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OPTIMAL CONVERGENCE RECOVERY FOR THE FOURIER-FINITE-ELEMENT APPROXIMATION OF MAXWELL'S EQUATIONS IN NONSMOOTH AXISYMMETRIC DOMAINS

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Abstract

Three-dimensional time-harmonic Maxwell's problems in axisymmetric domains $\hat{\Omega}$ with edges and conical points on the boundary are treated by means of the Fourier-finite-element method. The Fourier-fem combines the approximating Fourier series expansion of the solution with respect to the rotational angle using trigonometric polynomials of degree N ($N \to \infty$), with the finite element approximation of the Fourier coefficients on the plane meridian domain $\Omega_a \subset \mathbb{R}^2_+$ of $\hat{\Omega}$ with mesh size h ($h \to 0$). The singular behaviors of the Fourier coefficients near angular points of the domain Ω_a are fully described by suitable singular functions and treated numerically by means of the singular function method with the finite element method on graded meshes. It is proved that the rate of convergence of the mixed approximations in $H^1(\hat{\Omega})^3$ is of the order $O(h + N^{-1})$ as known for the classical Fourier-finite-element approximation of problems with regular solutions.

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1 Introduction

Standard techniques of approximation of boundary value problems in domains with boundary irregularities lose accuracy due to the presence of singularities in the solutions. Knowledge of the form of the singularities can be used to construct special adaptations of the numerical technique, for instance, the finite element method, in order to improve convergence. In practical applications, *local mesh refinements* (cf. [1, 15, 20]), the *singular function method* (cf. [25]), the *dual singular function method* (cf. [9, 13, 18]) and the *singular complement method* (cf. [2, 3]), are the most commonly used adaptive methods.

It is well known that solutions of Maxwell's problems in domains with reentrant edges and corners, cannot be approximated by means of the usual Lagrange finite elements, even with mesh grading. In this case, the edge elements method [19] would be a more appropriate algorithm. Thus, motivated by the desire to adapt the usual nodal H^1-conforming finite element method to the solution of Maxwell's problems in domains with reentrant edges and corners, Several authors have proposed the *singular function method* (cf. [5, 14, 16]) and the *singular complement method* (cf. [2, 3]) as good alternatives. However, in these papers the proposed algorithms are based on the H^1-regularity expansion of the solutions of the Maxwell problems considered. As a consequence, the rate of convergence of the approximate solutions is not optimal in the sense that it is lower than one would expect for problems with sufficiently smooth solutions.

This paper is motivated principally by the desire to develop an adaptation of the above mentioned methods with better convergence properties and to extend the approach to treat some Maxwell's problems in three-dimensional domains containing singularities. Thus, we consider in this paper time-harmonic Maxwell's problems in three-dimensional axisymmetric domains with edges and conical points on the boundary.

It has been shown (cf. [4, 15, 17, 20, 21, 22]) that boundary value problems in three-dimensional axisymmetric domains $\hat{\Omega}$ with non-axisymmetric data can be solved more efficiently by means of the Fourier-finite-element method (Fourier-fem). The underlying principle is the application of partial Fourier approximation (truncated partial Fourier series) using trigonometric polynomials of degree N ($N \to \infty$) with respect to the rotational angle, as first step. This step reduces the three-dimensional problem into a system of N two-dimensional problems on the plane meridian domain $\Omega_a \subset \mathbb{R}^2_+$ of $\hat{\Omega}$ and whose solutions are the Fourier coefficients of the solution of the three-dimensional problem. As second step, the N two-dimensional boundary value problems are solved numerically by the finite element method on Ω_a with mesh size h ($h \to 0$). The mathematical framework for the application of this method to the solution of time-harmonic Maxwell's equations has been studied in [22].

Just like the solution of the main Maxwell problem in three dimensions, the solutions of the reduced two-dimensional boundary value problems in the plane meridian domain Ω_a exhibit singular behaviors near corner of Ω_a and, as a consequence, cannot be approximated by the usual Lagrange finite elements if Ω_a is not convex (cf. [22]). This problem is addressed in this paper by first describing fully the asymptotic behavior of the Fourier coefficients in the vicinity of the angular points of the meridian domain Ω_a and treating the singularities by the singular function method with the usual nodal H^1-conforming finite element method on locally graded meshes. By means of error estimates we prove that the rate of convergence of the mixed approximations is of the oder $O(h+N^{-1})$ as known for the classical Fourier-fem.

This paper is organized in the following way. In Section 2 the boundary value problem in three dimensions and the associated function spaces are given. Moreover, the partial Fourier analysis with respect to the rotational angle, the reduced two-dimensional problems, the associated spaces in two dimensions and regularities results for the solutions are described. Section 3 is dedicated to the Fourier-Galerkin approximation. The triangulation of the domain Ω_a and the Fourier-finite-element subspaces are introduced and bounds of the Fourier-finite-element approximation error in the norm of H^1 under various regularity assumptions are proved. In particular, for domains with reentrant edges and conical points, it is shown that a combination of the singular function method with the finite element method on locally graded meshes for the approximation of the two-dimwnsional solutions would yield optimal convergence rate.

2 Analytical preliminaries

2.1 The boundary value problem and the geometry

Let $\hat{\Omega}$ be a bounded and simply connected open subset of $\mathbb{R}^3$ with at least Lipschitz continuous boundary $\hat{\Gamma} := \partial\hat{\Omega}$. Suppose $\hat{\Omega}$ represents an isotropic and homogeneous medium and consider on $\hat{\Omega}$ the electric field $\hat{\mathbf{E}} = (E_1, E_2, E_3)$ of time-harmonic Maxwell's equations

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\hat{\mathbf{E}} - \alpha^2\hat{\mathbf{E}} = \hat{\mathbf{f}} & \text{in} \quad \hat{\Omega} \\ \hat{\mathbf{E}} \wedge \hat{\mathbf{n}} = \mathbf{0} & \text{on} \quad \hat{\Gamma} \end{cases} \tag{2.1}$$

where α is a complex number with nonzero imaginary part and $\hat{\mathbf{f}} \in L_2(\hat{\Omega})^3$. We notice that equation (2.1) is not an elliptic system, however, it can be solved by considering the following equivalent elliptic system (cf. [5, 14, 16]):

$$\begin{cases} -\triangle\hat{\mathbf{E}} - \alpha^2\hat{\mathbf{E}} = \hat{\mathbf{f}} & \text{in} \quad \hat{\Omega} \\ \hat{\mathbf{E}} \wedge \hat{\mathbf{n}} = \mathbf{0} & \text{on} \quad \hat{\Gamma} \\ \operatorname{div}\hat{\mathbf{E}} = 0 & \text{on} \quad \hat{\Gamma} \end{cases} \tag{2.2}$$

Let (x_1, x_2, x_3) denote the Cartesian coordinates of the point $x \in \mathbb{R}^3$ and suppose that $\hat{\Omega}$ is axisymmetric with respect to the x_3-axis, that is, the set $\hat{\Omega} \setminus \Gamma_0$ (Γ_0 is the part of the x_3-axis contained in $\hat{\Omega}$) is generated by rotation of a bounded plane meridian domain $\Omega_a \subset \mathbb{R}^2_+$ about the x_3-axis. We denote by $\partial\Omega_a$ the boundary of Ω_a and $\Gamma_a := \partial\Omega_a \setminus \bar{\Gamma}_0$.

Let r, φ, z ($x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, $x_3 = z$) with $\varphi \in (-\pi, \pi]$ denote the cylindrical coordinates. Then the mappings $\hat{\Omega} \setminus \Gamma_0 \to \Omega := \Omega_a \times (-\pi, \pi]$ and $\hat{\Gamma} \to \Gamma := \Gamma_a \times (-\pi, \pi]$ are one-to-one. Accordingly, for any function $\hat{v}(x)$, $x \in \hat{\Omega} \setminus \Gamma_0$, some function $v(r, \varphi, z)$ on Ω is uniquely defined by

$$v(r, \varphi, z) := \hat{v}(r\cos\varphi, r\sin\varphi, z) \tag{2.3}$$

and for each vector field $\hat{\mathbf{v}}(x) = (\hat{v}_1(x), \hat{v}_2(x), \hat{v}_3(x))$, $x \in \hat{\Omega} \setminus \Gamma_0$, some vector field $\mathbf{v}(r, \varphi, z) = (v_r(r, \varphi, z), v_\varphi(r, \varphi, z), v_z(r, \varphi, z))$ is uniquely defined on Ω by

$$v_r := \hat{v}_1 \cos\varphi + \hat{v}_2 \sin\varphi, \quad v_\varphi := -\hat{v}_1 \sin\varphi + \hat{v}_2 \cos\varphi, \quad v_z := \hat{v}_3 \tag{2.4}$$

By (2.4) the images of the operators **curl** and div in cylindrical coordinates are

$$\begin{aligned} \mathbf{curl}_{r\varphi z}\, \mathbf{v} &:= \Big(\frac{1}{r}\frac{\partial v_z}{\partial \varphi} - \frac{\partial v_\varphi}{\partial z}, \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, \frac{1}{r}\frac{\partial (r v_\varphi)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \varphi}\Big)^T \\ \mathrm{div}_{r\varphi z}\, \mathbf{v} &:= \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z} \end{aligned} \tag{2.5}$$

We consider on Ω_a some corner point $E_a = (r_e, z_e) \in \Gamma_a$ with interior angle θ_0 and introduce in the (r, z)-plane local polar coordinates (R, θ) with respect to E_a by $r - r_e = R\cos(\theta + \theta_r)$, $z - z_e = R\sin(\theta + \theta_r)$. For a corner $C_a = (0, z_c) \in \bar{\Gamma}_a \cap \bar{\Gamma}_0$ with interior angle $\theta_0 \neq \pi/2$ we introduce local polar coordinates (R, θ) with respect to this point by $r = R\sin(\theta)$, $z - z_c = -R\cos(\theta)$. The rotation of $E_a \in \Gamma_a$ about the x_3-axis yields an axisymmetric edge on $\hat{\Gamma}$ and the angular point C_a is a conic point on $\hat{\Gamma}$. In Ω_a some circular sector neighborhood G_a of the angular points E_a and C_a, respectively, is defined by (see Figure 2.1 and Figure 2.2)

$$\bar{G}_a := \{(r, z) \in \bar{\Omega}_a : 0 \le R \le R_0, 0 \le \theta \le \theta_0\}, \quad G_a := \bar{G}_a \setminus \partial G_a, \quad \partial_0 G_a := \partial G_a \setminus \bar{\Gamma}_0 \tag{2.6}$$

where ∂G_a is the boundary of G_a. Let $\hat{G}$ denote the 3D domain generated by rotating G_a about the x_3-axis and $\partial \hat{G}$ its boundary. Then $G = G_a \times (-\pi, \pi]$ and $\partial_0 G := \partial_0 G_a \times (-\pi, \pi]$ are the images of $\hat{G}$ and $\partial \hat{G}$ in the (r, φ, z)-system.

