MZ–TH/06–14
DESY 06–130

Next-to-next-to-leading order $\mathcal{O}(\alpha^2\alpha_s^2)$ results for top quark pair production in photon–photon collisions: The one-loop squared contributions

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(Dated: November 4, 2006)

We calculate the one-loop squared contributions to the next-to-next-to-leading order $\mathcal{O}(\alpha^2\alpha_s^2)$ radiative QCD corrections for the production of heavy quark pairs in the collisions of unpolarized on–shell photons. In particular, we present analytical results for the squared matrix elements that correspond to the product of the one–loop amplitudes. All results of the perturbative calculation are given in the dimensional regularization scheme. These results represent the Abelian part of the corresponding gluon–induced next-to-next-to-leading order cross section for heavy quark pair hadroproduction.

PACS numbers: 12.38.Bx, 13.85.-t, 13.85.Fb, 13.88.+e

I. INTRODUCTION

The increasing precision of present and forthcoming experiments in high energy physics requires a corresponding precision of the theoretical predictions. The next-to-leading order (NLO) predictions for hadronic heavy quark pair production suffer from inherent theoretical errors because of the well–known large uncertainty in choosing the renormalization and factorization scales. These errors are expected to be greatly reduced at next-to-next-to-leading order (NNLO) and therefore the need for a NNLO calculation of hadronic heavy quark pair production in QCD is by now clearly understood.

A few years ago virtual two-loop and loop-by-loop corrections (we shall also refer to the "one-loop squared" contributions as the "loop-by-loop" contributions) were calculated by several groups in massless QCD (see e.g. [1] and references therein). The completion of a similar program for processes that involve massive quarks requires much more dedication and work since the inclusion of an additional mass scale dramatically complicates the whole situation. There are a number of publications where physicists work towards building up the necessary tools for calculating two-loop massive processes. For example, there are papers in which fully analytical forms of various master integrals are derived (see e.g. [2, 3, 4, 5]). Recently Bernreuther *et al.* calculated the two-loop vertex corrections to heavy quark pair production from vector and axial vector currents [6, 7]. These results were utilized to determine a partial result on the forward–backward asymmetry in the process $e^+e^- \to Q\overline{Q}$ involving the two-loop contributions folded with the Born term and the loop-by-loop contributions [8].

In general, there are four classes of contributions that need to be calculated for the NNLO corrections to the hadronic production of heavy quark pairs. The first class involves the pure two-loop contribution which has to be folded with the leading order (LO) term. The second class of diagrams consist of the so-called loop-by-loop contribution arising from the product of one-loop virtual matrix elements which, for the special case of $\gamma\gamma$ collisions, form the subject of this paper. Further there are the one-loop gluon emission contributions that are folded with the one–gluon emission graphs. The one-loop gluon emission contributions also include the interesting class of the so-called pentagon graphs. Finally, there are the squared two gluon emission contributions that are purely of tree–type.

In this paper, we concentrate on heavy quark pair production in photon-photon collisions which constitute the Abelian part of the gluon–induced hadroproduction of heavy quark pairs. From the technical point of view photon–induced heavy quark pair production is much simpler than the corresponding hadroproduction of heavy quark pairs. First of all there are no contributions from the subprocess $q\bar{q} \to Q\overline{Q}$. Second there are no contributions from three–gluon coupling graphs which implies that there will be no collinear singularities and therefore the highest singularity in the one–loop amplitudes will be an infrared (IR) singularity proportional to $(1/\varepsilon)$. This in turn implies that the Laurent series expansion of the one–loop amplitudes can be truncated at $\mathcal{O}(\varepsilon)$ when calculating the loop–by–loop contributions. This is quite a bonus since it is the $\mathcal{O}(\varepsilon^2)$ terms in the Laurent series expansion that really complicate things [5]. Whereas the $\mathcal{O}(\varepsilon^2)$ contributions in the one–loop amplitudes involve a multitude of multiple polylogarithms of maximal weight and depth four [5, 9] the $\mathcal{O}(\varepsilon)$ contributions needed in the present calculation involve at most trilog functions with their accompanying $\zeta(3)$ functions.

It has been emphasized by many physicists that running the ILC in the photon–photon mode is a very interesting option for the ILC (see e.g. [10, 11]). The high energy photons can be generated by Compton back–

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scattering of laser light on the high energy electron and positron bunches of the collider with practically no loss in energy and luminosity. Note that this reaction is also a clean channel for the investigation of various properties of heavy quarks (see e.g. [12]).

In this paper we report on a calculation of the NNLO squared one–loop matrix elements (loop–by–loop contribution) for the process $\gamma\gamma \to Q\overline{Q}$. The calculation is carried out in the dimensional regularization scheme [13] with space-time dimension $n = 4 - 2\varepsilon$. The $\mathcal{O}(\varepsilon)$ expansion for the one-loop scalar master integrals that enter the calculation have been determined by us in [5]. For the divergent and finite terms the relevant amplitude expressions were given in [14]. The order ε amplitudes have been written down in [15].

In a sequel to this paper we shall present results on the square of photoproduction amplitudes ($\gamma g \to Q\overline{Q}$). Further we shall calculate the squares of the non–Abelian gluon–induced $gg \to Q\overline{Q}$ amplitudes and the quark–induced $q\bar{q} \to Q\overline{Q}$ amplitudes which are needed for the loop-by-loop part of the NNLO description of heavy flavor production.

In our presentation we shall make use of our notation for the coefficient functions of the relevant scalar one–loop integrals calculated up to $\mathcal{O}(\varepsilon^2)$ in [5]. For the $\gamma\gamma \to Q\overline{Q}$ case one needs one scalar one–point function A, the four scalar two–point functions B_1, B_2, B_3 and B_4, the three scalar three–point functions C_2, C_5 and C_6, and one scalar four-point function D_1. As was mentioned before, the $\gamma\gamma \to Q\overline{Q}$ amplitudes are only IR–singular due to the absence of a three–gluon coupling contribution. And, in fact, one finds that the above set of multi–point amplitudes have at most a $1/\varepsilon$ singularity [5]. For example, employing the notation of [5], we define successive coefficient functions $D_1^{(j)}$ for the Laurent series expansion of the *complex* scalar four-point function D_1. One has

$$D_1 = iC_\varepsilon(m^2)\Big\{\frac{1}{\varepsilon}D_1^{(-1)} + D_1^{(0)} + \varepsilon D_1^{(1)} + \mathcal{O}(\varepsilon^2)\Big\}, \quad (1.1)$$

where $C_\varepsilon(m^2)$ is defined by

$$C_\varepsilon(m^2) \equiv \frac{\Gamma(1+\varepsilon)}{(4\pi)^2}\left(\frac{4\pi\mu^2}{m^2}\right)^\varepsilon. \quad (1.2)$$

We use this notation for both the real and the imaginary part of D_1, i.e. for $\mathrm{Re}D_1$ and $\mathrm{Im}D_1$. As also mentioned before we can truncate the Laurent series at $\mathcal{O}(\varepsilon)$ since this is sufficient for the $\gamma\gamma \to Q\overline{Q}$ case treated in this paper. Expansions similar to Eq. (1.1) hold for the scalar one–point function A, the scalar two–point functions B_i and the scalar three–point functions C_i.

