^n \rangle = \eta_{ab}\delta^{m,-n} \,, \qquad \langle P_a^m, P_b^n \rangle = \langle J_a^m, J_b^n \rangle = 0 \tag{4.15}$$

defines an invariant form since the bilinear expression

$$W := \sum_{n=-\infty}^{\infty} P^{a(n)} J_a^{(-n)} \tag{4.16}$$

commutes with all gauge group generators. For instance,

$$[W, P_b^k] = \varepsilon_{abc} \sum_{n=-\infty}^{\infty} P^{a(n)} P^{c(k-n)} = 0 \tag{4.17}$$

can be seen by performing an index shift $n \to n' = k - n$, which shows that the sum is symmetric in a and c.[4]

We turn now to the calculation of the action, the equations of motion and the explicit form of the gauge transformations, which is necessary to identify the Kaluza-Klein symmetries and fields. The gauge field takes values in the Kac-Moody algebra, i.e. it can be written as

$$\mathcal{A}_\mu = e_\mu^{a(n)} P_a^n + \omega_\mu^{a(n)} J_a^n \,. \tag{4.18}$$

Note, that in the description of ordinary Einstein gravity as Chern-Simons theory the gauge field ω_μ^a is interpreted as the spin-connection, which like in the Palatini formulation is determined only by the equations of motion to be the Levi-Civita connection. Here, instead, we have an infinite number of 'connections' and their meaning will be interpreted later.

With the invariant quadratic form defined in (4.15), the action reads

$$S_{\mathrm{CS}} = \int d^3x \, \varepsilon^{\mu\nu\rho} e_{\mu a}^{(n)} \left(\partial_\nu \omega_\rho^{a(-n)} - \partial_\rho \omega_\nu^{a(-n)} + \varepsilon^{abc} \omega_{\nu b}^{(m)} \omega_{\rho c}^{(-n-m)} \right) \,. \tag{4.19}$$

If we define 'generalized' curvatures

$$R^{a(n)} = d\omega^{a(n)} + \varepsilon^{abc} \omega_b^{(m)} \wedge \omega_c^{(n-m)} \,, \tag{4.20}$$

[4]Upon truncating the quadratic form to the zero-modes, this reduces to an invariant form of the Poincaré algebra, which was the one used in [38] to construct the Chern-Simons action describing pure Poincaré gravity in $D = 3$.

the action may be written in a more compact form as

$$S_{\rm CS} = \int e_a^{(n)} \wedge R^{a(-n)} \ . \tag{4.21}$$

The field equations implying vanishing field strength, $\mathcal{F}_{\mu\nu} = 0$, read in the given case

$$\begin{aligned} \partial_\mu e_\nu^{a(n)} - \partial_\nu e_\mu^{a(n)} + \varepsilon^{abc} e_{\mu b}^{(n-m)} \omega_{\nu c}^{(m)} + \varepsilon^{abc} \omega_{\mu b}^{(n-m)} e_{\nu c}^{(m)} = 0 \ , \\ \partial_\mu \omega_\nu^{a(n)} - \partial_\nu \omega_\mu^{a(n)} + \varepsilon^{abc} \omega_{\mu b}^{(n-m)} \omega_{\nu c}^{(m)} = 0 \ . \end{aligned} \tag{4.22}$$

Due to the mixing of the infinitely many 'spin connections', the torsion defined by $e_\mu^{a(0)}$ does no longer vanish by the equations of motion. This in turn implies that it is not transparent which part of the Einstein equation expresses the curvature and which part the energy-momentum tensor for the higher spin-2 fields. We will clarify this point later.

Next we evaluate the explicit form of the gauge transformations. Introducing the algebra-valued transformation parameter $u = \rho^{a(n)} P_a^n + \tau^{a(n)} J_a^n$, for the transformations given by $\delta \mathcal{A}_\mu = D_\mu u$ one finds

$$\begin{aligned} \delta e_\mu^{a(n)} &= \partial_\mu \rho^{a(n)} + \varepsilon^{abc} e_{\mu b}^{(n-m)} \tau_c^{(m)} + \varepsilon^{abc} \omega_{\mu b}^{(n-m)} \rho_c^{(m)} \ , \\ \delta \omega_\mu^{a(n)} &= \partial_\mu \tau^{a(n)} + \varepsilon^{abc} \omega_{\mu b}^{(n-m)} \tau_c^{(m)} \ . \end{aligned} \tag{4.23}$$

To see that these gauge transformations indeed include the spin-2 Kaluza-Klein transformations (4.8) for $M = 0$, let us define for a given Kaluza-Klein transformation parameterized by ξ_k^μ the gauge parameters

$$\rho^{a(n)} = \xi_k^\mu e_\mu^{a(n-k)} \ , \qquad \tau^{a(n)} = \xi_k^\mu \omega_\mu^{a(n-k)} \ . \tag{4.24}$$

Then the gauge transformation (4.23) takes the form

$$\begin{aligned} \delta e_\mu^{a(n)} &= \partial_\mu \xi_k^\rho e_\rho^{a(n-k)} + \xi_k^\rho \partial_\rho e_\mu^{a(n-k)} \\ &\quad + \xi_k^\rho \left(\partial_\mu e_\rho^{a(n-k)} - \partial_\rho e_\mu^{a(n-k)} + \varepsilon^{abc} e_{\mu b}^{(n-k-m)} \omega_{\rho c}^{(m)} + \varepsilon^{abc} \omega_{\mu b}^{(n-k-m)} e_{\rho c}^{(m)} \right) \ , \end{aligned} \tag{4.25}$$

where we have again performed an index shift. We see that the first term reproduces the correct Kaluza-Klein transformation in (4.8) with $M = 0$, while the last term vanishes by the equations of motion (4.22). On-shell the Kaluza-Klein transformations are therefore realized as gauge transformations. That the symmetry is realized only on-shell should not come as a surprise because this is already the case for the diffeomorphisms [38], which are now part of the Kaluza-Klein-symmetries.

Thus we have determined a theory which is by construction a consistent coupling of infinitely many spin-2 fields. One might also ask the question whether the theory can be consistently truncated to a finite number of spin-2 fields, i.e. where only $e_\mu^{a(n)}, n = -N, ..., N$ for any finite N remain. These issues will be discussed in sec. 4.6.

So far we have seen that the action permits a consistent spin-2 invariance. It remains to be checked that it can also be viewed as a deformation of a sum of free Pauli-Fierz Lagrangians (2.1), in particular that the first-order theory constructed here is equivalent to a second order action. To see this we introduce an expansion parameter κ and linearize the theory by writing

$$e_\mu^{a(0)} = \delta_\mu^a + \kappa h_\mu^{a(0)} + O(\kappa^2) \ , \qquad e_\mu^{a(\pm 1)} = \kappa h_\mu^{a(\pm 1)} + O(\kappa^2) \ . \tag{4.26}$$

We concentrate for simplicity reasons only on the case where just $e_\mu^{a(\pm 1)}$ are present. Even though this will turn out not to be consistent with the gauge symmetry in general, it yields correct results up to order $O(\kappa)$ as the corrections by the full equations are at least of order $O(\kappa^2)$. Using the equations of motion for $e_\mu^{a(\pm 1)}$ we can now express $\omega_\mu^{a(\pm 1)}$ in terms of them. One finds upon expanding up to $O(\kappa)$

$$\begin{aligned} \omega_\mu^{a(1)} \ = \ & \kappa\big[\varepsilon^{\nu ac}(\partial_\mu h_{\nu c}^{(1)} - \partial_\nu h_{\mu c}^{(1)}) + \delta_b^\nu (h_\mu^{b(1)}\omega_\nu^{a(0)} - h_\mu^{a(1)}\omega_\nu^{b(0)} \\ & + \omega_\mu^{b(0)} h_\nu^{a(1)} - \omega_\mu^{a(0)} h_\nu^{b(1)}) - \frac{1}{4}\varepsilon^{\sigma\rho d}(\partial_\rho h_{\sigma d}^{(1)} - \partial_\sigma h_{\rho d}^{(1)})\delta_\mu^a \\ & - \frac{1}{4}\delta^{\rho c}\delta^{\sigma d}(h_{\rho d}^{(1)}\omega_{\sigma c}^{(0)} - h_{\rho c}^{(1)}\omega_{\sigma d}^{(0)} + \omega_{\rho d}^{(0)} h_{\sigma c}^{(1)} - \omega_{\rho d}^{(0)} h_{\sigma c}^{(1)})\delta_\mu^a\big] + O(\kappa^2) \ , \end{aligned} \tag{4.27}$$

and analogously for $\omega_\mu^{a(-1)}$. The next step would be to insert these relations into the equation for $e_\mu^{a(0)}$ and solve the resulting expression for the 'spin connection' $\omega_\mu^{a(0)}$. However, due to the fact that $e_\mu^{a(\pm 1)}$ as well as $\omega_\mu^{a(\pm 1)}$ are yet of order $O(\kappa)$, for the approximation linear in κ we just get

$$0 = \varepsilon^{abc}\delta_{\mu b}\omega_{\nu c}^{(0)} + \varepsilon^{abc}\delta_{\nu c}\omega_{\mu b}^{(0)} + \kappa\left(\partial_\mu h_\nu^{a(0)} - \partial_\nu h_\mu^{a(0)} + \varepsilon^{abc}(h_{\mu b}^{(0)}\omega_{\nu c}^{(0)} + \omega_{\mu b}^{(0)} h_{\nu c}^{(0)})\right) \tag{4.28}$$

In the limit $\kappa \to 0$ this is the relation for vanishing torsion in the flat case and is therefore solved by $\omega_\mu^{a(0)} = 0$. Up to order $O(\kappa)$ we have the usual relation of vanishing torsion for the 'metric' $h_\mu^{a(0)}$ and it can therefore be solved as in the standard case. But to solve the equation we have to multiply with the inverse vielbein and therefore up to order $O(\kappa)$ the solution is just the linearized Levi-Civita connection. Altogether we have

$$\omega_\mu^{a(0)} = \kappa(\text{lin. Levi-Civita connection}) + O(\kappa^2) \ . \tag{4.29}$$

We can now insert this relation into the formulas (4.27) for $\omega_\mu^{a(\pm 1)}$ and get

$$\omega_\mu^{a(1)} = \kappa\big[\varepsilon^{\nu ac}(\partial_\mu h_{\nu c}^{(1)} - \partial_\nu h_{\mu c}^{(1)}) - \frac{1}{4}\varepsilon^{\sigma\rho d}(\partial_\rho h_{\sigma d}^{(1)} - \partial_\sigma h_{\rho d}^{(1)})\delta_\mu^a\big] + O(\kappa^2) \tag{4.30}$$

and analogously for $\omega_\mu^{a(-1)}$. Finally inserting this expression into the equation of motion for $\omega_\mu^{a(1)}$ to that order, i.e.

$$0 = \partial_\mu \omega_\nu^{a(1)} - \partial_\nu \omega_\mu^{a(1)} \ , \tag{4.31}$$

and multiplying with $\varepsilon^{\mu\nu\lambda}$ results in

$$0 = \partial_\mu \partial^\lambda h^{\mu\nu} - \partial^\nu \partial^\lambda \hat{h} + \eta^{\lambda\nu} \Box \hat{h} - \eta^{\lambda\nu} \partial_\mu \partial_\rho h^{\mu\rho} + \partial^\nu \partial_\mu h^{\mu\lambda} - \Box h^{\nu\lambda} \; . \tag{4.32}$$

This coincides exactly with the equation of motion derived from the original free spin-2 action (2.1) in the massless case. Here we have defined $h_{\mu\nu} := \delta^a_\mu h^{(1)}_{\nu a} + \delta^a_\nu h^{(1)}_{\mu a}$, which also results up to $O(\kappa)$ from the general formula for the higher spin-2 fields in a metric-like representation:

$$g^{(n)}_{\mu\nu} := e^{a(n-m)}_\mu e^{(m)}_{\nu a} \; . \tag{4.33}$$

Clearly, also for the linearized Einstein equation we get the free spin-2 equation, and therefore we can summarize our analysis by saying that the theory reduces in the linearization up to order $O(\kappa)$ to a sum of Pauli-Fierz terms. On the other hand, we know that the full theory has the Kaluza-Klein symmetries (4.8) which mix fields of different level and accordingly the full theory (4.19) has to include non-linear couplings. This in turn implies that the higher order terms in κ cannot vanish and the theory is therefore a true deformation of a pure sum of Pauli-Fierz terms. In summary, one can solve the equations of motion for $\omega^{a(n)}_\mu$ at least perturbatively, thus giving a second-order formulation. However, it would be much more convenient to have a deeper geometrical understanding for the $\omega^{a(n)}_\mu$. Such a geometrical interpretation indeed exists and is based on a notion of algebra-valued differential geometry developed by Wald [48], which we are going to discuss in the next section.

4.3.2 Geometrical interpretation of the spin-2 symmetry

Cutler and Wald analyzed in [47] the question of possible consistent extensions of a free spin-2 gauge invariance to a collection of spin-2 fields. This may be compared to the similar question of a consistent gauge symmetry for a collection of spin-1 fields. In this case one knows that the resulting theories are Yang-Mills theories determined by a non-abelian Lie algebra. Analogously it was shown in [47] that such a spin-2 theory is organized by an associative and commutative algebra $\mathfrak{A}$ (which should not be confused with a Lie algebra). Namely, the additional index which indicates the different spin-2 fields is to be interpreted as an algebra index, and therefore any collection of spin-2 fields can be viewed as a single spin-2 field, which takes values in a nontrivial algebra. An associative and commutative algebra $\mathfrak{A}$ can be characterized by its multiplication law, which is with respect to a basis given by a tensor $a^k{}_{nm}$ according to

$$(v \cdot w)^n = a^n{}_{mk} v^m w^k \; , \tag{4.34}$$

where $v, w \in \mathfrak{A}$. That the algebra is commutative and associative is encoded in the relations

$$a^k{}_{mn} = a^k{}_{(mn)} \; , \qquad a^k{}_{mn} a^n{}_{lp} = a^k{}_{np} a^n{}_{ml} \; . \tag{4.35}$$

With respect to such a given algebra $\mathfrak{A}$, the allowed gauge transformations can be written according to [47] as

$$\delta g^{(n)}_{\mu\nu} = \partial_{(\mu}\xi^{(n)}_{\nu)} - 2\Gamma^{\sigma\ n}_{\mu\nu\ l}\xi^{(l)}_{\sigma} =: \nabla_\mu \xi^{(n)}_\nu + \nabla_\nu \xi^{(n)}_\mu \ , \tag{4.36}$$

where the generalized Christoffel symbol is defined by

$$\Gamma^{\sigma\ n}_{\mu\nu\ l} = \frac{1}{2} g^{\sigma\rho\ k}_{\ \ l} \left(\partial_\mu g_{\rho\nu}{}^{n}_{\ k} + \partial_\nu g_{\rho\mu}{}^{n}_{\ k} - \partial_\rho g_{\mu\nu}{}^{n}_{\ k} \right) \ , \tag{4.37}$$

and

$$g_{\mu\nu\ n}^{\ \ k} = a^k_{\ nm} g^{(m)}_{\mu\nu} \ . \tag{4.38}$$

We see that ∇_μ has the formal character of a covariant derivative.

Moreover, it has been shown in [48] that beyond this formal resemblance to an ordinary metric-induced connection, there exists a geometrical interpretation in the following sense. As in pure general relativity, where the symmetry transformations are given by the diffeomorphisms acting on the fields via a pullback, the transformation rules (4.36) are the infinitesimal version of a diffeomorphism on a generalized manifold. This new type of manifold introduced in [48] generalizes the notion of an ordinary real manifold to 'algebra-valued' manifolds, where the algebra $\mathfrak{A}$ replaces the role of $\mathbb{R}$. To be more precise, such a manifold is locally modeled by a n-fold cartesian product $\mathfrak{A}^n$ in the same sense as an ordinary manifold is locally given by $\mathbb{R}^n$. On these manifolds one can correspondingly define a metric which looks from the point of view of the underlying real manifold like an ordinary, but algebra-valued metric. The diffeomorphisms of these generalized manifolds act infinitesimally on the metric exactly as written above via an algebra-valued generalization of a Lie-derivative. This Lie derivative acts, e.g., on algebra-valued $(1,1)$ tensor fields as

$$\mathcal{L}_\xi T^{\mu\ (n)}_{\ \nu} = a^n_{mk} \left(\xi^{\rho(m)} \partial_\rho T^{\mu\ (k)}_{\ \nu} - T^{\rho\ (m)}_{\ \nu} \partial_\rho \xi^{\mu(k)} + T^{\mu\ (m)}_{\ \rho} \partial_\nu \xi^{\rho(k)} \right) \ , \tag{4.39}$$

and in an obvious way on all higher-rank tensor fields. (For further details see [48].) Moreover, most of the constructions known from Riemannian geometry like the curvature tensor have their analogue here.

To check whether our theory fits into this general framework we first have to identify the underlying commutative algebra. Due to the fact that the theory contains necessarily an infinite number of spin-2 fields, the algebra has to be infinite-dimensional, too, and we will assume that the formalism applies also to this case.

We will argue that the algebra is given by the algebra of smooth functions on S^1, on which we had compactified, together with the point-wise multiplication of functions as the algebra structure.[5] With respect to the complete basis $\{e^{in\theta}, n = -\infty, ..., \infty\}$

[5] That the spin-2 couplings arising in Kaluza-Klein compactifications might be related to Wald's framework in this way has first been suggested by Reuter in [79], where he analyzed the reduction of a dimensionally continued Euler form in $D = 6$. Namely, the latter has the exceptional property of inducing an infinite tower of massless spin-2 fields due to the existence of an infinite-dimensional symmetry already in the higher-dimensional theory.

of functions on S^1, the multiplication is given due to elementary Fourier analysis by

$$(f \cdot g)^n = \sum_{m=-\infty}^{\infty} f^{n-m} \cdot g^m = \sum_{k,m=-\infty}^{\infty} \delta_{k+m,n} f^k g^m \,, \tag{4.40}$$

such that the algebra is characterized by

$$a^n{}_{km} = \delta_{k+m,n} \,. \tag{4.41}$$

This implies that the metric can be written according to (4.38) as

$$g_{\mu\nu}{}^n{}_k = a^n{}_{km} g^{(m)}_{\mu\nu} = g^{(n-k)}_{\mu\nu} \,. \tag{4.42}$$

Now it can be easily checked that the Kaluza-Klein transformations (4.8) for $M = 0$ applied to (4.33) can be written as

$$\delta g^{(n)}_{\mu\nu} = \nabla_\mu \xi^{(n)}_\nu + \nabla_\nu \xi^{(n)}_\mu \,, \tag{4.43}$$

i.e. they have exactly the required form. Here the connection ∇_μ is calculated as in (4.36) with respect to the algebra (4.41). For this we have assumed that indices are raised and lowered according to

$$\xi^{(n)}_\mu = g_{\mu\nu}{}^n{}_k \xi^{(k)\nu} \,, \tag{4.44}$$

while the inverse metric is defined through the relation

$$g^{\mu\rho}{}^n{}_k g_{\rho\nu}{}^k{}_m = \delta^\mu_\nu \delta^n_m \,. \tag{4.45}$$

With the help of this geometrical interpretation we are now also able to interpret the existence of an infinite number of 'spin-connections' $\omega^{a(n)}_\mu$. If we assume that the vielbeins are invertible in the sense of (4.45), one can solve the equations of motion (4.22) for the connections in terms of $e^{a(n)}_\mu$, as we have argued in sec. 4.3.1. Then one can define a generalized covariant derivative by postulating the vielbein to be covariantly constant,

$$\nabla_\mu e^{a(n)}_\nu = \partial_\mu e^{a(n)}_\nu - \Gamma^{\rho}{}_{\mu\nu}{}^n{}_m e^{a(m)}_\rho + \omega^{a(n-m)}_\mu{}_b e^{b(m)}_\nu = 0 \,. \tag{4.46}$$

Since the antisymmetric part $\nabla_{[\mu} e^{a(n)}_{\nu]}$ vanishes already by the equations of motion (4.22), this requirement specifies the symmetric part of $\nabla_\mu e^{a(n)}_\nu$. In turn, the algebra-valued metric (4.33) is covariantly constant with respect to this symmetric connection,

$$\nabla_\mu g^{(n)}_{\nu\rho} = \partial_\mu g^{(n)}_{\nu\rho} - \Gamma^{\sigma}{}_{\mu\nu}{}^n{}_m g^{(m)}_{\sigma\rho} - \Gamma^{\sigma}{}_{\mu\rho}{}^n{}_m g^{(m)}_{\sigma\nu} = 0 \,. \tag{4.47}$$

But this is on the other hand also the condition which uniquely fixes the Christoffel connection in (4.37) as a function of the algebra-valued metric [48]. Altogether, the equations of motion for the Chern-Simons action (4.19) together with (4.46) determine a symmetric connection, which is equivalent to the algebra-valued Christoffel connection (4.37) compatible with the metric (4.33).

