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SOLVABILITY OF URYSOHN AND URYSOHN-VOLTERRA EQUATIONS WITH HYSTERESIS IN WEIGHTED SPACES

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Abstract

The existence of solutions to nonlinear integral equations of the second kind with hysteresis, of Urysohn-Volterra and Urysohn types has been established. We develop the solvability theory of Urysohn-Volterra equation with hysteresis in weighted spaces proposed by the author [M.A. Darwish, On solvability of Urysohn-Volterra equations with hysteresis in weighted spaces, J. Integral Equations and Application, **14 (2)** (2002), 151-163].

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1. Introduction

There are various ways how hysteretic behavior of a system can be related to an integral equation. One particular setting, which has been studied by many authors, is using a convolution integral to describe the memory of a given system. The memory is characterized by the convolution kernel and thus the evolution depends on all past values of the state; typically, as one goes back in time, the influence of the past values of the present evolution decreases. There are, however, several hysteretic phenomena which cannot be treated in this way; in particular, it cannot be used to describe a hysteretic system whose hysteresis loops do not depend on the speed with which they are traversed. This property is called rate independence and is inherently nonlinear. In [2-6], we discuss systems where a Urysohn-Volterra integral equation is coupled to a rate independent hysteretic process. For more information about hysteresis, for instance, see [1, 8, 12].

In this paper we study nonlinear integral equations of the second kind with hysteresis, namely

$$y(t) = f(t) + \int_{-\infty}^{t} F(t, s, y(s), \mathcal{W}[S[y]](s))\, ds, \tag{1.1}$$

$$y(t) = f(t) + \int_{-\infty}^{\infty} F(t, s, y(s), \mathcal{W}[S[y]](s))\, ds. \tag{1.2}$$

Here $\mathcal{W}$ and S denote a hysteresis operator and a superposition operator of the form $S[y](t) = k(y(t))$, respectively. More precisely, we assume f and F to be given $n-$vector valued functions, while y is the unknown $n-$vector function. Equations (1.1) and (1.2) are known as Urysohn-Volterra and Urysohn integral equations, respectively [3].

Equation (1.1) is history-dependent so, in general, this problem requires that one gives an initial condition on $(-\infty, 0]$ and it may be treated with the techniques of standard Urysohn-Volterra equations with hysteresis, see Darwish [4]-[6]. Therefore, the nonuniqueness of solutions of equations (1.1) is an intrinsic feature which occurs even in the case of linear integral equations without hysteresis. For example, the equation

$$y(t) = e^{-t} + \frac{1}{2} \int_{-\infty}^{t} e^{-2\,(t-s)}\, y(s)\, ds \tag{1.3}$$

has the solution set $\{y(t) = 2\, e^{-t} + c\, e^{-3/2\, t} : c \text{ is a constant}\}$ (cf. [10]). However, it was recently observed by Darwish [3] that uniqueness of equation (1.1) occurs in some kind of weighted spaces.

The main object of this paper is to show the existence of solutions of equation (1.1) and develop the solvability theory of Urysohn-Volterra equation with hysteresis in weighted spaces proposed by Darwish [3]. More precisely, in Section 3 we relax the restrictions imposed on the function F in [3] and obtain the existence and uniqueness of solutions of equation (1.1). Also, our method is applicable to an Urysohn equation with hysteresis.

2. Preliminaries

In this section, we state some results needed in the proof of our main theorems.

Let $\mathrm{I} \subset \mathbb{R}$ and consider a weight function $w : \mathrm{I} \to \mathbb{R}_+$ be continuous and nondecreasing, $\mathbb{R}_+ = (0, +\infty)$. Define $\mathrm{C}_w \equiv \mathrm{C}_w(\mathrm{I}; \mathbb{R}^n) := \{\phi \mid \phi : \mathrm{I} \to \mathbb{R}^n \text{ continuous}\}$ with the following norm

$$\|\phi\|_w = \sup_{t\in\mathrm{I}} \frac{\|\phi(t)\|_{\mathbb{R}^n}}{w(t)}, \ \forall\, \phi \in \mathrm{C}_w$$

to be the underlying space for our problem. Then the spaces $(\mathrm{C}_w, \|.\|_w)$ is a Banach space.

Definition 1. (*Rate independent functionals*)
A functional $\mathcal{H} : C([0,T]; \mathbb{R}^n) \to \mathbb{R}^n$; is called rate independent if and only if $\mathcal{H}[u \circ \psi] = \mathcal{H}[u]$ holds for all $u \in C([0,T]; \mathbb{R}^n)$ and all admissible time transformations, i.e., continuous increasing functions $\psi : [0,T] \to [0,T]$ satisfying $\psi(0) = 0$ and $\psi(T) = T$.

Definition 2. ($Volterra - operator$)
Let X be a Banach space. An operator $F : C([0,T]; X) \to C([0,T])$ is called a Volterra-operator if for all $s \in [0,T]$ and for all $u, v \in C([0,T]; X)$ with $u(\sigma) = v(\sigma)$ for all $\sigma \in [0,s]$ implies $(Fu)(\sigma) = (Fv)(\sigma)$ for all $\sigma \in [0,s]$.

