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Piecewise Linear Regression Splines with Hyperbolic Covariates

John B. Cologne, Ph.D.; Richard Sposto, Ph.D.

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双曲共変数を利用した線形回帰の断片的結合[§]Piecewise Linear Regression Splines with
Hyperbolic CovariatesJohn B. Cologne¹ Richard Sposto^{1,2}

要 約

多段階的な線形反応があり、各段階の間を滑らかに移行するデータに、曲線をあてはめる問題について考察する。我々の提案は断片的な線形回帰を結合する代わりに、共変数として双曲線を代用し滑らかな曲線を求めることである。これは Griffiths と Miller および Watts と Bacon の二段階線形双曲反応モデルの拡張となり、二つ以上の線形部分に対応するための直感的かつ簡単な方法である。双曲共変数を使用して得られる回帰については、非線形回帰法をあてはめることにより、隣接した線形部分の曲率の度合を推定できる。これによって線形の場合に比べて、著しく複雑になることはない。反応の勾配が結合点で急に変化することについてどうしても納得できないような場合に、この追加作業は価値のあるものとなる。また、結合点(外挿するとき線形部分が交差する横軸の値)についても、その数およびおおよその位置がわかっているならば、推定できる。日本人女性コホートの初潮年齢の変化に関するデータを用いて、試験的にデータ解析を行うための利用例を示した。

[§] 本論文にはこの要約以外に訳文はない。承認 1991年7月22日。印刷 1992年9月。

¹放影研統計部, ²現在, メリーランド州ポトマック EMMES 会社。

Commentary and Review Series

Piecewise Linear Regression Splines with Hyperbolic Covariates[§]

John B. Cologne, Ph.D.¹; Richard Sposto, Ph.D.^{1,2}

Summary

Consider the problem of fitting a curve to data that exhibit a multiphase linear response with smooth transitions between phases. We propose substituting hyperbolas as covariates in piecewise linear regression splines to obtain curves that are smoothly joined. The method provides an intuitive and easy way to extend the two-phase linear hyperbolic response model of Griffiths and Miller and Watts and Bacon to accommodate more than two linear segments. The resulting regression spline with hyperbolic covariates may be fit by nonlinear regression methods to estimate the degree of curvature between adjoining linear segments. The added complexity of fitting nonlinear, as opposed to linear, regression models is not great. The extra effort is particularly worthwhile when investigators are unwilling to assume that the slope of the response changes abruptly at the join points. We can also estimate the join points (the values of the abscissas where the linear segments would intersect if extrapolated) if their number and approximate locations may be presumed known. An example using data on changing age at menarche in a cohort of Japanese women illustrates the use of the method for exploratory data analysis.

Introduction

We are interested in regression situations in which the data display multiple linear phases, or segments, with obvious join points. At times there may be subject matter knowledge that leads to the specification of a multiphase linear model as the true mechanistic model; in other instances, the multiphase linear model may simply be a convenient means of summarizing trends in the response, such as in the case of exploratory analyses. Piecewise linear regression splines^{1,2} may be employed if the join points are assumed known, although estimation of the exact values of the join points may still be desirable in some cases. However, the data may sometimes exhibit smooth transitions between the linear phases, or the investigator may expect such smooth transitions, so that a linear regres-

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¹Department of Statistics, RERF; ²presently at the EMMES Corporation, Potomac, Maryland.

sion spline imposes constraints on the fitted values (discontinuities in the first derivatives at the join points) that might lead to a poor or unacceptable fit to the data near the join points.