Wald also constructed an algebra-valued generalization of the Einstein-Hilbert action, whose relation to the Chern-Simons action (4.19) we are going to discuss now. This generalization is (written for three space-times dimensions) given by

$$S^m = \int a^m_{nl} R^n \varepsilon_{\mu\nu\rho}{}^l dx^\mu \wedge dx^\nu \wedge dx^\rho \ , \tag{4.48}$$

where R^n and $\varepsilon_{\mu\nu\rho}{}^l$ denote the algebra-valued scalar curvature and volume form, respectively [48]. Applied to the algebra (4.41) it yields

$$S^m = \int R^{m-n} \varepsilon_{\mu\nu\rho}{}^n dx^\mu \wedge dx^\nu \wedge dx^\rho \ . \tag{4.49}$$

Here the volume form is given by $\varepsilon_{\mu\nu\rho}{}^n = e^n \epsilon_{\mu\nu\rho}$, where

$$e^n = \frac{1}{3!} \epsilon^{\mu\nu\rho} \epsilon_{abc} e^{a(n-m-k)}_\mu e^{b(m)}_\nu e^{c(k)}_\rho \ , \tag{4.50}$$

such that the zero-component of (4.49) indeed coincides with the Chern-Simons action (4.21). One may wonder about the meaning of the other components of the algebra-valued action (4.49), whose equations of motion cannot be neglected for generic algebras. However, the quadratic form (4.15) which has been used to construct the Chern-Simons action (4.19), is actually not unique, but instead there is an infinite series of quadratic forms,

$$\langle P^m_a, J^n_b \rangle_k = \eta_{ab} \delta^{m,k-n} \ , \tag{4.51}$$

each of which is invariant and can therefore be used to define a Chern-Simons action. These will then be identical to the corresponding components of the algebra-valued action (4.49). But, as all of these actions imply the same equations of motion, namely $\mathcal{F}_{\mu\nu} = 0$, and are separately invariant under gauge transformations, there is no need to consider the full algebra-valued metric, but instead the zero-component is sufficient.

Moreover, also matter couplings can be described in this framework in a spin-2-covariant way. For instance, an algebra-valued scalar field ϕ^n can be coupled via

$$S^m_{\text{scalar}} = \int a^m_{nk} a^n_{lp} \partial_\lambda \phi^l \partial^\lambda \phi^p \varepsilon_{\mu\nu\rho}{}^k dx^\mu \wedge dx^\nu \wedge dx^\rho \ . \tag{4.52}$$

Furthermore, (4.52) is invariant under algebra-diffeomorphisms. The latter act via the Lie derivative in (4.39), such that the scalars transform with respect to the algebra (4.41) as

$$\delta_\xi \phi^n = a^n_{km} \xi^{\rho(k)} \partial_\rho \phi^m = \xi^{\rho(k)} \partial_\rho \phi^{n-k} \ , \tag{4.53}$$

i.e. as required by (4.8) in the phase $M \to 0$. Again, for the algebra (4.41) considered here, the zero-component of (4.52) is separately invariant and can be written as

$$S_{\text{scalar}} = \int d^3x \sqrt{g}^{-n} \partial_\mu \phi^l \partial^\mu \phi^{n-l} = \int d^3x d\theta \sqrt{g} \partial_\mu \phi \partial^\mu \phi \ . \tag{4.54}$$

Similarly, the Chern-Simons action can be rewritten by retaining a formal θ-integration and assuming all fields to be θ-dependent. For explicit computations it is accordingly often more convenient to work with θ-dependent expressions and therefore we will give subsequent formulas in both versions.

Finally let us briefly discuss the resolution of the aforementioned no-go theorems for consistent gravity/spin-2 couplings. In [45] it has been shown that Wald's algebra-valued spin-2 theories for arbitrary algebras generically contain ghost-like excitations. Namely, the algebra has to admit a metric specifying the kinetic Pauli-Fierz terms in the free-field limit and moreover has to be symmetric in the sense that lowering the upper index in a^k_{mn} by use of this metric results in a totally symmetric $a_{mnk} = a_{(mnk)}$. Requiring the absence of ghosts, i.e. assuming the metric to be positive-definite, restricts the algebra to a direct sum of one-dimensional ideals (which means $a^k_{mn} = 0$ whenever $m \neq n$). The theory reduces in turn to a sum of independent Einstein-Hilbert terms. For the infinite-dimensional algebra considered here the metric is given by the L^2-norm for square-integrable functions (see formula (D.58) in the appendix), which is clearly positive-definite. The action may instead be viewed as an integral over Einstein-Hilbert terms and is thus in agreement with [45].

4.3.3 Non-linear σ-model and its global symmetries

Apart from the spin-2 sector also the infinite tower of scalar fields ϕ^n will survive in the unbroken limit $M \to 0$. We have seen in the last section how spin-2 invariant couplings for scalar fields can be constructed. To fix the actual form of these couplings, we have to identify also the global symmetries in this limit, and in order to uncover the maximal global symmetry, we will dualize all degrees of freedom into scalars.

We note from (4.8) that in the unbroken phase the Virasoro algebra $\hat{v}$ parameterized by ξ^5_k reduces to an abelian gauge symmetry. As a general feature of ungauged limits, the full $\hat{v}$ will then turn out to be realized only as a global symmetry. More precisely, we expect an invariance under rigid transformations of the general form

$$\delta_{\xi^5}\chi^n = i\sum_k (n-(1-\Delta)k)\xi^5_k \chi^{n-k} \,, \tag{4.55}$$

where ξ^5_k is now space-time independent. One easily checks that these are representations of $\hat{v}$ (for details see appendix B). They will be labeled by the conformal dimension Δ. More precisely, the Kaluza-Klein fields e^a_μ, A_μ and ϕ transform as $\Delta = 1$, $\Delta = -1$ and $\Delta = 2$, respectively.

We start form the zero-mode action (4.3) and replace it by the algebra-valued generalization discussed in the last section. For the Einstein-Hilbert term we have already seen that this procedure yields the correct $\widehat{iso(1,2)}$ gauge theory, and therefore it is sufficient to focus on the scalar kinetic term and the Yang-Mills term. The action reads

$$S_{\text{matter}} = \int d^3x d\theta e \left(-\frac{1}{4}\phi^2 F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\phi^{-2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \right) \,, \tag{4.56}$$

where all fields are now θ-dependent or, equivalently, algebra-valued.

To dualize the $U(1)$ gauge fields A_μ^n into new scalars φ^n, we define the standard duality relation

$$\phi^2 F_{\mu\nu} = e\varepsilon_{\mu\nu\rho} g^{\rho\sigma} \partial_\sigma \varphi \ , \tag{4.57}$$

which is not affected by the θ-dependence of all fields. Thus, the abelian duality between vectors and scalars persists also in the algebra-valued case, and the degrees of freedom can be assigned to ϕ and φ. The Lagrangian for the scalar fields then takes in the unbroken limit $M \to 0$ the form

$$\mathcal{L}_{\text{scalar}} = \frac{1}{2} e g^{\mu\nu} \phi^{-2} (\partial_\mu \phi \partial_\nu \phi + \partial_\mu \varphi \partial_\nu \varphi) \ , \tag{4.58}$$

which coincides formally with the zero-mode action after a standard dualization, but now with all fields still being θ-dependent. From (4.57) one determines the transformation properties of the dual scalar φ under $\hat{v}$ and finds

$$\delta_{\xi^5} \varphi = \xi^5 \partial_5 \varphi + 2\varphi \partial_5 \xi^5 \ , \tag{4.59}$$

i.e. it transforms in the same representation as ϕ with $\Delta = 2$. (For the computation it is crucial to take into account that also e_μ^a transforms under $\hat{v}$.) Now one easily checks that the action is invariant under global Virasoro transformations.

Moreover, it is well known that the zero-mode scalar fields span a non-linear σ-model with coset space $SL(2,\mathbb{R})/SO(2)$ as target space, carrying the 'Ehlers group' $SL(2,\mathbb{R})$ as isometry group [80]. If one includes all Kaluza-Klein modes at $M = 0$, this symmetry is enhanced to an infinite-dimensional algebra, which we are going to discuss now. Defining the complex scalar field $Z = \varphi + i\phi$, the action (4.58) can be rewritten as

$$\mathcal{L}_{\text{scalar}} = \frac{1}{2} e g^{\mu\nu} \frac{\partial_\mu Z \partial_\nu \bar{Z}}{(Z - \bar{Z})^2} \ , \tag{4.60}$$

which is invariant under the $SL(2,\mathbb{R})$ isometries acting as

$$Z \to Z' = \frac{aZ + b}{cZ + d} \ , \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \ . \tag{4.61}$$

This invariance is not spoiled by the fact that Z is still θ-dependent and so the $SL(2,\mathbb{R})$ acts on the full tower of Kaluza-Klein modes, as can be seen by expanding (4.61) into Fourier modes. But moreover, also the $SL(2,\mathbb{R})$ group elements can depend on θ, and therefore an additional infinite-dimensional symmetry seems to appear.

To determine the algebra structure of this infinite-dimensional symmetry, let us first introduce a basis for $sl(2,\mathbb{R})$:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.62}$$

Infinitesimally, with transformation parameter $\alpha = \alpha(\theta)$ they act according to (4.61) as

$$\delta_\alpha(h)Z = -2\alpha Z \ , \qquad \delta_\alpha(e)Z = -\alpha \ , \qquad \delta_\alpha(f)Z = \alpha Z^2 \ , \tag{4.63}$$

or, expanded in Fourier components, as

$$\begin{aligned} &\delta_{\alpha^m}(h)Z^n = -2\alpha^m Z^{n-m} \ , \qquad \delta_{\alpha^m}(e)Z^n = -\delta^{mn}\alpha^m \ , \\ &\delta_{\alpha^m}(f)Z^n = \alpha^m Z^{n-m-l}Z^l \ . \end{aligned} \tag{4.64}$$

In particular, the real part of Z, i.e. the dual scalar φ, transforms as a shift under e-transformations, which will later on be promoted to local shift symmetries in the gauged theory.

We can now compute the closure of these symmetry variations with the Virasoro variations $\delta_{\xi^m}(Q)$. One finds

$$\begin{aligned} [\delta_{\xi^m}(Q), \delta_{\eta^n}(h)]Z^k &= -in\delta_{(\xi\eta)^{m+n}}(h)Z^k \ , \\ [\delta_{\xi^m}(Q), \delta_{\eta^n}(e)]Z^k &= i(-n-2m)\delta_{(\xi\eta)^{m+n}}(e)Z^k \ , \\ [\delta_{\xi^m}(Q), \delta_{\eta^n}(f)]Z^k &= i(-n+2m)\delta_{(\xi\eta)^{m+n}}(f)Z^k \ , \end{aligned} \tag{4.65}$$

where we have set

$$(\xi\eta)^{m+n} = \xi^m\eta^n \ . \tag{4.66}$$

Furthermore, the extended $sl(2,\mathbb{R})$ transformations close among themselves according to

$$\begin{aligned} [\delta_{\alpha^m}(h), \delta_{\beta^n}(e)]Z^k &= 2\delta_{(\alpha\beta)^{m+n}}(e)Z^k \ , \\ [\delta_{\alpha^m}(h), \delta_{\beta^n}(f)]Z^k &= -2\delta_{(\alpha\beta)^{m+n}}(f)Z^k \ , \\ [\delta_{\alpha^m}(e), \delta_{\beta^n}(f)]Z^k &= \delta_{(\alpha\beta)^{m+n}}(h)Z^k \ , \\ [\delta_{\alpha^m}(h), \delta_{\alpha^n}(h)]Z^k &= [\delta_{\alpha^m}(e), \delta_{\alpha^n}(e)]Z^k = [\delta_{\alpha^m}(f), \delta_{\alpha^n}(f)]Z^k = 0 \ . \end{aligned} \tag{4.67}$$

Altogether we can conclude that the following Lie algebra is a global symmetry of the ungauged theory

$$\begin{aligned} [Q^m, Q^n] &= i(m-n)Q^{m+n} \ , \quad [Q^m, e_n] = i(-n-2m)e_{m+n} \ , \\ [Q^m, h_n] &= -inh_{m+n} \ , \qquad [Q^m, f_n] = i(-n+2m)f_{m+n} \ , \\ [h_m, e_n] &= 2e_{m+n} \ , \qquad [h_m, f_n] = -2f_{n+m} \ , \\ [e_m, f_n] &= h_{m+n} \ , \qquad [e_m, e_n] = [h_n, h_m] = [f_m, f_n] = 0 \ . \end{aligned} \tag{4.68}$$

We see that the symmetry algebra includes not only the Virasoro algebra $\hat{v}$, but also the Kac-Moody algebra $\widehat{sl(2,\mathbb{R})}$, which transforms under $\hat{v}$. Note, that these transformation properties are not the standard ones known from the Sugawara construction (see appendix B and [76]). However, this algebra reduces to the standard form upon the change of basis given by $\hat{Q}^m = Q^m + mh^m$, such that it clearly defines a consistent Lie algebra.

In summary, we can think of the scalar fields ϕ^n and φ^n as parameterizing an infinite-dimensional σ-model coset space

$$\mathcal{M} = \frac{\widehat{SL(2,\mathbb{R})}}{\widehat{SO(2)}} \,. \tag{4.69}$$

Strictly speaking this is not full truth, since the metric used to contract indices is actually algebra-valued. Thus, here we have an algebra-valued generalization of a σ-model, which in turn is the reason that it does not only have the symmetries $\widehat{sl(2,\mathbb{R})}$, but instead the whole algebra $\hat{v} \ltimes \widehat{sl(2,\mathbb{R})}$ defined by (4.68).

In total, the ungauged phase of the effective Kaluza-Klein action without any truncation is therefore given by

$$\begin{aligned} S = \int d^3x \, \big(&- \varepsilon^{\mu\nu\rho} e^{(n)}_{\mu a} (\partial_\nu \omega^{a(-n)}_\rho - \partial_\rho \omega^{a(-n)}_\nu + \varepsilon^{abc} \omega^{(m)}_{\nu b} \omega^{(-n-m)}_{\rho c}) \\ &+ \frac{1}{2} e g^{\mu\nu} \phi^{-2} (\partial_\mu \phi \partial_\nu \phi + \partial_\mu \varphi \partial_\nu \varphi) \big) \,, \end{aligned} \tag{4.70}$$

where in the second term the algebra multiplication defined in (4.52) is implicit, or in other words, where all fields are θ-dependent and an integration over θ is assumed. The action is by construction invariant under spin-2 transformations. Moreover, we have already seen that the scalar couplings are also invariant under global Virasoro transformations. To see that this is also the case for the generalized Einstein-Hilbert term, we have to show that one can determine the transformation rule for $\omega_\mu^{a(n)}$ such that the action stays invariant. This is indeed possible, and one finds

$$\delta_{\xi^5} \omega_\mu^{a(n)} = i \sum_k (n-k) \xi^5_k \omega_\mu^{a(n-k)} \,, \qquad \delta_{\xi^5} \omega^a_\mu = \xi^5 \partial_\theta \omega^a_\mu \,. \tag{4.71}$$

Equivalently, they can be computed by solving the $\omega_\mu^{a(n)}$ in terms of the vielbeins by use of (4.22) and then applying a $\hat{v}$ transformation to this expression. Both results coincide. Note that instead the full algebra-valued action (4.49) transforms non-trivially under $\hat{v}$, namely as

$$\delta S^m = im \xi^5_n S^{(m-n)}. \tag{4.72}$$

However, as we have already seen in sec. 3.2, it is sufficient to include only the zero-component in (4.70), which is clearly invariant.

4.3.4 Dualities and gaugings

So far we have determined the unbroken phase of the Kaluza-Klein theory in a description where all propagating degrees of freedom reside in scalar fields. Before we turn to a gauging of a subgroup of the global symmetries we have to ask whether it is still possible to assign all degrees of freedom to scalars, since the introduction of gauge fields necessarily seems to enforce the appearance of local degrees of freedom

that are instead carried by vectors. However, in 2.2.1 we have reviewed the peculiar fact that in three-dimensional gauged supergravities all Yang-Mills-type gaugings are on-shell equivalent to Chern-Simons gaugings with an enlarged number of scalar fields [60, 61]. Thus all bosonic degrees of freedom can still appear as scalar fields. We are going to show that this duality also applies to the present case.

To begin with, we note that in the gauged theory all partial derivatives are replaced by covariant ones. For a given field χ transforming in a representation Δ under $\hat{v}$ the covariant derivative reads

$$D_\mu \chi^n = \partial_\mu \chi^n - ig \sum_k (n - (1-\Delta)k) A_\mu^k \chi^{n-k} \ , \tag{4.73}$$

where we have introduced the gauge coupling $g = M$. Indeed, it transforms by construction covariantly under local $\hat{v}$ transformations, $\delta_\xi(D_\mu \chi^n) = ig(n - (1 - \Delta)k)\xi_k D_\mu \chi^{n-k}$, if we assume as usual that A_μ^n transforms as a gauge field under the adjoint (i.e. as the Kaluza-Klein vector in (4.8) with $\Delta = -1$). Similarly, the non-abelian $\hat{v}$ field strength is given by

$$F_{\mu\nu}^n = \partial_\mu A_\nu^n - \partial_\nu A_\mu^n + ig \sum_m (n - 2m) A_\mu^{n-m} A_\nu^m \ . \tag{4.74}$$

These expressions are given for the Kaluza-Klein fields in θ-dependent notation by

$$\begin{aligned} D_\mu \phi &= \partial_\mu \phi - g A_\mu \partial_\theta \phi - 2g\phi \partial_\theta A_\mu \ , \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - g A_\mu \partial_\theta A_\nu + g A_\nu \partial_\theta A_\mu \ . \end{aligned} \tag{4.75}$$

The part of the gauged action containing scalar fields will be given by the covariantisation of the action (4.56) according to (4.75). One easily checks that this transforms into a total θ-derivative under local ξ_k^5-transformations, i.e. defines an invariant action. Furthermore, in appendix C we show by explicit reduction that exactly these terms appear, as well as an explicit $\hat{v}$ gauge invariant mass term for the spin-2 fields, i.e. the action reads (see also [81, 36])

$$\mathcal{L}_{\text{scalar}} = \frac{1}{2} e g^{\mu\nu} \phi^{-2} D_\mu \phi D_\nu \phi - \frac{1}{4} e \phi^2 g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + \mathcal{L}_m \ . \tag{4.76}$$

To show that this action is indeed on-shell equivalent to a Chern-Simons gauged theory we introduce as in 2.2.1 new gauge fields for each of the former Yang-Mills fields, or in other words, we enhance the gauge symmetry with nilpotent shift symmetries (see also [63]). To explain this dualization procedure, let us consider the Yang-Mills equation resulting from (4.76)

$$D^\mu(\phi^2 F_{\mu\nu}) = j_\nu \ , \tag{4.77}$$

where j_ν denotes the current induced by the charged fields. It implies integrability of the duality relation

$$\frac{1}{2} e^{-1} \varepsilon^{\mu\nu\rho} \phi^2 F_{\nu\rho} = D^\mu \varphi + g B^\mu =: \mathcal{D}^\mu \varphi \ , \tag{4.78}$$

where φ will be the scalar field carrying the former degrees of freedom of A_μ, and B_μ is the gauge field corresponding to the enlargement of the gauge group. From the previous section we know already the transformation properties of φ under $\hat{v}$, and one may check explicitly that it does not change in the gauged phase. In particular, also the dual vector B_μ will transform with $\Delta = 2$. In other words, it transforms under the dual of the adjoint representation of $\hat{v}$, which will later on turn out to be important.[6] To define the dual action we add instead of the Yang-Mills term a Chern-Simons-like term $B \wedge F$, where F denotes the non-abelian field strength, and get

$$\mathcal{L}_{\text{scalar}} = \frac{1}{2} e g^{\mu\nu} \phi^{-2} \left(D_\mu \phi D_\nu \phi + \mathcal{D}_\mu \varphi \mathcal{D}_\nu \varphi \right) - \frac{1}{2} g \varepsilon^{\mu\nu\rho} B_\mu F_{\nu\rho} + \mathcal{L}_m \ . \tag{4.79}$$

Indeed, varying with respect to B_μ one recovers the duality relation (4.78), and eliminating the dual scalar φ by means of this relation yields the Yang-Mills type theory (4.76). Thus we have shown that the degrees of freedom of the A^n_μ can be assigned to new scalars φ^n, if at the same time new topological gauge fields B^n_μ are introduced that promote the former global shift transformations (i.e. the e-transformations of $\widehat{sl(2,\mathbb{R})}$) to a local symmetry.

4.4 Broken phase of the Kaluza-Klein theory

Up to now we have determined the action (4.70) of the ungauged theory, which is invariant under global $\hat{v} \ltimes \widehat{sl(2,\mathbb{R})}$ transformations as well as local spin-2 transformations. We argued that in order to get the full Kaluza-Klein action one has to gauge the Virasoro algebra together with the shift symmetries of (4.68). In the next section we discuss the effect of this gauging on the topological fields. We will see that they combine into a single Chern-Simons theory. As we have seen in 2.2.1 this is quite analogous to gauged supergravities, where truncating to the topological fields results in the Chern-Simons theories of [54] for AdS-supergroups. The scalars will be discussed thereafter.

4.4.1 Local Virasoro invariance for topological fields

As usual the gauging proceeds in several steps. First of all, one has to replace all partial derivatives by covariant ones. Let us start with the generalized Einstein-Hilbert term. The covariant derivative for $\omega^{a(n)}_\mu$ in accordance with (4.71) reads

$$D_\mu \omega^{a(n)}_\nu = \partial_\mu \omega^{a(n)}_\nu - ig \sum_m (n-m) A^m_\mu \omega^{a(n-m)}_\nu \ , \tag{4.80}$$

or equivalently

$$D_\mu \omega^a_\nu = \partial_\mu \omega^a_\nu - A_\mu \partial_5 \omega^a_\nu \ . \tag{4.81}$$

[6] For a definition see appendix B.

The covariantized Einstein-Hilbert action is then invariant under local Virasoro transformations. In contrast it will no longer be invariant under all spin-2 transformations, but only under three-dimensional diffeomorphisms. This is due to the fact that the explicit ∂_5 appearing in the covariant derivatives will also act on the spin-2 transformation parameter. Thus, the gauging will deform the spin-2 transformations.