Recall that an operator $\mathcal{W} : C(\mathrm{I}; \mathbb{R}^n) \to C(\mathrm{I})$ is hysteresis if it has the Volterra property and the rate independent property. For more information about hysteresis operator, see [1] and the references therein.

Remark 1. *By definition hysteresis operators possess the Volterra property. This is actually what is needed here; the rate-independence itself does not play any role.*

Lemma 1. [3]
Let $F : C(\mathrm{I}; \mathbb{R}^n) \to C(\mathrm{I})$ be a Volterra-operator. Assume that F is Lipschitz continuous on every bounded subset of $C(\mathrm{I}; \mathbb{R}^n)$. Then for every $C > 0$ there exists $L > 0$ such that

$$|(Fy_2)(s) - (Fy_1)(s)| \le L \sup_{\substack{\tau \in \mathrm{I} \\ \tau \le s}} \|y_2(\tau) - y_1(\tau)\|_{\mathbb{R}^n}, \tag{2.1}$$

holds for all $s \in \mathrm{I}$ and all $y_i \in C(\mathrm{I}; \mathbb{R}^n)$ with $\|y_i\| \le C$, $i = 1,\ 2$.

3. The unique solvability

Let I be a (bounded or unbounded) closed subinterval of $\mathbb{R}$ and define C_w as above. To facilitate our discussion, let us first state the following assumptions:

- $(A1)$ $f \in \mathrm{C}_w$,
- $(A2)$ $F : \mathrm{I} \times \mathrm{I} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ continuous,
- $(A3)$ There exists a measurable function $m(t,s)$ defined on I^2, such that

$$\|F(t,s,y_2,\omega_2) - F(t,s,y_1,\omega_1)\| \le m(t,s)\ \{\|y_2 - y_1\| + |\omega_2 - \omega_1|\},$$

- $(A4)$ $\mathcal{W}oS : \mathrm{C}(\mathrm{I}; \mathbb{R}^n) \to \mathrm{C}(\mathrm{I})$ satisfies the Lipschitz condition

$$|\mathcal{W}[S[y_2]](s) - \mathcal{W}[S[y_1]](s)| \le L \sup_{s\in\mathrm{I}} \|y_2(s) - y_1(s)\|_{\mathbb{R}^n},$$

fore some $L > 0$,

(A5)

$$E = (1+L)\ \sup_{t\in \mathrm{I}} \int_{\mathrm{I}} \frac{w(s)}{w(t)}\ m(t,s)\ ds < 1.$$

Theorem 1. *Let assumptions* $(A1)-(A5)$ *be satisfied. Then the equation*

$$y(t) = f(t) + \int_{\mathrm{I}} F(t,s,y(s),\mathcal{W}[S[y]](s))\ ds,\ \ t\in \mathrm{I}, \tag{3.1}$$

has a unique solution in C_w.

Proof: Define the operator $\mathcal{F}$ on C_w by

$$(\mathcal{F}y)(t) = f(t) + \int_{\mathrm{I}} F(t,s,y(s),\mathcal{W}[S[y]](s))\ ds \tag{3.2}$$

which enjoys the property that any fixed point of $\mathcal{F}$ is a solution of (3.1). We shall prove that

(*i*) $\mathcal{F}$ maps C_w into itself,

(*ii*) $\mathcal{F}$ is a contraction mapping with constant $E<1$.

To show (*i*), for $y_i \in \mathrm{C}_w;\ i=1,\ 2$, we have

$$\begin{aligned}
\|\mathcal{F}y_2 - \mathcal{F}y_1\|_w &= \sup_{t\in\mathrm{I}} \frac{\|(\mathcal{F}y_2(t)) - (\mathcal{F}y_1(t))\|_{\mathbb{R}^n}}{w(t)} \\
&\le \sup_{t\in\mathrm{I}} \frac{1}{w(t)} \int_{\mathrm{I}} \|F(t,s,y_2(s),\mathcal{W}[S[y_2]](s)) - \\
&\qquad F(t,s,y_1(s),\mathcal{W}[S[y_1]](s))\|_{\mathbb{R}^n}\ ds \\
&\le \sup_{t\in\mathrm{I}} \int_{\mathrm{I}} \frac{1}{w(t)}\ m(t,s)[\|y_2(s)-y_1(s)\|_{\mathbb{R}^n} + \\
&\qquad |\mathcal{W}[S[y_2]](s) - \mathcal{W}[S[y_1]](s)|]\ ds \\
&\le (1+L)\sup_{t\in\mathrm{I}} \int_{\mathrm{I}} \frac{w(s)}{w(t)}\ m(t,s)\ ds\ \|y_2-y_1\|_w \\
&= E\ \|y_2-y_1\|_w.
\end{aligned}$$

This shows that $\mathcal{F}$ is a contraction with constant $E<1$ and maps C_w into itself since $f\in\mathrm{C}_w$. By Banach's contraction mapping principals $\mathcal{F}$ has a unique fixed point y^* that is the unique solution of equation (3.2) and for any initial point $y_0\in\mathrm{C}_w$, the sequence $\{y_n\}$ defined by

$$y_{n+1}(t) = f(t) + \int_{\mathrm{I}} F(t,s,y_n(s),\mathcal{W}[S[y_n]](s))\ ds$$

converges in C_w to y^*.