The paper is organized as follows. Section II contains an outline of our general approach as well as a discussion of the singularity structure of the NNLO squared matrix element. In Section III we discuss the structure of the finite part of our result. Our results are summarized in Section IV. Finally, in an Appendix we present results for the various coefficient functions that appear in the main text.

II. NOTATION AND THE SINGULARITY STRUCTURE OF THE SQUARED AMPLITUDES

The one–loop Feynman diagrams relevant for heavy flavor production by two on-shell photons

$$\gamma(p_1) + \gamma(p_2) \to Q(p_3) + \overline{Q}(p_4), \quad (2.1)$$

are depicted in Fig. 1.

The directions of the momenta correspond to the physical configuration, e.g. p_1 and p_2 are ingoing whereas p_3 and p_4 are outgoing. With m the heavy quark mass we define:

$$s \equiv (p_1+p_2)^2, \qquad t \equiv T - m^2 \equiv (p_1-p_3)^2 - m^2,$$
$$u \equiv U - m^2 \equiv (p_2-p_3)^2 - m^2. \quad (2.2)$$

so that the energy-momentum conservation reads $s+t+u=0$.

When squaring the amplitudes one sums over the spins of the final state heavy quarks. We have decided to also average over the spins in the initial state when presenting our results. In the spin–averaging over the initial photons we divide by a factor of four. One could as well divide by $(n-2)(n-2)$ where $(n-2) = 2(1-\varepsilon)$ are the spin degrees of freedom of a on–shell massless photon in DREG. This is a matter of convention. If one wants to convert to the convention where the spin–averaging factor is $(n-2)(n-2)$ one has to multiply our result by the overall factor

$$(1-\varepsilon)^{-2} = 1 + 2\varepsilon + 3\varepsilon^2 + \mathcal{O}(\varepsilon^3). \quad (2.3)$$

Also, our results have to be multiplied by the overall factor

$$\mathcal{C} = \left(g^2 e_Q^2 g_s^2 C_\varepsilon(m^2)\right)^2, \quad (2.4)$$

where g and g_s are the renormalized electromagnetic and strong coupling constants, respectively, and e_Q is the fractional charge of the heavy quark. $C_\varepsilon(m^2)$ is defined in (1.2).

When squaring amplitudes, we contract polarization vectors in the Feynman gauge. Note that we have used the on–shell conditions $p_1 \cdot \epsilon_1 = 0$ and $p_2 \cdot \epsilon_2 = 0$ in our amplitude calculation [14, 15] according to the framework set up in [16]. This has advantages in the non–Abelian case since one can omit ghost contributions when squaring the amplitudes. Using the above on–shell conditions already on the amplitude level means that one takes full advantage of gauge invariance when squaring the amplitudes.

As shown e.g. in [14, 15] the self–energy and vertex diagrams contain ultraviolet (UV) and IR poles after mass renormalization. It is well known that the renormalization of the wave function and QED-type vertex renormalization effectively imply multiplication of all the external

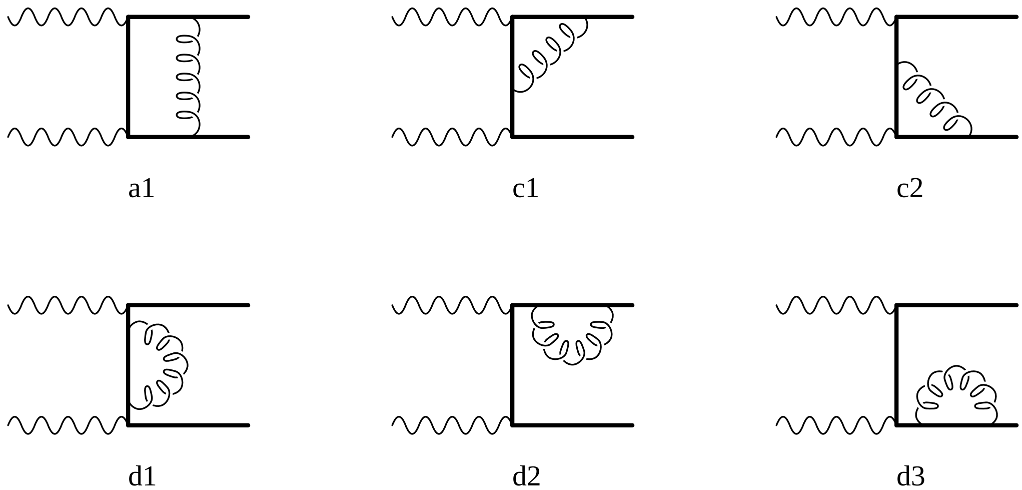

FIG. 1: The t-channel one-loop graphs contributing to the photon fusion amplitude. Wavy lines represent the photons, curly lines represent the gluons and the thick solid lines correspond to the heavy quarks.

massive quark self–energy graphs by one–half. We used this fact as a partial check of our calculation. Halving the external self–energies (in our case these are matrix elements for the graphs Fig. 1(d2) and Fig. 1(d3)) we have explicitly verified cancellation of UV divergences for all the relevant sets of squared amplitudes.

In order to fix our normalization we write down the differential cross section for $\gamma\gamma \to Q\overline{Q}$ in terms of our amplitudes squared $|M|^2$. One has

$$\frac{d\sigma_{\gamma\gamma\to Q\overline{Q}}}{dtdu} = \frac{1}{2s}\frac{d(\mathrm{PS})_2}{dtdu}|M|^2_{\gamma\gamma\to Q\overline{Q}}\,, \tag{2.5}$$

where the n–dimensional two–body phase space is given by

$$d(\mathrm{PS})_2 = \frac{m^{-2\varepsilon}}{8\pi s}\frac{(4\pi)^\varepsilon}{\Gamma(1-\varepsilon)}\left(\frac{tu-sm^2}{sm^2}\right)^{-\varepsilon}\delta(s+t+u)dtdu. \tag{2.6}$$

The final spin summed and initial spin averaged squared matrix element can be written down as a sum of three terms:

$$|M|^2_{\mathrm{Loop}\times\mathrm{Loop}} = \mathcal{C}C_F^2\Big(\frac{1}{\varepsilon^2}V^{(-2)} + \frac{1}{\varepsilon}V^{(-1)} + N_C V^{(0)}\Big), \tag{2.7}$$

where $C_F = 4/3$ and C has been defined in (2.4). Note that we used 1/4 for the spin-averaging factor as explained before (2.3). The first two terms contain IR poles of the second and the first order, respectively. The third term represents the finite part. All three terms are originally bilinear forms in the coefficient functions that define the Laurent series expansion of the scalar integrals (1.1). Some of these coefficient functions are zero and some of them are just numbers. In the latter case we have substituted these numbers for the coefficient functions in the three terms above. This has been done for the coefficient functions $A^{(j)}$, $B_1^{(-1)}$, $B_2^{(-1)}$, $B_3^{(j)}$ and $B_4^{(j)}$.