Furthermore, we have already seen that in order to guarantee that the resulting action will be equivalent to the original Yang-Mills gauged theory, one has to introduce a Chern-Simons term for the Kaluza-Klein vectors A^n_μ, whose propagating degrees of freedom are now carried by the dual scalars φ^n, as well as for the dual gauge fields B^n_μ. This implies that we do not have to gauge only the Virasoro algebra $\hat{v}$, but instead the whole subalgebra of (4.68), which is spanned by (Q^m, e_m), while the rigid symmetry given by h_m and f_m will be broken explicitly. Both gauge fields combine into a gauge field for this larger algebra. Moreover, in contrast to $\hat{v}$ itself this algebra carries a non-degenerate invariant quadratic form, namely

$$\langle Q^m, e_n \rangle = \delta^{n,-m} \,, \tag{4.82}$$

such that a Chern-Simons action can be defined. The existence of this form is due to the fact that e_m transforms actually under the co-adjoint action of $\hat{v}$, as we have argued in 3.4. We will see that the Chern-Simons action with respect to this quadratic form indeed reproduces the correct $B \wedge F$-term in (4.79).

It is tempting to ask, whether all topological fields, i.e. the gravitational fields together with the gauge fields for the Virasoro and shift symmetry, can be combined into a Chern-Simons theory for a larger algebra. The latter would have to combine the affine Poincaré algebra with the algebra spanned by (Q^m, e_m). Naively one would think that the semi-direct product $\hat{v} \ltimes \widehat{iso(1,2)}$ defined in (4.11) and extended by e_m according to (4.68) is the correct choice. However, it does not reproduce the right Kaluza-Klein symmetry transformations, and moreover, the algebra seems not to admit a non-degenerate and invariant quadratic form. To see that a Chern-Simons formulation nevertheless exists, we observe that varying the total action consisting of the sum of $\hat{v}$-covariantized Einstein-Hilbert action and $B \wedge F$ with respect to A_μ, we get the non-abelian field strength for B_μ plus terms of the form $e^a_\mu \partial_5 \omega_{\nu a}$. Thus, a Chern-Simons interpretation is only possible if the latter terms are contained in the field strength of B_μ, or in other words, if the algebra also closes according to $[P, J] \sim e$.

Demanding consistency with the Jacobi identities and requiring that e^a_μ and ω^a_μ transform under the correct representation of $\hat{v}$, the following Lie algebra is then uniquely fixed up to a free parameter α:

$$\begin{aligned}
&[P^m_a, J^n_b] = \varepsilon_{abc} P^{c(m+n)} + i\alpha n \eta_{ab} e_{m+n} \,, \quad [J^m_a, J^n_b] = \varepsilon_{abc} J^{c(m+n)} \,, \\
&[P^m_a, P^n_b] = 0 \,, \\
&[Q^m, Q^n] = ig(m-n) Q^{m+n} \,, \qquad [Q^m, P^n_a] = ig(-m-n) P^{m+n}_a \,, \\
&[Q^m, J^n_a] = -ign J^{m+n}_a \,, \qquad [Q^m, e_n] = ig(-n-2m) e_{m+n} \,, \\
&[P^m_a, e_n] = [J^m_a, e_n] = [e_m, e_n] = 0 \,.
\end{aligned} \tag{4.83}$$

Here we have rescaled the Q^m with the gauge coupling constant g for later convenience.

We see that one gets an algebra which looks similar to the one proposed in [33] (see (4.11)), except that it does not contain simply the semi-direct product of $\hat{v}$ with the affine Poincaré algebra, since the P_a^n and J_a^n transform in different representations of $\hat{v}$. But in contrast to sec. 4.3.3, where we observed a similar phenomenon for the global symmetry algebra, there seems not to exist an obvious change of basis which reduces the algebra to the standard form. Namely, because of the different index structure the Q^m can be shifted neither by P_a^m nor J_a^m. In fact, that the algebra is consistent even in this non-standard form is possible only because of the nilpotency of translations, i.e. $[P_a^m, P_b^n] = 0$.

Furthermore, we observe that the algebra admits a central extension e_m of the Poincaré algebra even at the classical level. (Even though, strictly speaking, it is only a central extension for the Poincaré subalgebra, since the e_m do not commute with the Q^m.) Remarkably, it is exactly this modification of the algebra that allows the existence of an invariant quadratic form. Namely, the bilinear expression

$$W = P^{a(-m)} J_a^{(m)} + \frac{\alpha}{g} Q^m e_{-m} \tag{4.84}$$

(in particular, $\langle Q^m, e_n \rangle = \frac{g}{\alpha} \delta^{m,-n}$) is invariant under (4.83). The total Chern-Simons action constructed with respect to this quadratic form, with the gauge field written as

$$\mathcal{A}_\mu = e_\mu^{a(n)} P_a^n + \omega_\mu^{a(n)} J_a^n + A_\mu^n Q^n + B_\mu^n e_n \ , \tag{4.85}$$

is then indeed given by

$$S_{CS} = \int d^3x d\theta \big(\varepsilon^{\mu\nu\rho} e_{\mu a} (D_\nu \omega_\rho^a - D_\rho \omega_\nu^a + \varepsilon^{abc} \omega_{\nu b} \omega_{\rho c}) + \frac{g}{\alpha} \varepsilon^{\mu\nu\rho} B_\mu F_{\nu\rho} \big) \ , \tag{4.86}$$

i.e. consists of the $\hat{v}$-covariantized Einstein-Hilbert term and the Chern-Simons action for A_μ and B_μ.

Let us briefly comment on the reality constraints on (4.83). Naively one would take (4.83) as real Lie algebra, and correspondingly the gauge fields in (4.85) would also be real. However, the reality condition $(Q^*)^m = Q^m$ (and similarly for all other generators) is not consistent, since taking the complex conjugate of (4.83) changes relative signs. Instead, only the reality constraint $(Q^*)^m = Q^{-m}$ can be consistently imposed. This is on the other hand also in accordance with the reality condition for the original Kaluza-Klein fields in (4.2), and therefore the fields in (4.85) fulfill exactly the correct reality constraint.

The equations of motion for the Chern-Simons action in (4.86) again imply vanishing field strength,

$$\mathcal{F}_{\mu\nu} = R_{\mu\nu}^{a(n)} J_a^n + T_{\mu\nu}^{a(n)} P_a^n + F_{\mu\nu}^n Q^n + G_{\mu\nu}^n e_n = 0 \ , \tag{4.87}$$

whose components can in turn be written as

$$\begin{aligned}
R^{a(n)}_{\mu\nu} &= \partial_\mu \omega^{a(n)}_\nu - \partial_\nu \omega^{a(n)}_\mu + \varepsilon^{abc}\omega^{(n-m)}_{\mu b}\omega^{(m)}_{\nu c} \\
&+ ig\sum_m (n-m)\omega^{a(n-m)}_\mu A^m_\nu - ig\sum_m m A^{n-m}_\mu \omega^{a(m)}_\nu \ , \\
T^{a(n)}_{\mu\nu} &= D_\mu e^{a(n)}_\nu - D_\nu e^{a(n)}_\mu + \varepsilon^{abc} e^{(n-m)}_{\mu b}\omega^{(m)}_{\nu c} + \varepsilon^{abc}\omega^{(n-m)}_{\mu b} e^{(m)}_{\nu c} \ , \\
F^n_{\mu\nu} &= \partial_\mu A^n_\nu - \partial_\nu A^n_\mu + ig\sum_m (n-2m) A^{n-m}_\mu A^m_\nu \ , \\
G^n_{\mu\nu} &= \partial_\mu B^n_\nu - \partial_\nu B^n_\mu + ig\sum_m (m-2n) A^{n-m}_\mu B^m_\nu + ig\sum_m (n+m) B^{n-m}_\mu A^m_\nu \\
&+ i\alpha \sum_m m e^{a(n-m)}_\mu \omega^{(m)}_{\nu a} - i\alpha \sum_m (n-m)\omega^{a(n-m)}_\mu e^{(m)}_{\nu a} \ .
\end{aligned} \tag{4.88}$$

Here $D_\mu e^{a(n)}_\nu$ denotes the $\hat{v}$-covariant derivative on $e^{a(n)}_\mu$, which is in θ-notation given by

$$D_\mu e^a_\nu = \partial_\mu e^a_\nu - A_\mu \partial_5 e^a_\nu - e^a_\nu \partial_5 A_\mu \ . \tag{4.89}$$

Moreover, all quantities can be rewritten in θ-dependent notation, e.g. the non-abelian field strength for B_μ is given by

$$\begin{aligned}
G_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu + 2(B_\mu \partial_5 A_\nu - B_\nu \partial_5 A_\mu) - A_\mu \partial_5 B_\nu + A_\nu \partial_5 B_\mu \\
&+ \alpha \left(e^a_\mu \partial_\theta \omega_{\nu a} - e^a_\nu \partial_\theta \omega_{\mu a} \right) \ .
\end{aligned} \tag{4.90}$$

The gauge transformations for gauge parameter $u = \rho^{a(n)} P^n_a + \tau^{a(n)} J^n_a + \xi^5_n Q^n + \Lambda^n e_n$ can be written as

$$\begin{aligned}
\delta e^{a(n)}_\mu &= \partial_\mu \rho^{a(n)} + \varepsilon^{abc} e^{(n-m)}_{\mu b}\tau^{(m)}_c + \varepsilon^{abc}\omega^{(n-m)}_{\mu b}\rho^{(m)}_c \\
&- ign A^{n-m}_\mu \rho^{a(m)} + ign e^{a(n-m)}_\mu \xi^5_m \ , \\
\delta \omega^{a(n)}_\mu &= \partial_\mu \tau^{a(n)} + \varepsilon^{abc}\omega^{(n-m)}_{\mu b}\tau^{(m)}_c \\
&- ig\sum_m m A^{n-m}_\mu \tau^{a(m)} + ig\sum_m (n-m)\omega^{a(n-m)}_\mu \xi^5_m \\
\delta A^n_\mu &= \partial_\mu \xi^n_5 + ig\sum_m (n-2m)\xi^5_m A^{n-m}_\mu \ , \\
\delta B^n_\mu &= \partial_\mu \Lambda^n + ig\sum_m (m-2n)\Lambda^m A^{n-m}_\mu + ig\sum_m (n+m)\xi^5_m B^{n-m}_\mu \\
&+ i\alpha \sum_m m e^{a(n-m)}_\mu \tau^{(m)}_a + i\alpha \sum_m (m-n)\omega^{a(n-m)}_\mu \rho^{(m)}_a \ .
\end{aligned} \tag{4.91}$$

Let us now check, whether the Kaluza-Klein symmetries are included in these gauge transformations. First of all, it reproduces the correct transformation rule for B_μ under $\hat{v}$, as can be seen by rewriting the last equation of (4.91) in θ-dependent notation

$$\begin{aligned}
\delta B_\mu &= \partial_\mu \Lambda - 2g\Lambda \partial_\theta A_\mu - g A_\mu \partial_\theta \Lambda + g\xi^5 \partial_\theta B_\mu + 2g B_\mu \partial_\theta \xi^5 \\
&+ \alpha e^a_\mu \partial_\theta \tau_a - \alpha \rho_a \partial_\theta \omega^a_\mu \ .
\end{aligned} \tag{4.92}$$

By comparing (4.91) with (4.8) we also see that the Virasoro gauge transformations parameterized by ξ^5 are correctly reproduced for e^a_μ and A_μ. To compare with the spin-2 transformations we define in analogy to (4.24) the transformation parameter

$$\rho^a = \xi^\rho e^a_\rho \ , \quad \tau^a = \xi^\rho \omega^a_\rho \ , \quad \xi^5 = \xi^\rho A_\rho \ . \tag{4.93}$$

Then one finds for the vielbein

$$\delta_\xi e^a_\mu = \xi^\rho \partial_\rho e^a_\mu + \partial_\mu \xi^\rho e^a_\rho + g A_\rho \partial_\theta \xi^\rho e^a_\mu - g A_\mu \partial_\theta \xi^\rho e^a_\rho - \xi^\rho T^a_{\rho\mu} \ , \tag{4.94}$$

which implies that on-shell, i.e. for $T^a_{\mu\nu} = 0$, the gauge transformations coincide with the Kaluza-Klein symmetries in (4.6) and (4.7). With the same transformation parameter and $\Lambda = \xi^\rho B_\rho$ we find for A_μ and B_μ the following transformation rules (again up to field strength terms)

$$\begin{aligned} \delta_\xi A_\mu &= \xi^\rho \partial_\rho A_\mu + \partial_\mu \xi^\rho A_\rho - g A_\mu \partial_\theta \xi^\rho A_\rho \ , \\ \delta_\xi B_\mu &= \xi^\rho \partial_\rho B_\mu + \partial_\mu \xi^\rho B_\rho - g A_\mu \partial_\theta \xi^\rho B_\rho + 2g B_\mu \partial_\theta \xi^\rho A_\rho + \alpha e^a_\mu \partial_\theta \xi^\rho \omega_{\rho a} \ , \end{aligned} \tag{4.95}$$

which reproduces for A_μ the same transformation as in (4.8), up to the ϕ-dependent term (which, of course, cannot be contained in a Chern-Simons formulation).

As in the case of the pure gravity-spin-2 theory, the topological phase of the Kaluza-Klein theory is given by a Chern-Simons theory, and moreover the Kaluza-Klein symmetry transformations are on-shell equivalent to the non-abelian gauge transformations determined by (4.83). Even though this equivalence holds only on-shell, the Kaluza-Klein transformations are separately an (off-shell) symmetry, since $\delta_\xi \mathcal{A}_\mu = \xi^\rho \mathcal{F}_{\rho\mu}$ leaves the Chern-Simons action invariant, as can be easily checked with (4.13).

Finally, let us check that spin-2 transformations together with the Virasoro transformations build a closed algebra, as it should be at least on-shell, since they were constructed as Yang-Mills gauge transformations. For the vielbein, e.g., one finds[7]

$$[\delta_\xi, \delta_{\eta^5}] e^a_\mu = \delta_{(\eta\xi)} e^a_\mu - \delta_{(\xi\eta)^5} e^a_\mu \ , \tag{4.96}$$

with the parameter given by

$$(\eta\xi)^\rho = \eta^5 \partial_5 \xi^\rho \ , \qquad (\xi\eta)^5 = \xi^\rho \partial_\rho \eta^5 \ . \tag{4.97}$$

The same formula holds for A_μ and B_μ. But, for B_μ one also has to check the closure of the shift symmetries with spin-2 and here one finds

$$[\delta_\xi, \delta_\Lambda] B_\mu = -\delta_{\tilde{\Lambda}} B_\mu - 2\Lambda \partial_5 \xi^\rho F_{\rho\mu} \ , \tag{4.98}$$

where

$$\tilde{\Lambda} = \xi^\rho \partial_\rho \Lambda + 2\Lambda \partial_5 \xi^\rho A_\rho \ . \tag{4.99}$$

Therefore the algebra closes only on-shell, i.e. if $F_{\mu\nu} = 0$.

[7] As before, we indicate Virasoro transformations by a subscript 5 on the transformation parameter.

4.4.2 Virasoro-covariantisation for scalars

To summarize the results of the last section, we have seen that in the gauged phase the spin-2 transformations of sec. 3 are no longer a symmetry due to the substitution of partial derivatives by covariant ones. Therefore the spin-2 transformations have to be deformed by g-dependent terms. For the topological fields we have seen that a Chern-Simons formulation exists, which in turn yields modified spin-2 transformations, which are consistent by construction.

Next let us focus on the scalar fields. For them we have already noted the form of the covariant derivative in (4.75), and the same formula holds for φ, but with the difference that it also has to be covariant with respect to the local shift symmetries gauged by B_μ. The latter act as $\delta_\Lambda\varphi = -g\Lambda$, i.e. the covariant derivative reads in θ-notation

$$\mathcal{D}_\mu\varphi = \partial_\mu\varphi - A_\mu\partial_5\varphi - 2\varphi\partial_5 A_\mu + gB_\mu \ . \tag{4.100}$$

Altogether, replacing the partial derivatives in (4.70) by covariant ones and adding the Chern-Simons action constructed in the last section as well as an explicit mass term which is known to appear (see appendix C), results in

$$\begin{aligned} S_{\mathrm{KK}} = \int d^3x d\theta \big[\varepsilon^{\mu\nu\rho}\big(&- e^a_\mu(D_\nu\omega_{\rho a} - D_\rho\omega_{\nu a} + \varepsilon_{abc}\omega^b_\nu\omega^c_\rho) - \frac{1}{2}gB_\mu F_{\nu\rho}\big) \\ &+ \frac{1}{2}eg^{\mu\nu}\phi^{-2}(D_\mu\phi D_\nu\phi + \mathcal{D}_\mu\varphi\mathcal{D}_\nu\varphi) + \mathcal{L}_m\big] \ . \end{aligned} \tag{4.101}$$

Here we have determined the free parameter of the algebra (4.83) to be $\alpha = 2$ in order to get the correct Chern-Simons term for A_μ and B_μ discussed in sec. 3.4. There we have already observed that varying this action with respect to B_μ one recovers the duality relation (4.78). In turn, the equations of motion for (4.101) and for the Yang-Mills gauged action are equivalent. This can be seen directly by imposing the gauge $\varphi = 0$ in (4.101) and then integrating out B_μ, which results exactly in the Kaluza-Klein action containing the Yang-Mills term in (4.76). Moreover, varying with respect to ω^a_μ still implies $T^a_{\mu\nu} = 0$. This shows that ω^a_μ can be expressed in terms of e^a_μ as is standard, but with the exception that all derivatives on e^a_μ are now $\hat{v}$-covariant. In the second-order formulation this means that the Einstein-Hilbert part looks formally the same as in sec. 4.3.4, but with all Christoffel symbols now containing $\hat{v}$-covariant derivatives. This is on the other hand also what one gets by direct Kaluza-Klein reduction in second-order form [36, 34]. Thus we have shown, that (4.101) is on-shell equivalent to the Kaluza-Klein action which results from dimensional reduction.

In view of the fact that (4.101) is manifestly $\hat{v}$ and shift invariant it remains the question how the spin-2 symmetries are realized. As for the case of the topological fields, also the σ-model action for the scalar fields will no longer be invariant under the unmodified spin-2 transformations for the same reasons. To find the deformed transformation rule for the scalars, one way is to check the closure of the algebra. The unmodified spin-2 transformations do not build a closed algebra with the local $\hat{v}$ transformations. But, if we deform the spin-2 transformation to

$$\delta_\xi\phi = \xi^\rho\partial_\rho\phi + 2g\phi\partial_\theta\xi^\rho A_\rho, \tag{4.102}$$

the algebra closes according to

$$[\delta_\xi, \delta_{\eta^5}]\phi = \delta_{(\eta\xi)}\phi - \delta_{(\xi\eta)^5}\phi \,, \tag{4.103}$$

i.e. exactly like in the case of the topological fields with the parameters given in (4.97). The Kaluza-Klein transformations can therefore be entirely reconstructed by requiring closure of the algebra. The same transformation holds for the dual scalar φ.

In the presence of matter fields we have to be careful about the closure of the algebra also on the gauge fields. Namely, for the pure Chern-Simons theory shift with spin-2 transformations in (4.99) close on-shell (as it should), but for the theory constructed here the field strength does not vanish. Thus one way to get a closing algebra is to extend the transformation rule according to

$$\delta' B_\mu = -2\varphi\partial_\theta\xi^\rho F_{\mu\rho} \,, \tag{4.104}$$

and all transformations close off-shell.

Therefore we see that in the full theory the transformation rules for the vectors A_μ and B_μ get extended by scalar field dependent terms. That is on the other hand also what we already know from the symmetry variations in (4.8) for A^n_μ, and these terms will be needed in order for the full action to be spin-2 invariant. This is in complete analogy to the construction of gauged supergravities, where the procedure of gauging is only consistent with supersymmetry, if additional couplings like mass terms are added, while the supersymmetry variations are supplemented by scalar-dependent terms. However, in the present case the invariance of the Yang-Mills gauged Kaluza-Klein theory is guaranteed by construction, which in turn implies that the on-shell equivalent dual theory (4.101) is also invariant (if one assumes transformation rules for B_μ, which are on-shell given by the variation of the left-hand side of (4.78)). In view of our aim to construct the Kaluza-Klein theories for more general backgrounds, it would however be important to find a systematic procedure to determine the scalar-dependent corrections for gaugings of arbitrary diffeomorphism Lie algebras. This we will leave for future work, but here let us just show how the scalar-dependent correction in (4.104) ensures the invariance under spin-2 for a subsector.