Consider the data in Figure 1, which we adapted from Figure C of Smith¹ (the fitted curve will be described later). The response Y is chemical conductivity in an acid-base titration experiment; the regressor variable x is the volume of base added. The slope of the response changes at the point where one equivalent amount of base has been added. Smith fit the piecewise linear spline

$$f(x) = \alpha_0 + \alpha_1 x + \gamma(x - t)_+ \quad (1)$$

where $f(x) = E(Y/x)$ and $(a)_+$ is the so-called "plus" function defined by

$$(a)_+ = \begin{cases} a & a \geq 0 \\ 0 & a < 0 \end{cases}$$

She used a hypothesized equivalence point of 5.0395 mL as the join point. By testing the statistical significance of a discontinuity between the two segments at the join point, she was able to draw inference concerning the fit of the theoretical model to the data. Two features of Figure 1 are noteworthy: 1) Although the data display two linear components, these appear to be smoothly joined by a small area of curvature; and 2) separate lines fit through the two segments do not intersect at the theoretical join point, so a data-based estimate of the join point is desirable.

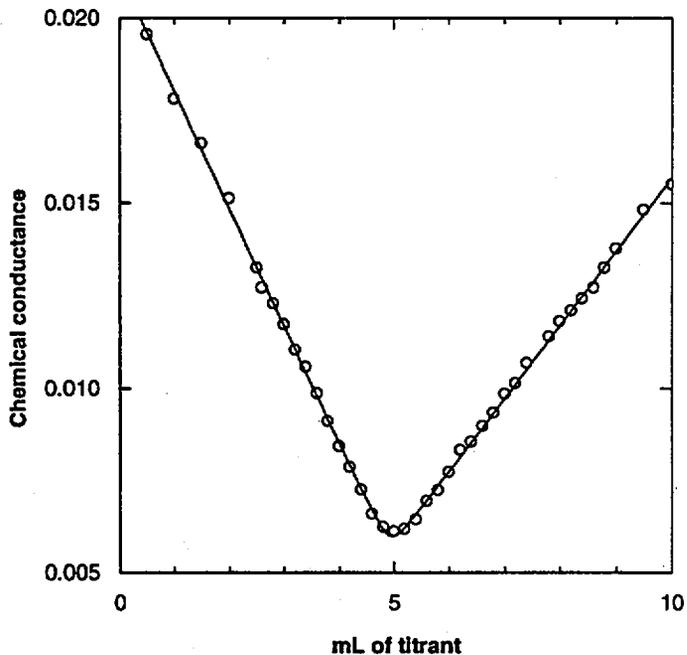


Figure 1. Chemical conductivity data (adapted from Figure C of Smith¹). The points represent observed chemical conductivity at given titrant levels. The curve is the two-phase linear regression spline with hyperbolic covariate (text equation [5] with t estimated; Table 1).

Griffiths and Miller³ and Watts and Bacon⁴ independently proposed fitting hyperbolas to two-phase linear models when the linear segments are smoothly joined. The expected response function for such a two-phase model is

$$f(x) = \beta_0 + \beta_1(x - t) + \beta_2 \sqrt{(x - t)^2 + e(\beta_3)} \quad (2)$$

(we use “ $e(\beta_3)$ ” in order to avoid an inequality constraint when estimating β_3). Because the limit of (2) as $\beta_3 \rightarrow -\infty$ is the piecewise linear spline (1), (2) generalizes (1) to include piecewise linear models that are smoothly joined.

In the next section we generalize the piecewise linear spline by replacing the plus function $(a)_+$ with a hyperbolic function as the regression spline covariate to estimate curvature at the join point. We show that, in the case of only two linear segments, this approach is exactly equivalent to the hyperbolic response model (2). Extending the use of hyperbolic covariates to include more than two linear segments is straightforward and intuitive. We also estimate the join points in cases in which their exact values are unknown or in which it is desirable to draw inference concerning hypothesized values. Finally, we illustrate the method with an exploratory analysis of changing age at menarche among a group of Japanese women.