The most singular first term in (2.7) is proportional to the leading order term of the process $\gamma\gamma \to Q\overline{Q}$, e.g.

$$\frac{1}{\varepsilon^2}V^{(-2)} = \frac{4}{\varepsilon^2}\left|\left(2m^2-s\right)C_6^{(-1)} - 1\right|^2 \frac{1}{(ge_Q)^4}|M_{\mathrm{LO}}|^2, \tag{2.8}$$

where the Taylor series expansion of the square of the LO amplitude $|M_{\mathrm{LO}}|^2$ can be calculated to be

$$\frac{1}{(ge_Q)^4}|M_{\mathrm{LO}}|^2 = 2N_C\Big\{\frac{t^2+u^2}{tu} + 4\frac{m^2 s}{tu} - 4\left(\frac{m^2 s}{tu}\right)^2 +\varepsilon\; 2\left(1-\frac{s^2}{tu}\right) + \varepsilon^2\frac{s^2}{tu}\Big\}. \tag{2.9}$$

Note that the original amplitude expression contains the coefficient functions $D_1^{(-1)}$ and $D_1^{(-1)}\big|_{t\leftrightarrow u}$ which have been substituted for by the $(t \leftrightarrow u)$ symmetric coefficient function $C_6^{(-1)}$ using the relations $D_1^{(-1)} = C_6^{(-1)}/t$ and $D_1^{(-1)}\big|_{t\leftrightarrow u} = C_6^{(-1)}/u$ [5]. It is only after these substitutions that the factorization of $|M_{\mathrm{LO}}|^2$ in (2.8) becomes apparent. We remind the reader that the coefficient functions are complex functions. This has to be taken into account when calculating the three contributions in Eq. (2.7).

For the second term in (2.7) we obtain:

$$\tfrac{1}{\varepsilon}V^{(-1)} = \tfrac{8}{\varepsilon}N_C\mathrm{Re}\Big\{\Big((2m^2-s)C_6^{(-1)} - 1\Big)\Big[h + B_1^{(0)}h_{B_1^0} + \tfrac{1}{2}B_2^{(0)}h_{B_2^0} \tag{2.10}$$

$$+C_2^{(0)}h_{C_2^0}+\tfrac{1}{2}C_5^{(0)}h_{C_5^0}+\tfrac{1}{2}C_6^{(0)}h_{C_6^0}+D_1^{(0)}h_{D_1^0}+(t\leftrightarrow u)\Big]^*\Big\}.$$

We have chosen to label the coefficient functions h_i according to the coefficients of the Laurent series expansion which they multiply. Note we have again reexpressed the coefficient functions $D_1^{(-1)}$ and $D_1^{(-1)}\big|_{t\leftrightarrow u}$ via $C_6^{(-1)}$ in order to obtain (2.10) in a factorized form.

The relevant coefficients read:

$$\begin{aligned}
h &= 2m^4(16m^4s+4m^2st+28m^2su-8m^2tu+2stu-19tu^2-30m^2tu^2/T)/(Utu^3)\\
&\quad -(56m^8u+29m^6st-17m^4t^2U-105m^4tuT+7m^2t^3T+m^2t^2uT-2m^2t^2uU-3st^3u)/(s\beta^2tuTU),\\
h_{B_1^0} &= (4m^4s^2-16m^4sT+8m^2tTu-3st^2(2m^2-u)-t^4(2m^2-u)/T)/(t^3u),\\
h_{B_2^0} &= -4D/(tu\beta^2),\\
h_{C_2^0} &= 2(6m^4st-2m^4su+8m^4tu-2m^2st^2-4m^2t^3-2st^3+t^2u^2)/(t^2u),\\
h_{C_5^0} &= -s(8m^4-3s^2+2tu)/(tu),\\
h_{C_6^0} &= -(8m^4s+2m^2s^2+8m^2tu-s^3-2stu)/(tu),\\
h_{D_1^0} &= (16m^6s+24m^4st-8m^4u^2+2m^2stu+2s^2t^2-stu^2)/(tu),
\end{aligned}\tag{2.11}$$

where $\beta=\sqrt{1-4m^2/s}$. The terms multiplied by 1/2 in (2.10) are $(t\leftrightarrow u)$ symmetric which, when adding the $(t\leftrightarrow u)$ term, add up to the full contribution.

We emphasize that the real part of the second factor in square brackets in (2.10) is nothing but the finite part of the NLO contribution (up to an overall trivial factor, see below) calculated in [17] by summing the two finite terms (i.e. not multiplying the IR pole $1/\varepsilon$) in Eqs. (16) and (22) of [17]. We mention that one can also obtain the imaginary part of the second factor in square brackets from [17] by using the procedure described in the last section of [14]. When comparing with the results in [17] one has to of course substitute the explicit expressions for the coefficient functions given in [5].

In the following we want to exhibit a remarkable structure of the NLO and NNLO single pole contributions. For the NLO case we have

$$|M^{\rm pole}_{\rm NLO}|^2=\frac{4}{\varepsilon}g_s^2C_\varepsilon(m^2)C_F \times {\rm Re}\left\{\left(2m^2-s\right)C_6^{(-1)}-1\right\}|M_{\rm LO}|^2; \tag{2.12}$$

Let us now write Eq. (2.10) in the form

$$\frac{1}{\varepsilon}V^{(-1)}=\frac{8}{\varepsilon}N_C({\rm Re}A\ {\rm Re}B+{\rm Im}A\ {\rm Im}B), \tag{2.13}$$

where A and B stand for the round and square brackets, respectively, i.e.

$$\begin{aligned}
A &= (2m^2-s)C_6^{(-1)}-1,\\
B &= h+B_1^{(0)}h_{B_1^0}+\frac{1}{2}B_2^{(0)}h_{B_2^0}+C_2^{(0)}h_{C_2^0}\\
&\quad +\frac{1}{2}C_5^{(0)}h_{C_5^0}+\frac{1}{2}C_6^{(0)}h_{C_6^0}+D_1^{(0)}h_{D_1^0}+(t\leftrightarrow u).
\end{aligned}\tag{2.14}$$

One observes the remarkable pattern

$$|M^{\rm pole}_{\rm NNLO}|^2\equiv \mathcal{C}C_F^2\frac{1}{\varepsilon}V^{(-1)}=\frac{2}{\varepsilon}g_s^2C_\varepsilon(m^2)C_F \times\left\{{\rm Re}A\ |M^{\rm finite}_{\rm NLO}|^2+4(ge_Q)^4g_s^2C_\varepsilon(m^2)C_FN_C{\rm Im}A\ {\rm Im}B\right\}. \tag{2.15}$$

We also mention that when calculating separately the $t-,u-$channel contributions and their interference the second power of the denominator $D=m^2s-tu$ would still be present in the results for the $1/\varepsilon$ contributions. Only after adding up all contributions the two remaining powers of this denominator for the $1/\varepsilon$ contributions cancel. It is in this way that the factorized structure of the amplitude is obtained. In the previous NLO calculation by one of us [17], the cancellation of all the powers of the denominator D as well as the factorized structure was obtained separately for the $t\otimes t$ or $t\otimes u$ contributions. Of course, when adding up all the channels, we obtain the same result as in [17] for the complete gauge invariant set of diagrams.

Note that the factorized form of Eq. (2.15) holds only when one retains the full ε dependence in the Born term as given in (2.9).