For this it will be convenient to separate from the spin-2 transformations those parts which represent already a symmetry for each term separately. To do so we remember that to realize the spin-2 transformations on the topological fields as gauge transformations we had to switch on also the Virasoro transformations with parameter $\xi^5 = \xi^\rho A_\rho$. Now we will turn the logic around and apply a spin-2 transformation followed by a Virasoro transformation with parameter $\xi^5 = -\xi^\rho A_\rho$. Since Virasoro invariance is manifest, this is a symmetry if and only if spin-2 is a symmetry. One may easily check that on e^a_μ and ϕ (as well as φ) this transformation is given by

$$\begin{aligned} \delta_\xi\phi &= \xi^\rho D_\rho\phi \,, \\ \delta_\xi e^a_\mu &= \xi^\rho D_\rho e^a_\mu + D_\mu\xi^\rho e^a_\rho \,. \end{aligned} \tag{4.105}$$

Here we have used (4.89) and also introduced a Virasoro covariant derivative for the spin-2 transformation parameter (of which we may think as transforming as

$\delta_{\eta^5}\xi^\mu = \eta^5\partial_5\xi^\mu$),

$$D_\mu\xi^\rho = \partial_\mu\xi^\rho - A_\mu\partial_5\xi^\rho \ . \tag{4.106}$$

We see that we get transformation rules which look formally like a diffeomorphism symmetry, except that all appearing derivatives are $\hat{v}$-covariant. In the following we will refer to these transformations as 'gauged diffeomorphisms'. In contrast, the gauge fields A_μ and B_μ transform as

$$\begin{aligned}\delta_\xi A_\mu &= \xi^\rho F_{\rho\mu} \\ \delta_\xi B_\mu &= \xi^\rho D_\rho B_\mu + D_\mu\xi^\rho B_\rho + 2\,e^a_\mu\partial_\theta\xi^\rho\omega_{\rho a} \ .\end{aligned} \tag{4.107}$$

It remains the question whether actions can be constructed that are manifestly invariant under these transformations. To analyze this let us start with an action constructed from a scalar Lagrangian given by

$$S = \int d^3x d\theta\, e\,\mathcal{L} \ , \tag{4.108}$$

and moreover being invariant under local Virasoro transformations. Put differently, this means that the Lagrangian varies as $\delta_{\xi^5}\mathcal{L} = \xi^5\partial_5\mathcal{L} - 2\mathcal{L}\partial_5\xi^5$ under $\hat{v}$ (because then it transforms together with the vielbein determinant, whose symmetry variation reads $\delta_{\xi^5}e = \xi^5\partial_5 e + 3e\partial_5\xi^5$, into a total θ-derivative). By use of the $\hat{v}$-covariant derivative given by

$$D_\mu\mathcal{L} = \partial_\mu\mathcal{L} - A_\mu\partial_5\mathcal{L} + 2\mathcal{L}\partial_5 A_\mu \ , \tag{4.109}$$

we can then evaluate the variation of the action under gauged diffeomorphisms and find

$$\begin{aligned}\delta_\xi S &= \int d^3x d\theta\,[(\xi^\rho D_\rho e + eD_\rho\xi^\rho)\mathcal{L} + e\xi^\rho D_\rho\mathcal{L}] = \int d^3x d\theta D_\rho(e\xi^\rho\mathcal{L}) \\ &= \int d^3x d\theta\,[\partial_\rho(e\xi^\rho\mathcal{L}) - \partial_5(e\xi^\rho A_\rho\mathcal{L})] = 0 \ .\end{aligned} \tag{4.110}$$

Thus, if one constructs an action from a Lagrangian that transforms as a scalar under gauged diffeomorphisms, then the action is invariant under these gauged diffeomorphisms if and only if it is also invariant under local Virasoro transformations. The latter requirement is satisfied in our theory by construction. Thus it remains to be checked whether the Lagrangian transforms as a scalar. However, using (4.105), (4.107) and $[D_\mu, D_\nu]\phi = -2\phi\partial_5 F_{\mu\nu} - \partial_5\phi F_{\mu\nu}$, one proves that the covariant derivative $D_\mu\phi$ transforms under gauged diffeomorphisms as

$$\delta_\xi(D_\mu\phi) = D_\mu\xi^\rho D_\rho\phi + \xi^\rho D_\rho D_\mu\phi - 2\phi\partial_5\xi^\rho F_{\rho\mu} \ , \tag{4.111}$$

i.e. it does not transform like a one-form, but requires an additional piece proportional to $F_{\mu\nu}$, which again shows that corrections have to be added to the transformation rules.

Let us consider the subsector of the theory where we rescale

$$e^a_\mu \to \kappa e^a_\mu \,, \qquad \varphi \to \kappa^{-1/2}\varphi \,, \tag{4.112}$$

and then take the limit $\kappa \to 0$. The action then reads

$$S_{\kappa\to 0} = \frac{1}{2}\int d^3x d\theta \left(-g\varepsilon^{\mu\nu\rho}B_\mu F_{\nu\rho} + e g^{\mu\nu}\phi^{-2}\mathcal{D}_\mu\varphi\mathcal{D}_\nu\varphi\right) \,. \tag{4.113}$$

However, in view of the fact that the term $\sim e^a_\mu \partial_\theta \xi^\rho \omega_{\rho a}$ in (4.107) disappears in this limit and with the additional contribution (4.104) in the B_μ variation, the extra term in (4.111) is canceled, and the kinetic term for φ is therefore separately invariant. The Chern-Simons term on the other hand transforms according to (4.13) as

$$\delta_\xi S_{\kappa\to 0} = -g\int \varepsilon^{\mu\nu\rho}F_{\sigma\mu}F_{\nu\rho}\varphi\partial_5\xi^\sigma = 0 \,, \tag{4.114}$$

where we have used that a totally antisymmetric object in four indices vanishes in $D = 3$. Thus we have shown that the scalar field modification in (4.104) is sufficient in order to restore the spin-2 invariance of this subsector of the theory.

4.5 Spin-2 symmetry for general matter fields

As we have discussed in the last section the gauging of global symmetries requires a deformation of the spin-2 transformations, which in turn induces a spin-2 mass term. This is in analogy to the gauging of supergravity, where the spin-3/2 transformations (i.e. the supersymmetry variations) have to be modified by g-dependent terms, which similarly induces gravitino mass terms. However, as we have seen in sec. 2.2.1, the gauging of supergravity generically induces also a potential for the scalar fields. Therefore one may wonder whether a similar phenomenon can happen for spin-2 theories. In fact, so far we discussed only the theory corresponding to a Kaluza-Klein reduction of the Einstein-Hilbert term. For the gauging of more general theories – exhibiting the Kaluza-Klein action for general matter couplings already in the higher-dimensional theory – we will however see that a scalar potential naturally appears.

To show this let us discuss the simplest example of a scalar field η in $D = 4$ coupled to gravity,

$$\mathcal{L} = -ER + \frac{1}{2}Eg^{MN}\partial_M\eta\partial_N\eta \,. \tag{4.115}$$

The ungauged phase of this theory simply consists of the algebra-valued Einstein-Hilbert term in sec. 4.3.1 coupled to a θ-dependent (or algebra-valued) kinetic scalar term of canonical form $\frac{1}{2}\partial_\mu\eta\partial^\mu\eta$ (compare eq. (4.52)). This theory is invariant under the local spin-2 symmetries in 4.3.1, and rigid Virasoro transformations, which act on η as

$$\delta_\xi\eta = \xi^\rho\partial_\rho\eta \,, \qquad \delta_{\xi^5}\eta = \xi^5\partial_\theta\eta \,. \tag{4.116}$$

Thus η transforms in the $\Delta = 0$ representation of $\hat{v}$. The gauging of $\hat{v}$ again requires a minimal substitution, which reads in the given case

$$\partial_\mu \eta \longrightarrow D_\mu \eta = \partial_\mu \eta - g A_\mu \partial_\theta \eta \ . \tag{4.117}$$

Under the gauged diffeomorphism of sec. 4.4.2 this covariant derivative transforms as

$$\delta_\xi (D_\mu \eta) = \xi^\rho D_\rho D_\mu \eta + D_\mu \xi^\rho D_\rho \eta \ , \tag{4.118}$$

which can be shown along the lines of (4.111). We see that it transforms like a 1-form. As we have argued in the last section the corresponding action just for η will therefore be invariant under the gauged diffeomorphisms $\delta_\xi \eta = \xi^\rho D_\rho \eta$. However, we know that the invariance of the Einstein-Hilbert term requires an additional ϕ-dependent variation for A_μ (see (4.7)). This in turn will spoil the invariance of the scalar couplings. Namely, under $\delta'_\xi A_\mu = -\phi^{-2} \partial_5 \xi^\rho g_{\rho\mu}$ one has

$$\delta'_\xi \left(e g^{\mu\nu} D_\mu \eta D_\nu \eta \right) = 2 e \phi^{-2} \partial_5 \xi^\rho D_\rho \eta \partial_5 \eta \ . \tag{4.119}$$

To compensate for this one can add a scalar potential of the form

$$e V(\phi, \eta) = e \phi^{-2} (\partial_5 \eta)^2 \ , \tag{4.120}$$

which transforms as

$$\delta_\xi (eV) = \partial_\rho \left(e \xi^\rho \phi^{-2} (\partial_5 \eta)^2 \right) + 2 e \phi^{-2} \partial_5 \xi^\rho D_\rho \eta \partial_5 \eta \ , \tag{4.121}$$

and therefore cancels (4.119) up to a total derivative. Altogether we have shown that the deformed spin-2 transformations require a scalar potential, and the matter couplings in the gauged phase are given by

$$S[\phi, \eta] = \frac{1}{2} \int d^3x d\theta e \left(g^{\mu\nu} D_\mu \eta D_\nu \eta - V(\phi, \eta) \right) \ , \tag{4.122}$$

with the potential in (4.120). One may also check explicitly that the Kaluza-Klein reduction of (4.115) leads exactly to (4.122).

4.6 Consistent truncations and extended supersymmetry

After constructing the consistent gravity–spin-2 couplings appearing in Kaluza-Klein theories in the broken and unbroken phase, one may ask the following question: Is a truncation to a finite subset of spin-2 fields possible? To answer this let us first review in which sense the truncation to the massless modes is justified.

Since in standard Kaluza-Klein compactifications the higher Kaluza-Klein modes are much heavier than the zero-modes, they can be integrated out in an effective description. For this one usually assumes that this is equivalent to just setting the

massive modes equal to zero. Apart from the question whether this is really the correct approach of 'integrating out' degrees of freedom, it is not guaranteed that this is a consistent truncation in the Kaluza-Klein sense. The latter requires that the truncated theory is still compatible with the higher-dimensional equations of motion. More precisely, this demands that each solution of the truncated theory can be lifted to a solution of the full theory. To illustrate the latter, let us consider the massless theory resulting from an S^1 compactification (see (4.3) above). Setting the dilaton ϕ to a constant (as was done originally by Kaluza) implies by its equations of motion $F^{\mu\nu}F_{\mu\nu} = 0$. This is of course not consistent with a generic solution of the Yang-Mills equations for $\phi = \text{const}$. Thus the truncation of the dilaton is not consistent in the Kaluza-Klein sense, even though it is perfectly consistent by itself (just describing the usual Einstein-Maxwell system). In contrast it is usually assumed that the truncation of the massive modes is consistent. This can be seen directly for compactifications on tori. For them the zero-modes are simply characterized by the requirement of being independent of the internal coordinates. Therefore the field equations and symmetry variations in (4.8) do not mix zero-modes with massive modes, and the truncation is consistent. However, for compactifications on generic manifolds this is a highly non-trivial statement, and in general actually not true [82, 83]. In fact, an explicit proof requires several elaborate field redefinitions and a large number of miraculous identities have to be satisfied [84, 85].[8]

Let us now comment on the question whether truncations to a finite number of spin-2 fields might be possible. First of all, such a truncation is known not to be consistent in the strict Kaluza-Klein sense [82]. Nevertheless it has been proposed in the early literature on Kaluza-Klein theories that this truncation might still be consistent by itself, thus providing a circumvention of the no-go theorem also for the case of a finite number of spin-2 fields [92]. The situation has been clarified in [93] and can be rephrased by use of the analysis in 4.3.1 as follows. The truncation to a finite number of spin-2 fields containing, say, all fields with level $|n| \leq N$ for fixed N, is not consistent in the Kaluza-Klein sense since the corresponding subset of the Kac-Moody algebra is simply not a subalgebra. This is actually just a different manifestation of the fact that the naive truncation of the corresponding Chern-Simons theory would allow for solutions that cannot be lifted to solutions of the full theory. However, one might still hope to get a theory which is consistent by itself. This would be the case if and only if the subset resulting from the Kac-Moody algebra by setting all generators with $|n| > N$ to zero would result in a consistent Lie algebra (albeit not being a subalgebra). This turns out not to be the case. Rather one finds that the Jacobi identity is violated, e.g.

$$[[J_a^{(1)}, J_b^{(1)}], J_c^{(-1)}] + [[J_c^{(-1)}, J_a^{(1)}], J_b^{(1)}] + [[J_b^{(1)}, J_c^{(-1)}], J_a^{(1)}] = 2\eta_{c[a}J_{b]}^{(1)} \ . \quad (4.123)$$

Correspondingly, the resulting Chern-Simons theory would be inconsistent.[9] Moreover, (4.123) shows that these inconsistencies appear in the ungauged but also in the

[8] Such a proof has been done explicitly for the case of Kaluza-Klein reduction of 11-dimensional supergravity on $AdS_4 \times S^7$ in [86], based on results in [87]. See also [88, 89, 90, 91].

[9] Note that the truncation to, e.g., $n = 1$ does result in a consistent Lie algebra, on which, however, the quadratic form (4.15) degenerates. Instead there exists an alternative invariant form,

gauged phase, as can be seen from (4.11) and (4.83). In analogy to the given reasoning it has been argued in [93] that the consistency problems related to Kaluza-Klein truncations are basically the same as those related to 'higher-spin' couplings. Put differently, a 'higher-spin' theory resulting from Kaluza-Klein reduction represents a consistent Kaluza-Klein truncation if and only if it is consistent by itself.

Let us now discuss the similar problem for massive spin-3/2 couplings. In general the infinite tower of spin-3/2 fields appearing in Kaluza-Klein theories would require also a Kac-Moody-like extension of a superalgebra as symmetry group. The resulting field theory would than have an infinite number of supercharges ($\mathcal{N} = \infty$), among which all but finitely many are spontaneously broken.[10] Even though we will not analyze the structure of these superextensions in this thesis, we can already draw some conclusions from the theory constructed in sec. 3.6. There we have seen that Kaluza-Klein supergravity on $AdS_3 \times S^3 \times S^3$ consists of a tower of $\mathcal{N} = 8$ supermultiplets (in accordance with the supersymmetry that is preserved by the background), which contains two spin-3/2 multiplets. While the multiplets containing fields up to spin-1 are described by gauged $\mathcal{N} = 8$ supergravities, each of the two spin-3/2 multiplets required already $\mathcal{N} = 16$ supersymmetry. Thus the appropriate symmetry algebra will be some infinite-dimensional extension of the $\mathcal{N} = 8$ AdS superalgebra, which contains two $\mathcal{N} = 16$ superalgebras as consistent truncations. Since it is natural to assume that the above reasoning for spin-2 couplings also applies to spin-3/2 systems, we are lead to expect that a truncation is consistent in the Kaluza-Klein sense if it is consistent by itself. Since the $\mathcal{N} = 16$ theory defined in 3.6 is by construction consistent, this can already be interpreted as evidence that the corresponding truncation is consistent in the strict Kaluza-Klein sense. This in turn would mean that the infinite-dimensional extension of the $\mathcal{N} = 8$ algebra contains at least two $\mathcal{N} = 16$ superalgebras as (consistent) subalgebras.

whose Chern-Simons theory leads to the ghost-like gravity – spin-2 coupled system discussed in [46, 48].

[10] For instance, the topological subsector of this Kaluza-Klein supergravity could then presumably be constructed as a Chern-Simons theory for the corresponding super-Kac-Moody algebra.

Chapter 5

Applications for the AdS/CFT correspondence

5.1 The AdS/CFT dictionary

In the introduction we mentioned the AdS/CFT correspondence as a realization of the holographic principle and emphasized the importance of massive Kaluza-Klein modes. After discussing the construction of effective actions for massive Kaluza-Klein states, we are going to discuss potential applications for the AdS/CFT correspondence.

First of all we have to explain the AdS/CFT duality more precisely. Maldacena originally conjectured the correspondence by considering a number N of $D3$ branes, which are on the one hand described by a $U(N)$ Born-Infeld gauge theory, and on the other hand as a (solitonic) solution of supergravity. In the limit $\alpha' \to 0$ both descriptions leave free gravity in the bulk together with $\mathcal{N} = 4$ super-Yang-Mills theory in $3+1$ dimensions on the one side and type IIB supergravity on $AdS_5 \times S^5$ (as the near-horizon limit) on the other side [17, 18]. Then, Maldacena concluded, both theories have to be equivalent. However, what does it mean exactly that these theories are 'dual'? This has been clarified by Witten in [94] and by Gubser, Klebanov and Polyakov in [95], which we will briefly explain in the following.

Let us first try to understand in which sense a Minkowski space can be interpreted as the boundary of AdS. As in [94] we are going to discuss for simplicity reasons the duality in the case of euclidean AdS and Minkowski spaces. The $d+1$-dimensional AdS space can be defined as an open unit ball $\sum_{i=0}^{d} y_i^2 < 1$ in $\mathbb{R}^{d+1}$ (with coordinates $y_0, ..., y_d$) with metric

$$ds^2 = \frac{4}{(1-|y|^2)^2} \sum_{i=0}^{d} dy_i^2 \ . \tag{5.1}$$

The boundary of this space, namely the sphere S^d defined by $\sum_{i=0}^{d} y_i^2 = 1$, can in turn be viewed as the conformal compactification of euclidean d-dimensional Minkowski space (where a point at infinity has been added). However, the metric defined in (5.1)

does not induce a metric on the boundary, since it becomes singular for $|y| = 1$. In order to get a metric on the boundary which is related to (5.1) one may pick a function f on the closed unit ball (i.e. on the open ball together with its boundary), which has a first order zero on the boundary (as, e.g., $f(y) = 1 - |y|^2$), and then consider the metric $d\tilde{s}^2 = f^2 ds^2$. The latter extends to a well defined metric on the boundary. But, there exists no natural choice for the function f required for defining this metric, and therefore the latter is only well-defined up to conformal transformations. In fact, any rescaling of f will induce a conformal rescaling of the metric on the boundary. Thus, the given metric on AdS defines a conformal structure on the boundary.

We have seen in which way the boundary of euclidean AdS may be viewed as (the conformal compactification of) euclidean Minkowski space. And moreover we have argued that a given metric on AdS induces a conformal structure on the Minkowski space. This is in agreement with the claim that a gravity theory on AdS is dual to a conformal field theory on the boundary. Let us now examine the question how exactly two such theories might be related. As a first simple example we consider a free massless scalar field on AdS, i.e. a field ϕ obeying the Laplace equation $\nabla^\mu \nabla_\mu \phi = 0$.[1] It is a well-known fact that for a given function ϕ_0 on the sphere there exists a unique solution of the Laplace equation on the ball which reduces to ϕ_0 on the boundary. Thus in the case of a scalar field there exists a one-to-one correspondence with functions ϕ_0 on S^d – which will later be interpreted as sources in the CFT – and solutions of the Laplace equation on AdS. Similar results can be derived for gauge fields and also for the metric itself. For the latter the result is known as the Graham-Lee theorem. In analogy to the discussion above it states that any conformal structure on S^d is induced by a unique metric on AdS solving the Einstein equations with negative cosmological constant. In total we can conclude that for all massless fields appearing in a supergravity theory there exists a unique solution of their equations of motion obeying a given set of boundary conditions, which are in turn interpreted as the data of a conformal field theory.

Let us note that in accordance with this picture the symmetries on both sides of the duality match, since the conformal group in D dimensions, $SO(2, D)$, coincides with the AdS isometry group in $D + 1$ dimensions. However, the $D = 3$ case is exceptional as the AdS_3 group $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ at the boundary gets enhanced to two copies of the Virasoro algebra [96], corresponding to the infinite-dimensional conformal symmetry in two dimensions.

As a next step we have to understand how the dynamics on both sides of the correspondence might be related. As the boundary data ϕ_0 should be interpreted as sources in the CFT, there will be couplings to a conformal field/operator $\mathcal{O}$ of the form $\int_{S^d} \phi_0 \mathcal{O}$. The precise form of the correspondence, as developed in [94, 95], now claims that the correlation functions in the CFT are encoded in the supergravity

[1] The notion of mass is actually more subtle on AdS than recognized by this equation. For a more precise definition see the discussion of AdS representation in sec. 3.1.2.

action S via the relation[2]

$$\langle \exp \int_{S^d} \phi_0 \mathcal{O} \rangle_{\mathrm{CFT}} = \exp(-S(\phi)) \ . \tag{5.2}$$

Here the supergravity action is evaluated on those solutions of its equations of motion that obey the required boundary conditions.

So far we have discussed the AdS/CFT correspondence in case of massless fields in the supergravity theory. On the CFT side these fields correspond to sources ϕ_0 that have conformal dimension zero. This is simply due to the fact that the function f defining the conformal class mentioned above does not show up at all in the definition of ϕ_0. Since ϕ_0 has conformal dimension zero, it follows that the conformal dimension of the CFT field $\mathcal{O}$ is d.

Let us now turn to the case of a scalar field satisfying the massive Klein-Gordon equation $(\nabla^\mu \nabla_\mu + m^2)\phi = 0$. For the analysis of this equation it is convenient to choose a coordinate system in which we introduce z according to $|y| = \tanh(\frac{z}{2})$ as a new coordinate. The boundary at $|y| = 1$ then corresponds to $z \to \infty$. For large z the Klein-Gordon equation can be written as

$$\left(-e^{-dz} \frac{d}{dz} e^{dz} \frac{d}{dz} + m^2 \right) \phi = 0 \ . \tag{5.3}$$

Making the ansatz $\phi \sim e^{(\Delta - d)z}$ this equation reduces to

$$\Delta = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2} \ , \tag{5.4}$$

where the mass is given in units of the AdS length scale L_0. Therefore, in the massless case there are the two independent solutions $\phi \sim 1$ and $\phi \sim e^{-dz}$. The existence of the constant solution is in turn the reason for the existence of a unique solution satisfying the given boundary conditions defined by ϕ_0. (Roughly speaking, in an expansion of ϕ into harmonics each partial wave yields a given constant at infinity and adding these up one gets the unique solution of the Laplace equation [94].) In contrast, the massive case yields two independent solutions of the form $\phi \sim e^{\Delta_\pm z}$. Thus, one cannot find a solution of the massive equations of motion that approaches a given constant at infinity and so there exists no unique solution on AdS. As for the definition of the conformal structure on the boundary we may again take a function f that has a simple zero at the boundary. Then one can look for solutions of the equations of motion that behave instead like

$$\phi \sim f^{d-\Delta} \phi_0 \ . \tag{5.5}$$

(An obvious example would be $f \sim e^{-z}$ which has a zero for $z \to \infty$.) The definition of ϕ_0 in (5.5) depends on the choice of f. As we have seen above, a rescaling

[2]Of course, this equation has to be suitably regularized, since generically both sides are simply infinite [94]. See, e.g., [97, 98] for the notion of 'holographic renormalization'.