Hyperbolic covariates

Model (2) can be derived from a slightly different approach than that taken by Griffiths and Miller³ or Watts and Bacon,⁴ which proves useful in easily generalizing to more than two linear segments. Rather than modeling $f(x)$ as a hyperbola, consider using a specific hyperbola in place of $(x - t)_+$ in equation (1). In particular, define

$$s(x; \delta, t) = \frac{(x - t)}{2} + \sqrt{\frac{(x - t)^2}{4} + e^\delta} \quad (3)$$

Figure 2 displays function (3) at $t = 0$ for various values of δ . Using $s(x; \delta, t)$ in place of $(x - t)_+$, (1) becomes

$$f(x) = \alpha_0 + \alpha_1 x + \gamma s(x; \delta, t) \quad (4)$$

Because

$$\lim_{\delta \rightarrow \infty} s(x; \delta, t) = (x - t)_+ ,$$

model (1) is a special case of the more general model (4). Continuity in $s(x; \delta, t)$ smoothes out the response in the region joining the segments to the left and right of t . Model (4) is equivalent to the hyperbolic response model (2): Starting with model (4),

$$\begin{aligned}
 f(x) &= \alpha_0 + \alpha_1 x + \frac{\gamma}{2}(x-t) + \gamma \sqrt{\frac{(x-t)^2}{4} + e^\delta} \\
 &= (\alpha_0 + \alpha_1 t) + \left(\alpha_1 + \frac{\gamma}{2} \right) (x-t) + \frac{\gamma}{2} \sqrt{(x-t)^2 + e^{\delta + \ln 4}} \\
 &= \beta_0 + \beta_1(x-t) + \beta_2 \sqrt{(x-t)^2 + e^{\beta_3}} \quad ,
 \end{aligned}$$

where $\beta_0 = (\alpha_0 + \alpha_1 t)$, $\beta_1 = (\alpha_1 + \gamma/2)$, $\beta_2 = (\gamma/2)$, and $\beta_3 = [\delta + \ln(4)]$.

Now we wish to extend model (4) to more than two segments. Suppose we have $P + 1$ linear segments with $P \geq 1$ join points t_p ($p = 1, \dots, P$). Define the model

$$f(x) = \alpha_0 + \alpha_1 x + \sum_{p=1}^P \gamma_p s(x; \delta_p, t_p) \tag{5}$$

with

$$s(x; \delta_p, t_p) = \frac{(x-t_p)}{2} + \sqrt{\frac{(x-t_p)^2}{4} + e^{\delta_p}} \quad .$$

Model (5) requires estimating a curvature parameter (δ_p) at each join point t_p . The join points may be fixed or estimated (although, again, if they are estimated, their number and approximate locations are presumed known). Equation (5) is a generalization of the piecewise linear regression spline

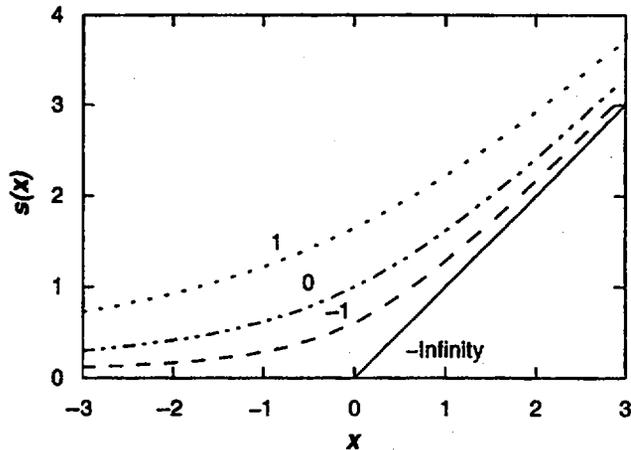


Figure 2. Examples of the hyperbolic covariate $s(x) = \frac{x}{2} + \sqrt{\frac{x^2}{4} + e^\delta}$ (text equation [3]) for various values of the curvature parameter δ .

$$f(x) = \alpha_0 + \alpha_1 x + \sum_{p=1}^P \gamma_p (x - t_p)_+ \quad , \quad (6)$$

which smoothes the transitions between segments while retaining the piecewise linear nature of the response.