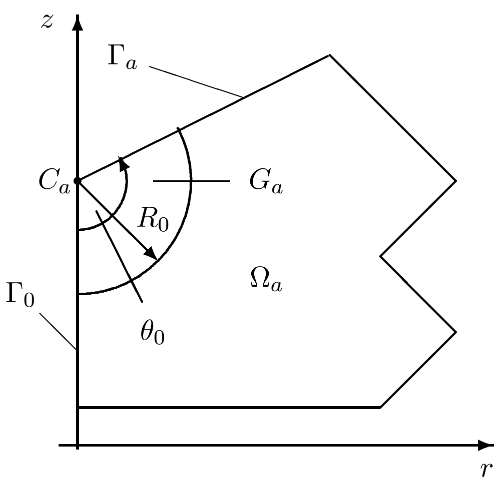

Figure 2.1

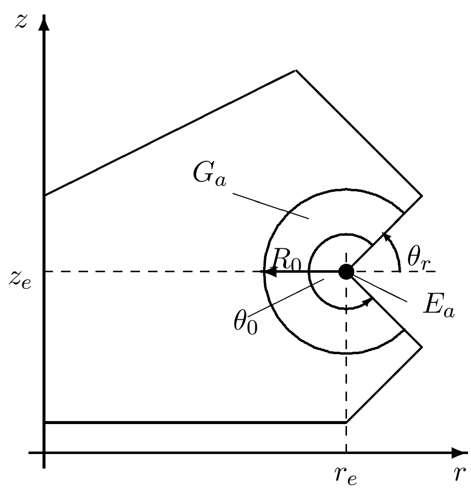

Figure 2.2

For the variational formulation of the boundary value problem (2.2) in cylindrical coordinates, we introduce on Ω the following function spaces (see [22] for more details)

$$\begin{aligned}
L_2^*(\Omega) &:= \{v = v(r,\varphi,z) : \int_\Omega |v|^2 dr d\varphi dz < \infty, v \ 2\pi\text{-periodic w.r.t. } \varphi\} \\
X^0_{1/2}(\Omega) &:= \{v = v(r,\varphi,z) : r^{1/2} v \in L_2^*(\Omega)\} \\
\mathcal{V}(\mathbf{curl}_{r\varphi z}, \Omega) &:= \{\mathbf{v} \in (X^0_{1/2}(\Omega))^3 : \ \mathbf{curl}_{r\varphi z}\,\mathbf{v} \in (X^0_{1/2}(\Omega))^3\} \\
\mathcal{V}_0(\mathbf{curl}_{r\varphi z}\,, \Omega) &:= \{\mathbf{v} \in \mathcal{V}(\mathbf{curl}_{r\varphi z}, \Omega) : \ v_z n_r - v_r n_z = 0, \ v_\varphi = 0 \quad \text{on} \quad \Gamma\} \\
\mathcal{V}_0(\mathbf{curl}_{r\varphi z}, \text{div}_{r\varphi z}) &:= \{\mathbf{v} \in \mathcal{V}_0(\mathbf{curl}_{r\varphi z}\,, \Omega) : \quad \text{div}_{r\varphi z}\mathbf{v} \in X^0_{1/2}(\Omega)\} \qquad (2.7)\\
V_N(\Omega) &:= \{\mathbf{v} \in (X^0_{1/2}(\Omega))^3 : \frac{\partial v_r}{\partial r}, \frac{\partial v_\varphi}{\partial r}, \frac{\partial v_z}{\partial r}, \frac{\partial v_r}{\partial z}, \frac{\partial v_\varphi}{\partial z}, \frac{\partial v_z}{\partial z}, \frac{1}{r}\frac{\partial v_r}{\partial \varphi} - \frac{1}{r} v_\varphi, \\
&\qquad \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{1}{r} v_r, \ \frac{1}{r}\frac{\partial v_z}{\partial \varphi} \in X^0_{1/2}(\Omega); \ v_z n_r - v_r n_z = 0, \ v_\varphi = 0 \quad \text{on} \quad \Gamma\}
\end{aligned}$$

equipped with the norms

$$\begin{aligned}
\|v\|_{X^0_{1/2}(\Omega)} &:= \Big\{\int_\Omega |v|^2 r dr d\varphi dz\Big\}^{1/2} \\
\|\mathbf{v}\|_{\mathcal{V}(\mathbf{curl}_{r\varphi z}, \Omega)} &:= \{\|\mathbf{v}\|^2_{X^0_{1/2}(\Omega)^3} + \|\mathbf{curl}_{r\varphi z}\,\mathbf{v}\|^2_{X^0_{1/2}(\Omega)^3}\}^{1/2} \\
\|\mathbf{v}\|_{\mathcal{V}_0(\mathbf{curl}_{r\varphi z}, \text{div}_{r\varphi z})} &:= \{\|\mathbf{v}\|^2_{\mathcal{V}(\mathbf{curl}_{r\varphi z}, \Omega)} + \|\text{div}_{r\varphi z}\,\mathbf{v}\|^2_{X^0_{1/2}(\Omega)}\}^{1/2} \qquad (2.8)\\
\|\mathbf{v}\|_{V_N(\Omega)} &:= \Big\{\|\mathbf{v}\|^2_{X^0_{1/2}(\Omega)^3} + \Big\|\frac{\partial \mathbf{v}}{\partial r}\Big\|^2_{X^0_{1/2}(\Omega)^3} + \Big\|\frac{\partial \mathbf{v}}{\partial z}\Big\|^2_{X^0_{1/2}(\Omega)^3} + \Big\|\frac{1}{r}\frac{\partial v_z}{\partial \varphi}\Big\|^2_{X^0_{1/2}(\Omega)} \\
&\qquad + \Big\|\frac{1}{r}(\frac{\partial v_r}{\partial \varphi} - v_\varphi)\Big\|^2_{X^0_{1/2}(\Omega)} + \Big\|\frac{1}{r}(\frac{\partial v_\varphi}{\partial \varphi} + v_r)\Big\|^2_{X^0_{1/2}(\Omega)}\Big\}^{1/2}
\end{aligned}$$

In cylindrical coordinates, the weak solution of (2.2) is obtained as follows:
Find $\mathbf{u} \in \mathcal{V}_0(\mathbf{curl}_{r\varphi z}, \text{div}_{r\varphi z})$ such that

$$b(\mathbf{u},\mathbf{v}) = h(\mathbf{v}) \quad \text{for any} \quad \mathbf{v} \in \mathcal{V}_0(\mathbf{curl}_{r\varphi z}, \text{div}_{r\varphi z}), \quad \text{where} \qquad (2.9)$$

$$b(\mathbf{u},\mathbf{v}) := \int_\Omega \Big\{\mathbf{curl}_{r\varphi z}\,\mathbf{u}\cdot\mathbf{curl}_{r\varphi z}\,\mathbf{v} + \text{div}_{r\varphi z}\,\mathbf{u}\ \text{div}_{r\varphi z}\,\mathbf{v} - \alpha^2\mathbf{u}\cdot\mathbf{v}\Big\} r dr d\varphi dz \qquad (2.10)$$

$$h(\mathbf{v}) := \int_\Omega \mathbf{f}\cdot\mathbf{v}\, r dr d\varphi dz \qquad (2.11)$$

Theorem 2.1 ([5, 8, 16]). *Suppose $\alpha^2 \neq 0$ is not an eigenvalue of the Dirichlet-Laplace operator on Ω. Then the sesquilinear form $b(\mathbf{u},\mathbf{v})$ from (2.10) is coercive in $\mathcal{V}_0(\mathbf{curl}_{r\varphi z}, div_{r\varphi z})$, and for any $\mathbf{f} \in X^0_{1/2}(\Omega)^3$, there exists a unique solution $\mathbf{u} \in \mathcal{V}_0(\mathbf{curl}_{r\varphi z}, div_{r\varphi z})$ of the variational problem (2.9). Moreover, the solution $\mathbf{u}$ satisfies the a priori estimate*

$$\|\mathbf{u}\|_{\mathcal{V}_0(\mathbf{curl}_{r\varphi z}, div_{r\varphi z})} \leq C\|\mathbf{f}\|_{X^0_{1/2}(\Omega^3} \qquad (2.12)$$

Remark 2.1 ([5, 8, 16]). *The problem:*

$$\textit{Find } \mathbf{u} \in V_N(\Omega): \quad b(\mathbf{u}, \mathbf{v}) = h(\mathbf{v}) \quad \textit{for any} \quad \mathbf{v} \in V_N(\Omega) \tag{2.13}$$

where $V_N(\Omega)$, $b(\cdot,\cdot)$ and $h(\cdot)$ are taken from (2.7), (2.10) *and* (2.11), *respectively, has a unique solution. In fact, the sesquilinear form $b(\mathbf{u}, \mathbf{v})$ is still coercive in $V_N(\Omega)$. That is, there is a constant $C > 0$ such that*

$$b(\mathbf{u}, \mathbf{u}) \geq C\|\mathbf{u}\|^2_{\mathcal{V}_N(\Omega)} \tag{2.14}$$

If the domain Ω is convex (or sufficiently regular), then the spaces $\mathcal{V}_0(\mathbf{curl}_{r\varphi z},\ div_{r\varphi z})$ and $V_N(\Omega)$ coincide and in this case Problems (2.9) *and* (2.13) *are equivalent, in the sense that they have the same solution which can be approximated by the usual Fourier-finite-element method (cf. [4, 15, 17, 20]). However, if Ω is not convex, then the space $V_N(\Omega)$ is a closed subspace of $\mathcal{V}_0(\mathbf{curl}_{r\varphi z},\ div_{r\varphi z})$ with infinite codimension, and in this case problems* (2.9) *and* (2.13) *are no longer equivalent, and the solution $\mathbf{u} \in \mathcal{V}_0(\mathbf{curl}_{r\varphi z},\ div_{r\varphi z})$ of* (2.9) *can no longer be approximated by the usual Fourier-finite-element method.*

2.2 Partial Fourier analysis

For functions $\mathbf{u} \in \mathcal{V}_0(\mathbf{curl}_{r\varphi z},\ \mathrm{div}_{r\varphi z})$, we employ partial Fourier analysis with respect to the rotational angle φ using the orthogonal and complete system $\{1, \sin\varphi, \cos\varphi, \cdots, \sin n\varphi, \cos n\varphi, \cdots\}$ in $L_2(-\pi, \pi)$. We have (cf. [4, 17, 20, 21])

$$\begin{aligned} u_r &= \sum_{n=0}^{\infty}\Big(u^s_{rn}(r,z)\cos n\varphi + u^a_{rn}(r,z)\sin n\varphi\Big) \\ u_\varphi &= \sum_{n=0}^{\infty}\Big(u^s_{\varphi n}(r,z)(-\sin n\varphi) + u^a_{\varphi n}(r,z)\cos n\varphi\Big) \\ u_z &= \sum_{n=0}^{\infty}\Big(u^s_{zn}(r,z)\cos n\varphi + u^a_{zn}(r,z)\sin n\varphi\Big) \end{aligned} \tag{2.15}$$

where the symmetric $\mathbf{u}^s_n = (u^s_{rn}, u^s_{\varphi n}, u^s_{zn})$ and the antisymmetric $\mathbf{u}^a_n = (u^a_{rn}, u^a_{\varphi n}, u^a_{zn})$ Fourier coefficients are defined almost everywhere on Ω_a.