$f \to e^w f$ induces a conformal transformation on the boundary. Since ϕ cannot be affected by such a transformation, ϕ_0 has to transform at the same time according to $\phi_0 \to e^{w(\Delta-d)}\phi_0$. Thus we can conclude that the associated conformal operator $\mathcal{O}$ has conformal dimension Δ. In summary, massive fields in the supergravity theory correspond to fields in the CFT whose conformal dimension is given in terms of the mass as a real root of (5.4).[3]

So far we discussed the AdS/CFT correspondence in case of an unbroken conformal symmetry on the CFT side. However, if both theories are equivalent, the duality should also persist in case that certain scalars get a vev and break some part of the symmetry spontaneously [101]. Even more, a duality is expected to hold also in case of an explicit breaking of the conformal symmetry by the addition of mass terms [102]. Those deformations take the form

$$\mathcal{L}' = \mathcal{L}_{\mathrm{CFT}} + m\mathcal{O} \,, \tag{5.6}$$

where $\mathcal{O}$ generically denotes some operator in the CFT. Correspondingly, the conformal symmetry will be broken as well as some supercharges and possibly some part of the gauge group. Since the conformal transformations correspond to isometries in the AdS bulk, a breaking of the conformal symmetry will also lead to geometrical deformations of the AdS space. Those deformations are given by so-called domain wall solutions, which have a reduced isometry group [103]. As the dual field theory is no longer scale-invariant, RG flows are possible. The domain-wall solutions provide in turn an effective tool for the analysis of RG flows and thus of more realistic aspects of field theories.

Apart from that the focus has more recently turned to another type of generalized AdS/CFT duality. These are the so-called marginal deformations, which have been considered in an interesting work by Lunin and Maldacena [104]. These deformations preserve the conformal symmetry completely, but break some amount of supersymmetry and the gauge symmetry. Therefore the gravity duals are still AdS theories, but with a reduced internal symmetry.

In the following we are going to discuss those marginal deformations for the gravity dual on $AdS_3 \times S^3 \times S^3$. This is motivated by the fact that the dual conformal field theory is much less understood than, e.g., in the $AdS_3 \times S^3$ case. To begin with, let us briefly summarize where the different versions of the AdS_3/CFT_2 duality come from. The $AdS_3 \times S^3$ geometry is – instead of the system of D3 branes mentioned at the beginning of this section – realized as the near-horizon limit of a system of parallel D1 and D5 branes. The dual field theory is relatively well understood and given by a non-linear σ-model, whose target space is a symmetric product orbifold $\mathsf{Sym}^N(M_4)$. In contrast, the $AdS_3 \times S^3 \times S^3$ background arises as the near-horizon geometry of the so-called double D1-D5 system. Its dual field theory is required to realize a *large* $\mathcal{N} = 4$ superconformal algebra. The latter contains two instead of one affine $\widehat{SU(2)}$ subalgebras, corresponding to the isometries on the second S^3.

[3]Stability of supergravity on an Anti-de Sitter space does not require m^2 to be strictly positive, but just bounded from below, in accordance with the condition $m^2 \geq -d^2/4$ implied by the reality of (5.4). See [99, 100] and also the discussion in [101].

Surprisingly, this complicates the direct determination of the dual CFT. See [66, 105] for possible approaches to this problem. A better understanding of the gravity dual might be helpful and so we turn now to the discussion of marginal deformations in terms of the theories constructed in chapter 3.

In sec. 5.2 we discuss the effect of turning on some scalars in the spin-3/2 multiplet, which will further break half of the supercharges and some part of the gauge group. Similarly we discuss in sec. 5.3 a marginal deformation in the Yang-Mills multiplet, which will break the gauge group to its diagonal.

5.2 Marginal $\mathcal{N} = (4,0)$ deformations

In this section we will focus on marginal deformations in the spin-3/2 multiplet on $AdS_3 \times S^3 \times S^3$, whose effective Kaluza-Klein supergravity was constructed in chapter 3.

There we have computed the scalar potential for the gauge group singlets for the spin-3/2 multiplet $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})_S$ on $AdS_3 \times S^3 \times S^3$ and have moreover shown that the ground state with all scalars having vanishing vacuum expectation value breaks already half of the supersymmetry. In order to study deformations of the dual CFT, it would be interesting to analyze different ground states, which break further symmetries and where in addition some of the scalar fields get a vev. For this one has to compute and to minimize the scalar potential on these fields. In order to be sure that such a minimum of a truncated sector is also a ground state of the full theory, the truncation has to include all scalars that are singlets under a subgroup of the symmetry group [106].

Here this is the case, e.g., for the six scalars transforming as $(0,0,1,0) \oplus (0,0,0,1)$ (in addition to the two singlets we have already discussed), because these are singlets under $SO(4)_L$. Let us give a vev to all of these six scalar fields. Then in turn we can use a $SO(3)^+ \times SO(3)^-$ rotation to bring them into the form[4]

$$\langle \vec{\phi}_+ \rangle = \begin{pmatrix} \phi_3 \\ 0 \\ 0 \end{pmatrix}, \qquad \langle \vec{\phi}_- \rangle = \begin{pmatrix} \phi_6 \\ 0 \\ 0 \end{pmatrix}. \tag{5.7}$$

This ground state breaks spontaneously $SO(3) \times SO(3) \to SO(2) \times SO(2)$, which implies that four of the gauge fields will become massive in a Higgs effect. This in turn requires the existence of four scalar fields that can act as Goldstone bosons. For the choice (5.7) of the ground state these Goldstone bosons are given by the fluctuations around (5.7) in the 2/3 component, whereas the fluctuations around the 1-component describe the Higgs fields, i.e. we get in total two Higgs fields and four Goldstone bosons, exactly as required. Moreover, we expect the scalar potential not to depend on the Goldstone bosons, i.e. on the 2/3 component. But we could have

[4] We will show explicitly that even in the case of a non-linearly realized symmetry such a transformation always exists.

chosen each component as the Higgs field initially, while the other two would then act as Goldstone bosons, not entering the scalar potential. This in turn implies that the potential will also not depend on these scalars, in particular not on the Higgs field, which therefore stays massless. This one can also verify directly, by computing the T-tensor for $\vec{\phi}_+$ and $\vec{\phi}_-$ and proceeding as in sec. 3.3.4.

5.2.1 Non-linear realization of $SO(3)^+ \times SO(3)^-$

In the following we are going to analyze how the spontaneously broken $SO(3)^+ \times SO(3)^-$ symmetry is realized. For this we compute first the metric for the σ-model scalar manifold, which is spanned by the eight scalar fields that are singlets under $SO(4)_L$ – denoted by ϕ_1,...,ϕ_8. This can be done by computing the corresponding $E_{8(8)}$-valued group element $\mathcal{V}$ and extracting from the non-compact part P_μ^A of the current $\mathcal{V}^{-1}D_\mu\mathcal{V}$ the metric via

$$\frac{1}{2}g_{ij}(\phi)D^\mu\phi^i D_\mu\phi^j = \frac{1}{4}P^{\mu A}P_\mu^A\,, \qquad i,j=1,...,8\,. \tag{5.8}$$

One finds a metric of the form

$$ds^2 = 15\left(d\phi_1^2 + d\phi_2^2 + 2\cosh^2(\frac{\phi_1-\phi_2}{2})ds_6^2\right), \tag{5.9}$$

which is a warped product of a flat two-dimensional space with the six-dimensional space given by

$$\begin{aligned} ds_6^2 &= \cosh^2\phi_4\cosh^2\phi_5\cosh^2\phi_6\cosh^2\phi_7\cosh^2\phi_8 d\phi_3^2 \\ &+ \cosh^2\phi_5\cosh^2\phi_6\cosh^2\phi_7\cosh^2\phi_8 d\phi_4^2 \\ &+ \cosh^2\phi_6\cosh^2\phi_7\cosh^2\phi_8 d\phi_5^2 \\ &+ \cosh^2\phi_7\cosh^2\phi_8 d\phi_6^2 \\ &+ \cosh^2\phi_8 d\phi_7^2 \\ &+ d\phi_8^2\,. \end{aligned} \tag{5.10}$$

We are going to show that this space is nothing else than the coset space $SO(1,6)/SO(6)$ or in other words the euclidean six-dimensional Anti-de Sitter space. The latter can be defined as a hypersurface in a seven-dimensional Minkowski space spanned by $X_0,...,X_6$ and carrying the $SO(1,6)$ invariant metric

$$ds^2 = -dX_0^2 + dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2 + dX_5^2 + dX_6^2\,. \tag{5.11}$$

This six-dimensional hypersurface is given by

$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 = -1\,. \tag{5.12}$$

If we parametrize the surface (5.12) by the following coordinates

$$\begin{aligned}
X_0 &= \cosh\psi\cosh\omega\cosh t\cosh\phi\cosh\theta\cosh\chi\ ,\\
X_1 &= \cosh\psi\sinh\omega\cosh\phi\cosh\theta\cosh\chi\ ,\\
X_2 &= \sinh\psi\cosh\phi\cosh\theta\cosh\chi\ ,\\
X_3 &= \cosh\psi\cosh\omega\sinh t\cosh\phi\cosh\theta\cosh\chi\ ,\\
X_4 &= \sinh\phi\cosh\theta\cosh\chi\ ,\quad X_5 = \sinh\theta\cosh\chi\ ,\quad X_6 = \sinh\chi\ .
\end{aligned} \tag{5.13}$$

and identify $\phi_3 = t$, $\phi_4 = \omega$, $\phi_5 = \psi$, $\phi_6 = \phi$, $\phi_7 = \theta$ and $\phi_8 = \chi$, the metric which is induced by (5.11) on (5.12) is given exactly by (5.10).

This manifold contains two three-dimensional submanifolds, which correspond to the two triplets in $(0,0,1,0)\oplus(0,0,0,1)$. Namely, if we set $\phi = \theta = \chi = 0$, this defines a submanifold with metric

$$ds_3^2 = \cosh^2\phi_5\cosh^2\phi_4 d\phi_3^2 + \cosh^2\phi_5 d\phi_4^2 + d\phi_5^2\ . \tag{5.14}$$

The latter is also a coset space, namely $SO(1,3)/SO(3)$, which can be seen in complete analogy to the manifold considered above: If we define a three-dimensional hypersurface in a four-dimensional Minkowski space, which is parametrized by (5.13) with $\phi = \theta = \chi = 0$, the induced metric is given by (5.14). The same is true for the second triplet, i.e. the submanifold given by $t = \omega = \psi = 0$ is also $SO(1,3)/SO(3)$.

Now we can also examine how the $SO(3)^{\pm}$ acts as an isometry on the scalar fields.[5] It is sufficient to discuss the case of one triplet of scalar fields, i.e. the case where the coset space is reduced to $SO(1,3)/SO(3)$. Then the isometry group is clearly given by $SO(1,3)$ and its subgroup $SO(3)$ is characterized by the requirement that X_0 remains invariant. This implies in particular that the combination $\cosh\psi\cosh\omega\cosh t$ is invarariant under the non-linear action of $SO(3)$, as can be seen from (5.13).[6] Furthermore it is the only invariant, since it is equal to $X_1^2 + X_2^2 + X_3^2 + 1$ and apart from the latter there are no other independent $SO(3)$ invariants. Similarly we can derive the following: Making an $SO(3)$ transformation on $\vec{X} = (X_1, X_2, X_3)$ to bring it into the form $\vec{X} = (0, 0, |\vec{X}|)$ implies for the coordinates of $SO(1,3)/SO(3)$

$$t = \mathrm{Arsinh}\left(\frac{1}{\sqrt{60}}|\vec{X}|\right)\ ,\qquad \omega = 0\ ,\qquad \psi = 0\ . \tag{5.15}$$

This in turn implies that it is sufficient to evaluate all expressions only on one of the scalar fields, say $\phi_3 = t$, because by use of the $SO(3)$ symmetry the others can be set to zero (as one would expect for Goldstone bosons). Afterwards all expressions containing only t can be 'covariantized' by use of the rule

$$t \longrightarrow \mathrm{Arsinh}\sqrt{\cosh^2\psi\cosh^2\omega\cosh^2 t - 1}\ . \tag{5.16}$$

In the following we will therefore evaluate all expressions only for the case $\phi_4 = \phi_5 = 0$ and similarly for the second triplet.

[5] Note, that these scalar fields do not build a linear representation of $SO(3)$.

[6] One may also check explicitly that the action of the non-linear $SO(3)$ coset space symmetries explained in sec. 2.2 leaves this combination invariant.

5.2.2 Resulting $\mathcal{N} = (4,0)$ spectrum

That the $SO(4)_R$ symmetry is broken implies that also some amount of the supersymmetry of the right factor of the supergroup will be broken, because $\mathcal{N} = 4$ supersymmetry in the AdS background is not consistent without the required internal symmetries. To determine the amount of unbroken supersymmetry, we use that the number of solutions of $\langle \delta\psi_\mu \rangle = 0$ in the Anti-de Sitter background is given by the number of eigenvalues of A_1 that are equal to $\pm\frac{1}{2}$. One finds

$$\begin{array}{rl} \frac{3}{2} + 2h_+ & (\#2) \\ \frac{3}{2} + 2h_- & (\#2) \\ -\frac{1}{2} - 2h_+ & (\#2) \\ -\frac{1}{2} - 2h_- & (\#2) \\ -\frac{3}{2} & (\#4) \\ \frac{1}{2} & (\#4)\,, \end{array} \tag{5.17}$$

with

$$h_\pm \;=\; -\frac{1}{2} + \frac{1}{2\sqrt{1+\alpha^2}} \Big[\cosh^2\hat{\phi}_3 + \alpha^2 \cosh^2\phi_6 \pm 2\alpha \sinh\hat{\phi}_3 \sinh\phi_6 \Big]^{\frac{1}{2}}\,. \tag{5.18}$$

We have also made a field redefinition given by

$$\sinh\hat{\phi}_3 = \sinh\phi_3 \cosh\phi_6\,. \tag{5.19}$$

This implies that all supercharges of the right factor are spontaneously broken, i.e. the whole supergroup is broken to

$$D^1(2,1|\alpha)_L \times D^1(2,1|\alpha)_R \to D^1(2,1|\alpha)_L \times SL(2,\mathbb{R})_R \times SO(2)^+_R \times SO(2)^-_R\,, \tag{5.20}$$

and the theory has a residual $\mathcal{N} = (4,0)$ supersymmetry.

We will now turn to the question, how the supermultiplets rearrange under the reduced symmetry. So far everything has been expressed in terms of short multiplets of $D^1(2,1|\alpha)$. But due to the fact that some additional fields get massive we also have to expect the appearance of long multiplets. The general structure of these multiplets in terms of their highest-weight states h_0 and $h_0 = \frac{1}{2}$ is given in tab. 5.1 Let us

h	(ℓ^+,ℓ^-)
h_0	$(0,0)$
$h_0+\frac{1}{2}$	$(\frac{1}{2},\frac{1}{2})$
h_0+1	$(1,0)\oplus(0,1)$
$h_0+\frac{3}{2}$	$(\frac{1}{2},\frac{1}{2})$
h_0+2	$(0,0)$

h	(ℓ^+,ℓ^-)
$\frac{1}{2}$	$(\frac{1}{2},\frac{1}{2})$
1	$(1,0)\oplus(0,1)\oplus(0,0)$
$\frac{3}{2}$	$(\frac{1}{2},\frac{1}{2})$
2	$(0,0)$

Table 5.1: Long and short $\mathcal{N} = (4,0)$ multiplets

denote them by $[h_0]_l$ and $[0]_s$, respectively. Due to the fact that no supersymmetry of the right factor survives, the $16^2 = 256$ degrees of freedom will assemble into 16 supermultiplets, each of them containing 16 degrees of freedom. The precise form of the resulting $\mathcal{N} = (4,0)$ multiplets, in particular the shifted values for the conformal dimensions, can be extracted from the mass spectrum of the spin-3/2 and vector fields. The former we have already computed in (5.17), while the latter can be extracted from the eigenvalues of the non-compact part of the T-tensor. Altogether one finds that the spin-$\frac{3}{2}$ multiplet $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})_S$ decomposes under $\mathcal{N} = (4,4) \to \mathcal{N} = (4,0)$ into

$$\begin{aligned}(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})_S \to\ & 2[h^+]_l \otimes (h^+ + \tfrac{1}{2}) \ \oplus\ 2[h^-]_l \otimes (h^- + \tfrac{1}{2}) \\ & \oplus\ 2[h^+]_l \otimes (h^+ + \tfrac{3}{2}) \ \oplus\ 2[h^-]_l \otimes (h^- + \tfrac{3}{2}) \\ & \oplus\ 2[h^3]_l \otimes (h^3 + 1) \ \oplus\ 2[h^4]_l \otimes (h^4 + 1) \\ & \oplus\ [0]_s \otimes (2) \ \oplus\ 3[0]_s \otimes (1)\ ,\end{aligned} \tag{5.21}$$

where each supermultiplet of $D^1(2,1|\alpha)_L$ is tensored with a representation of $SL(2,\mathbb{R})$, characterized by its conformal dimension h. Here the multiplets in the first two lines are massive spin-3/2 multiplets, containing one gravitino, 4 vectors, 7 spin-1/2 fermions and 4 scalars. The multiplets in the third and fourth line are massive spin-1 multiplets, containing 2 vectors, 8 spin-1/2 fermions and 6 scalars. Finally, the multiplets in the last line are a massive spin-3/2 multiplet, containing four gravitinos, 7 vectors, 4 spin-1/2 fermions and one scalar, and three massive spin-1 multiplets, each of them containing one vector, 8 spin-1/2 fermions and 7 scalars. Altogether the field content is given apart from the supergravity multiplet by 12 massive spin-3/2 fields, 50 massive vectors, 116 spin-1/2 fermions and 78 scalars. This is exactly the field content expected from the original multiplet $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})_S$, if one takes into account that 4 scalars are eaten by some vectors, whereas 4 fermions get eaten by the gravitinos. Explicitly, the conformal dimensions are given by (5.18) and moreover by

$$\begin{aligned}h^3 &= -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 16\sinh^2(\hat{\phi}_3)\frac{\alpha^2}{1+\alpha^2}}\ , \\ h^4 &= -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 16\sinh^2(\phi_6)\frac{1}{1+\alpha^2}}\ ,\end{aligned} \tag{5.22}$$

With the explicit form of the $\mathcal{N} = (4,0)$ supermultiplets at hand we can also check another symmetry, namely the discrete symmetry which interchanges the two spheres $S^{\pm}$, or in other words which exchanges the two scalar field triplets. For the ratio α of the two spheres radii this symmetry acts as $\alpha \to 1/\alpha$, and we have to check whether the spectrum reflects this symmetry – as it should, since $D^1(2,1|\alpha) \cong D^1(2,1|\frac{1}{\alpha})$. Under this symmetry one has

$$\frac{1}{1+\alpha^2} \to \frac{\alpha^2}{1+\alpha^2}\ , \qquad \frac{2\alpha}{1+\alpha^2} \to \frac{2\alpha}{1+\alpha^2}\ . \tag{5.23}$$

We can therefore conclude that this discrete symmetry acts on the scalar fields as

$$\phi_6 \to -\hat{\phi}_3\ , \quad \hat{\phi}_3 \to \phi_6\ , \qquad \text{or} \qquad \sinh\phi_6 \to -\sinh\phi_3\cosh\phi_6\ , \tag{5.24}$$

because then the conformal dimensions transform into each other,

$$h^+ \to h^-, \qquad h^3 \to h^4, \tag{5.25}$$

such that the spectrum is invariant. Moreover, it is also possible to describe this symmetry in terms of the $SO(6) \subset SO(1,6)$ isometries, which are realized on the full coset space $SO(1,6)/SO(6)$. Namely, for the 'embedding coordinates' X_1, ..., X_6 the transformation (5.24) implies

$$X_1^2 + X_2^3 + X_3^2 = X_4'^2 + X_5'^2 + X_6'^3 \,, \tag{5.26}$$

where the primed coordinates correspond to the transformed scalar fields. After fixing the $SO(3)^+ \times SO(3)^-$ symmetry – which we have done by rotating everything into the ϕ_3, ϕ_6 - direction – this specifies the transformation which rotates both vectors X^+ and X^- into each other, or to be more precise, $X_3 \to X_4$ and $X_4 \to -X_3$.

5.2.3 Lifting the deformation to $D = 10$

Finally we will discuss the question, whether the considered deformation corresponds to a higher-dimensional geometry, or in other words whether the deformed theory can also be obtained from a Kaluza-Klein reduction on a deformation of $AdS_3 \times S^3 \times S^3$. Since, as we argued in sec. 4.6, there is some evidence that the spin-3/2 multiplet describes a consistent truncation, such a 10-dimensional solution has to exist. Constructing this solution would actually yield further evidence for the consistency of the truncation.