Inference concerning the curvature parameter and join point

We can estimate the parameters of model (5) using nonlinear regression methods.⁵ Assume for the moment that t is known. To obtain the nonlinear least squares estimates of the parameters, we need the derivatives

$$\frac{\partial f}{\partial \alpha_0} = 1 \quad ,$$

$$\frac{\partial f}{\partial \alpha_1} = x \quad ,$$

and

$$\left. \begin{aligned} \frac{\partial f}{\partial \gamma_p} &= s(x; \delta_p, t_p) \\ \frac{\partial f}{\partial \delta_p} &= \frac{2\gamma_p e^{\delta_p}}{\sqrt{\frac{(x - t_p)^2}{4} + e^{\delta_p}}} \end{aligned} \right\} p = 1, \dots, P \quad .$$

Now suppose that the value of t_p is not known exactly (but t_p is assumed to exist) or that we wish to make inference concerning a hypothesized value t_p^0 . (Note that this does not include situations in which we wish to make inference concerning whether there should be a join point; see the Discussion.) We may easily incorporate estimation of t_p into the nonlinear regression problem. Adding t_p as a further parameter, we include the derivative

$$\frac{\partial f}{\partial t_p} = -\gamma_p \left[\frac{1}{2} + \frac{x - t_p}{4\sqrt{\frac{(x - t_p)^2}{4} + e^{\delta_p}}} \right] .$$

Inference concerning t_p may be based on the likelihood ratio test comparing model (5) with t_p estimated to model (5) with t_p fixed at $t_p = t_p^0$. In order to avoid boundary problems with $\delta = -\infty$ when making inference concerning t , we presume that the true value of δ is finite. This is consistent with our motivation for studying smoothly joined models, because we are interested in situations in which it is believed that the true underlying model involves a smooth transition between the linear segments.

Examples

Example 1: chemical conductivity. Consider again the data in Figure 1. Smith¹ proposed the model

$$f(x) = \alpha_0 + \alpha_1 x + \gamma (x - t^0)_+ + \psi I_{(x \geq t^0)} \quad (7)$$

where $I_{(a)}$ is an indicator function that is 1 when $a \geq 0$ and 0 otherwise, to make inference concerning the theoretical join point $t^0 = 5.0395$. The first two sections of Table 1 show least squares estimates of parameters for the continuous piecewise linear regression spline (1) and for the discontinuous model (7). Note that the spline model (1) contains a discontinuity in the first derivative at t^0 . This does not conform to the observed data, which display a smooth transition between the two linear segments. The null hypothesis $H_0: \psi = 0$ is rejected, and thus the data do not support the theoretical equivalence of 5.0395. Although the discontinuous spline model (7) therefore provides a better fit to the data than the continuous spline, it ignores the smooth joining between linear segments and it is also less realistic in describing the outcome because of its discontinuity.

Using model (7), we may compute the true equivalence point

$$t^* = t^0 - \frac{\psi}{\gamma} ,$$

where the fitted lines would intersect if extrapolated. Inference concerning t^* could be based upon the asymptotic variance of its maximum likelihood or nonlinear least squares estimator $\hat{t}^* = t^0 - \hat{\psi}/\hat{\gamma}$ using the delta method and the covariance between $\hat{\gamma}$ and $\hat{\psi}$ (Table 1). We estimate $\hat{t}^* = 4.951$. An intuitive alternative approach, however, is to estimate the join point directly using model (4) and to perform the likelihood ratio test of $H_0: t = t^0$. This approach may be preferable in that it provides a direct estimate of the join point and is more true to the observed data in accommodating the smooth transition between the two linear segments.