For the study of functions $w = w(r,z)$ and $\mathbf{w} = (w_1(r,z), w_2(r,z), w_3(r,z))$ defined on the plane meridian domain Ω_a, we introduce the following weighted Sobolev spaces (cf. [22])

$$
\begin{aligned}
L_2(\Omega_a) &:= \Big\{w = w(r,z) : \int_{\Omega_a} |w|^2 drdz < \infty\Big\} \\
X(\Omega_a) &:= \big\{w = w(r,z) : r^{1/2} w \in L_2(\Omega_a)\big\} \\
W^{k,2}_\alpha(\Omega_a) &:= \big\{w = w(r,z) : r^\alpha D^\beta w \in L_2(\Omega_a),\ 0 \le |\beta| \le k\big\} \quad \text{for} \quad k = 1,2;\ \alpha \text{ real} \\
X^{1,2}_{1/2}(\Omega_a) &:= \big\{w \in W^{1,2}_{1/2}(\Omega_a) :\ r^{-1} w \in X(\Omega_a)\big\} \\
\mathcal{W}^n(\Omega_a) &:= \Big\{\mathbf{w} \in (X(\Omega_a))^3 :\ \frac{n}{r} w_3 - \frac{\partial w_2}{\partial z}, \frac{\partial w_1}{\partial z} - \frac{\partial w_3}{\partial r}, \frac{1}{r}\frac{\partial (r w_2)}{\partial r} - \frac{n}{r} w_1, \in X(\Omega_a)\Big\} \qquad (2.16) \\
\mathcal{W}^n_0(\Omega_a) &:= \big\{\mathbf{w} \in \mathcal{W}^n(\Omega_a) :\ w_2 = 0,\ w_3 n_r - w_1 n_z = 0 \quad \text{on} \quad \Gamma_a\big\} \\
\mathcal{U}^n(\Omega_a) &:= \Big\{\mathbf{w} \in \mathcal{W}^n(\Omega_a) : \frac{1}{r}\frac{\partial (r w_1)}{\partial r} - \frac{n}{r} w_2 + \frac{\partial w_3}{\partial z} \in X(\Omega_a)\Big\} \\
\mathcal{U}^n_0(\Omega_a) &:= \big\{\mathbf{w} \in \mathcal{U}^n(\Omega_a) :\ w_2 = 0,\ w_3 n_r - w_1 n_z = 0 \quad \text{on} \quad \Gamma_a\big\} \\
V^n(\Omega_a) &:= \begin{cases} \big\{\mathbf{w} \in X^{1,2}_{1/2}(\Omega_a) \times X^{1,2}_{1/2}(\Omega_a) \times W^{1,2}_{1/2}(\Omega_a)\big\} & \text{if} \quad n = 0 \\ \big\{\mathbf{w} \in W^{1,2}_{1/2}(\Omega_a) \times W^{1,2}_{1/2}(\Omega_a) \times X^{1,2}_{1/2}(\Omega_a) : r^{-1}(w_1 - w_2) \in X(\Omega_a)\big\} & \text{if} \quad n = 1 \\ \big\{\mathbf{w} \in X^{1,2}_{1/2}(\Omega_a) \times X^{1,2}_{1/2}(\Omega_a) \times X^{1,2}_{1/2}(\Omega_a)\big\} & \text{if} \quad n \ge 2 \end{cases} \\
V^n_0(\Omega_a) &:= \{\mathbf{w} \in V^n(\Omega_a) : \quad w_2 = 0,\ w_3 n_r - w_1 n_z = 0 \quad \text{on} \quad \Gamma_a\}
\end{aligned}
$$

equipped with the norms

$$
\begin{aligned}
\|w\|_{L_2(\Omega_a)} &:= \Big(\int_{\Omega_a} |w|^2 drdz\Big)^{1/2}, \quad \|w\|_{X(\Omega_a)} := \|r^{1/2} w\|_{L_2(\Omega_a)} \\
\|w\|_{W^{k,2}_\alpha(\Omega)} &:= \Big(\sum_{|\beta| \le k} \|r^\alpha D^\beta w\|^2_{L_2(\Omega_a)}\Big)^{1/2} \\
\|w\|_{X^{1,2}_{1/2}(\Omega_a)} &:= \Big\{\Big\|\frac{w}{r}\|^2_{X(\Omega_a)} + \Big\|\frac{\partial w}{\partial r}\|^2_{X(\Omega_a)} + \Big\|\frac{\partial w}{\partial z}\|^2_{X(\Omega_a)}\Big\}^{1/2} \\
\|\mathbf{w}\|_{\mathcal{W}^n(\Omega_a)} &:= \Big\{\|\mathbf{w}\|^2_{X(\Omega_a)^3} + \Big\|\frac{n}{r} w_3 - \frac{\partial w_2}{\partial z}\Big\|^2_{X(\Omega_a)} + \Big\|\frac{\partial w_1}{\partial z} - \frac{\partial w_3}{\partial r}\Big\|^2_{X(\Omega_a)} \\
&\quad + \Big\|\frac{1}{r}\frac{\partial (r w_2)}{\partial r} - \frac{n}{r} w_1\Big\|^2_{X(\Omega_a)}\Big\}^{1/2} \qquad (2.17) \\
\|\mathbf{w}\|_{\mathcal{U}^n(\Omega_a)} &:= \Big\{\|\mathbf{w}\|^2_{\mathcal{W}^n(\Omega_a)} + \Big\|\frac{1}{r}\frac{\partial (r w_1)}{\partial r} - \frac{n}{r} w_2 + \frac{\partial w_3}{\partial z}\Big\|^2_{X(\Omega_a)}\Big\}^{1/2} \\
\|\mathbf{w}\|_{V^n(\Omega_a)} &:= \begin{cases} \Big\{\|w_1\|^2_{X^{1,2}_{1/2}(\Omega_a)} + \|w_2\|^2_{X^{1,2}_{1/2}(\Omega_a)} + \|w_3\|^2_{W^{1,2}_{1/2}(\Omega_a)}\Big\}^{1/2}, & n = 0 \\ \Big\{\|w_1\|^2_{W^{1,2}_{1/2}(\Omega_a)} + \|w_2\|^2_{W^{1,2}_{1/2}(\Omega_a)} + \|w_3\|^2_{X^{1,2}_{1/2}(\Omega_a)} + \Big\|\frac{(w_1 - w_2)}{r}\Big\|^2_{X(\Omega_a)}\Big\}^{1/2}, & n = 1 \\ \Big\{\|w_1\|^2_{X^{1,2}_{1/2}(\Omega_a)} + \|w_2\|^2_{X^{1,2}_{1/2}(\Omega_a)} + \|w_3\|^2_{X^{1,2}_{1/2}(\Omega_a)}\Big\}^{1/2}, & n \ge 2 \end{cases}
\end{aligned}
$$

For $\mathbf{v} = (v_r(r,z), v_z(r,z))$ and $\psi = \psi(r,z)$ we introduce in the Cartesian system (r,z) the notations

$$
\mathrm{curl}_{rz}\mathbf{v} = \frac{\partial v_z}{\partial r} - \frac{\partial v_r}{\partial z}, \ \mathrm{div}_{rz}\mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z}
$$

Remark 2.2. *Suppose that* $\bar{\Omega}_a \cap \bar{\Gamma}_0 = \emptyset$*, i.e., the axisymmetric domain* $\hat{\Omega}$ *is hollow, then weighting factors of the form* r^α $(\alpha \neq 0)$ *are not needed for the boundedness of the norms in Definition 2.3. By interchanging the first and second components of* $\mathbf{w} = (w_1, w_2, w_3)^T$ *we establish the following equivalence (in terms of equivalent norms).*

(i) The spaces $\mathcal{U}_0^n(\Omega_a)$ *are equivalent to* $H_0^1(\Omega_a) \times \mathcal{H}_0(curl_{rz},\ div_{rz})$

(ii) The spaces $V_0^n(\Omega_a)$ *are equivalent to* $H_0^1(\Omega_a) \times H_N(\Omega_a)$

In the following theorem, norms of functions defined on the three-dimensional domain Ω are expressed by means of generalized completeness relationships by norms of their Fourier coefficients defined on Ω_a.

Theorem 2.2 ([20, 21, 22]). (i) *Let* $\mathbf{u} = (u_r, u_\varphi, u_z) \in X_{1/2}^0(\Omega)^3$ *and let* $\mathbf{u}_n^s = (u_{rn}^s, u_{\varphi n}^s, u_{zn}^s)$, $\mathbf{u}_n^a = (u_{rn}^a, u_{\varphi n}^a, u_{zn}^a)$ $(n \in \mathbb{N}_0 := \{0, 1, 2, \cdots\})$ *denote its Fourier coefficients with* $\mathbf{u}_0^s = (u_{r0}^s, 0, u_{z0}^s)$ *and* $\mathbf{u}_0^a = (0, u_{\varphi 0}^a, 0)$*. Then (use* $t = r, \varphi, z$*)*

$$\|u_t\|_{X_{1/2}^0(\Omega)}^2 = 2\pi\Big\{\|u_{t0}^s\|_{X(\Omega_a)}^2 + \|u_{t0}^a\|_{X(\Omega_a)}^2\Big\} + \pi\sum_{n=1}^{\infty}\Big(\|u_{tn}^s\|_{X(\Omega_a)}^2 + \|u_{tn}^a\|_{X(\Omega_a)}^2\Big) < \infty \quad (2.18)$$

(ii) *If* $\mathbf{u} \in \mathcal{V}(\mathbf{curl}_{r\varphi z},\ \Omega)$*, then the sums*

$$\begin{aligned} a0 \;&:=\; 2\pi\Big\{\|u_{r0}^s\|_{X(\Omega_a)}^2 + \|u_{z0}^s\|_{X(\Omega_a)}^2 + \|u_{\varphi 0}^a\|_{X(\Omega_a)}^2 + \Big\|\frac{\partial u_{\varphi 0}^a}{\partial z}\Big\|_{X(\Omega_a)}^2 \\ &+\; \Big\|\frac{\partial u_{r0}^s}{\partial z} - \frac{\partial u_{z0}^s}{\partial r}\Big\|_{X(\Omega_a)}^2 + \Big\|\frac{1}{r}\frac{\partial(r u_{\varphi 0}^a)}{\partial r}\Big\|_{X(\Omega_a)}^2\Big\} \end{aligned} \quad (2.19)$$

$$\begin{aligned} a1 \;&:=\; \pi\sum_{n=1}^{\infty}\sum_{e\in\{s,a\}}\Big\{\|\mathbf{u}_n^e\|_{X(\Omega_a)^3}^2 + \Big\|\frac{n}{r}u_{zn}^e - \frac{\partial u_{\varphi n}^e}{\partial z}\Big\|_{X(\Omega_a)}^2 \\ &+\; \Big\|\frac{\partial u_{rn}^e}{\partial z} - \frac{\partial u_{zn}^e}{\partial r}\Big\|_{X(\Omega_a)}^2 + \Big\|\frac{1}{r}\frac{\partial(r u_{\varphi n}^e)}{\partial r} - \frac{n}{r}u_{rn}^e\Big\|_{X(\Omega_a)}^2\Big\} \end{aligned} \quad (2.20)$$

are bounded and

$$\|\mathbf{u}\|_{\mathcal{V}(\mathbf{curl}_{r\varphi z},\Omega)}^2 = a0 + a1 \quad (2.21)$$

(iii) *If* $\mathbf{u} \in \mathcal{V}_0(\mathbf{curl}_{r\varphi z},\ \mathrm{div}_{r\varphi z})$*, then the series*

$$a2 := 2\pi\Big\{\Big\|\frac{1}{r}\frac{\partial(r u_{r0}^s)}{\partial r} + \frac{\partial u_{z0}^s}{\partial z}\Big\|_{X(\Omega_a)}^2\Big\} + \pi\sum_{n=1}^{\infty}\sum_{e\in\{s,a\}}\Big\{\Big\|\frac{1}{r}\frac{\partial(r u_{rn}^e)}{\partial r} - \frac{n}{r}u_{\varphi n}^e + \frac{\partial u_{zn}^e}{\partial z}\Big\|_{X(\Omega_a)}^2\Big\}$$

is bounded and

$$\|\mathbf{u}\|_{\mathcal{V}(\mathbf{curl}_{r\varphi z},\mathrm{div}_{r\varphi z})}^2 = a0 + a1 + a2 \quad (2.22)$$

(iv) If $\mathbf{u} \in V_N(\Omega)$ *then*

$$\begin{aligned}
\|\mathbf{u}\|^2_{V_N(\Omega)} &= 2\pi\Big\{\|u^s_{r0}\|^2_{W^{1,2}_{1/2}(\Omega_a)} + \|u^s_{z0}\|^2_{W^{1,2}_{1/2}(\Omega_a)} + \|u^a_{\varphi 0}\|^2_{W^{1,2}_{1/2}(\Omega_a)} + \Big\|\frac{1}{r}u^s_{r0}\Big\|^2_{X(\Omega_a)} \\
&+ \Big\|\frac{1}{r}u^a_{\varphi 0}\Big\|^2_{X(\Omega_a)}\Big\} + \pi\sum_{n=1}^{\infty}\sum_{e\in\{s,a\}}\Big\{\|\mathbf{u}^e_n\|^2_{W^{1,2}_{1/2}(\Omega_a)^3} + \Big\|\frac{1}{r}(u^e_{rn} - nu^e_{\varphi n})\Big\|^2_{X(\Omega_a)} \\
&+ \Big\|\frac{1}{r}(nu^e_{rn} - u^e_{\varphi n})\Big\|^2_{X(\Omega_a)} + n^2\Big\|\frac{1}{r}u^e_{zn}\|^2_{X(\Omega_a)}\Big\} < \infty \qquad (2.23)
\end{aligned}$$

The symbol $\sum_{e\in\{s,a\}}$ *means summation over* s *and* a*, respectively.*

We will use subsequently the notation u^e_n to mean some property holds for the Fourier coefficient u^s_n as well as u^a_n.