The deformed background geometry can be identified through its isometries. Namely, due to the fact that the gauge group is broken from $SO(4)_L \times SO(3)_R^+ \times SO(3)_R^-$ to $SO(4)_L \times SO(2)_R^+ \times SO(2)_R^-$, the isometry group will be reduced similarly. Put differently, both 3-spheres will be replaced by a three-dimensional manifold whose isometry group is $SO(3) \times SO(2)$ and which is a smooth deformation of a S^3. Such a geometry indeed exists and is given by a so-called 'squashed' sphere [82, 107, 108]. If the latter has radius R its line element is given by

$$\begin{aligned} ds^2 &= \frac{R^2}{4}\left[\sigma_1^2 + \sigma_2^2 + \frac{1}{1+q}\sigma_3^2\right] \\ &= \frac{R^2}{4}\left[d\theta^2 + \sin^2\theta d\phi^2 + \frac{1}{1+q}\left(d\psi^2 + 2\cos\theta d\psi d\phi + \cos^2\theta d\phi^2\right)\right], \end{aligned} \tag{5.27}$$

where σ_i are the left-invariant one forms on S^3 and $q \in (-1, \infty)$ is the 'squashing' parameter. This geometry is by definition invariant under the left $SO(3)_L$, but breaks the right $SO(3)_R$ to $SO(2)_R$ for $q \neq 0$. Thus the isometry group is $SO(3)_L \times SO(2)_R$ for both spheres and the case $q = 0$ corresponds to a undeformed 3-sphere. This manifold is topologically still a 3-sphere, but with a squashed S^1 fibration over S^2. In order to show that this geometry arises as a solution of type IIB supergravity, one would have to define a similar deformation of the 3-form flux, which gives rise to (5.27) via the Einstein equations. We will leave this for future work.

5.3 Marginal $\mathcal{N} = (3,3)$ deformations

In this section we are going to discuss marginal deformations, which result in a partial supersymmetry breaking $\mathcal{N} = (4,4) \to \mathcal{N} = (3,3)$.[7] These explorations are motivated by a system of intersecting D-branes, which has been considered in [66]. More specifically, the original $AdS_3 \times S^3 \times S^3$ background is the near horizon limit of a double D1/D5 system, containing in particular a D5/D5' system of branes. In case that the number of D5 and D5' branes are equal, $Q_5^+ = Q_5^-$, such a configuration can be deformed via joining the D5- and D5'-branes into a single set of Q_5 D5-branes along a 4-manifold. This manifold is in turn characterized by a parameter ρ, of which we therefore may think as a deformation parameter, describing the deformation away from the original background. The deformed system breaks the $SO(4) \times SO(4)$ to a diagonal subgroup [66].

In the effective supergravities such a deformation corresponds to giving a vev to certain scalars in such a way that the gauge symmetry gets spontaneously broken (i.e. here to the diagonal subgroup). If the required scalars are contained in one of the lowest multiplets of chapter 3, this spontaneous symmetry breaking should be visible within one of the effective supergravities discussed there. In the following we are going to argue that the deformation considered in [66] can indeed by seen in one of the YM multiplets.

5.3.1 Deformations in the Yang-Mills multiplet

We have to identify a deformation in the YM multiplets that breaks the gauge group $SO(4) \times SO(4)$ to a diagonal subgroup, breaking the supersymmetry at the same time as $\mathcal{N} = (4,4) \to \mathcal{N} = (3,3)$. More specifically, we consider a breaking of the gauge group $SO(4)_L \times SO(4)_R$ to $SO(3)_L^{(D)} \times SO(3)_R^{(D)}$, where $SO(3)_L^{(D)}$ and $SO(3)_R^{(D)}$ represent the diagonal of the two factors in $SO(4)_L$ and $SO(4)_R$, respectively. Moreover, we set $\alpha = 1$ in the following.

One sees from tab. 3.5 that under this subgroup each YM multiplet contains two scalar singlets, i.e. we have a four-dimensional manifold of scalars invariant under $SO(3)_L^{(D)} \times SO(3)_R^{(D)}$. At the origin, these scalars come in two pairs with square masses 0 and 3, i.e. they correspond to operators of conformal dimensions 2 and 3. In particular, there are two marginal operators.

Let us now consider the truncation of the Lagrangian on this four-dimensional target space manifold. The effective action was given by a $\mathcal{N} = 8$ supergravity, and the scalar fields take values in the coset space $SO(8,8)/SO(8) \times SO(8)$. Accordingly we can parametrize them by a $SO(8,8)$ matrix $\mathcal{S}$ as

$$\mathcal{S} \;=\; \exp \begin{pmatrix} 0 & 0 & v_1 & w_2 \\ 0 & 0 & w_1 & v_2 \\ v_1 & w_1 & 0 & 0 \\ w_2 & v_2 & 0 & 0 \end{pmatrix} , \tag{5.28}$$

[7] This section is based on work done with Marcus Berg and Henning Samtleben [109].

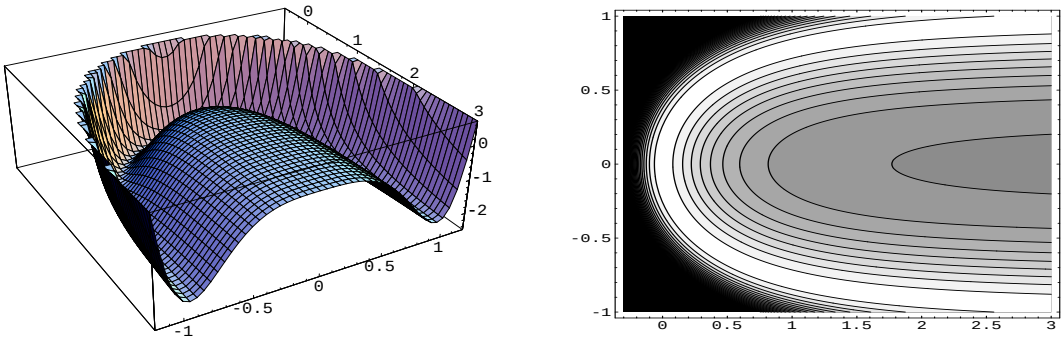


Figure 5.1: Flat valley in the potential, the $\mathcal{N} = (4,4)$ origin is located at $(0,0)$. Coordinates here are $x = \sin\phi \sinh z\,, y = \cos\phi \sinh z$.

where each entry represents a multiple of the 4×4 unit matrix. In particular, the two gauge group singlets are parametrized by v_1, v_2, while the truncation to a single YM multiplet is given by $w_1 = w_2 = 0$. In the following, we further truncate to the two-dimensional subspace defined by $v_1 = v_2$, $w_1 = -w_2$. Similar to the analysis above, Lagrangian and scalar potential can now be computed. In terms of the new variables

$$z^2 = v_1^2 + w_1^2\,, \quad \phi = \arctan(w_1/v_1)\,, \tag{5.29}$$

the Lagrangian reads

$$e^{-1}\mathcal{L} = \partial_\mu z\,\partial^\mu z + \sinh^2 z\,\partial_\mu\phi\,\partial^\mu\phi - V\,, \tag{5.30}$$

with the scalar potential

$$V = -2 + 8\sinh^2 z\,(\sinh z - \cos\phi\cosh z)^2\,(1 + 2\cosh 2z - 2\cos\phi\sinh 2z)\,. \tag{5.31}$$

This scalar potential is bounded from below by $V \geq -2$ and takes this value along the curve

$$z = \operatorname{arctanh}(\cos\phi)\,, \tag{5.32}$$

which thus constitutes a flat direction in the potential, see Figure 5.1. Explicit computation shows that this extends to a flat direction in the four-dimensional target space (5.28) and thus of the full scalar potential. This deformation breaks the gauge group down to $SO(3)^{(D)}_L \times SO(3)^{(D)}_R$.

5.3.2 Resulting $\mathcal{N} = (3,3)$ spectrum

Let us now turn to the analysis of the residual supersymmetries. As above they can be extracted from the gravitino mass spectrum. This is determined by the eigenvalues of the tensor A_1, for which one finds

$$m_i = \pm\tfrac{1}{2} \quad (\#3)\,, \qquad m_i = \pm\tfrac{1}{2}\sqrt{8\cosh 2z - 7} \quad (\#1)\,. \tag{5.33}$$

Supersymmetry is thus broken from $\mathcal{N}=(4,4)$ down to $\mathcal{N}=(3,3)$. For the corresponding conformal dimensions $\Delta=\frac{1}{2}+|m|$ this yields

$$\Delta_{\text{gravitino}} = \{\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2},1+\tfrac{1}{2}\sqrt{8\cosh 2z-7},1+\tfrac{1}{2}\sqrt{8\cosh 2z-7}\}\,. \quad (5.34)$$

In order to determine the reorganization into $\mathcal{N}=(3,3)$ supermultiplets, we also have to compute some bosonic masses. In the present case it is actually possible to diagonalize the full scalar potential around the deformation to get the scalar masses. One finds

$$m_i^2 = \begin{cases} -1 & (\#9) \\ 0 & (\#34) \\ 3 & (\#1) \\ \frac{1}{2}\left(-1+4\cosh 2z-3\sqrt{8\cosh 2z-7}\right) & (\#1) \\ \frac{1}{2}\left(-5+4\cosh 2z-\sqrt{8\cosh 2z-7}\right) & (\#9) \\ \frac{1}{2}\left(-5+4\cosh 2z+\sqrt{8\cosh 2z-7}\right) & (\#9) \\ \frac{1}{2}\left(-1+4\cosh 2z-3\sqrt{8\cosh 2z-7}\right) & (\#1) \end{cases} \quad (5.35)$$

For the associated conformal dimensions $\Delta=1+\sqrt{1+m^2}$ this implies

$$\Delta_i = \begin{cases} 1 & (\#9) \\ 2 & (\#34) \\ 3 & (\#1) \\ \frac{1}{2}\left(-1+\sqrt{8\cosh 2z-7}\right) & (\#1) \\ \frac{1}{2}\left(1+\sqrt{8\cosh 2z-7}\right) & (\#9) \\ \frac{1}{2}\left(3+\sqrt{8\cosh 2z-7}\right) & (\#9) \\ \frac{1}{2}\left(5+\sqrt{8\cosh 2z-7}\right) & (\#1) \end{cases} \quad (5.36)$$

From these values we can infer the entire spectrum in terms of $\mathcal{N}=(3,3)$ supermultiplets. The spectrum is organized under the supergroup $OSp(3|2,\mathbb{R})_{\text{L}}\otimes OSp(3|2,\mathbb{R})_{\text{R}}$, whose supermultiplets we have to shortly review in the following.

A short $OSp(3|2,\mathbb{R})$ supermultiplet is defined by its highest weight state $(\ell)^{h_0}$, where ℓ labels the $SO(3)$ spin and $h_0=\ell/2$ is the charge under the Cartan subgroup $SO(1,1)\subset SL(2,\mathbb{R})$. The corresponding supermultiplet, which we will denote by $(\ell)_{\text{S}}$, is generated from the highest weight state by the action of two out of the three supercharges and carries 8ℓ degrees of freedom. Its $SO(3)^{\pm}$ representation content is summarized in Table 5.2.

The generic long multiplet $(\ell)_{\text{long}}$ instead is built from the action of all three supercharges on the highest weight state and correspondingly carries $8(2\ell+1)$ degrees

h	
h_0	(ℓ)
$h_0+\frac{1}{2}$	$(\ell)+(\ell-1)$
h_0+1	$(\ell-1)$

Table 5.2: The generic short supermultiplet $(\ell)_{\text{S}}$ of $OSp(3|2,\mathbb{R})$, with $h_0=\ell/2$.

h_L \ h_R	$\frac{1}{2}$	1	$\frac{3}{2}$
$\frac{1}{2}$	$(1;1)$	$(1;0)+(1;1)$	$(1;0)$
1	$(0;1)+(1;1)$	$(0;0)+(0;1)+(1;0)+(1;1)$	$(0;0)+(1;0)$
$\frac{3}{2}$	$(0;1)$	$(0;0)+(0;1)$	$(0;0)$

Table 5.3: The short $\mathcal{N}=(3,3)$ multiplet $(1;1)_{\rm S}$.

h_L \ h_R	h	$h+\frac{1}{2}$	$h+1$	$h+\frac{3}{2}$
h	$(0;0)$	$(0;1)$	$(0;1)$	$(0;0)$
$h+\frac{1}{2}$	$(1;0)$	$(1;1)$	$(1;1)$	$(1;0)$
$h+1$	$(1;0)$	$(1;1)$	$(1;1)$	$(1;0)$
$h+\frac{3}{2}$	$(0;0)$	$(0;1)$	$(0;1)$	$(0;0)$

Table 5.4: The long $\mathcal{N}=(3,3)$ multiplet $(0;0)_{\rm long}$.

of freedom. Its highest weight state satisfies the unitarity bound $h \geq \ell/2$. In case this bound is saturated, the long multiplet decomposes into two short multiplets (5.2) according to

$$(\ell)_{\rm long} \;=\; (\ell)_{\rm S} \oplus (\ell{+}1)_{\rm S}\;. \tag{5.37}$$

A semishort $\mathcal{N}=4$ multiplet $(\ell^+,\ell^-)_{\rm S}$ breaks according to

$$(\ell^+,\ell^-)_{\rm S} \;=\; (\ell^+{+}\,\ell^-)_{\rm S} + (\ell^+{+}\,\ell^-{-}1)_{\rm long} + \;\ldots\; + (|\ell^+{-}\,\ell^-|)_{\rm long}\;, \tag{5.38}$$

into semishort and long $\mathcal{N}=3$ multiplets.

The $\mathcal{N}=(3,3)$ spectrum can now be summarized as follows. Let us first note, that the two YM-multiplets $(0,1;0,1)_{\rm S}$ and $(1,0;1,0)_{\rm S}$ reduce to the same short $\mathcal{N}=(3,3)$ supermultiplet $(1;1)_{\rm S}$, given in table 5.3. Comparing the conformal dimensions (5.36) to table 5.3 we observe that along the deformation a linear combination of the two YM multiplets remains in the short multiplet $(1;1)_{\rm S}$, whereas the other fields combine into the long massive $\mathcal{N}=(3,3)$ supermultiplet $(0;0)_{\rm long}$ with $h=\frac{1}{4}\,(-1+\sqrt{8\cosh 2z-7})$ summarized in table 5.4. In total, the two $\mathcal{N}=(4,4)$ YM multiplets decompose into $\mathcal{N}=(3,3)$ multiplets according to

$$(0,1;0,1)_{\rm S} \;\oplus\; (1,0;1,0)_{\rm S} \longrightarrow (1,1)_{\rm S} \;\oplus\; (0,0)_{\rm long}\;. \tag{5.39}$$

As the deformation is switched off, we find the $\mathcal{N}=(3,3)$ multiplet shortening:

$$(0;0)_{\rm long} \;\to\; (0;0)_{\rm S} + (1;0)_{\rm S} + (0;1)_{\rm S} + (1;1)_{\rm S}\;, \tag{5.40}$$

where $(0;0)_{\mathrm{S}}$, $(1;0)_{\mathrm{S}}$, and $(0;1)_{\mathrm{S}}$ denote unphysical multiplets without propagating degrees of freedom, e.g.:

h_L \ h_R	0	$\frac{1}{2}$	1
0	$(0;0)$		$-(0;0)$
$\frac{1}{2}$			
1	$-(0;0)$		$(0;0)$

h_L \ h_R	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$(1;0)$		$-(1;0)$
1	$(0;0)+(1;0)$		$-(0;0)-(1;0)$
$\frac{3}{2}$	$(0;0)$		$-(0;0)$

Negative states have to be interpreted here as in sec. 3.1.2. For further details see [109].

Chapter 6

Outlook and Discussion

In this thesis we analyzed massive states in Kaluza-Klein theories through their spontaneously broken symmetries. We focused in particular on the local 'higher-spin' symmetries which are required for the consistency of the spin-3/2 and spin-2 couplings. For Kaluza-Klein supergravity on $AdS_3 \times S^3 \times S^3 \times S^1$ we discussed the effective theories for the lowest (spin-1/2 and spin-1) supermultiplets and for a spin-3/2 multiplet. While the former can be described as gauged $\mathcal{N} = 8$ supergravities – in accordance with the amount of supersymmetry that is preserved by the background – the latter requires an enhancement of supersymmetry to $\mathcal{N} = 16$. We constructed this theory as a new example of a gauged maximally-supersymmetric AdS_3 supergravity. It was shown that this theory does not possess a phase, where all supercharges are unbroken. Rather we found that the vacuum at the origin of the scalar potential in fig. 3.1 breaks already half of the supersymmetry, giving rise to eight massive spin-3/2 fields via a super-Higgs mechanism. Thus we confirmed the general expectation of sec. 2.1 that the massive spin-3/2 states appearing in Kaluza-Klein supergravity have to be accompanied by spontaneously broken supersymmetries. However, the puzzle remains how an infinite tower of spin-3/2 fields could be coupled consistently to gravity. According to the given reasoning this needs an infinite number of supercharges, which naively would then lead to states of arbitrary high spin, in conflict with the fact that Kaluza-Klein supergravities contain only fields up to spin 2. The constructed theory suggests, however, the following resolution. The aforementioned no-go theorem actually relies on the assumption that the theory admits a phase, where the entire supersymmetry is unbroken, such that the field content can be organized into representations of the required superalgebra. If this theory, instead, does not possess such a phase, the no-go theorem simply does not apply.

The analogous problems for spin-2 states were discussed in chapter 4, where we focused on Kaluza-Klein compactification of four-dimensional gravity on an S^1. There we showed that the unbroken phase, in which the spin-2 fields are massless, corresponds to the 'decompactification limit', i.e. the phase where the radius of the internal manifold goes to infinity. Correspondingly, there is an infinite-dimensional gauge symmetry, which ensured consistency of the gravity – spin-2 couplings. The resulting three-dimensional theory has been constructed as a Chern-Simons theory based on

the affine Poincaré algebra. We have moreover shown the existence of a geometrical interpretation for these spin-2 theories, which can be applied to arbitrary dimensions. This is an extension of the Riemannian geometry underlying ordinary general relativity to a notion of algebra-valued differential geometry. While the Chern-Simons formulation is special to $D = 3$, the latter geometrical formulation exists in any dimension. Correspondingly, we show in appendix D that the Chern-Simons theory related to an arbitrary internal manifold is equivalent to the generalized Einstein-Hilbert action based on algebra-valued differential geometry. Finally we discussed the 'broken phase', in which the spin-2 fields are supposed to become massive via a novel spin-2-Higgs mechanism. It has been shown that this phase results from a gauging of a subgroup of the rigid symmetry group, which in turn deforms the spin-2 symmetries and induces a spin-2 mass term. The entire construction had a deep analogy to the gauging of supergravity: First of all, the spin-2 symmetries in (4.8) are 'supersymmetries' in the sense that they transform fields of different spin into each other, even though they do this in a purely bosonic theory. Like in supergravity (see the discussion in 2.1) here the consistency relies on the fact that also the metric transforms under the spin-2 symmetries. Concerning the gauging we observed that it is possible to start in the ungauged phase from a formulation, where all degrees of freedom are carried by scalars. These scalars span in turn a (generalized) non-linear σ-model manifold carrying an enhanced global symmetry, in this case the affine extension of the Ehlers group. The gauge vectors in turn appeared as purely topological gauge fields and combined with the spin-2 fields and the metric into a Chern-Simons theory for an extended algebra. This parallels the case of gauged supergravities, where the compact gauge vectors combine with the metric and the gravitinos into one of the Chern-Simons theories based on AdS supergroups discussed in sec. 2.2.[1] Moreover, we argued in sec. 4.5 that generically the gauging is only consistent with the spin-2 symmetries if a scalar potential is introduced, which also parallels the gauging of supergravity.

In chapter 5 we considered marginal deformations of the theories constructed in chapter 3. We analyzed the effects of switching on certain scalar fields in the spin-3/2 multiplet $(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2})_S$, which results in a spontaneous partial supersymmetry breaking $\mathcal{N} = (4, 4) \to \mathcal{N} = (4, 0)$ as well as a breaking of part of the gauge group. We have moreover determined the geometry of the moduli space spanned by the scalars participating in this symmetry breaking. Similarly we examined a marginal deformation in the spin-1 multiplets, which implied supersymmetry breaking to $\mathcal{N} = (3, 3)$. The required Higgs and super-Higgs effects were analyzed as well as the resulting spectrum and the reorganization into representations of the reduced AdS supergroups. These explorations provide the first step of a more detailed analysis of the dual field theories.

The presented work can be extended into various directions. First of all it would be interesting to find the 10-dimensional ancestors for the marginal deformations

[1] Similarly, in [39] consistent couplings of an infinite tower of higher-spin fields to gravity have been constructed as a Chern-Simons theory of a higher-spin algebra. The Chern-Simons theory of sec. 4.4.1 provides the analogue for an infinite tower of spin-2 fields.

discussed in chapter 5. We introduced already a possible 10-dimensional geometry in sec. 5.2.3, which would give rise to the required breaking of the gauge group. It remains to be checked that this metric can be embedded into an exact solution of type IIB supergravity. The analogous question for the $\mathcal{N} = (3,3)$ deformation will be discussed in a forthcoming paper [109].

Furthermore, there are a variety of directions, in which the investigation of the spin-2 symmetries of chapter 4 can be extended. First, the analysis of the spin-2 symmetries in the gauged phase has only been performed for the pure Chern-Simons theory (in which the Kaluza-Klein symmetries are realized as Yang-Mills gauge transformations) and a particular subsector of the matter-coupled theory. In order to apply this program to more complicated internal manifolds, a systematic analysis of possible gaugings and the resulting deformations of the spin-2 symmetries is necessary. In addition, it would be interesting to study these questions also for higher-dimensional geometries. A first step for this can be found in appendix D.