The bottom two sections of Table 1 show estimated parameters from the fitted spline model (4) with hyperbolic covariate, under $H_0: t = t^0$ and $H_1: t \neq t^0$ (where, again, $t^0 = 5.0395$). Under H_1 , we obtained $\hat{t} = 4.918$, which is slightly smaller than the value of \hat{t}^* obtained from the discontinuous model (7). Assuming normally distributed errors, the log likelihood ratio statistic is

$$L = -2 \log \left(\frac{\hat{\sigma}_1^2}{\hat{\sigma}_0^2} \right)^{\frac{n}{2}} = 39 \log \left[\frac{35(3.93 \times 10^{-8})}{34(1.55 \times 10^{-8})} \right] = 37.4 ,$$

with $p < .001$. Thus the regression spline with hyperbolic covariate to accommodate curvature at the join point confirms Smith's finding that the data do not support the theoretical join point, but based on a more reasonable model. The plotted line in Figure 1 represents the model under H_1 —namely, two smoothly joined segments with the join point estimated.

Table 1. Parameter estimates for the first example

Parameter	Estimate	SE
Continuous linear regression spline—Model 1		
α_0	0.02087	0.0001190
α_1	-0.003027	0.00003089
γ	0.005085	0.00005003
	$s^2 = 4.18 \times 10^{-8}$ *	
Discontinuous linear regression spline—Model 7		
α_0	0.02106	0.0001039
α_1	-0.003109	0.00003039
γ	0.005093	0.0004000
ψ	0.0004527	0.00009775
	$s^2 = 2.667 \times 10^{-8}$	
	$\text{Cov}(\gamma, \delta) = 1.859 \times 10^{-10}$	
Hyperbolic covariate regression spline—Model 4—with fixed join point ($t = 5.0395$)		
α_0	0.02092	0.0001211
α_1	-0.003057	0.00003846
γ	0.005142	0.00006699
δ	-4.914	0.9938
	$s^2 = 3.93 \times 10^{-8}$	
Hyperbolic covariate regression spline—Model 4—with estimated join point		
α_0	0.02120	0.00008622
α_1	-0.003181	0.00003038
γ	0.005173	0.00004253
δ	-4.983	0.6546
t	4.918	0.01613
	$s^2 = 1.55 \times 10^{-8}$	

*Residual mean square.

Example 2: age at menarche. The data in Figure 3 show mean age at menarche by year of birth among a cohort of Japanese women (the fitted curve is described below). Individual data (age at menarche vs. birthdate) are too numerous to show (the total sample size is 7972). Except for small numbers at the endpoints, the number of women in each birth year in this sample is approximately evenly distributed before and after 1945, with approximately 30–50 women per birth year prior to 1945 and around 200 per birth year after 1945. This cohort is from studies of reproductive effects of radiation exposure in Hiroshima and Nagasaki (we ignore questions of radiation effects in this report, however). A portion of these data has been described elsewhere.⁶

With the exception of birth years between approximately 1925 and 1950, the trend in age at menarche appears to consist of two major linear components. In particular, age at menarche seems to have declined more rapidly in women born after 1950 than in those born before 1925. This might be due to more rapid changes in socioeconomic conditions (including diet and life-style) in Japan after World War II, but this is only speculation; the cause deserves further study but is not immediately relevant to our present purposes. For birth years in the period 1925–50, there is a rather complex relationship between year of birth and age at menarche (this period corresponds to girls reaching the average age at menarche between approximately 1940 and 1965). Qualitatively, the response increases slightly for birth years near 1930, then drops rapidly until about 1945, then increases slightly again before resuming a steady decline in those born after 1950.