Remark 2.3 ([15]). *If the function* $v(r,\varphi,z)$ *and only some of its derivatives belong to* $X^0_{1/2}(\Omega)$*, then corresponding completeness relations of the type* (2.18) *hold for these derivatives. For example, let* $\frac{\partial^l v}{\partial\varphi^l} \in X^0_{1/2}(\Omega)$ $(l = 0,1)$*, then*

$$\|v\|^2_{X^0_{1/2}(\Omega)} + \Big\|\frac{\partial v}{\partial\varphi}\Big\|^2_{X^0_{1/2}(\Omega)} = 2\pi\|v^s_0\|^2_{X(\Omega_a)} + \pi\sum_{n=1}^{\infty}(1+n^2)\big\{\|v^s_n\|^2_{X(\Omega_a)} + \|v^a_n\|^2_{X(\Omega_a)}\big\} < \infty \quad (2.24)$$

The solution $\mathbf{u}$ of the variational equations (2.9) can be represented by Fourier series according to (2.15) with Fourier coefficients $\mathbf{u}^e_n(r,z)$, $(r,z)\in\Omega_a$, $n\in\mathbb{N}_0$, being the unique solutions of an infinite sequence of two-dimensional boundary value problems. For simplicity let us introduce the notations

$$\begin{aligned}
\mathbf{curl}^n_{rz}\mathbf{u}^e_n &:= \Big(\frac{n}{r}u^e_{zn} - \frac{\partial u^e_{\varphi n}}{\partial z}, \frac{\partial u^e_{rn}}{\partial z} - \frac{\partial u^e_{zn}}{\partial r}, \frac{1}{r}\frac{\partial(ru^e_{\varphi n})}{\partial r} - \frac{n}{r}u^e_{rn}\Big) \qquad (2.25)\\
\mathrm{div}^n_{rz}\mathbf{u}^e_n &:= \frac{1}{r}\frac{\partial(ru^e_{rn})}{\partial r} - \frac{n}{r}u^e_{\varphi n} + \frac{\partial u^e_{zn}}{\partial z}
\end{aligned}$$

Theorem 2.3 ([20, 21, 22]). *(i) Let* $\mathbf{u}^e_n$*,* $\mathbf{v}^e_n$ *and* $\mathbf{f}^e_n$ $(n\in\mathbb{N})$ *denote the Fourier coefficients of the functions* $\mathbf{u}$*,* $\mathbf{v} \in \mathcal{V}_0(\mathbf{curl}_{r\varphi z}, \mathrm{div}_{r\varphi z})$ *and* $\mathbf{f} \in X^0_{1/2}(\Omega)^3$ *respectively. Then the functionals* $b(\mathbf{u},\mathbf{v})$ *and* $h(\mathbf{v})$ *from* (2.10) *and* (2.11)*, respectively, can be represented in the form*

$$\begin{aligned}
b(\mathbf{u},\mathbf{v}) &= 2\pi\big\{b_0(\mathbf{u}^s_0,\mathbf{v}^s_0) + b_0(\mathbf{u}^a_0,\mathbf{v}^a_0)\big\} + \pi\sum_{n=1}^{\infty}\big\{b_n(\mathbf{u}^s_n,\mathbf{v}^s_n) + b_n(\mathbf{u}^a_0,\mathbf{v}^a_n)\big\} \qquad (2.26)\\
h(\mathbf{v}) &= 2\pi\big\{h^s_0(\mathbf{v}^s_0) + h^a_0(\mathbf{v}^a_0)\big\} + \pi\sum_{n=1}^{\infty}\big\{h^s_n(\mathbf{v}^s_n) + h^a_n(\mathbf{v}^a_n)\big\} \qquad (2.27)\\
b_n(\mathbf{u}^e_n,\mathbf{v}^e_n) &:= \int_{\Omega_a}\big\{\mathbf{curl}^n_{rz}\mathbf{u}^e_n\cdot\mathbf{curl}^n_{rz}\mathbf{v}^e_n + \mathrm{div}^n_{rz}\mathbf{u}^e_n\mathrm{div}^n_{rz}\mathbf{v}^e_n - \alpha^2\mathbf{u}^e_n\cdot\mathbf{v}^e_n\big\}rdrdz \qquad (2.28)\\
h^e_n(\mathbf{v}^e_n) &:= \int_{\Omega_a}\mathbf{f}^e_n\cdot\mathbf{v}^e_n rdrdz \qquad (2.29)
\end{aligned}$$

(ii) For $\mathbf{f} \in X^0_{1/2}(\Omega)^3$*, let* $\mathbf{u} \in \mathcal{V}_0(\mathbf{curl}_{r\varphi z}, \mathrm{div}_{r\varphi z})$ *be the solution of the three-dimensional problem* (2.9)*. If* $\mathbf{u}^e_n$ *and* $\mathbf{f}^e_n$ $(n \in \mathbb{N}_0, e \in \{s, a\})$ *are the Fourier coefficients of* $\mathbf{u}$ *and* $\mathbf{f}$*, then* $\mathbf{u}^e_n$ *are unique solutions of the two-dimensional variational equations*

$$n \in \mathbb{N}_0 : \text{ find } \mathbf{u}^e_n \in \mathcal{U}^n_0(\Omega_a) : \; b_n(\mathbf{u}^e_n, \mathbf{w}) = h^e_n(\mathbf{w}) \quad \text{for any} \quad \mathbf{w} \in \mathcal{U}^n_0(\Omega_a) \tag{2.30}$$

where $b_n(\cdot,\cdot)$ *and* $h^e_n(\cdot)$ *are from* (2.28) *and* (2.29)*, respectively.*
(iii) The solutions $\mathbf{u}^e_n$ *of* (2.30) *satisfy the a priori estimates*

$$\|\mathbf{u}^e_n\|^2_{\mathcal{U}^n_0(\Omega_a)} \le C\|\mathbf{f}^e_n\|^2_{X(\Omega_a)^3}, \quad n \in \mathbb{N}_0 \tag{2.31}$$

(iv) If the solutions $\mathbf{u}^e_n \in \mathcal{U}^n_0(\Omega_a)$ *of the two-dimensional boundary value problems* (2.30) *satisfy additionally the regularity assumption* $\mathbf{u}^e_n \in V^n_0(\Omega_a)$*, then the norms* $\|\mathbf{u}^e_n\|_{V^n_0(\Omega_a)}$ *and* $\|\mathbf{u}^e_n\|_{\mathcal{U}^n_0(\Omega_a)}$ *are equivalent, and* $\mathbf{u}^e_n$ *also satisfy the a priori estimates*

$$\|\mathbf{u}^e_n\|_{V^n_0(\Omega_a)} \le C_2\|\mathbf{f}^e_n\|_{X(\Omega_a)^3}, \quad n \in \mathbb{N}_0 \tag{2.32}$$

$$\|\mathbf{u}^e_n\|_{V^n_0(\Omega_a)} \le C_3\Big\{\|\mathbf{u}^e_n\|_{W^{1,2}_{1/2}(\Omega_a)^3} + n^2\Big\|\frac{1}{r}\mathbf{u}^e_n\Big\|_{X(\Omega_a)^3}\Big\} \le \frac{C_4}{n^2}\|\mathbf{f}^e_n\|_{X(\Omega_a)^3} \text{ for } n \ge 2 \tag{2.33}$$

Remark 2.4. *In practice, a finite number* $N > 0$ *of the two-dimensional boundary value problems* (2.30) *are solved numerically by means of the finite element method and an approximation of the solution* $\mathbf{u} \in \mathcal{V}_0(\mathbf{curl}_{r\varphi z}, div_{r\varphi z})$ *of the three-demensional boundary value problem* (2.9) *is obtained via Fourier synthesis.*

2.3 Effects of edges on the 2D solutions

In this subsection we investigate the effects of corners $E_a \in \Gamma_a$ on the regularity of the solutions $\mathbf{u}^e_n \in \mathcal{U}^n_0(\Omega_a)$ of the two-dimensional boundary value problems (2.30).

Since the problem of regularity of solutions of boundary value problems is a local one, we suppose temporarily that the domain Ω_a has only one corner with vertex $E_a = (r_k, z_k) \in \Gamma_a$ and interior angle $\theta_0 \ne \pi$, and that the angles at $\bar{\Gamma}_a \cap \bar{\Gamma}_0$ are right angles. We introduce a smooth cut-off function $\eta(r, z) = \tilde{\eta}(R) \in \mathbb{C}^\infty[0, \infty)$ by

$$\tilde{\eta}(R) := \begin{cases} 1 & \text{for} \quad 0 \le R \le R_0/3 \\ 0 & \text{for} \quad R \ge 2R_0/3 \end{cases} \tag{2.34}$$

where R and R_0 are from (2.6).