Moreover, since we are ultimately interested in compactifications on AdS spaces, the Kac-Moody algebra based on the Poincaré group should be generalized such that it contains the affine extension of the AdS group. However, as $AdS_3 \times S^1$ is not a solution of four-dimensional AdS-gravity, this enforces already the introduction of a non-flat compact manifold. For instance, one could analyze the $AdS_3 \times S^3$ case, in which the diffeomorphism group of S^3 should appear as a gauge group. One may expect the appearance of an extended algebra of the form (4.83) and an associated Chern-Simons description.

Finally, it would be crucial to study the supersymmetric case. We argued already in sec. 4.6 that this could presumably be achieved via the introduction of a super-Kac-Moody algebra, whose Chern-Simons theory would possess an infinite number of supersymmetries ($\mathcal{N} = \infty$). Applied to the case of, e.g., $AdS_3 \times S^3$, this algebra should carry an AdS supergroup as subgroup. Apart from the finite number of supersymmetries preserved by the background, all supercharges will be broken spontaneously.

In analogy to the spin-2 case one could think that any truncation to a finite number of spin-3/2 fields larger than expected from the background supergroup (corresponding to the 'zero-modes') is inconsistent. However, the theory constructed in chapter 3 shows already that this is not true in general. As discussed in 4.6, the infinite-dimensional symmetry underlying the entire Kaluza-Klein tower on $AdS_3 \times S^3 \times S^3$ has to contain not only the $\mathcal{N} = 8$ supergroup determined by the background, but also $\mathcal{N} = 16$ algebras associated to the additional massive spin-3/2 fields. In contrast to compactifications on tori, the existence of consistent subalgebras larger than the symmetry group of the zero-modes can therefore not be excluded a priori. It is amusing to speculate about the possibility that consistent subalgebras may exist which have even more than 32 real supercharges. For instance, the next massive spin-3/2 multiplet on $AdS_3 \times S^3 \times S^3$, namely $(1,1;1,1)_S$ in (3.29), contains 72 massive spin-3/2 fields, thus requiring $\mathcal{N} = 80$ supersymmetry! Even though this sounds unlikely at first sight, there seems to be no convincing argument excluding the existence of such a truncation.

Chapter 7

Appendices

A The different faces of $E_{8(8)}$

The maximal supergravity theories in three dimensions are organized under the exceptional group $E_{8(8)}$. In particular, their scalar sector is given by a coset space σ model with target space $E_{8(8)}/SO(16)$. In this appendix, we describe the Lie algebra $\mathfrak{e}_{8(8)}$ in different decompositions relevant for the embedding of the gauge group and for the construction of the embedding tensor in the main text.

A.1 $E_{8(8)}$ in the $SO(16)$ basis

The 248-dimensional Lie algebra of $E_{8(8)}$ may be characterized starting from its 120-dimensional maximal compact subalgebra $\mathfrak{so}(16)$, spanned by generators $X^{IJ} = X^{[IJ]}$ with commutators

$$[X^{IJ}, X^{KL}] \;=\; \delta^{JK} X^{IL} - \delta^{IK} X^{JL} - \delta^{JL} X^{IK} + \delta^{IL} X^{JK} \,, \tag{A.1}$$

where $I, J = 1, \ldots, 16$ denote $SO(16)$ vector indices. The 128-dimensional non-compact part of $\mathfrak{e}_{8(8)}$ is spanned by generators Y^A which transform in the fundamental spinorial representation of $SO(16)$, i.e. which satisfy commutators

$$[X^{IJ}, Y^A] \;=\; -\tfrac{1}{2}\,\Gamma^{IJ}_{AB}\, Y^B \,, \qquad [Y^A, Y^B] \;=\; \tfrac{1}{4}\,\Gamma^{IJ}_{AB}\, X^{IJ} \,. \tag{A.2}$$

Here $A, B = 1, \ldots, 128$ label the spinor representation of $SO(16)$ and $\Gamma^{IJ} = \Gamma^{[I}\Gamma^{J]}$ denotes the antisymmetrized product of $SO(16)$ Γ-matrices. Moreover, we use indices $\dot{A}, \dot{B} = 1, \ldots, 128$ to label the conjugate spinor representation of $SO(16)$. In the main text, indices $\mathcal{M}, \mathcal{N} = 1, \ldots, 248$ collectively label the full Lie algebra of $E_{8(8)}$, i.e. $\{t^{\mathcal{M}}\} = \{X^{IJ}, Y^A\}$ with

$$[\, t^{\mathcal{M}}, t^{\mathcal{N}} \,] \;=\; f^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}}\, t^{\mathcal{K}} \,. \tag{A.3}$$

The Cartan-Killing form finally is given by

$$\eta^{\mathcal{M}\mathcal{N}} \;=\; \frac{1}{60}\,\mathrm{tr}\,(t^{\mathcal{M}}\, t^{\mathcal{N}}) \;=\; \frac{1}{60}\, f^{\mathcal{M}\mathcal{K}}{}_{\mathcal{L}}\, f^{\mathcal{N}\mathcal{L}}{}_{\mathcal{K}} \,. \tag{A.4}$$

A.2 $E_{8(8)}$ in the $SO(8,8)$ basis

Alternatively, $\mathfrak{e}_{8(8)}$ may be built starting from its maximal subalgebra $\mathfrak{so}(8,8)$ spanned by 120 generators $X^{\mathtt{IJ}}$ with commutators

$$[X^{\mathtt{IJ}}, X^{\mathtt{KL}}] \;=\; \eta^{\mathtt{JK}} X^{\mathtt{IL}} - \eta^{\mathtt{IK}} X^{\mathtt{JL}} - \eta^{\mathtt{JL}} X^{\mathtt{IK}} + \eta^{\mathtt{IL}} X^{\mathtt{JK}} \,, \tag{A.5}$$

where $\mathtt{I}, \mathtt{J}, \ldots$ now denote vector indices of $SO(8,8)$ and $\eta_{\mathtt{IJ}} = \mathrm{diag}(-1,\ldots,-1,1,\ldots,1)$ is the $SO(8,8)$ invariant tensor. Similarly to the above, the full $\mathfrak{e}_{8(8)}$ is obtained by adding 128 generators $\hat{Q}_{\mathtt{A}}$, $\mathtt{A} = 1,\ldots,128$, transforming in the spinor representation of $SO(8,8)$

$$[X^{\mathtt{IJ}}, \hat{Q}_{\mathtt{A}}] \;=\; -\tfrac{1}{2}\Gamma^{\mathtt{IJ}}{}_{\mathtt{A}}{}^{\mathtt{B}}\, \hat{Q}_{\mathtt{B}} \,, \qquad [\hat{Q}_{\mathtt{A}}, \hat{Q}_{\mathtt{B}}] \;=\; \tfrac{1}{4}\,\eta_{\mathtt{IK}}\eta_{\mathtt{JL}}\eta_{\mathtt{BC}}\,\Gamma^{\mathtt{KL}}{}_{\mathtt{A}}{}^{\mathtt{C}}\, X^{\mathtt{IJ}} \,. \tag{A.6}$$

Here $\Gamma^{\mathtt{IJ}}{}_{\mathtt{A}}{}^{\mathtt{B}}$ denote the (rescaled) $SO(8,8)$-generators in the spinor representation, i.e.

$$\Gamma^{\mathtt{IJ}}{}_{\mathtt{A}}{}^{\mathtt{B}} = \tfrac{1}{2}(\Gamma^{\mathtt{I}}{}_{\mathtt{A}}{}^{\dot{\mathtt{C}}}\, \overline{\Gamma}^{\mathtt{J}}{}_{\dot{\mathtt{C}}}{}^{\mathtt{B}} - \Gamma^{\mathtt{J}}{}_{\mathtt{A}}{}^{\dot{\mathtt{C}}}\, \overline{\Gamma}^{\mathtt{I}}{}_{\dot{\mathtt{C}}}{}^{\mathtt{B}}) \,, \tag{A.7}$$

where the gamma matrices satisfy

$$\Gamma^{\mathtt{I}}{}_{\mathtt{A}}{}^{\dot{\mathtt{C}}}\, \overline{\Gamma}^{\mathtt{J}}{}_{\dot{\mathtt{C}}}{}^{\mathtt{B}} + \Gamma^{\mathtt{J}}{}_{\mathtt{A}}{}^{\dot{\mathtt{C}}}\, \overline{\Gamma}^{\mathtt{I}}{}_{\dot{\mathtt{C}}}{}^{\mathtt{B}} = 2\eta^{\mathtt{IJ}}\, \delta_{\mathtt{A}}{}^{\mathtt{B}} \,, \tag{A.8}$$

with the transpose $\overline{\Gamma}$, and where $\mathtt{A}, \mathtt{B}, \ldots$, denote spinor indices and $\dot{\mathtt{A}}, \dot{\mathtt{B}}, \ldots$, conjugate spinor indices of $SO(8,8)$. It is important to note that in contrast to the $SO(16)$ decomposition described above, spinor indices in these equations are raised and lowered not with the simple δ-symbol but with the corresponding $SO(8,8)$ invariant tensors $\eta_{\mathtt{AB}}$, $\eta_{\dot{\mathtt{A}}\dot{\mathtt{B}}}$ of indefinite signature (cf. (A.11) below).

A.3 $E_{8(8)}$ in the $SO(8) \times SO(8)$ basis

According to (3.41), (3.41) in the main text, the two decompositions of sections A.1 and A.2 may be translated into each other upon further breaking down to $SO(8)_L \times SO(8)_R$. To this end, we use the decomposition $SO(8,8) \to SO(8)_L \times SO(8)_R$ with

$$\begin{aligned} \mathbf{16}_V \;&\to\; (\mathbf{8}_V, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}_V) \,, \\ \mathbf{128}_S \;&\to\; (\mathbf{8}_S, \mathbf{8}_C) \oplus (\mathbf{8}_C, \mathbf{8}_S) \,, \qquad \mathbf{128}_C \;\to\; (\mathbf{8}_S, \mathbf{8}_S) \oplus (\mathbf{8}_C, \mathbf{8}_C) \,, \end{aligned} \tag{A.9}$$

corresponding to the split of $SO(8,8)$ indices:

$$\mathtt{I} = (\hat{a}, b), \qquad \mathtt{A} = (\alpha\dot{\beta}\,, \dot{\gamma}\delta) \,, \qquad \dot{\mathtt{A}} = (\alpha\beta, \dot{\gamma}\dot{\delta}) \,. \tag{A.10}$$

Here, $\hat{a}, \hat{b}, \ldots$ and $a, b, \ldots$ denote vector indices for the left and the right $SO(8)$ factor, respectively, while $\alpha, \beta, \ldots$ and $\dot{\alpha}, \dot{\beta}, \ldots$ denote spinor and conjugate spinor indices, respectively, for both $SO(8)$ factors. The invariant tensors $\eta^{\mathtt{IJ}}$, $\eta_{\mathtt{AB}}$ and $\eta_{\dot{\mathtt{A}}\dot{\mathtt{B}}}$ in this $SO(8)$ notation take the form

$$\begin{aligned} \eta^{\mathtt{IJ}} &= \begin{pmatrix} -\delta^{\hat{a}\hat{b}} & 0 \\ 0 & \delta^{ab} \end{pmatrix} \,, \qquad \eta_{\mathtt{AB}} = \begin{pmatrix} \eta_{\alpha\dot{\alpha},\beta\dot{\beta}} & 0 \\ 0 & \eta_{\dot{\alpha}\beta,\dot{\gamma}\delta} \end{pmatrix} = \begin{pmatrix} \delta_{\alpha\beta}\delta_{\dot{\alpha}\dot{\beta}} & 0 \\ 0 & -\delta_{\dot{\alpha}\dot{\gamma}}\delta_{\beta\delta} \end{pmatrix} \,, \\ \eta_{\dot{\mathtt{A}}\dot{\mathtt{B}}} &= \begin{pmatrix} \eta_{\alpha\beta,\gamma\delta} & 0 \\ 0 & \eta_{\dot{\alpha}\dot{\beta},\dot{\gamma}\dot{\delta}} \end{pmatrix} = \begin{pmatrix} \delta_{\alpha\gamma}\delta_{\beta\delta} & 0 \\ 0 & -\delta_{\dot{\alpha}\dot{\gamma}}\delta_{\dot{\beta}\dot{\delta}} \end{pmatrix} \,. \end{aligned} \tag{A.11}$$

It is straightforward to verify, that the $SO(8,8)$ gamma matrices (A.8) can be expressed in terms of the $SO(8)$ gamma matrices $\Gamma^a_{\alpha\dot\gamma}$ as (see also [110])

$$\begin{aligned}
\Gamma^a{}_{\beta\dot\gamma}{}^{\delta\epsilon} &= \delta_{\beta\delta}\Gamma^a_{\epsilon\dot\gamma}\,, \qquad \Gamma^a{}_{\dot\alpha\beta}{}^{\dot\gamma\dot\delta} = -\delta_{\dot\alpha\dot\gamma}\Gamma^a_{\beta\dot\delta}\,,\\
\Gamma^{\hat a}{}_{\dot\alpha\beta}{}^{\gamma\delta} &= \delta_{\beta\delta}\Gamma^a_{\gamma\dot\alpha}\,, \qquad \Gamma^{\hat a}{}_{\alpha\dot\beta}{}^{\dot\gamma\dot\delta} = -\delta_{\dot\beta\dot\delta}\Gamma^a_{\alpha\dot\gamma}\,.
\end{aligned} \tag{A.12}$$

With the results from the previous section, $\mathfrak{e}_{8(8)}$ can now explicitly be given in the $\mathfrak{so}(8)_L \oplus \mathfrak{so}(8)_R$ basis. Generators split according to $\{X^{ab}, X^{\hat a\hat b}, X^{a\hat b}, \hat Q_{\alpha\dot\beta}, \hat Q_{\dot\gamma\delta}\}$ with the commutation relations

$$\begin{aligned}
[X^{ab}, X^{cd}] &= \delta^{bc}X^{ad} - \delta^{ac}X^{bd} - \delta^{bd}X^{ac} + \delta^{ad}X^{bc}\,,\\
[X^{\hat a\hat b}, X^{\hat c\hat d}] &= -\delta^{\hat b\hat c}X^{\hat a\hat d} + \delta^{\hat a\hat c}X^{\hat b\hat d} + \delta^{\hat b\hat d}X^{\hat a\hat c} - \delta^{\hat a\hat d}X^{\hat b\hat c}\,,\\
[X^{ab}, X^{c\hat d}] &= \delta^{bc}X^{a\hat d} - \delta^{ac}X^{b\hat d}\,,\\
[X^{\hat a\hat b}, X^{c\hat d}] &= \delta^{\hat a\hat d}X^{c\hat b} - \delta^{\hat b\hat d}X^{c\hat a}\,,\\
[X^{a\hat b}, X^{c\hat d}] &= \delta^{\hat b\hat d}X^{ac} - \delta^{ac}X^{\hat b\hat d}\,,\\
[X^{ab}, \hat Q_{\alpha\dot\beta}] &= -\tfrac12\overline\Gamma^{[a}_{\dot\beta\epsilon}\Gamma^{b]}_{\epsilon\dot\delta}\hat Q_{\alpha\dot\delta}\,, \qquad [X^{ab}, \hat Q_{\dot\alpha\beta}] = -\tfrac12\Gamma^{[a}_{\beta\dot\zeta}\overline\Gamma^{b]}_{\dot\zeta\delta}\hat Q_{\dot\alpha\delta}\,,\\
[X^{\hat a\hat b}, \hat Q_{\alpha\dot\beta}] &= \tfrac12\Gamma^{[a}_{\alpha\dot\epsilon}\overline\Gamma^{b]}_{\dot\epsilon\dot\gamma}\hat Q_{\gamma\dot\beta}\,, \qquad [X^{\hat a\hat b}, \hat Q_{\dot\alpha\beta}] = \tfrac12\overline\Gamma^{[a}_{\dot\alpha\epsilon}\Gamma^{b]}_{\epsilon\dot\gamma}\hat Q_{\dot\gamma\beta}\,,\\
[X^{a\hat b}, \hat Q_{\alpha\dot\beta}] &= \tfrac12\Gamma^a_{\delta\dot\beta}\Gamma^b_{\alpha\dot\gamma}\hat Q_{\dot\gamma\delta}\,, \qquad [X^{a\hat b}\,, \hat Q_{\dot\alpha\beta}] = \tfrac12\Gamma^a_{\beta\dot\delta}\Gamma^b_{\gamma\dot\alpha}\hat Q_{\gamma\dot\delta}\,,\\
[\hat Q_{\alpha\dot\alpha}, \hat Q_{\beta\dot\beta}] &= \tfrac14\delta_{\alpha\beta}\overline\Gamma^{[a}_{\dot\alpha\gamma}\Gamma^{b]}_{\gamma\dot\beta}X^{ab} - \tfrac14\delta_{\dot\alpha\dot\beta}\Gamma^{[a}_{\alpha\dot\gamma}\overline\Gamma^{b]}_{\dot\gamma\beta}X^{\hat a\hat b}\,,\\
[\hat Q_{\alpha\dot\alpha}, \hat Q_{\dot\beta\beta}] &= -\tfrac12\Gamma^a_{\beta\dot\alpha}\Gamma^b_{\alpha\dot\beta}X^{a\hat b}\,,\\
[\hat Q_{\dot\alpha\alpha}, \hat Q_{\dot\beta\beta}] &= \tfrac14\delta_{\alpha\beta}\overline\Gamma^{[a}_{\dot\alpha\gamma}\Gamma^{b]}_{\gamma\dot\beta}X^{\hat a\hat b} - \tfrac14\delta_{\dot\alpha\dot\beta}\Gamma^{[a}_{\alpha\dot\gamma}\overline\Gamma^{b]}_{\dot\gamma\beta}X^{ab}\,.
\end{aligned} \tag{A.13}$$

Moreover, the Cartan-Killing form (A.4) can be computed in the $SO(8)\times SO(8)$ basis by use of this explicit form of the structure constants. The result is

$$\begin{aligned}
&\eta^{ab,cd} = -\delta^{[ab],[cd]}\,, \qquad \eta^{\hat a\hat b,\hat c\hat d} = -\delta^{[\hat a\hat b],[\hat c\hat d]}\,, \qquad \eta^{a\hat b,c\hat d} = \delta^{ac}\delta^{\hat b\hat d}\,,\\
&\eta_{\alpha\dot\beta,\gamma\dot\delta} = \delta_{\alpha\gamma}\delta_{\dot\beta\dot\delta}\,, \qquad \eta_{\dot\alpha\beta,\dot\gamma\delta} = -\delta_{\dot\alpha\dot\gamma}\delta_{\beta\delta}\,,
\end{aligned} \tag{A.14}$$

while all other components vanish.

Finally let us identify explicitly the $SO(16)$ subalgebra in this $SO(8,8)$ basis. With respect to the $SO(8)\times SO(8)$ decomposition of $SO(16)$ in (3.37) the indices split according to

$$I = (\dot\alpha, \beta)\,, \qquad A = (\alpha\dot\beta, \hat a b)\,, \qquad \dot A = (\alpha a, \hat b\dot\beta)\,. \tag{A.15}$$

Correspondingly, the $SO(16)$ generators X^{IJ} decompose into $X^{\alpha\beta}$, $X^{\dot\alpha\dot\beta}$ and $X^{\alpha\dot\beta}$, and can be written in terms of the compact $E_{8(8)}$ generators by

$$X^{\alpha\beta} = \tfrac12\Gamma^{ab}_{\alpha\beta}X^{ab}\,, \qquad X^{\dot\alpha\dot\beta} = -\tfrac12\Gamma^{\hat a\hat b}_{\dot\alpha\dot\beta}X^{\hat a\hat b}\,, \qquad X^{\alpha\dot\beta} = \hat Q_{\dot\beta\alpha}\,. \tag{A.16}$$

That these satisfy the $SO(16)$ algebra can be verified explicitly by use of standard gamma matrix identities. The noncompact generators Y^A are identified as

$$Y^{\alpha\dot{\beta}} = \hat{Q}_{\alpha\dot{\beta}} \,, \qquad Y^{\hat{a}b} = X^{b\hat{a}} \,. \tag{A.17}$$

One immediately verifies that this split into compact and noncompact generators is in agreement with the eigenvalues of the Cartan-Killing form (A.14).