Remark 2. *In* [3]*, Darwish considered the closed subset*

$$D = \{y\in\mathrm{C}_w\ :\ \|y-f\|\le b\},$$

where b *is a number such that* $\|f\|\le b$*. He could only prove the existence for equation (3.1) for* $E\le\frac{1}{2}$ *and uniqueness if*

$$\tilde{E} = \int_{\mathrm{I}} \sup_{t\in\mathrm{I}} \left\{ \frac{w(s)}{w(t)}\ m(t,s) \right\}\ ds < 1.$$

So our theorem improves upon his Theorems 3.1 and 3.2 in [3].

Now we turn to consider the Urysohn-Volterra equation

$$y(t) = f(t) + \int_a^t F(t,s,y(s),\mathcal{W}[S[y]](s))\ ds \tag{3.3}$$

in C_w, where a can be finite or $-\infty$. We write $\mathrm{I} = [a, \infty)$.

Theorem 2. *Let $f \in \mathrm{C}_w(\mathrm{I}; \mathbb{R}^n)$. Assume that F satisfies all the assumptions as in Theorem 1 on $a \le s \le t$, $t \in \mathrm{I}$, and, instead of* (A5) *let us assume that*

$$0 < \tilde{E} = (1+L) \int_a^t \sup_{s<t} \left\{ \frac{w(s)}{w(t)}\, m(t,s) \right\},\ ds < \infty. \tag{3.4}$$

Then equation (3.3) has a unique solution in C_w.

Proof: Define the operator $\mathcal{F}$ on C_w by

$$(\mathcal{F}y)(t) = f(t) + \int_a^t F(t, s, y(s), \mathcal{W}[S[y]](s))\, ds. \tag{3.5}$$

It is easy to see that $\mathcal{F}$ maps C_w into itself. Also, it can be proved by induction that for all $x,\ y \in \mathrm{C}_w$ and any integers $n \ge 1$,

$$\begin{aligned} |\mathcal{F}^n y(t) - \mathcal{F}^n x(t)| \ &\le\ (1+L) \int_a^t \int_a^{s_n} ... \int_a^{s_2} m(t, s_n)\, m(s_n, s_{n-1}) \ ... \\ &\qquad m(s_2, s_1)\, |y(s_1) - x(s_1)|\, ds_1 ... ds_n. \end{aligned}$$

Set

$$Q(s) = (1+L) \sup_{s<t} \left\{ \frac{w(s)}{w(t)}\, m(t,s) \right\}.$$

Then we get

$$\begin{aligned} \|\mathcal{F}^n y - \mathcal{F}^n x\|_w \ &=\ \sup_{t\in \mathrm{I}} \frac{1}{w(t)}\, \|\mathcal{F}^n y(t) - \mathcal{F}^n x(t)\|_{\mathbb{R}^n} \\ &\le\ (1+L) \sup_{t\in \mathrm{I}} \frac{1}{w(t)} \int_a^t \int_a^{s_n} ... \int_a^{s_2} m(t, s_n)\, m(s_n, s_{n-1}) \ ... \\ &\qquad m(s_2, s_1)\, |y(s_1) - x(s_1)|\, ds_1 ... ds_n \\ &\le\ \left(\sup_{t\in \mathrm{I}} \int_a^t Q(s_n)\, ds_n \int_a^{s_n} Q(s_{n-1})\, ds_{n-1} \ ... \right. \\ &\qquad \left. \int_a^{s_2} Q(s_1)\, ds_1 \right) \|y - x\|_w \\ &\le\ \frac{1}{n!} \left(\int_{\mathrm{I}} Q(s)\, ds \right)^n \|y - x\|_w \\ &\le\ \beta_n\, \|y - x\|_w, \end{aligned}$$

where $\beta_n = \frac{1}{n!} \left(\int\limits_{\mathrm{I}} Q(s)\, ds \right)^n$. Therefore, $\mathcal{F}^n$ is a contraction whenever n is large enough. This implies that equation (3.3) has a unique solution $y \in \mathrm{C}_w$ by using a generalization of Banach's Contraction Principal (cf. [7]). Moreover, for any $y_1 \in \mathrm{C}_w$, the sequence

$$y_{n+1} = f(t) + \int_a^t F(t, s, y_n(s), \mathcal{W}[S[y_n]](s))\, ds \tag{3.6}$$

converges in C_w to y.

Remark 3. *Darwish* [3] *only showed that the sequence $\{y_n\}$ defined by* (3.6) *with $y_0 = f$ converges to a solution of equation* (3.3)*. The uniqueness was verified separately.*

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