Remark 2.5. *We observe that* $\mathbf{u}^e_n = (1-\eta)\mathbf{u}^e_n + \eta\mathbf{u}^e_n$*. Thus, the global regularity of the solutions* $\mathbf{u}^e_n$ $(n \in \mathbb{N}_0)$ *of* (2.30) *on* Ω_a *is determined by the regularity of the functions* $(1-\eta)\mathbf{u}^e_n$ *and* $\eta\mathbf{u}^e_n$*. From the singularity theory of solutions of elliptic boundary value problems (cf. [10, 11, 12, 13]), we have that for* $\mathbf{f}^e_n \in X(\Omega_a)^3$ *the solution* $\mathbf{u}^e_n$ *of* (2.30) *satisfies* $\mathbf{u}^e_n \in$

$W^{2,2}_{1/2}(\Omega_a \cap N_a)^3$ $(n \in \mathbb{N}_0)$ *for any compact subset* N_a *of* Ω_a *that does not contain irregular boundary points of* Ω_a *and the estimate*

$$\|\mathbf{u}^e_n\|_{W^{2,2}_{1/2}(\Omega_a\cap N_a)^3} \le C\{\|\mathbf{f}^e_n\|_{X(\Omega_a)^3} + \|\mathbf{u}^e_n\|_{\mathcal{U}^n_0(\Omega_a)}\} \tag{2.35}$$

holds. Furthermore, it can be shown (cf. [12, 13, 20]) that the functions $\mathbf{u}^e_{n\eta} := \eta\mathbf{u}^e_n$ *are unique solutions of boundary value problems of the form:*
Find $\mathbf{u}^e_{n\eta} \in \mathcal{U}^n_0(G_a)$ $(n \in \mathbb{N}_0)$ *such that*

$$(\mathbf{curl}^n_{rz}\mathbf{u}^e_{n\eta}, \mathbf{curl}^n_{rz}\mathbf{w}) + (div^n_{rz}\mathbf{u}^e_{n\eta}, div^n_{rz}\mathbf{w}) - \alpha^2(\mathbf{u}^e_{n\eta}, \mathbf{w}) = (\mathbf{f}^e_{n\eta}, \mathbf{w}),\ \forall \mathbf{w} \in \mathcal{U}^n_0(G_a) \tag{2.36}$$

where the new right hand side $\mathbf{f}^e_{n\eta}$ *now depends on* η, $\mathbf{f}^e_n$, $\mathbf{u}^e_n$ *and some first order derivatives of* $\mathbf{u}^e_n$. *Thus,* $\mathbf{f}^e_{n\eta} \in X(G_a)^3$ *is satisfied and*

$$\|\mathbf{f}^e_{n\eta}\|_{X(G_a)^3} \le C\|\mathbf{f}^e_n\|_{X(\Omega_a)^3} \tag{2.37}$$

We will drop subsequently the subscript η and will write $\mathbf{u}^e_n$, $\mathbf{f}^e_n$, and so on, instead of $\mathbf{u}^e_{n\eta}$, $\mathbf{f}^e_{n\eta}$, and so on.

Lemma 2.1. *For* $\mathbf{f}^e_n \in X(G_a)^3$, *the solution* $\mathbf{u}^e_n \in \mathcal{U}^n_0(G_a)$ $(n \in \mathbb{N}_0)$ *of the two-dimensional boundary value problem* (2.36) *admits the following properties.*

(i) If $0 < \theta_0 < \frac{\pi}{2}$, *then* $\mathbf{u}^e_n$ *satisfies the relations*

$$\mathbf{u}^e_n \in \mathcal{U}^n_0(G_a) \cap W^{2,2}_{1/2}(G_a)^3, \quad \|\mathbf{u}^e_n\|_{W^{2,2}_{1/2}(G_a)^3} \le C\|\mathbf{f}^e_n\|_{X(G_a)^3},\ n \in \mathbb{N}_0 \tag{2.38}$$

(ii) If $\frac{\pi}{2} \le \theta_0 < 2\pi$, *then there exist unique real numbers* γ^e_{nk} $(k = 1, 2, 3)$ *and* γ^e_n *such that* $\mathbf{u}^e_n = (u^e_{rn}(r, z), u^e_{\varphi n}(r, z), u^e_{zn}(r, z))$ *can be split in the form*

$$\begin{aligned}
&\mathbf{u}^e_n = \mathbf{w}^e_n + \mathbf{s}^e_n, \quad \mathbf{w}^e_n \in W^{2,2}_{1/2}(G_a)^3\\
&s^e_{rn}(r, z) := \sum_{k=1}^{3}\sum_{0<\lambda_k\le 2} \gamma^e_{nk}R^{\lambda_k-1}\sin\lambda_k\theta, \quad \lambda_k := \frac{k\pi}{\theta_0}\\
&s^e_{\varphi n}(r, z) := \gamma^e_n R^{\lambda_1}\sin\lambda_1\theta\\
&s^e_{zn}(r, z) := \sum_{k=1}^{3}\sum_{0<\lambda_k\le 2} \gamma^e_{nk}R^{\lambda_k-1}\cos\lambda_k\theta, \quad \lambda_k := \frac{k\pi}{\theta_0}\\
&\sum_{k=1}^{3}\sum_{0<\lambda_k\le 2} |\gamma^e_{nk}| + |\gamma^e_n| + \|\mathbf{w}^e_n\|_{W^{2,2}_{1/2}(G_a)^3} \le C\|\mathbf{f}^e_n\|_{X(G_a)^3}
\end{aligned} \tag{2.39}$$

with R *and* θ *from* (2.6) *and* η *from* (2.34).

Proof. By integration by parts formula, one verifies that the variational problems (2.36) are the generalized formulations of the two-dimensional boundary value problems

$$-\frac{\partial^2 u_{rn}^e}{\partial r^2}-\frac{\partial^2 u_{rn}^e}{\partial z^2}-\frac{1}{r}\frac{\partial u_{rn}^e}{\partial r}+\frac{1+n^2}{r^2}u_{rn}^e-\frac{2n}{r^2}u_{\varphi n}^e-\alpha^2 u_{rn}^e=f_{rn}^e \quad \text{in} \quad G_a \tag{2.40}$$

$$-\frac{\partial^2 u_{\varphi n}^e}{\partial r^2}-\frac{\partial^2 u_{\varphi n}^e}{\partial z^2}-\frac{1}{r}\frac{\partial u_{\varphi n}^e}{\partial r}+\frac{1+n^2}{r^2}u_{\varphi n}^e-\frac{2n}{r^2}u_{rn}^e-\alpha^2 u_{\varphi n}^e=f_{\varphi n}^e \quad \text{in} \quad G_a \tag{2.41}$$

$$-\frac{\partial^2 u_{zn}^e}{\partial r^2}-\frac{\partial^2 u_{zn}^e}{\partial z^2}-\frac{1}{r}\frac{\partial u_{zn}^e}{\partial r}+\frac{n^2}{r^2}u_{zn}^e-\alpha^2 u_{zn}^e=f_{zn}^e \quad \text{in} \quad G_a \tag{2.42}$$

$$u_{rn}^e n_z-u_{zn}^e n_r=0, \quad \frac{\partial u_{rn}^e}{\partial r}+\frac{1}{r}u_{rn}^e+\frac{\partial u_{zn}^e}{\partial z}=0, \quad u_{\varphi n}^e=0 \quad \text{on} \quad \partial G_a \tag{2.43}$$

Since $\bar{G}_a \cap \bar{\Gamma}_0=\emptyset$, Remark 2.2 implies that $\frac{1}{r^2}u_{rn}^e$, $\frac{1}{r^2}u_{\varphi n}^e$, $\frac{1}{r^2}u_{zn}^e$, $\frac{1}{r}\frac{\partial u_{rn}^e}{\partial r}$, $\frac{1}{r}\frac{\partial u_{rn}^e}{\partial z}$, $\frac{1}{r}\frac{\partial u_{\varphi n}^e}{\partial r}$, $\frac{1}{r}\frac{\partial u_{\varphi n}^e}{\partial z}$, $\frac{1}{r}\frac{\partial u_{zn}^e}{\partial r}$, $\frac{1}{r}\frac{\partial u_{zn}^e}{\partial z} \in L_2(G_a)$. Thus, solving the problems (2.40) − (2.43) is equivalent to solving the following two boundary value problems separately.

$$-\triangle_{rz}\tilde{\mathbf{u}}_n^e-\alpha^2\tilde{\mathbf{u}}_n^e= \quad \text{in} \quad G_a, \quad \tilde{\mathbf{f}}_n^e \in L_2(G_a)^2 \tag{2.44}$$

$$\tilde{u}_{rn}^e n_z-\tilde{u}_{zn}^e n_r=0, \quad \frac{\partial u_{rn}^e}{\partial r}+\frac{\partial u_{zn}^e}{\partial z}=\tilde{t}_n^e \quad \text{on} \quad \partial G_a \tag{2.45}$$

and

$$-\triangle_{rz}\tilde{u}_{\varphi n}^e=\tilde{f}_{\varphi n}^e \quad \text{in} \quad G_a, \quad \tilde{u}_{\varphi n}^e=0 \quad \text{on} \quad \partial G_a, \quad \tilde{f}_{\varphi n}^e \in L_2(G_a) \tag{2.46}$$

where

$$\begin{aligned}
&\triangle_{rz}:=\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}, \quad \tilde{\mathbf{u}}_n^e=(\tilde{u}_{rn}^e,\tilde{u}_{zn}^e), \quad \tilde{\mathbf{f}}_n^e=(\tilde{f}_{rn}^e,\tilde{f}_{zn}^e)\\
&\tilde{f}_{rn}^e:=f_{rn}^e-\frac{1+n^2}{r^2}u_{rn}^e+\frac{1}{r}\frac{\partial u_{rn}^e}{\partial r}+\frac{2n}{r^2}u_{\varphi n}^e,\\
&\tilde{f}_{zn}^e:=f_{zn}^e-\frac{n^2}{r^2}u_{zn}^e+\frac{1}{r}\frac{\partial u_{zn}^e}{\partial r}, \quad \tilde{t}_n^e:=\frac{1}{r}u_{rn}^e\\
&\tilde{f}_{\varphi n}^e:=f_{\varphi n}^e+\frac{2n}{r^2}u_{rn}^e+\alpha^2 u_{\varphi n}^e-\frac{1+n}{r^2}u_{\varphi n}^e-\frac{1}{r}\frac{\partial u_{\varphi n}^e}{\partial r}
\end{aligned}$$

But (2.44), (2.45) is the regularized Maxwell's equations in a circular sector and (2.46) is the Dirichlet problem for the Poisson equation in a circular sector. Thus, the proof of relations (2.38), (2.39) follows from the proof of relations which express the regularity properties of solutions of problems of the type (2.44), (2.45) Cf. [23]), and problems of the type (2.46), see e.g. [12, 13]. ■

Theorem 2.4. *For* $\mathbf{f}_n^e \in X(\Omega_a)^3$*, let* $\mathbf{u}_n^e \in \mathcal{U}_0^n(\Omega_a)$ $(n \in \mathbb{N}_0)$ *be the solution of the two-dimensional boundary value problem* (2.30)*.*

(i) If $0<\theta_0<\frac{\pi}{2}$*, then* $\mathbf{u}_n^e$ *satisfies the relations*

$$\mathbf{u}_n^e \in \mathcal{U}_0^n(\Omega_a)\cap W_{1/2}^{2,2}(\Omega_a)^3, \quad \|\mathbf{u}_n^e\|_{W_{1/2}^{2,2}(\Omega_a)^3} \le C\|\mathbf{f}_n^e\|_{X(\Omega_a)^3}, \; n \in \mathbb{N}_0$$

(ii) If $\frac{\pi}{2} \leq \theta_0 < 2\pi$, then there exist unique real numbers γ_{nk}^e $(k = 1, 2, 3)$ and γ_n^e such that $\mathbf{u}_n^e = (u_{rn}^e(r,z), u_{\varphi n}^e(r,z), u_{zn}^e(r,z))$ can be split in the following form.

$$\mathbf{u}_n^e = \mathbf{w}_n^e + \mathbf{s}_n^e, \quad \mathbf{w}_n^e \in W_{1/2}^{2,2}(\Omega_a)^3 \tag{2.47}$$

$$s_{rn}^e(r,z) := \tilde{\eta}(R) \sum_{k=1}^{3} \sum_{0<\lambda_k\leq 2} \gamma_{nk}^e R^{\lambda_k - 1} \sin \lambda_k \theta, \quad \lambda_k := \frac{k\pi}{\theta_0}$$

$$s_{\varphi n}^e(r,z) := \tilde{\eta}(R) \gamma_n^e R^{\lambda_1} \sin \lambda_1 \theta \tag{2.48}$$

$$s_{zn}^e(r,z) := \tilde{\eta}(R) \sum_{k=1}^{3} \sum_{0<\lambda_k\leq 2} \gamma_{nk}^e R^{\lambda_k - 1} \cos \lambda_k \theta, \quad \lambda_k := \frac{k\pi}{\theta_0}$$

$$\sum_{k=1}^{3} \sum_{0<\lambda_k\leq 2} |\gamma_{nk}^e| + |\gamma_n^e| + \|\mathbf{w}_n^e\|_{W_{1/2}^{2,2}(\Omega_a)^3} \leq C \|\mathbf{f}_n^e\|_{X(\Omega_a)^3}$$

(iii) If there are J corners on Γ_a with interior angles $\pi/2 \leq \theta_j < 2\pi$ $(\theta_j \neq \pi,\ j = 1, \cdots, J)$, then the singular part $\mathbf{s}_n^e$ in (2.47) is to be replaced by the sum of the corresponding singular parts $\mathbf{s}_{jn}^e$ $(j = 1, \cdots, J)$. The relations in (2.48) continue to be valid.