We would like to mention that the nice and simple factorized form of (2.15) could have been anticipated from the fact that in the one-loop matrix elements the single pole always multiplies the LO term. Consider e.g. Eq. (7) of the recent publication [18] where the predictions for an overall singular structure of massless one-loop amplitudes are extended to the case of massive partons. In our case with only two colored massive particles, the second term in Eq. (9) of [18], i.e. containing the infrared factor which controls color correlations, completely factorizes. The second single pole structure comes from the function of Eq. (18) of [18]. Together, they give a full IR structure that is proportional to

$$2C_F\left\{\frac{2m^2-s}{s\beta}\left(\ln x+i\pi\right)-1\right\}. \tag{2.16}$$

Considering the fact that the $1/\varepsilon$ pole in our integral function C_6 is multiplied by $(\ln x+i\pi)/s\beta$, we indeed reproduce in our Eq. (2.15) the infrared structure derived (e.g. by formal squaring of Eq. (7)) from general considerations in [18]. In the next section we present the finite parts of our squared amplitudes. Quite naturally, these results cannot be derived from general principles and must be obtained from an explicit calculation.

III. STRUCTURE OF THE FINITE PART

In this section we present the finite part of our result. It can be written as

$$\begin{aligned}
V^{(0)} = {}& \tfrac{1}{2}f + \mathrm{Re}\Big\{B_1^{(0)}\Big[f_{B_1^0} + B_1^{(0)} f_{B_1^0B_1^0} + \tfrac{1}{2}B_{1u}^{(0)} f_{B_1^0B_{1u}^0} + B_2^{(0)} f_{B_1^0B_2^0} + C_2^{(0)} f_{B_1^0C_2^0} + C_{2u}^{(0)} f_{B_1^0C_{2u}^0} \\
& + C_5^{(0)} f_{B_1^0C_5^0} + C_6^{(0)} f_{B_1^0C_6^0} + D_1^{(0)} f_{B_1^0D_1^0} + D_{1u}^{(0)} f_{B_1^0D_{1u}^0}\Big]^* + B_1^{(1)} f_{B_1^1} + B_2^{(0)}\Big[\tfrac{1}{2}f_{B_2^0} + \tfrac{1}{2}B_2^{(0)} f_{B_2^0B_2^0} \\
& + C_2^{(0)} f_{B_2^0C_2^0} + \tfrac{1}{2}C_5^{(0)} f_{B_2^0C_5^0} + \tfrac{1}{2}C_6^{(0)} f_{B_2^0C_6^0} + D_1^{(0)} f_{B_2^0D_1^0}\Big]^* + \tfrac{1}{2}B_2^{(1)} f_{B_2^1} + C_2^{(0)}\Big[f_{C_2^0} + C_2^{(0)} f_{C_2^0C_2^0} + \tfrac{1}{2}C_{2u}^{(0)} f_{C_2^0C_{2u}^0} \\
& + C_5^{(0)} f_{C_2^0C_5^0} + C_6^{(0)} f_{C_2^0C_6^0} + D_1^{(0)} f_{C_2^0D_1^0} + D_{1u}^{(0)} f_{C_2^0D_{1u}^0}\Big]^* + C_2^{(1)} f_{C_2^1} + \tfrac{1}{2}C_5^{(0)}\Big[f_{C_5^0} + C_5^{(0)} f_{C_5^0C_5^0} + C_6^{(0)} f_{C_5^0C_6^0} \\
& + 2D_1^{(0)} f_{C_5^0D_1^0}\Big]^* + \tfrac{1}{2}C_5^{(1)} f_{C_5^1} + \tfrac{1}{2}C_6^{(0)}\Big[f_{C_6^0} + C_6^{(0)} f_{C_6^0C_6^0} + 2D_1^{(0)} f_{C_6^0D_1^0}\Big]^* + \tfrac{1}{2}C_6^{(1)} f_{C_6^1} + D_1^{(-1)}\Big[f_{D_1^{-1}} + B_1^{(0)} f_{D_1^{-1}B_1^0} \\
& + B_1^{(1)} f_{D_1^{-1}B_1^1} + B_2^{(0)} f_{D_1^{-1}B_2^0} + B_2^{(1)} f_{D_1^{-1}B_2^1} + C_2^{(0)} f_{D_1^{-1}C_2^0} + C_2^{(1)} f_{D_1^{-1}C_2^1} + C_5^{(0)} f_{D_1^{-1}C_5^0} + C_5^{(1)} f_{D_1^{-1}C_5^1} + C_6^{(0)} f_{D_1^{-1}C_6^0} \\
& + C_6^{(1)} f_{D_1^{-1}C_6^1} + D_1^{(0)} f_{D_1^{-1}D_1^0} + D_1^{(1)} f_{D_1^{-1}D_1^1}\Big]^* + D_{1u}^{(-1)}\Big[B_1^{(0)} f_{D_{1u}^{-1}B_1^0} + B_1^{(1)} f_{D_{1u}^{-1}B_1^1} + C_2^{(0)} f_{D_{1u}^{-1}C_2^0} \\
& + C_2^{(1)} f_{D_{1u}^{-1}C_2^1} + D_1^{(0)} f_{D_{1u}^{-1}D_1^0} + D_1^{(1)} f_{D_{1u}^{-1}D_1^1}\Big]^* + D_1^{(0)}\Big[f_{D_1^0} + D_1^{(0)} f_{D_1^0D_1^0} + \tfrac{1}{2}D_{1u}^{(0)} f_{D_1^0D_{1u}^0}\Big]^* + D_1^{(1)} f_{D_1^1}\Big\} + (t\leftrightarrow u),
\end{aligned} \tag{3.1}$$

where the subscript "u" is an operational definition prescribing a $(t\leftrightarrow u)$ interchange in the argument of that function, i.e. $B_{1u}^{(0)}=B_1^{(0)}\big|_{t\leftrightarrow u}$ etc... Note that all three functions $C_6^{(-1)}$, $D_1^{(-1)}$ and $D_{1u}^{(-1)}$ appear when calculating (3.1), as well as some spurious poles $1/\varepsilon$ multiplying them. However, when substituting $C_6^{(-1)}$ in terms of $D_1^{(-1)}$ and $D_1^{(-1)}\big|_{t\leftrightarrow u}$ these spurious poles cancel out.

Again the terms multiplied by 1/2 in (3.1) are $(t\leftrightarrow u)$ symmetric which, when adding the $(t\leftrightarrow u)$ term indicated at the end of (3.1), add up to the full contribution.

Several coefficient functions in (3.1) are trivially related to the corresponding ones in (2.10):

$$\begin{aligned}
& f_{B_1^1}=-8h_{B_1^0}, \qquad f_{B_2^1}=-8h_{b_2^0}, \qquad f_{C_2^1}=-8h_{C_2^0}, \\
& f_{C_5^1}=-8h_{C_5^0}, \qquad f_{C_6^1}=-8h_{C_6^0}, \qquad f_{D_1^1}=-8h_{D_1^0}.
\end{aligned} \tag{3.2}$$

The results for the other coefficient functions appearing in (3.1) can be found in the Appendix.