A.4 $E_{8(8)}$ in the $SO(4) \times SO(4)$ basis

To explicitly describe the embedding of the gauge group $G_0 = G_c \ltimes (\hat{T}_{34}, T_{12})$ described in section 3.3.1, we finally need the decomposition of $E_{8(8)}$ under the $SO(4)_L \times SO(4)_R$ from (3.18). This is obtained from the previous section upon further decomposition according to (3.35), (3.36). In $SO(8)_R$ indices a, α, $\dot{\alpha}$, this corresponds to the splits

$$a = ([ij], 0, \bar{0}) \,, \qquad \alpha = (i,j) \,, \qquad \dot{\alpha} = (i,j) \,, \tag{A.18}$$

and similarly for $SO(8)_L$. Here, $i, j, \ldots$ denote $SO(4)$ vector indices. The $SO(8)$ gamma matrices can then be expressed in terms of the invariant $SO(4)$ tensors δ^{ij} and ε^{ijkl} as

$$\Gamma^{ij} = \begin{pmatrix} \varepsilon^{ij} & 2\delta^{ij} \\ -2\delta^{ij} & \varepsilon^{ij} \end{pmatrix} \,, \qquad \Gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \,, \qquad \Gamma^{\bar{0}} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \,, \tag{A.19}$$

with the 4×4 matrices

$$\mathbf{1}_{kl} \;=\; \delta_{kl} \,, \qquad (\varepsilon^{ij})_{kl} \;=\; \varepsilon^{ijkl} \,, \qquad (\delta^{ij})_{kl} \;=\; \delta^{ij}_{kl} \;=\; \delta^{i[k}\delta^{l]j} \,. \tag{A.20}$$

It is straightforward to check that the matrices (A.19) satisfy the standard Clifford algebra, making use of the relations

$$\begin{aligned} \delta^{ij}(\delta^{mn})^t + \delta^{mn}(\delta^{ij})^t + \varepsilon^{ij}(\varepsilon^{mn})^t + \varepsilon^{mn}(\varepsilon^{ij})^t \;&=\; 2\delta^{ij,mn}\,\mathbf{1}, \\ \varepsilon^{ij}(\delta^{mn})^t + \varepsilon^{mn}(\delta^{ij})^t - \delta^{ij}(\varepsilon^{mn})^t - \delta^{mn}(\varepsilon^{ij})^t \;&=\; 0, \end{aligned} \tag{A.21}$$

which can be proved using the identity $\varepsilon^{[ijkl}\delta^{m]}_n = 0$. Next we have to decompose these Γ-matrices into selfdual and anti-selfdual parts, corresponding to (A.18),

$$\Gamma^{ij}_{\pm} = \frac{1}{\sqrt{2}}\big(\Gamma^{ij} \pm \frac{1}{2}\varepsilon^{ijkl}\Gamma^{kl}\big), \tag{A.22}$$

such that $\tilde{\Gamma}^{ij}_{\pm} := \frac{1}{2}\varepsilon^{ijkl}\Gamma^{kl}_{\pm} = \pm\Gamma^{ij}_{\pm}$. Inserting the representation (A.19) of Γ-matrices into the structure constants in (A.13) yields the decomposition of $\mathfrak{e}_{8(8)}$ in the $\mathfrak{so}(4)_L \oplus \mathfrak{so}(4)_R$ basis.

B Kac-Moody and Virasoro algebras

Here we give a short introduction into the subject of Kac-Moody algebras and the Virasoro algebra. For the special case that their central charges vanish they have a simple geometrical meaning, and therefore we discuss this case first.

A Kac-Moody algebra $\hat{\mathfrak{g}}$ is associated to an ordinary finite-dimensional Lie group[1] G as the Lie algebra of the so-called loop group. The latter is defined as the set of smooth maps from the unit circle S^1 into the group G. Parametrizing the S^1 by an angle θ as in the main text, the loop group $\mathcal{G}$ consists of periodic maps

$$\theta \longrightarrow \gamma(\theta) \in G \,, \tag{B.23}$$

whose group structure is given by point-wise multiplication. It defines an infinite-dimensional Lie group. To determine the Lie algebra $\hat{\mathfrak{g}}$ of this group, we introduce a basis t^a for the Lie algebra $\mathfrak{g}$ of G. The loop group elements can then be written as

$$\gamma(\theta) = \exp[\alpha_a(\theta) t^a] \,, \tag{B.24}$$

or, near the identity, as

$$\gamma(\theta) \approx 1 + t^a \sum_{n=-\infty}^{\infty} \alpha_a^n e^{in\theta} \,, \tag{B.25}$$

where we have performed a Fourier expansion of α. This relation implies that the generators of the loop group can be identified with

$$t_n^a = t^a e^{in\theta} \,, \tag{B.26}$$

since then the expansion (B.25) reads

$$\gamma \approx 1 + \sum_{n,a} \alpha_a^n t_n^a \,. \tag{B.27}$$

With (B.26) the Lie algebra $\hat{\mathfrak{g}}$ can be computed to be

$$[t_m^a, t_n^b] = f^{ab}{}_c \, t_{m+n}^c \,, \qquad m, n = -\infty, ..., \infty \,, \tag{B.28}$$

where $f^{ab}{}_c$ are the structure constants of G. (B.28) defines the so-called Kac-Moody algebra or the affine extension of G. The Lie algebra $\mathfrak{g}$ of G is embedded as the subalgebra $\hat{\mathfrak{g}}_0$ spanned by the generators t_0^a. Moreover, the generators satisfy $(t_n^a)^\dagger = t_{-n}^a$.

Next let us turn to the Virasoro algebra. While the loop group can be defined as the group of maps from S^1 into a Lie group G, the Virasoro algebra $\hat{v}$ can be introduced as the Lie algebra of the Diffeomorphism group of S^1. This diffeomorphism

[1] This one is usually, but not in this thesis, taken to be semi-simple and compact.

group $\mathcal{V} = \mathrm{Diff}(S^1)$ consists of smooth and invertible maps $S^1 \to S^1$, whose group multiplication is defined by composition

$$(\xi_1 \cdot \xi_2)(\theta) = \xi_1(\xi_2(\theta)) \ . \tag{B.29}$$

Infinitesimally they are given by

$$\theta \longrightarrow \theta - \alpha(\theta) \ , \tag{B.30}$$

where α is periodic in θ. On functions f on S^1 these diffeomorphisms act as

$$f(\theta) \longrightarrow f(\theta) - \alpha(\theta)\frac{d}{d\theta}f(\theta) \ . \tag{B.31}$$

Upon expanding α in Fourier components, one can read of the generators of $\hat{v}$:

$$Q_n = -e^{in\theta}\frac{d}{d\theta} \ . \tag{B.32}$$

The resulting Lie algebra is then given by

$$[Q_m, Q_n] = i(m-n)Q_{m+n} \ . \tag{B.33}$$

This is the so-called Virasoro algebra $\hat{v}$. In analogy to $\hat{\mathfrak{g}}$ the generators fulfill the reality constraint $Q_n^\dagger = Q_{-n}$.

Since both type of algebras, the Kac-Moody algebras and the Virasoro algebra, are defined geometrically with regard to S^1, there exists an obvious way to interrelate both algebras. In fact, any diffeomorphism $\xi \in \mathcal{V}$ can act on a loop group element $\gamma \in \mathcal{G}$ as

$$(\xi \cdot \gamma)(\theta) = \gamma(\xi(\theta)) \ . \tag{B.34}$$

This defines a semi-direct product $\hat{v} \ltimes \hat{\mathfrak{g}}$ with elements (ξ, γ), whose Lie algebra can be computed by use of (B.26) and (B.32) to be given by (B.28), (B.33) and

$$[Q_m, t_n^a] = -n\ t_{m+n}^a \ . \tag{B.35}$$

Let us now turn to the more general case of non-vanishing central charges. One may check explicitly that consistent Lie algebras (in the sense of fulfilling the Jacobi identities) can be defined with the following central extensions. The latter are defined as generators c that commute with all generator, e.g. $[Q_m, c] = 0$. The centrally extended Virasoro and Kac-Moody algebras then read

$$\begin{aligned} [Q_m, Q_n] &= i(m-n)Q_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n} \ , \\ [t_m^a, t_n^b] &= f^{ab}{}_c\ t_{m+n}^c + km\delta^{ab}\delta_{m,-n} \ . \end{aligned} \tag{B.36}$$

In the case of vanishing central charge $c = 0$ the Virasoro algebra is also called Witt algebra. Moreover, the semi-direct product structure (B.35) is consistent also with the central extensions in (B.36). The structure of this semi-direct product can also be obtained via the so-called Sugawara construction, in which the Virasoro generators are realized as bilinears of Kac-Moody generators [76]. It should be noted that for generic Lie groups G this is a unique product between the associated Kac-Moody algebra and $\hat{v}$, in contrast to the affine Poincaré algebra $\widehat{iso(1,2)}$ discussed in the main text.

Representations of $\hat{v}$

Let us briefly summarize the representations of the Witt algebra, which are relevant in the main text [36]. First, the adjoint representation can clearly be defined also for infinite-dimensional Lie algebras and is here given by

$$\delta\chi^n = -\sum_{k,m} \xi^k (t_k)^n{}_m \chi^m = -\sum_{k,m} \xi^k f^n_{km} \chi^m = i \sum_k (n-2k) \xi^k \chi^{n-k} \,, \tag{B.37}$$

where f^n_{km} are the structure constants of $\hat{v}$ defined by (B.33). In analogy to this a much broader class of representations can be defined according to

$$\delta\chi^n = i \sum_k (n - (1-\Delta)k) \xi^k \chi^{n-k} \,. \tag{B.38}$$

One can check explicitly, that these are representations of $\hat{v}$:

$$[\delta_{\xi^m}, \delta_{\xi^n}]\chi^k = i(m-n)\delta_{\xi^{m+n}}\chi^k \,, \qquad \xi^{m+n} = \xi^m \xi^n \,. \tag{B.39}$$

The representations of $\hat{v}$ can therefore be labeled by a number Δ, which we call conformal dimension in analogy to conformal field theories. The adjoint representation is included in (B.38) as $\Delta = -1$.

Among these representations is the dual of the adjoint representation, which can be characterized as follows. For each representation ρ on a vector space V one has the dual representation ρ^* on the dual space V^*, which is defined by the requirement $\langle \rho^*(g)(v^*), \rho(g)(v) \rangle = \langle v^*, v \rangle$, where $\langle \, , \rangle$ denotes the natural pairing between vectors in V and V^* and g is a group element. This implies $\rho^*(g) = {}^t\rho(g^{-1})$ or at the level of the Lie algebra $\rho^*(X) = -{}^t\rho(X)$, where $X \in \mathfrak{g}$. Since the adjoint representation is given by $(t_m)^n_k = f^n_{mk}$, the co-adjoint representation matrices read $(t^*_k)_n{}^m = -(t_k)^m_n = -f^m_{kn}$. Applied to the Witt algebra, the co-adjoint action is

$$\delta\chi^*_n = -\sum_{k,m} \xi^k (t^*_k)_n{}^m \chi^*_m = i \sum_k (k-n) \xi_k \chi^*_{n+k} \,, \tag{B.40}$$

or, defining $\chi^n := \chi^*_{-n}$,

$$\delta\chi^n = i \sum_k (n+k) \xi^k \chi^{n-k} \,, \tag{B.41}$$

which coincides with the representation (B.38) for $\Delta = 2$.

C Kaluza-Klein action on $\mathbb{R}^3 \times S^1$ with Yang-Mills type gauging

As we have already emphasized, the inclusion of all Kaluza-Klein modes in the effective action for reductions on S^1 (or in general on arbitrary tori) can also be done explicitly, and in fact have been done for reductions to $D = 4$ (see [81, 36, 35, 34, 111]). Here we will show this for the $D = 4 \to D = 3$ reduction of four-dimensional gravity, which yields the Kaluza-Klein theory with Yang-Mills gauge fields.

In practice the computation is significantly simplified by use of the vielbein formalism. More specifically, the Einstein-Hilbert action

$$S_{\text{EH}} = -\int d^4x E R = -\int d^4x E E_A^M E_B^N R_{MN}^{\ \ AB} \tag{C.42}$$

can be computed from the components of the spin-connection $\omega_M^{\ AB}$ by use of

$$R_{MN}^{\ \ AB} = 2\partial_{[M}\omega_{N]}^{\ \ AB} + 2\omega_{[M}^{\ \ AC}\omega_{N]C}^{\ \ \ B} \,, \tag{C.43}$$

where the spin connection in flat indices is given by

$$\begin{aligned} \omega_{ABC} &= \frac{1}{2}(\Omega_{ABC} - \Omega_{BCA} + \Omega_{CAB}) \,, \\ \Omega_{ABC} &= 2E_{[A}^M E_{B]}^N \partial_M E_{NC} \,. \end{aligned} \tag{C.44}$$

It is convenient to express the Einstein-Hilbert action entirely in terms of Ω_{ABC}. Inserting (C.44) into (C.42) and performing several partial integrations one gets

$$S_{\text{EH}} = -\int d^4x E \left[-\frac{1}{4}\Omega^{ABC}\Omega_{ABC} + \frac{1}{2}\Omega^{ABC}\Omega_{BCA} + \Omega_{C\ \ B}^{\ B}\Omega^{C\ \ A}_{\ A} \right] \,. \tag{C.45}$$

Computing the Ω_{ABC} by use of the vielbein (4.1) and its inverse

$$E_A^M = \begin{pmatrix} \phi^{1/2} e_a^\mu & -\phi^{1/2} e_a^\rho A_\rho \\ 0 & \phi^{-1/2} \end{pmatrix} \,, \tag{C.46}$$

one gets the following (still θ-dependent) coefficients

$$\begin{aligned} \Omega_{abc} &= \phi^{1/2}\left[\hat{\Omega}^{(3)}_{abc} - e^\mu_{[a}\eta_{b]c}\phi^{-1}D_\mu\phi\right] \,, \\ \Omega_{ab}^{\ \ 5} &= \phi^{3/2} e_a^\mu e_b^\nu F_{\mu\nu} \,, \\ \Omega_{5b}^{\ \ 5} &= -\frac{1}{2}\phi^{-1/2} e_b^\mu D_\mu\phi \,, \\ \Omega_{5b}^{\ \ c} &= \phi^{-1/2} e_b^\mu D_5 e_\mu^c \,. \end{aligned} \tag{C.47}$$

All expressions appear already in a $\hat{v}$-covariant fashion.[2] Specifically,

$$\hat{\Omega}^{(3)}_{abc} = 2e^\mu_{[a}e^\nu_{b]}D_\mu e_{\nu c} \,, \tag{C.48}$$

[2]Note, that this is not the case for the $\omega_M^{\ AB}$, where one index has been transformed into a space-time index by use of the vielbein (4.1).

with the $\hat{v}$-covariant derivative $D_\mu e_{\nu c}$ defined in (4.89). Furthermore, we defined following [36] a covariantized x^5-derivative, which is given by

$$D_5 e^a_\mu = \partial_5 e^a_\mu - \frac{1}{2}\phi^{-1}\partial_5\phi\, e^a_\mu \,. \tag{C.49}$$

The denotation 'covariant derivative' is justified by the fact that $D_5 e^a_\mu$ transforms in contrast to $\partial_5 e^a_\mu$ covariantly under local $\hat{v}$ transformations:

$$\delta_{\xi^5}(D_5 e^a_\mu) = \xi^5\partial_5(D_5 e^a_\mu) + 2\partial_5\xi^5 D_5 e^a_\mu \,. \tag{C.50}$$

This means, that $D_5 e^a_\mu$ transforms covariantly, even though it transforms under a different representation as e^a_μ itself.

Inserting (C.47) into the Einstein-Hilbert action in the form (C.45) one finds after some computations

$$S_{\text{EH}} = \int d^3x d\theta e \left[-R^{(3),\text{cov}} - \frac{1}{4}\phi^2 F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\phi^{-2}g^{\mu\nu}D_\mu\phi D_\nu\phi + \mathcal{L}_m\right] \,, \tag{C.51}$$

where $R^{(3),\text{cov}}$ denotes the generalized Ricci scalar with respect to the covariantized connection in (C.47). Moreover, $\mathcal{L}_m$ contains the spin-2 mass term, which is induced by the gauging, and reads

$$\mathcal{L}_m = \frac{1}{4}\phi^{-2}g^{\mu\nu}g^{\rho\sigma}(D_5 g_{\mu\rho}D_5 g_{\nu\sigma} - D_5 g_{\mu\nu}D_5 g_{\rho\sigma}) - e^{a\mu}e^{b\nu}F_{\mu\nu}e^\rho_b D_5 e_{\rho a} \,. \tag{C.52}$$

One may check explicitly that $\mathcal{L}_m$ is invariant under local $\hat{v}$ transformations. In particular, the power of ϕ in front of the spin-2 mass term can be entirely determined from the requirement that the action stays invariant.

That (C.52) gives mass to the spin-2 fields in the Kaluza-Klein vacuum (4.9) can be seen as follows. The mass term in the free spin-2 theory (2.1) already given by Pauli and Fierz reads

$$\mathcal{L}_{\text{mass}} = \frac{M^2}{2}\left(h^{\mu\nu}h_{\mu\nu} - (\eta^{\mu\nu}h_{\mu\nu})^2\right) . \tag{C.53}$$

In fact, expanding (C.52) around the Kaluza-Klein vacuum (4.9)

$$g^n_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h^n_{\mu\nu}(x) \,, \tag{C.54}$$

and integrating out θ one gets the leading contribution

$$\frac{1}{2\kappa^2}\int d\theta\mathcal{L}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} M_n^2\eta^{\mu\nu}\eta^{\rho\sigma}\left(h^n_{\mu\rho}h^{-n}_{\nu\sigma} - h^n_{\mu\nu}h^{-n}_{\rho\sigma}\right) + O(\kappa^2) \,, \tag{C.55}$$

where

$$M_n^2 = \left(\frac{n}{R}\right)^2 \,. \tag{C.56}$$

This in turn shows that in lowest order in κ (C.52) reduces to an infinite sum of Pauli-Fierz spin-2 mass terms (C.53). Furthermore it can be shown that the vector and scalar modes A^n_μ and ϕ^n can be absorbed via a field redefinition into the spin-2 fields, such that in total the latter become massive in a Higgs effect. (At the linearized level this analysis has been performed, e.g., in [81].)

D Spin-2 theory for arbitrary internal manifold

In the main text we have shown that the Chern-Simons gauge theory of the affine Poincaré algebra describes a consistent gravity-spin-2 coupling. This is on the other hand also equivalent to Wald's algebra-valued generalization of the Einstein Hilbert action, where the algebra is given by the algebra of smooth functions on S^1. We are going to show that this picture generalizes to the case of an arbitrary internal manifold.

Let M be an arbitrary compact Riemannian manifold and $\{e_m\}$ a complete set of spherical harmonics (where generically m now denotes a multi-index), which we also take as a basis for the algebra of smooth functions on M. The infinite-dimensional extension of the Poincaré algebra is no longer given by a Kac-Moody algebra, but instead is spanned by generators $P_a^n = P_a \otimes e_n$ and $J_a^n = J_a \otimes e_n$, which satisfy the Lie algebra (compare the algebra in [112])

$$\begin{aligned} &[P_a^m, P_b^n] = 0 \ , \qquad [J_a^m, J_b^n] = \varepsilon_{abc} J^c \otimes (e_m \cdot e_n) \ , \\ &[J_a^m, P_b^n] = \varepsilon_{abc} P^c \otimes (e_m \cdot e_n) \ , \end{aligned} \tag{D.57}$$

Here $\cdot$ denotes ordinary multiplication of functions. Note, that this algebra reduces for the case $M = S^1$ to the Kac-Moody algebra $\widehat{iso(1,2)}$ in (4.11). There exists also an inner product on the space of functions, which is given by

$$(e_m, e_n) = \int_M \mathrm{dvol}_M e_m e_n \ , \tag{D.58}$$

such that a non-degenerate quadratic form on (D.57) exists:

$$\langle P_a^m, J_b^n \rangle = \eta_{ab}(e_m, e_n) \ . \tag{D.59}$$

In complete analogy to sec. 4.3.1 a Chern-Simons theory can then be defined, whose equations of motion read

$$\begin{aligned} \partial_\mu e_\nu^{a(n)} - \partial_\nu e_\mu^{a(n)} + a_{mk}^n \varepsilon^{abc} \left(e_{\mu b}^{(m)} \omega_{\nu c}^{(k)} + \omega_{\mu b}^{(m)} e_{\nu c}^{(k)} \right) = 0 \ , \\ \partial_\mu \omega_\nu^{a(n)} - \partial_\nu \omega_\mu^{a(n)} + a_{mk}^n \varepsilon^{abc} \omega_{\mu b}^{(m)} \omega_{\nu c}^{(k)} = 0 \ , \end{aligned} \tag{D.60}$$

while they are invariant under

$$\delta e_\mu^{a(n)} = \partial_\mu \rho^{a(n)} + a_{mk}^n \left(\varepsilon^{abc} e_{\mu b}^{(m)} \tau_c^{(k)} + \varepsilon^{abc} \omega_{\mu b}^{(m)} \rho_c^{(k)} \right) , \tag{D.61}$$

where a_{mk}^n defines the algebra structure with respect to the basis $\{e_m\}$. If one defines the transformation parameter to be

$$\rho^{a(n)} = a_{mk}^n \xi^{\mu(m)} e_\mu^{a(k)}, \qquad \tau^{a(n)} = a_{mk}^n \xi^{\mu(m)} \omega_\mu^{a(k)}, \tag{D.62}$$

one can show using the equations of motion and the associativity (4.35) of the algebra, that

$$\delta e_\mu^{a(n)} = a_{mk}^n \left(\xi^{\rho(m)} \partial_\rho e_\mu^{a(k)} + \partial_\mu \xi^{\rho(m)} e_\rho^{a(k)} \right) . \tag{D.63}$$

For the algebra-valued metric defined by $g^n_{\mu\nu} = a^n_{mk} e^{a(m)}_\mu e^{(k)}_{\nu a}$ this implies

$$\begin{aligned} \delta_\xi g^n_{\mu\nu} &= \partial_\mu \xi^{\rho(l)} g_{\rho\nu}{}^n{}_l + \partial_\nu \xi^{\rho(l)} g_{\rho\mu}{}^n{}_l + \xi^{\rho(l)} \partial_\rho g_{\mu\nu}{}^n{}_l \\ &= \nabla_\mu \xi^n_\nu + \nabla_\nu \xi^n_\mu \,, \end{aligned} \tag{D.64}$$

where again (4.35) has been used. Altogether, the gauge transformations of the Chern-Simons theory for (D.57) coincide with the algebra-diffeomorphisms. Thus we have shown that also for arbitrary internal manifolds the Chern-Simons description based on the algebra (D.57) is equivalent to Wald's algebra-valued multi-graviton theory. This is turn confirms, that the spin-2 theories appearing in Kaluza-Klein actions resulting from compactifications to arbitrary dimensions can also be treated within Wald's framework.

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