Proof. The representations (2.47) and (2.48) follow from the splitting $\mathbf{u}_n^e = (1-\eta)\mathbf{u}_n^e + \eta \mathbf{u}_n^e$ of the solution of (2.30), Remark 2.5 and Lemma 2.1. ■

2.4 Effects of conical points on the 2D solutions

In this subsection, we assume for simplicity that the axisymmetric domain $\hat{\Omega}$ has no edges and that $\hat{\Omega}$ has one conical point only on the rotation axis with angle $0 < \theta_0 < \pi$ $(\theta_0 \neq \pi/2)$, see Figure 2.1. In [24], the asymptotic behavior of the Fourier coefficients $\mathbf{u}_n^e$ $(n \in \mathbb{N}_0)$ of the solution $\hat{\mathbf{u}}$ of time-harmonic Maxwell's problems in three-dimensional axisymmetric domains with conical points on the rotation axis, has been studied. For $\hat{\mathbf{f}} \in L_2(\hat{\Omega})^3$, it is proved in [24] that, depending on the solid angle θ_0 at the conical point, only the Fourier coefficient $\mathbf{u}_1^e$ can exhibit singularities. In fact, the following results were proved.

Theorem 2.5. *For $\mathbf{f} \in X_{1/2}^0(\Omega)^3$, let $\mathbf{u} \in \mathcal{V}_0(\mathbf{curl}_{r\varphi z}, div_{r\varphi z})$ be the solution of (2.9), and let $\mathbf{u}_n^e$ and $\mathbf{f}_n^e$ $(n \in \mathbb{N}_0)$ denote the Fourier coefficients of $\mathbf{u}$ and $\mathbf{f}$, respectively.*

(i) For any $\theta_0 \in (0, \pi)$, the Fourier coefficients $\mathbf{u}_n^e$, $n \in \mathbb{N}_0$, satisfy the relation $\mathbf{u}_n^e \in W_{1/2}^{1,2}(\Omega_a)^3$.

(ii) For any $\theta_0 \in (0, \pi)$, the Fourier coefficients $\mathbf{u}_n^e$, $n \in \mathbb{N}_0 \setminus \{1\}$, satisfy the relations

$$\mathbf{u}_n^e \in W_{1/2}^{2,2}(\Omega_a)^3, \quad \|\mathbf{u}_n^e\|_{W_{1/2}^{2,2}(\Omega_a)^3} \leq \|\mathbf{f}_n^e\|_{X(\Omega_a)^3}, \quad n \in \mathbb{N}_0 \setminus \{1\} \tag{2.49}$$

(iii) There exists an angle $\theta^* \in (\frac{\pi}{2}, \pi)$, such that if $\theta_0 \in (0, \theta^*)$, then the Fourier coefficient $\mathbf{u}_1^e$ also satisfies the relations in (2.49).

(iv) If $\theta_0 \in (\theta^*, \pi)$ then there exists a real number γ^e such that the Fourier coefficient $\mathbf{u}_1^e$ can be split in the form

$$\mathbf{u}_1^e = \mathbf{w}_1^e + \mathbf{s}_1^e, \quad \mathbf{s}_1^e := \tilde{\eta}(R)\gamma^e R^\alpha \mathbf{U}_1(\alpha, \theta), \quad \mathbf{w}_1^e \in W_{1/2}^{2,2}(\Omega_a)^3 \tag{2.50}$$

$$|\gamma^e| + \|\mathbf{w}_1^e\|_{W_{1/2}^{2,2}(\Omega_a)^3} \leq \|\mathbf{f}_1^e\|_{X(\Omega_a)^3} \tag{2.51}$$

In (2.50)*, $\mathbf{U}_1(\alpha, \theta) := (P_\alpha(\cos\theta), P_\alpha(\cos\theta), P_\alpha^{-1}(\cos\theta))$, where $P_\alpha^{-k}(\cos\theta)$ $(k = 0, 1)$ are the associated Legendre functions of the first kind and α is the solution of the transcendental equation*

$$P_\alpha(\cos\theta_0) = 0, \quad 0 < \alpha < \frac{1}{2}$$

The notations R, θ and $\tilde{\eta}$ have the same meaning as in (2.6) *and* (2.34)*, respectively.*

Remark 2.6. *We observe from Theorem 2.5 that conical points on the domain Ω do not affect the required regularity of the Fourier coefficients $\mathbf{u}_n^e$ $(n \in \mathbb{N}_0)$ if the domain Ω is convex.*

3 The Fourier-finite-element method and error estimates

In this section we consider the discretization of the boundary value problem (2.9) by the Fourier-finite-element method using piecewise linear polynomial shape functions and the estimation of the error in the H^1-norm. Owing to the fact that the solution $\mathbf{u} \in \mathcal{V}_0(\mathbf{curl}_{r\varphi z}, \mathrm{div}_{r\varphi z})$ of (2.9) on nonconvex domains Ω cannot be approximated by the usual Fourier-finite-element method (cf. Remark 2.1), we distinguish subsequently the discretization of problem (2.9) on convex domains and nonconvex domains, since different strategies are needed.

3.1 Locally refined meshes

We want to approximate a finite number of the two-dimensional boundary value problems (2.30) by the finite element method. For this purpose, the plane meridian domain Ω_a is partitioned in the usual way (cf. [6]) into a set of shape regular and admissible triangulation $\mathcal{T}_h = \{T\}$, where $h \in (0, 1)$ is the discretization parameter. For problems containing singularities we use graded mesh refinement, as described below, to improve error estimates.

In the vicinity $\bar{G}_a$ of each singular point $K_a \in \{E_a, C_a\}$ (cf. (2.6)) of the domain Ω_a, a graded refinement of the mesh $\mathcal{T}_h$ is defined with the help of a grading parameter $0 < \kappa \leq 1$, a grading function $R_j = R_j(h)$ and a step size h_j, by

$$R_j(h) = \frac{2}{3} R_0 (jh)^{\frac{1}{\kappa}} \ (j = 0, 1, \cdots, J), \quad h_j = R_j(h) - R_{j-1}(h) \ (j = 1, 2, \cdots, J) \tag{3.1}$$

where $J := [h^{-1}]$ is the integer part of h^{-1}, R_0 is from (2.6) and the parameter κ is to be chosen in relation with the singular exponents as would be explained in Lemma 3.3. The following relationships which will be needed subsequently can easily be verified from (3.1).

Lemma 3.1. *For h, h_j, R_j and κ $(0 < \kappa \leq 1)$ the following relations hold.*

$$
\begin{aligned}
c_1 h R_j^{1-\kappa} \leq h_j \leq c_2 h R_j^{1-\kappa}, \qquad & c_3 R_j \frac{1}{j} \leq h_j \leq c_4 R_j \frac{1}{j},\ j = 1, 2, \cdots, J \\
h_{j-1} \leq h_j \leq c_5 h_{j-1}, \qquad & R_{j-1} \leq R_j \leq c_6 R_{j-1},\ j = 2, 3, \cdots, J
\end{aligned} \tag{3.2}
$$

The triangulation $\mathcal{T}_h$ is further refined in the vicinity $\bar{G}_a$ of each irregular point K_a such that the relative size h_T of each triangle $T \in \mathcal{T}_h$ depends on its distance $R_T := \text{dist}\,(T, K_a) := \inf_{p=(r,z)\in T} |K_a - p|$ from the vertex K_a in the following way.

Assumption 3.1. *The triangulation $\mathcal{T}_h$ is refined locally in the vicinity $\bar{G}_a$ of each singular point K_a by means of the grading parameter κ such that the following assumptions are satisfied.*

$$
\begin{aligned}
&\rho_1 h^{\frac{1}{\kappa}} \leq h_T \leq \rho_1^{-1} h^{\frac{1}{\kappa}} \text{ for } T \in \mathcal{T}_h : R_T = 0 \\
&\rho_2 h R_T^{1-\kappa} \leq h_T \leq \rho_2^{-1} h R_T^{1-\kappa} \text{ for } T \in \mathcal{T}_h : 0 < R_T < R_J \\
&\rho_3 h \leq h_T \leq \rho_3^{-1} h \text{ for } T \in \mathcal{T}_h : R_J \leq R_T
\end{aligned} \tag{3.3}
$$

with some constants ρ_i, $0 < \rho_i \leq 1$ $(i = 1, 2, 3)$ and R_J from (3.1).

We observe that if $\kappa = 1$, then there is no actual grading and the mesh is quasi-uniform. Owing to Assumption 3.1 and Lemma 3.1 the asymptotic behaviour of the parameter h_T near the vertex K_a is determined by the relations

$$
\begin{aligned}
c_1 h_j \leq h_T \leq c_1^{-1} h_j \quad & \text{for} \quad T \in \mathcal{T}_h : R_{j-1} \leq R_T \leq R_j (j = 1, 2, \cdots, J) \\
c_2 h \leq h_T \leq c_2^{-1} h \quad & \text{for} \quad T \in \mathcal{T}_h : R_J \leq R_T
\end{aligned} \tag{3.4}
$$

with $0 < c_i \leq 1$ $(i = 1, 2)$, and h_j, R_j from (3.1).

For subsequent error analysis, we distinguish the following subsets of the triangulation $\mathcal{T}_h$ contained in $\bar{G}_a$.

$$
\begin{aligned}
&D_{0h} := \{T \in \mathcal{T}_h : R_T = 0\}, \quad D_{1h} := \{T \in \mathcal{T}_h : 0 < R_T < R_1\} \\
&D_{jh} := \{T \in \mathcal{T}_h : R_{j-1} \leq R_T < R_j\}, \quad D_h := \cup_{j=0}^{J} D_{jh}
\end{aligned} \tag{3.5}
$$

Clearly, $D_h \subset \bar{G}_a$ is satisfied and the sets D_{jh} $(j = 0, 1, \cdots, J)$ are pairwise disjoint. Let n_j denotes the total number of triangles contained in D_{jh}, then the relation

$$
n_0 \leq C, \quad n_j \leq Cj, \quad j = 1, \cdots, J \tag{3.6}
$$

can be verified (cf. [15]), where the constant C is independent of h.

3.2 The Fourier-Galerkin approximation for convex domains

For convex domains $\Omega \subset \mathbb{R}^3$, the solution $\mathbf{u}$ of (2.9) belongs to the space $V_N(\Omega)$ (cf. Remark 2.1) and in this case problem (2.13) can be solved instead.