In the finite contribution Eq. (3.1) one can see the interplay of the product of powers of ε resulting from the Laurent series expansion of the scalar integrals (cf. Eq. (1.1)) on the one hand, and powers of ε resulting from doing the spin algebra in DREG on the other hand. For example, one has a contribution from $D_1^{(-1)}B_1^{(0)}$ as well as a contribution from $D_1^{(-1)}B_1^{(1)}$. Terms of the type $D_1^{(-1)}B_1^{(0)}$, where the superscripts corresponding to ε–powers do not compensate, would be absent in the regularization schemes where traces are effectively taken in four dimensions, i.e. in the so-called four-dimensional schemes or in Dimensional Reduction.

Finally we note that obtaining our results as well as casting them into the above compact forms (also for the coefficient functions presented in the Appendix) was done with the help of the REDUCE Computer Algebra System [19].

IV. CONCLUSIONS

We have presented $\mathcal{O}(\alpha^2\alpha_s^2)$ NNLO analytical results for the loop-by-loop contributions for heavy quark pair production in $\gamma\gamma$–collisions. The present paper deals with unpolarized photons in the initial state. Using the backscattering technique it is not difficult to obtain polarized photon beams of high intensity at the $\gamma\gamma$ option of the ILC by colliding the low energy laser light with polarized electron and positron beams. Since our amplitude results [15] contain the full spin information of the process an extension of the present paper to the case of polarized $\gamma\gamma$ production of heavy quark pairs would not be very difficult.

The present results form an Abelian subset of the non–Abelian gluon–induced loop-by-loop contributions to heavy quark pair production. This calculation constitutes a first step in obtaining the analytical results for the exact NNLO corrections to the heavy quark production processes in QCD. Analytical results in electronic format for all the terms in Eq. (2.7) are readily available [20]. The next step would be to provide similar results for the two remaining channels of heavy flavor production that were discussed in the Introduction. We reserve this task for future work.

Acknowledgments

Z.M. would like to thank the Particle Theory group of the Institut für Physik, Universität Mainz, for hospitality. The work of Z.M. was supported by a DFG (Germany) grant under contract 436 GEO 17/2/06. M.R. was supported by the Helmholtz Gemeinschaft under contract No. VH-NG-105.

APPENDIX

In this appendix we present the coefficient functions for the finite part of the loop-by-loop contributions appearing in Eq. (3.1).

In order to keep our expressions as compact as possible we introduce the notation:

$$z_t \equiv 2m^2 + t, \qquad z_u \equiv 2m^2 + u, \qquad D \equiv m^2 s - tu. \tag{A.1}$$

The coefficients of the finite part read:

$$\begin{aligned}
f = & -2(D\beta^2 TU2m^2 s(128m^{10}(t^2+u^2)^2 - 208m^8 s(t^2+u^2)^2 + 8m^6(10t^6 + 39t^5u + 34t^4u^2 + 114t^3u^3 + 34t^2u^4 \\
& +39tu^5 + 10u^6) - 8m^4 stu(13t^4 + 4t^3u + 66t^2u^2 + 4tu^3 + 13u^4) + m^2t^2u^2(36t^4 + 75t^3u \\
& +190t^2u^2 + 75tu^3 + 36u^4) - st^3u^3(7s^2 + 8tu)) \\
& +DTU2m^2t^2u^2(12m^2(t^5+u^5) + tu(4t^4 + 7t^3u + 42t^2u^2 + 7tu^3 + 4u^4)) \\
& +s\beta^2 TU384m^8t^4u^4 + TUt^4u^4(2(t^4T + u^4U) - 4tu(t^2T + u^2U) - 6m^2tu(t^2+u^2) - 12t^2u^2(T+U) \\
& -s^5 + 12st^2u^2) - t^4u^4(t-u)(t^4U(3t-u) + u^4T(t-3u)))/(D\beta^2T^2U^2st^4u^4),
\end{aligned}$$

$$\begin{aligned}
f_{B_1^0} = & 4(-T^2UD(512m^8T(t^2+u^2) + 448m^8(t^3+u^3) - 704m^8stu + 464m^6t^4 + 1776m^6t^3u + 1216m^6t^2u^2 \\
& +688m^6tu^3 + 80m^6u^4 + 184m^4t^4u + 768m^4t^3u^2 + 288m^4t^2u^3 + 72m^4tu^4 + 30m^2t^4u^2 - 23Tt^3u^3 \\
& +2m^2t^2u^4 - 22t^3u^4) \\
& +T^2Um^4tu(740m^2t^4 + 144m^2t^2u^2 + 12m^2u^4 - 31t^3u^2 + 33t^2u^3) \\
& -T^2\beta^2st^2(32m^8t^3 + 32m^6u^3U + 12m^4t^3u^2 - 8m^4su^4 - 16t^3u^4) \\
& +T^2t^4u^3(247m^4t - 4t^3 + 3u^3) - TDm^4t^5(48m^2t + 174m^2u + 190tu) \\
& -T2t^5u^2(18m^4t^2 - m^4tu + 2m^2t^2u + m^2u^3 - 2t^3u) + Dt^7u^2(3m^2 + 4u) + m^2t^7u^4)/(D\beta^2T^2Ust^4u^2),
\end{aligned}$$

$$\begin{aligned}
f_{B_1^0B_1^0} = & -2(DT(64m^{10} + 8m^8(29t + 5u) + 304m^6t^2 + 96m^6tu + 136m^4t^3 + 68m^4t^2u - 12t^5 - 9t^4u) \\
& +Tt^2(2m^6u^2 + 22m^4t^2T + m^2t^2(t^2 + 30tT + 5tu - 2u^2) + 5t^5) + t^8)/(DT^2t^4),
\end{aligned}$$

$$\begin{aligned}
f_{B_1^0B_{1u}^0} = & 4(-D(64m^{12} - 168m^{10}s + 8m^8(19s^2 + 22tu) + 40m^6(t^3+u^3) - 398m^6stu + 98m^4tu(t^2+u^2) \\
& +286m^4t^2u^2 - 49m^2st^2u^2 + 2t^3u^3) + 6m^4tu(2m^4(t^2+u^2) - t^2u^2))/(DTUt^2u^2),
\end{aligned}$$

$$f_{B_1^0B_2^0} = -8z_t(2D(m^2z_t + t^2) - 3t^2(4m^2T + t^2))/(D\beta^2Tt^2), \tag{A.2}$$

$$f_{B_1^0C_2^0} = 8(D^2T2(6m^6s - 2m^4t(5t + 2u) - t^2z_t(4t + u)) - DTt^2(2z_t(8m^6 - 7m^2t^2 - 4t^3) - t^2u^2)$$

$$+Dm^2t^5u - 3t^6z_t^3)/(D^2Tt^3),$$

$$f_{B_1^0C_{2u}^0} = 8(D^2Tm^2(12m^4s - 4m^2(3t^2+7tu+3u^2) - u(15t^2+tu+8u^2)) \\ -DTm^2u(16m^4uz_u + m^2t(13t^2+31u^2) + 19t^3u + 10tu^3) + DU2t^2u(m^2t^2+3m^2u^2+tu^2) \\ +Dm^2t^2u^2(7m^2u+t^2) - t^4u^2(2z_t(6m^2U+u^2) - tuz_u))/(D^2Tt^2u),$$