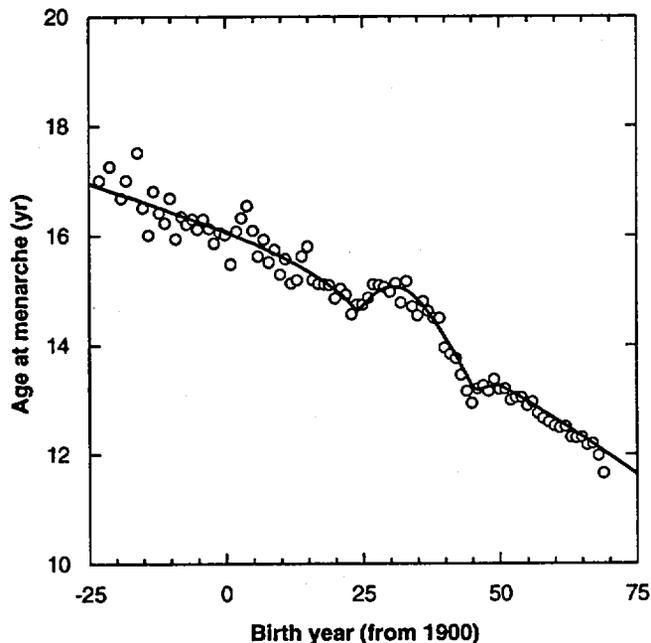


Figure 3. Mean age at menarche in a cohort of Japanese women by birth year. The points indicate observed mean age at menarche by calendar year. The curve is the fitted multiphase linear regression spline with hyperbolic covariates (text equation [8]; Table 2).

Although we cannot postulate a plausible sociobiological model for the behavior in age at menarche described above, the data demonstrate several approximately linear segments. We thus fit the model

$$f(x) = \alpha_0 + \alpha_1 + \sum_{p=1}^4 \gamma_p s(x; \delta_p, t_p) \quad (8)$$

with $P = 4$ join points taken to be 1925, 1930, 1945, and 1950 (we used these four values as starting values in a nonlinear least squares estimation of the t_p). Table 2 shows the estimated parameters for this model; the resulting fitted model is plotted in Figure 3. Only one curvature term contributed to the fit; this was the one for the second covariate, with the join point estimated at around 1931. The other curvature parameters were large in magnitude and negative, so that their contribution to the model (after exponentiation) was essentially zero. It is interesting that curvature from the one segment with measurable curvature induced nonlinearity in a nonadjacent segment; note in Figure 3 how the left-most segment curves downward toward the center of the plot. Although potentially a nuisance, this apparent contamination does not seem to degrade the

Table 2. Parameter estimates for the second example

Parameter	Estimate	SE
α_0	16.33	0.1636
α_1	-0.0307	0.009514
γ_1	0.2474	0.05596
γ_2	-0.4426	0.06246
γ_3	0.2248	0.06696
γ_4	-0.07647	0.05743
t_1	24.04	0.5007
t_2	30.63	2.164
t_3	45.34	0.9729
t_4	49.63	1.359
δ_1	— ^a	— ^b
δ_2	2.965	0.5019
δ_3	—	—
δ_4	—	—

$$s^2 = 1.671^c$$

^aThe estimated coefficient was less than -100, making $\exp(\delta_p)$ essentially zero.

^bThe SE could not be estimated by the usual asymptotic methods.

^cResidual mean square.

model fit in the present example. One could also consider introducing an additional join point around 1920.

Discussion

We have shown that multiphase linear regression models may be generalized to incorporate smooth transitions by using piecewise linear regression splines with hyperbolic covariates. In so doing, we have extended the two-phase model of Griffiths and Miller³ and Watts and Bacon.⁴ The result is a model that can provide a more acceptable fit to data that reflect gradual transitions between linear segments. If there is curvature over much of the length of a particular segment, the piecewise linear model might not hold, and an alternative response function might provide a better fit. On the other hand, if adjoining segments are linear but there are insufficient observations in the region of curvature at the join point, then the curvature cannot be estimated (A. J. Miller, personal communication).

The method is applicable to observational studies involving linear trends, where occasional perturbances periodically induce changes in the slope of the response. As an example, we have presented the results of one such study, in which multiple changes in socioeconomic conditions before, during, and after World War II may have been the cause of changes in the rate of decline in average age at menarche among Japanese women. The use of models with smooth transitions may be appealing to investigators because, although the perturbation may act abruptly, there may be no reason to assume that the slope of the measured response changes in an equally abrupt manner. In social and biological studies, natural variation among members of the study population will almost certainly guarantee differences in reaction to the perturbation, thereby smoothing the transition into the next phase. In our example, curvature was detected at only one transition point, contrary to our expectations. It is possible that the underlying response in fact changed abruptly, but it is also possible that variability in the data was too large to estimate the curvature parameters with any precision.