Let $\mathbf{R}_n^s$ and $\mathbf{R}_n^a$ denote the 3×3-diagonal matrices given by

$$\mathbf{R}_n^s := \text{diag}[\cos n\varphi, -\sin n\varphi, \cos n\varphi]; \quad \mathbf{R}_n^a := \text{diag}[\sin n\varphi, \cos n\varphi, \sin n\varphi] \tag{3.7}$$

As first step, an approximation of the solution $\mathbf{u} \in V_N(\Omega)$ of (2.13) is defined by Fourier series truncation (Fourier approximation) by

$$\mathbf{u}_N = \sum_{n=0}^{N} (\mathbf{R}_n^s \mathbf{u}_n^s + \mathbf{R}_n^a \mathbf{u}_n^a) \quad \text{for} \quad N > 0 \tag{3.8}$$

The undetermined $2N + 2$ Fourier coefficients $\mathbf{u}_n^s$ and $\mathbf{u}_n^a$ $(n = 0, 1, 2, \cdots, N)$ are solutions of the two-dimensional variational equations (2.30) for $0 \le n \le N$. These solutions will be approximated, in the second step, by the finite element method using piecewise linear polynomial shape functions. Thus, we define the finite element subspaces as follows.

$$\begin{aligned}
R_h &:= \{v_h(r,z) : v_h \in \mathbb{C}(\bar{\Omega}_a) : v_h|_T \in \mathbb{P}_1(T) \text{ for any } T \in \mathcal{T}_h\} \\
V_h^0 &:= \{\mathbf{v}_h = (u_h, v_h, w_h) \in R_h^3 : v_h = 0,\ u_h n_z - w_h n_r = 0 \text{ on } \bar{\Gamma}_a,\ u_h = v_h = 0 \text{ on } \Gamma_0\} \\
V_h^1 &:= \{\mathbf{v}_h = (u_h, v_h, w_h) \in R_h^3 : v_h = 0,\ u_h n_z - w_h n_r = 0 \text{ on } \bar{\Gamma}_a,\ u_h = v_h,\ w_h = 0 \text{ on } \Gamma_0\} \\
V_h^2 &:= \{\mathbf{v}_h = (u_h, v_h, w_h) \in R_h^3 : v_h = 0,\ u_h n_z - w_h n_r = 0 \text{ on } \bar{\Gamma}_a,\ u_h = v_h = w_h = 0 \text{ on } \Gamma_0\} \\
V_h^N &:= \Big\{\mathbf{v}_{hN} : \mathbf{v}_{hN} = \sum_{n=0}^{N} (\mathbf{R}_n^s \mathbf{v}_{nh}^s + \mathbf{R}_n^a \mathbf{v}_{nh}^a),\ \mathbf{v}_{0h}^e \in V_h^0,\ \mathbf{v}_{1h}^e \in V_h^1,\ \mathbf{v}_{nh}^e \in V_h^2,\ 2 \le n \le N\Big\}
\end{aligned} \tag{3.9}$$

where $\mathbb{P}_1(T)$ denotes the space of all polynomials of degree ≤ 1 on T. The inclusions $V_h^0 \subset V_0^0(\Omega_a)$, $V_h^1 \subset V_0^1(\Omega_a)$, $V_h^2 \subset V_0^n(\Omega_a)$ $(n \ge 2)$ and $V_h^N \subset V_N(\Omega)$ are obvious.

The Fourier-Galerkin approximation of the solution $\mathbf{u} \in V_N(\Omega)$ of (2.13) is obtained as follows. Find $\mathbf{u}_{hN} \in V_h^N$ such that

$$b(\mathbf{u}_{hN}, \mathbf{v}) = h(\mathbf{v}) \quad \text{for all} \quad \mathbf{v} \in V_h^N \tag{3.10}$$

where the functionals $b(\cdot,\cdot)$ and $h(\cdot)$ are taken from (2.10) and (2.11), respectively. By Lax/Milgram and Cea's lemmas, there exists a unique solution $\mathbf{u}_{hN} \in V_h^N$ which satisfies the a priori estimate

$$\|\mathbf{u} - \mathbf{u}_{hN}\|_{V_N(\Omega)} \le \|\mathbf{u} - \mathbf{v}_{hN}\|_{V_N(\Omega)} \quad \text{for any} \quad \mathbf{v}_{hN} \in V_h^N \tag{3.11}$$

By analogy to Theorem 2.3 one verifies that the solution $\mathbf{u}_{hN}$ of (3.10) admits the decomposition

$$\mathbf{u}_{hN} = \sum_{n=0}^{N} (R_n^s \mathbf{u}_{nh}^s + R_n^a \mathbf{u}_{nh}^a) \tag{3.12}$$

with the coefficients $\mathbf{u}^e_{nh}$ being the unique solutions of the variational (Galerkin) equations

$$\begin{array}{rlrcl} n=0: & \text{find} \quad \mathbf{u}^e_{0h} \in V^0_h : b_0(\mathbf{u}^e_{0h}, \mathbf{w}) & = & h^e_0(\mathbf{w}) & \text{for} \quad \mathbf{w} \in V^0_h \\ n=1: & \text{find} \quad \mathbf{u}^e_{1h} \in V^1_h : b_1(\mathbf{u}^e_{1h}, \mathbf{w}) & = & h^e_1(\mathbf{w}) & \text{for} \quad \mathbf{w} \in V^1_h \\ 2 \le n \le N: & \text{find} \quad \mathbf{u}^e_{nh} \in V^2_h : b_n(\mathbf{u}^e_{nh}, \mathbf{w}) & = & h^e_n(\mathbf{w}) & \text{for} \quad \mathbf{w} \in V^2_h \end{array} \tag{3.13}$$

where the functionals $b_n(\cdot,\cdot)$ and $h^e_n(\cdot)$ are from (2.28) and (2.29), respectively.

3.3 Estimates of the interpolation error

For estimating the error in (3.11), we define a projection $\mathbf{r}_{hN}\mathbf{u} : V_N(\Omega) \to V^N_h$ such that for $\mathbf{u} \in V_N(\Omega)$

$$\mathbf{r}_{hN}\mathbf{u} = \sum_{n=0}^{N} (\mathbf{R}^s_n \mathbf{u}^s_{nh} + \mathbf{R}^a_n \mathbf{u}^a_{nh}) \quad \text{with} \quad \mathbf{u}^e_{nh} = \mathbf{\Pi}_h \mathbf{s}^e_n + \mathbf{\Pi}_h \mathbf{w}^e_n, \quad n = 0, 1, \cdots, N \tag{3.14}$$

where the symbol Π_h denotes the usual Lagrange interpolation operator (cf. [6])) and $\mathbf{u}^e_n = \mathbf{s}^e_n + \mathbf{w}^e_n$ with $\mathbf{w}^e_n \in W^{2,2}_{1/2}(\Omega_a)^3$, corresponds to the splitting of the Fourier coefficients in regular and singular parts according to Theorem 2.4. For estimating the interpolation error for the regular part, we use the following lemma.

Lemma 3.2 ([17] Proposition 6.1). *Let the triangulation $\mathcal{T}_h$ $(h \in (0,1))$ satisfy Assumption 3.1 with the grading parameter $0 < \kappa \le 1$. Then there exist constants $C > 0$ independent of h and v such that*

$$\|v - \Pi_h v\|_{W^{1,2}_{1/2}(\Omega_a)} \le Ch\|v\|_{W^{2,2}_{1/2}(\Omega_a)} \quad \textit{for any} \quad v \in W^{2,2}_{1/2}(\Omega_a) \tag{3.15}$$

$$\|r^{-1}(v - \Pi_h v)\|_{X(\Omega_a)} \le Ch\|v\|_{W^{2,2}_{1/2}(\Omega_a)} \quad \textit{for any} \quad v \in W^{0,2}_{-1/2}(\Omega_a) \cap W^{2,2}_{1/2}(\Omega_a) \tag{3.16}$$

Now, let us consider the interpolation error $\mathbf{s}^e_n - \mathbf{\Pi}_h \mathbf{s}^e_n$ of the singular part. Obviously, the singular functions $\mathbf{s}^e_n$ are continuous on Ω_a and their interpolates are well defined. Since $\mathbf{s}^e_n$ vanishes for $R \ge 2/3R_0$, it is sufficient to consider the error $\mathbf{s}^e_n - \mathbf{\Pi}_h \mathbf{s}^e_n$ on $\bar{G}_a$ where weights of the type r^α $(\alpha \ne 0)$ can be neglected (cf. Remark 2.2).

Lemma 3.3. *Let the triangulation $\mathcal{T}_h$ satisfy Assumption 3.1 with $0 < \kappa \le 1$. Then there exist constants $C > 0$ independent of h and n such that*

$$\|\mathbf{s}^e_n - \mathbf{\Pi}_h \mathbf{s}^e_n\|^2_{L_2(G_a)^3} \le Ch^{2\lambda}\|\mathbf{f}^e_n\|^2_{X(\Omega_a)^3} \tag{3.17}$$

$$|\mathbf{s}^e_n - \mathbf{\Pi}_h \mathbf{s}^e_n|^2_{H^1(G_a)^3} \le C\beta^2(h,\kappa)\|\mathbf{f}^e_n\|^2_{X(\Omega_a)^3} \tag{3.18}$$

$$\textit{where} \quad \beta^2(h,\kappa) := \begin{cases} h^{\frac{2(\lambda-1)}{\kappa}} & \textit{for} \quad \kappa > \lambda - 1 \\ h^2|\ln h| & \textit{for} \quad \kappa = \lambda - 1 \\ h^2 & \textit{for} \quad 0 < \kappa < \lambda - 1 \end{cases}, \quad \lambda := \frac{\pi}{\theta_0} \tag{3.19}$$

Proof. The proof of this lemma is similar to the proof of Lemma 5.2 in [20] and so we keep it brief. The global interpolation error $\mathbf{s}_n^e - \mathbf{\Pi}_h \mathbf{s}_n^e$ can be expressed in terms of the local interpolation error $\mathbf{s}_n^e - \mathbf{\Pi}_T \mathbf{s}_n^e$, where $\mathbf{\Pi}_T \mathbf{s}_n^e := \mathbf{\Pi}_h \mathbf{s}_n^e|_T$, and with the help of the partitions D_{jh} $(j = 0, 1, \cdots, J)$ from (3.5) as follows.

$$|\mathbf{s}_n^e - \mathbf{\Pi}_h \mathbf{s}_n^e|^2_{H^1(G_a)^3} = \sum_{T\in D_{0h}} |\mathbf{s}_n^e - \mathbf{\Pi}_T \mathbf{s}_n^e|^2_{H^1(T)^3} + \sum_{j=1}^{J} \sum_{T\in D_{jh}} |\mathbf{s}_n^e - \mathbf{\Pi}_T \mathbf{s}_n^e|^2_{H^1(T)^3} \tag{3.20}$$