$$f_{B_1^0C_5^0} = 4(D^2T(8m^4(5t+u) - 4m^2s(t+3u) - 4t(st+2u^2)) + D^2t^2(14t^2+10tu+13u^2) \\ -DTt^2(8m^4(11t+19u) + 48m^2t^2 + 11t^3) - D(47m^2t^4u + 12m^2t^3u^2 + 4m^2t^2u^3 - t^5u) \\ +T(32m^6t^2u^2 + 32m^4t^2u^3 + 12m^2t^4(s^2-3tu) + 3t^7) - 32m^{12}s^2 + 3m^2t^4u^2(4m^2+u))/(D^2Tt^2),$$

$$f_{B_1^0C_6^0} = 4\beta^2s(D(8m^8s + 8m^6tu + 8m^4t^2(3t+4u) + m^2t^2(31t^2+28tu-9u^2) - t^3u^2) \\ -4m^8s(t^2-u^2) + 4m^2t^3(3st^2 - 2t^2T - u^3) + st^5(3t+5u))/(D^2Tt^2),$$

$$f_{B_1^0D_1^0} = 4(D^2T16m^6(11t+3u) + DT(64m^{10}s - 4m^6(11t^3+10t^2u+17tu^2+2u^3) \\ +2m^4t(51t^3-4t^2u-33tu^2-6u^3) + 2m^2t^3(31t^2+32tu+7u^2) + t^3(18t^3+15t^2u+10tu^2+3u^3)) \\ -Dt^5(7t^2+su) - 3t^6(16m^4T + m^2t(6t-u) + t^2T))/(D^2Tt^2),$$

$$f_{B_1^0D_{1u}^0} = 4(D^2T4m^2(16m^6 + 4m^4(7t+3u) + m^2(16t^2+27tu+u^2) + t^3) \\ -DT(64m^8u^2 + 2m^4u(23t^3-6t^2u+15tu^2-4u^3) + m^2tu(6t^3+15t^2u+tu^2-2u^3) + st^2u^2(t+2u)) \\ -Dm^4t(4t^4+3t^3u-26t^2u^2-13tu^3+4u^4) + 3m^4t^3u(T(t^2+15u^2) + u(4st-tu+u^2)))/(D^2Tt^2),$$

$$f_{B_2^0} = 8(8m^2(2m^2s^2 - 2m^2tu - stu)/(t^2u^2) + (24m^6s\beta^2/D - 28m^2s\beta^2 - 11D + 3tu)/(TU))/\beta^2,$$

$$f_{B_2^0B_2^0} = 4(4m^2s - 3t^2 + 2tu - 3u^2)/(D\beta^2),$$

$$f_{B_2^0C_2^0} = -8(D2(10m^4t + 2m^4u + 3m^2t^2 + m^2tu + 2t^2u) + \beta^2st^3(10m^2+3t) - tu(4m^2uz_t + t^2(t-u)))/(D^2\beta^2),$$

$$f_{B_2^0C_5^0} = 4s(D4(3m^2s - t^2 - u^2) - 6m^2s(t^2+u^2) + 3(t^2+u^2)^2)/D^2,$$

$$f_{B_2^0C_6^0} = 4s(D4(3m^2s - (t-u)^2) + 3t^3(4m^2+t) + 3u^3(4m^2+u) + 6t^2u^2)/D^2,$$

$$f_{B_2^0D_1^0} = 4(D^28m^2z_u + D4m^2t^2(5s\beta^2+2t) - \beta^26m^2st^2(3t^2+u^2) + 10m^2s^2t^3 - st^3(3s^2-2tu))/(D^2\beta^2),$$

$$f_{C_2^0} = -8(D^2(8m^{10}u(42t(t^2+u^2) + t^2u + 15u^3) + 2m^8u(149t^4+595t^3u+95t^2u^2+109tu^3+12u^4) \\ +2m^6t(4t^5+35t^4u+159t^3u^2+328t^2u^3+81tu^4+17u^5) \\ +2m^4t^2u(25t^4+322t^3u+132t^2u^2+75tu^3+10u^4) \\ +2m^2t^3u(7t^4+13t^3u+9t^2u^2+46tu^3+8u^4) + t^4u^2(10t^3+20t^2u+17tu^2+7u^3)) \\ +D\beta^22m^2st^2(40m^8u^3 + 100m^6t^2u^2 + 8m^4t^5 + 241m^4t^3u^2 + 67m^4tu^4 + 35m^2t^3u^3 + 29t^5u^2 + 62t^3u^4) \\ -Dm^2(32m^{12}u(t^2+u^2)(t+3u) - 128m^{10}t^5 - t^5u^2(49t^3-33t^2u+118tu^2+66u^3)) \\ +\beta^28m^2st(5m^{12}(t^4-u^4) + 2m^{10}t^3(t^2-11u^2) - 6m^8tu^5 - t^3u^4(16m^6+43m^4t+11m^2t^2+9t^3)) \\ -48m^2t^6u^4(t-u)^2)/(D^2\beta^2TUst^3u^2),$$

$$f_{C_2^0C_2^0} = 4(D^2(4m^8s^2 + t^2(16m^6t + 50m^4t^2 - 2m^4u^2 - 28m^2t^3 - 8t^4 - 2t^2u^2)) \\ +D\beta^2st^2(34m^6t^2 - 2m^6u^2 + 20m^4t^3 + 2m^4stu - 8m^2t^4 - 4t^5) \\ +D4t^5(m^2t^2+3m^2u^2+2tu^2) - \beta^43s^2t^8)/(D^3t^2),$$

$$f_{C_2^0 C_{2u}^0} = 8(D^3 4(m^6 s - m^4(s^2+7tu) + tu(s^2-tu)) + D^2 t^2 u^2 (76m^4 + (t-u)^2)$$
$$-D2m^2t^2u^2(64m^6+4m^2tu-5stu) - 3m^4t^2u^2(t-u)^4)/(D^3tu),$$

$$f_{c_2^0 C_5^0} = 4(D^3 8m^4(7t+u) - D^2 4m^2 t(35m^2t^2 + m^2u^2 + 9t^3 - 24t^2u - 4tu^2 - u^3)$$
$$-DT8m^2t^2(40m^4s + 11m^2t^2 - 24m^2u^2 + tu^2)$$
$$-D2m^2t^2(4m^4(13t^2+8u^2) + t^3(45t+59u) - u^2(5t^2-tu-4u^2))$$
$$+2m^2t^3(14t^5+10t^4u+42t^3u^2+15t^2u^3+18tu^4+u^5) - st^4(3s^4-2t^3u-2su^3))/(D^3t),$$

$$f_{C_2^0 C_6^0} = 4\beta^2 s(D^2 4(m^4(9t+u) + m^2(4t^2-u^2)) + DT48t^2(m^2u+t^2)$$
$$-D2(m^2t(5t(t^2+u^2)+2su^2) - t^2(14st^2+t^3-10stu+su^2)) - 3t^3(t-u)^4)/D^3,$$