Hudson⁷ discussed the estimation of join points in piecewise regression models using linear least squares. He showed that the overall least squares solution depends upon whether a join point coincides with an observed value of the abscissa and whether the fitted curves are constrained to have equal derivatives at a join point. In the present case, we do not believe a similar problem exists because 1) we are not explicitly imposing the equal derivative constraint on neighboring segments and 2) we can assume that the join point coincides with an observed abscissa value with probability zero (this ignores finite machine precision, of course; however, even with finite numbers, the probability that the true join point exactly equals one of the observed abscissa values should be very small). We do caution the reader, however, that the selection of good starting values for the join point is extremely crucial. In our experience with the above examples and several small simulation studies (data not shown), we have found that starting values that are far from the true join point can lead to local solutions to the nonlinear least squares algorithm that fit the data quite poorly. As is well known, it is important to check the fitted model—for example, by plotting fitted

values superimposed over the observed values—before accepting the output of a nonlinear regression algorithm.

In the age-at-menarche example, curvature at one join point influenced the fit in a nearby but nonadjacent segment. This may be explained by spillover of the difference between $s(x; t_p, \delta_p)$ and $(x - t_p)_+$ beyond a neighboring join point. Although the fitted model thus fortuitously picked up some nonlinearities in the present example, such behavior might not in general be desirable. The extent of such an influence on neighboring segments will obviously depend upon the data being analyzed. In our example, the two smoothly joined segments could have been collapsed into a single segment and fit with a second degree polynomial. However, this would require some messy algebraic calculations⁸ in order to preserve linearity in segments to the right of those in question.

We hasten to point out that, as with all regression models, we have made certain assumptions in applying the piecewise linear model. One is that the segments are truly linear; this can be tested in the regression spline model (6) by adding quadratic terms to the regression spline. But the resulting inference will be only approximate if there is significant curvature between adjoining segments. This is because the null model (6) would not be strictly valid since it ignores the curvature. Another assumption concerns the number and placement of the join points, or "knots," of the spline. Assuming the correct number of join points is highly critical, because this will impact inferences made concerning the fit of the presumed model to the data. In the present report we have assumed that the number and approximate locations of the join points are known, although this ignores the fact that this knowledge may be based upon inspection of the data. This should not pose a problem with exploratory analyses, however.

As pointed out by Smith,¹ it is possible to test the need for additional join points at fixed locations using the regression spline (6). Yet we do not presently have any means of making inference concerning the number of join points in the more general setting in which the degree of curvature is to be estimated. Friedman and Silverman⁹ discussed estimation of the number and placement of knots in regression splines by minimizing the average squared residual. Maximum likelihood-based inference is not possible when new join points are to be estimated, because the points are undefined under the null hypothesis that further regression parameters are equal to zero; see Feder¹⁰ for some results concerning inference in this situation. With our model, one additional parameter (curvature) is similarly undefined at each potential join point. Further research into the problem of making inferences concerning the number and placement of knots would be useful. One approach might be to use computationally intensive methods (e.g., bootstrapping) in place of more formal inferential methods.

Finally, the present methodology has been developed for a specific model (the piecewise linear spline with smoothly joined segments). However, the use of hyperbolic covariates may have broader applicability as a general method of smoothing discontinuities near knots. Other possible applications would be welcomed.

Acknowledgments

This research was motivated in part by the age-at-menarche data discussed briefly in Example 2. We are extremely grateful to Dr. Midori Soda and to the Department of Clinical Studies at the Radiation Effects Research Foundation for their efforts in obtaining these data and for making them available to us. We also thank Dr. Randy Carter for his valuable comments based on a critical reading of an earlier draft.

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