$$f_{C_2^0 D_1^0} = 4(D^2(16m^8(3t^2-2tu-u^2) + 4m^6(5t^3-12t^2u-11tu^2-2u^3) + 8m^4t(8t^3-8t^2u-5tu^2-u^3)$$
$$-4m^2t^2(29t^3+12t^2u+7tu^2+u^3) - 2t^3(2t^3+13t^2u+5tu^2+u^3))$$
$$-D2t^4(128m^8+32m^6t-24m^4t^2+3m^2t^2u-17t^3T) - 3t^7(16m^2tz_u + (t-u)(s^2-8tu)))/(D^3t),$$

$$f_{C_2^0 D_{1u}^0} = 4(D^2(16m^8(3t^2-2tu-u^2) + 4m^6(10t^3-13t^2u-u^3) - 4m^4t(2t^3+3t^2u-16tu^2+3u^3)$$
$$+4m^2(st^2(2s^2-tu)-tu^4))$$
$$-D16m^4t^2u(8m^2uz_t + (t-u)(t^2+u^2)) - 6m^4t^2u^2(5t^4+10t^2u^2+u^4) - 3s^3t^3u^2(2m^2s-tu))/(D^3t),$$

$$f_{C_5^0} = 4(D^2(64m^{10}(t^2+u^2) - m^6(40s^4+96s^2tu+144t^2u^2) + m^4(24s^5+44tu(t^3+u^3)+548st^2u^2)$$
$$-4m^2tu(3s^4+55tu(t^2+u^2)+25t^2u^2) + st^2u^2(17s^2-14tu))$$
$$-Dm^2((48m^8-25t^2u^2)(t-u)^4 - 32m^2t^2u^2(2m^2-s)(8m^4-7tu) + t^3u^3(21s^2-116tu))$$
$$-24m^2t^4u^4(8m^4+s^2-6tu))/(D^2TUt^2u^2),$$

$$f_{C_5^0 C_5^0} = -(D^3 896m^4 - D^2 2(8m^4(27s^2-80tu) + 34m^2s^3 + 24m^2stu - 49(t^4+u^4) - 16s^2tu + 26t^2u^2)$$
$$+D4(128m^4t^2u^2 - 7(t^5-u^5)(t-u) + 12tu(t^4+u^4-t^2u^2) - 22t^2u^2(t^2+u^2)) + 3(t^2+u^2)^2(t-u)^4)/D^3,$$

$$f_{C_5^0 C_6^0} = 2\beta^2 s^2(D2(40m^6s - 2m^4(3s^2-4tu) - 3m^2s(7s^2-16tu) + 11s^4 - tu(49s^2-50tu)) - 3(t^2+u^2)(t-u)^4)/D^3,$$

$$f_{C_5^0 D_1^0} = 2(D^3 64m^6 - D^2(32m^6(18t^2-5tu+u^2) - 24m^4(13t^3-7t^2u+5tu^2+u^3)$$
$$-4m^2(47t^4+77t^3u+34t^2u^2+11tu^3+3u^4) + 2t(2t^4-t^3u-3t^2u^2-21tu^3-u^4))$$
$$+D2t^2(256m^6t^2+288m^4t^2u-8m^2(9t^4-tu^3+u^4) - t^3(17t^2+59u^2)) - 3t^2(t-u)^5(uz_t-t^2))/D^3,$$

$$f_{C_6^0} = 4(-D^2(8m^6(3s^4-8s^2tu+22t^2u^2) + 4m^4stu(22s^2-139tu) + 4m^2t^3u^3 - st^2u^2(4s^2+33tu))$$
$$+D(64m^{12}s(t^2+u^2) - 16m^{10}(3t^4+2t^2u^2+3u^4) + 336m^6t^2u^2(tT+uU) - 298m^4st^2u^2(t^2+u^2)$$
$$+4m^2t^4u^4 - st^3u^3(7s^2-3tu)) + 8m^8s^7 - 192m^4t^4u^4(2m^2-s) - 8st^4u^4(s^2+3tu))/(D^2TUt^2u^2),$$

$$f_{C_6^0 C_6^0} = \beta^2 s(D2(16m^6s^2 + 2m^4s(3s^2+8tu) - m^2s^2(35s^2-106tu) + 8m^2t^2u^2 + s(t^2+u^2)(14s^2-47tu))$$
$$-3s(t-u)^6)/D^3,$$

$$f_{C_6^0 D_1^0} = -2\beta^2 s(D^3 16m^4 - D^2 2(4m^4t(7t-5u) - 2m^2(7t^3+6t^2u+2tu^2+u^3) + t(5t^3-23t^2u+4tu^2-2u^3))$$
$$-D2t(48m^4t^2u + 43m^2t^4 + m^2u(7t^3-12t^2u+17tu^2-3u^3) + 14t^5) + 3t^3(t-u)^5)/D^3,$$

$$f_{D_1^{-1}} = 8(2m^2-s)(-DTU(64m^6(t^2+u^2)-8m^4(5t^3+3t^2u-15tu^2-u^3)+4m^2t(3t^3+8tu^2-3u^3)$$
$$+t^2u(2t^2-5tu-8u^2))$$
$$+TUu^2(32m^2t^2(4m^4+tz_t)-m^2t^2u(13t-3u)+2t^4(t+4u))-t^2u^2(Tu^3(m^2-2t)+2t^4U))/(DTUt^2u^2),$$

$$f_{D_1^{-1}B_1^0} = 8(2m^2-s)(D2m^2(2m^4(3t-u)+11m^2t^2-2st^2)+T4m^2t^2(2st-u^2)+st^3(3Tu+t^2))/(DTt^2),$$

$$f_{D_{1u}^{-1}B_1^0} = 8(2m^2-s)(-D(4m^6(t+5u)+2m^4(3t^2+15tu+u^2)+m^2t(3t^2+u^2)-t^2u(3t+2u))$$
$$-t^4(4m^4-tu))/(DTt^2),$$

$$f_{D_1^{-1}B_1^1} = 8(2m^2-s)(DT2m^4(8m^2+17t)-T2m^2(m^4u(t+2u)+7m^2t^2T+t^2(t^2-4u^2))+D3st^3-4m^2t^4u)/(DTt^2),$$

$$f_{D_{1u}^{-1}B_1^1} = 8(2m^2-s)(Dm^2(2m^2(3t^2+4tu+2u^2)+t(4t^2+3tu+5u^2))$$
$$+T(16m^8s+20m^6st-12m^4t^2u-4m^2t^2u^2-t^3u^2)+m^2t^4u)/(DTt^2),$$

$$f_{D_1^{-1}B_2^0} = -8(2m^2-s)(D4(m^2s\beta^2+2(m^2s-t^2))+t(t-u)^3)/(D\beta^2s),$$

$$f_{D_1^{-1}B_2^1} = 16(2m^2-s)z_t/\beta^2,$$

$$f_{D_1^{-1}C_2^0} = 8(2m^2-s)(D^24m^4(3t-u)-D2t(m^4(9t^2-2tu+u^2)-m^2t(6t^2+15tu+u^2)-t^2(6t^2+9tu+2u^2))$$
$$+8m^2t^3z_t(2m^2u-t^2)-t^5(s^2-4u^2))/(D^2t),$$

$$f_{D_{1u}^{-1}C_2^0} = -8(2m^2-s)(D^24m^4(t+5u)-D2(m^4u(9t^2-2tu+u^2)+m^2t(2t^3-10t^2u-11tu^2+u^3)+t^3u(4s-u))$$
$$+t^2u(32m^4uT-4m^2t(2t-u)(t-u)-s^2t^2))/(D^2t),$$

$$f_{D_1^{-1}C_2^1} = 16(2m^2-s)(D(2m^6(t^2-4tu-u^2)+4m^4t^2(t-u)+m^2t^2(t-u)^2-t^3u^2)+\beta^2st^4(4m^4-tu))/(D^2t),$$

$$f_{D_{1u}^{-1}C_2^1} = 16(2m^2-s)(D(2m^6(3t^2+u^2)+2m^4(t^3-u^3)+m^2t(2s(t^2+u^2)-tu^2)-t^4u)$$
$$+z_tt^3u(m^4+u^2)-3m^4st^3u\beta^2+m^2tuT(2m^2u^2+t^3))/(D^2t),$$

$$f_{D_1^{-1}C_5^0} = -4(2m^2-s)(D4(2m^4t(3t-5u)+m^2(3t^3-2t^2u+3tu^2+2u^3)+t(5t^3+2t^2u+2tu^2-3u^3))$$
$$+8m^2t^2u((2m^2-s)(t-3u)+2u^2)-t(t-u)^5)/D^2,$$

$$f_{D_1^{-1}C_5^1} = 8(2m^2-s)(D(8m^6s-8m^4t(3t-2u)-m^2(4t^3-13t^2u-12tu^2-5u^3)-stu^2)$$
$$+2m^4u^2(10t^2+u^2)-10m^2t^3uT+s^3t^2u)/D^2,$$

$$f_{D_1^{-1}C_6^0} = 4(2m^2-s)(D(8m^4(6t^2-5tu+u^2)-2m^2t(29t^2-15tu-3u^2)-4t(5t^3+t^2u-tu^2-5u^3))$$
$$+2m^2t(t^3T-31u^3T-5t^3u+15t^2u^2+6tu^3-u^4)+t(t-u)^5)/D^2,$$

$$f_{D_1^{-1}C_6^1} = -8(2m^2-s)\beta^2s(D^22m^2-D(m^2(su+4t^2-8tu)-t^2u)+\beta^2st^3u-m^2tu(t-u)^2)/D^2,$$

$$f_{D_1^{-1}D_1^0} = 4t(2m^2-s)(-D(4m^4(7t^2-2u^2)-2m^2(28t^3+23t^2u+11tu^2+2u^3)-2t(6t^3+11t^2u+8tu^2+2u^3))$$
$$+4m^4tuT(9t-7u)-10m^2t^4T+14m^2t^2u^2(m^2-t)+t^3(s^3+2u^2(s-t)))/D^2,$$

$$f_{D_{1u}^{-1}D_1^0} = 4t(2m^2-s)(D(12m^4u(3t-u)+2m^2(2t^3-27t^2u-4tu^2-u^3)-18t^3u)-tu^2z_t(32m^4-17tu)$$

$$-4st^2u^3\beta^2 - 2tu^2T(3t^2-4u^2) + t^2u(t^3-t^2u-u^3))/D^2,$$

$$f_{D_1^{-1}D_1^1} = -8(2m^2-s)(D^2 16m^6 - Dm^2(4m^4(11t^2+5tu+2u^2)+4m^2t(3t^2+tu+2u^2)-t(4t^3+3tu^2+u^3)) \\ +m^4t^2z_t(9t^2+7u^2) - 4m^2t^3u^2T + st^2u(s^2T+tu(m^2-t)))/D^2,$$

$$f_{D_{1u}^{-1}D_1^1} = -8(2m^2-s)(D^2 16m^6 - D(4m^6(6t^2+tu+3u^2)+2m^4u^2(5t+u)-m^2t(2t^3+7t^2u-u^3)) \\ -4m^4t^3uT + 14m^4t^2u^2(2m^2-s) - 4m^2t^2u^3z_t - 2m^2t^5u - s^2t^3u^2)/D^2,$$

$$f_{D_1^0} = 4(D^2T(512m^{12}(t^2+u^2)+256m^{10}t(4t^2+3tu+7u^2)+32m^8(8t^4+37t^3u+67t^2u^2+9tu^3+21u^4)) \\ -D^2(8m^8(7t^5+126t^4u-126t^3u^2-24t^2u^3-15tu^4-22u^5)+4m^6t(2t^5-2t^4u-8t^3u^2-5t^2u^3-63tu^4-74u^5) \\ +m^4 2tu(13t^5+433t^4u-291t^3u^2-19t^2u^3-6tu^4-4u^5)+m^2 2t^3u(7t^4+70t^3u+52tu^3+2u^4) \\ +t^3u^2(10t^4+30t^3u+37t^2u^2+24tu^3+7u^4)) \\ +D\beta^2 s4m^6t(128m^4u^4+m^2tu(91t^3+243u^3)-2t^2(2t^4-2t^3u-49u^4)) \\ -Dm^2u^2(1024m^{10}Tu^2+8m^4tu(5t^4+3u^4)-2m^2t^5(75t^2-61tu+346uU+522u^2) \\ -t^3(9t^5+75t^3u^2-11t^2u^3+64tu^4+8u^5)) \\ -\beta^2m^2st^3u^4(196m^6u-472m^4t^2-29m^2t^2u-24t^4-33t^2u^2) \\ +m^2u^4(16m^6Tu^4-144m^2t^7-t^5(184t^2U-4tu^2+64Tu^2-49u^3)))/(D^2\beta^2 sTUt^2u^2),$$

$$f_{D_1^0D_1^0} = -(D^2(128m^{10}s+128m^8(3st-u^2)+4m^4t(5t^3+45t^2u-63tu^2-3u^3)+4m^2t^3(28t^2+41tu+26u^2) \\ +2t^2(19t^4+24t^3u+15t^2u^2+6tu^3+u^4)) \\ +D(16m^8(t(t^2+u^2)(29t+14u)+2u^2(s^2-tu))-52m^6t(t^4-u^4)-4m^4t^3(66t^3+41t^2u+tu^2+66u^3) \\ -4m^2t^3(23t^4+5u^4)-16t^8)+66m^4t^6(t^2+u^2)+60m^2t^6Tu^2+24m^2t^6(t^3+u^3)+3t^6(t^4+u^4))/D^3,$$

$$f_{D_1^0D_{1u}^0} = -2(D^3 128m^6(m^2-2s)+D^2 2m^2(128m^6tu+8m^4s(6t^2+6u^2-tu)-m^2(9s^4-17t^4-17u^4+106t^2u^2) \\ +2(2t^5+t^4u+27t^3u^2+27t^2u^3+tu^4+2u^5)) \\ +D2m^2(256m^6t^2u^2+48m^2t^3u^3-t^2u^2(26t^3+9stu+26u^3)) \\ -3t^2u^2(30m^4tu(t^2+u^2)-2m^2(t^4T+u^4U)-s^4tu))/D^3.$$

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[20] All the relevant results are available in REDUCE format. The results can be retrieved trough a web page http://www-zeuthen.desy.de/theory/research.html or can be obtained directly from the authors.