From the relation

$$|\mathbf{s}_n^e - \mathbf{\Pi}_T \mathbf{s}_n^e|^2_{H^1(T)^3} = |s_{rn}^e - \Pi_T s_{rn}^e|^2_{H^1(T)^3} + |s_{\varphi n}^e - \Pi_T s_{\varphi n}^e|^2_{H^1(T)^3} + |s_{zn}^e - \Pi_T s_{zn}^e|^2_{H^1(T)^3} \tag{3.21}$$

we estimate subsequently only the error $|s_{rn}^e - \Pi_T s_{rn}^e|^2_{H^1(T)^3}$, since the other terms can be estimated by analogy. First, we consider $|s_{rn}^e - \Pi_T s_{rn}^e|^2_{H^1(T)}$ on the set D_{0h} consisting of those triangles $T \in \mathcal{T}_h$ which have as a vertex the singular point K_a and on which $\mathbf{s}_n^e \notin H^2(T)^2$. With the help of the inequality

$$|s_{rn}^e - \Pi_T s_{rn}^e|_{H^1(T)} \le |s_{rn}^e|_{H^1(T)} + |\Pi_T s_{rn}^e|^2_{H^1(T)}$$

and the explicit representation of the functions s_{rn}^e and $\Pi_T s_{rn}^e$ (see Theorem 2.4) one easily verifies the estimate

$$|s_{rn}^e - \Pi_T s_{rn}^e|^2_{H^1(T)} \le C|\gamma_{n1}^e|^2 h^{\frac{2(\lambda-1)}{\kappa}} \quad \text{for} \quad T \in D_{0h} \tag{3.22}$$

with γ_{n1}^e from Theorem 2.4. For triangles $T \in D_{jh}$ $(j = 1, 2, \cdots, J)$, which do not have the singular point K_a as a vertex, we use the fact that $s_{rn}^e \in H^2(T)$ and employ the well known local interpolation error estimate

$$|s_{rn}^e - \Pi_T s_{rn}^e|^2_{H^1(T)} \le C h_T^2 |s_{rn}^e|^2_{H^2(T)} \quad \text{for} \quad T \in D_{jh},\ j = 1, 2, \cdots, J \tag{3.23}$$

where the constant C is independent of T. The norm on the right hand side of relation (3.23) can further be bounded as follows.

$$|s_{rn}^e|^2_{H^2(T)} \le C|\gamma_{n1}^e|^2 h_T^2 \mathrm{meas}(T) \Big(\inf_{(R,\theta)\in T} R\Big)^{2(\lambda-3)} \le C|\gamma_{n1}^e|^2 h_T^2 h_T^2 \Big(\inf_{(R,\theta)\in T} R\Big)^{2(\lambda-3)} \tag{3.24}$$

Combining (3.23) and (3.24) and taking into account relations (3.1) – (3.5) we get the estimates

$$\begin{aligned}
|s_{rn}^e - \Pi_T s_{rn}^e|^2_{H^1(T)} &\le C|\gamma_{n1}^e|^2 j^{\frac{4}{\kappa}-4} (j-1)^{\frac{2(\lambda-3)}{\kappa}} h^{\frac{2(\lambda-1)}{\kappa}} \quad \text{for} \quad T \in D_{jh},\ j = 2, \cdots, J \\
|s_{rn}^e - \Pi_T s_{rn}^e|^2_{H^1(T)} &\le C|\gamma_{n1}^e|^2 h^{\frac{2(\lambda-1)}{\kappa}} \quad \text{for} \quad T \in D_{1h}
\end{aligned} \tag{3.25}$$

Summing up (3.25) over all triangles $T \in D_{jh}$ $(j = 1, 2, \cdots, J)$ and taking into account relation (3.6) we get the estimate

$$\sum_{j=1}^{J} \sum_{T\in D_{jh}} |s_{rn}^e - \Pi_T s_{rn}^e|^2_{H^1(T)} \le C|\gamma_{n1}^e|^2 h^{\frac{2(\lambda-1)}{\kappa}} \Big(1 + \sum_{j=2}^{J} j^{\frac{4}{\kappa}-3} (j-1)^{\frac{2(\lambda-3)}{\kappa}}\Big) \tag{3.26}$$

Owing to the inequalities

$$1+\sum_{j=2}^{J} j^{\frac{4}{\kappa}-3}(j-1)^{\frac{2(\lambda-3)}{\kappa}} \le \sum_{j=1}^{J} j^{\frac{2(\lambda-1)}{\kappa}-3} \le C \begin{cases} 1 & \text{for} \quad \kappa > \lambda-1 \\ \ln J & \text{for} \quad \kappa = \lambda-1 \\ J^{\frac{2(\lambda-1)}{\kappa}-2} & \text{for} \quad 0<\kappa<\lambda-1 \end{cases} \tag{3.27}$$

and the fact that $J \le Ch^{-1}$ we get the estimate

$$\sum_{j=1}^{J} \sum_{T \in D_{jh}} |s_{rn}^e - \Pi_T s_{rn}^e|^2_{H^1(T)} \le C|\gamma_{n1}^e|^2 \beta^2(h,\kappa) \tag{3.28}$$

with $\beta^2(h,\kappa)$ from (3.19). Finally, assertion (3.18) is a consequence of the relations (3.20), (3.21), (3.22) and (3.28). Assertion (3.17) can be obtained by analogy. ■

3.4 Error estimates in $H^1(\hat{\Omega})^3$

In this subsection, we estimate the error $\mathbf{u}-\mathbf{u}_{hN}$ generated by replacing the solution $\mathbf{u} \in V_N(\Omega)$ of problem (2.13) by its Fourier-finite-element approximation $\mathbf{u}_{hN}$ from (3.10).

Theorem 3.1. *Let $\mathbf{u} \in V_N(\Omega)$ be the solution of* (2.13) *with $\mathbf{f} \in X^0_{1/2}(\Omega)^3$ and let $\mathbf{u}_N$ be its Fourier approximation defined by* (3.8). *Then there exists a constant $C>0$ independent of $N>0$, $\mathbf{u}$ and $\mathbf{f}$ such that*

$$\|\mathbf{u}-\mathbf{u}_N\|_{V_N(\Omega)} \le CN^{-1}\|\mathbf{f}\|_{X^0_{1/2}(\Omega)^3} \tag{3.29}$$

Proof. It follows from the completeness relation (2.23) and the a priori estimates (2.33) the inequalities

$$\begin{aligned} \|\mathbf{u}-\mathbf{u}_N\|^2_{V_N(\Omega)} &\le C \sum_{n=N+1}^{\infty} \sum_{e\in\{s,a\}} \Big\{ \|\mathbf{u}_n^e\|^2_{W^{1,2}_{1/2}(\Omega_a)^3} + n^2 \Big\| \frac{1}{r}\mathbf{u}_n^e \Big\|^2_{X(\Omega_a)^3} \Big\} \\ &\le CN^{-2} \sum_{n=N+1}^{\infty} \sum_{e\in\{s,a\}} n^2 \Big\{ \|\mathbf{u}_n^e\|^2_{W^{1,2}_{1/2}(\Omega_a)^3} + n^2 \Big\| \frac{1}{r}\mathbf{u}_n^e \Big\|^2_{X(\Omega_a)^3} \Big\} \le CN^{-2}\|\mathbf{f}\|_{X^0_{1/2}(\Omega)^3} \end{aligned}$$ ■

Theorem 3.2. *For $\mathbf{f} \in X^0_{1/2}(\Omega)^3$, let $\mathbf{u} \in V_N(\Omega)$ be the solution of* (2.13) *and $\mathbf{u}_{hN}$ its Fourier-finite-element approximation defined by* (3.10). *Suppose that $\partial\mathbf{f}/\partial\varphi \in X^0_{1/2}(\Omega)^3$ and that the triangulation $\mathcal{T}_h$ satisfies Assumption 3.1. Then there exists a constant $C>0$ independent of h, $N>0$ and $\mathbf{f}$ such that*

$$\|\mathbf{u}-\mathbf{u}_{hN}\|_{V_N(\Omega)} \le C(N^{-1}+\beta(h,\kappa))\Big(\|\mathbf{f}\|_{X^0_{1/2}(\Omega)^3} + \Big\|\frac{\partial \mathbf{f}}{\partial\varphi}\Big\|_{X^0_{1/2}(\Omega)^3} \Big) \tag{3.30}$$

where $\beta(h,\kappa)$ is taken from (3.19).

Proof. We get from (3.11) using the triangle inequality the estimate

$$\|\mathbf{u}-\mathbf{u}_{hN}\|_{V_N(\Omega)} \le \|\mathbf{u}-\mathbf{u}_N\|_{V_N(\Omega)} + \|\mathbf{u}_N - \mathbf{r}_{hN}\mathbf{u}\|_{V_N(\Omega)} \tag{3.31}$$

With the help of the completeness relation (2.23), Lemma 3.2, Lemma 3.3 and Remark 2.3 one verifies the estimates

$$\begin{aligned}
\|\mathbf{u}_N - \mathbf{r}_{hN}\mathbf{u}\|^2_{V_N(\Omega)} &\leq C\Big(\Big\|\frac{1}{r}(u^s_{r0} - \Pi_h u^s_{r0})\Big\|^2_{X(\Omega_a)} + \Big\|\frac{1}{r}(u^a_{\varphi 0} - \Pi_h u^a_{\varphi 0})\Big\|^2_{X(\Omega_a)} \\
&+ \sum_{n=0}^{N}\sum_{e\in\{s,a\}}\Big\{\|\mathbf{u}^e_n - \Pi_h\mathbf{u}^e_n\|^2_{W^{1,2}_{1/2}(\Omega_a)^3} + n^2\Big\|\frac{1}{r}(\mathbf{u}^e_n - \Pi_h\mathbf{u}^e_n)\Big\|^2_{X(\Omega_a)^3}\Big\}\Big) \\
&\leq C\Big(\Big\|\frac{1}{r}(w^s_{r0} - \Pi_h w^s_{r0})\Big\|^2_{X(\Omega_a)} + \Big\|\frac{1}{r}(w^a_{\varphi 0} - \Pi_h w^a_{\varphi 0})\Big\|^2_{X(\Omega_a)} \\
&+ \sum_{n=1}^{N}\sum_{e\in\{s,a\}}\Big\{\|\mathbf{w}^e_n - \Pi_h\mathbf{w}^e_n\|^2_{W^{1,2}_{1/2}(\Omega_a)^3} + n^2\Big\|\frac{1}{r}(\mathbf{w}^e_n - \Pi_h\mathbf{w}^e_n)\Big\|^2_{X(\Omega_a)^3} \\
&+ \Big\|s^s_{r0} - \Pi_h s^s_{r0}\Big\|^2_{L_2(\Omega_a)} + \Big\|s^a_{\varphi 0} - \Pi_h s^a_{\varphi 0}\Big\|^2_{L_2(\Omega_a)} \\
&+ \sum_{n=1}^{N}\sum_{e\in\{s,a\}}\Big\{\|\mathbf{s}^e_n - \Pi_h\mathbf{s}^e_n\|^2_{H^1(\Omega_a)^3} + n^2\Big\|\mathbf{s}^e_n - \Pi_h\mathbf{s}^e_n\Big\|^2_{L_2(\Omega_a)^3}\Big\}\Big) \\
&\leq C\beta^2(h,\kappa)\sum_{n=0}^{\infty}\sum_{e\in\{s,a\}}(1+n^2)\|\mathbf{f}^e_n\|^2_{X(\Omega_a)^3} \\
&\leq C\beta^2(h,\kappa)\Big(\|\mathbf{f}\|^2_{X^0_{1/2}(\Omega)^3} + \Big\|\frac{\partial\mathbf{f}}{\partial\varphi}\Big\|^2_{X^0_{1/2}(\Omega)^3}\Big) \qquad (3.32)
\end{aligned}$$

Theorem 3.2 follows finally from Theorem 3.1 and relations (3.31) and (3.32). ■

Remark 3.1. *The additional smoothness assumption on the function* $\mathbf{f}$ *in Theorem 3.2, i.e. the condition* $\partial\mathbf{f}/\partial\varphi \in (X^0_{1/2}(\Omega))^3$*, is only needed to uncouple the discretization parameters* h *and* N *as in* (3.30)*. One could also employ the method introduce by Mercier/Raugel [17], in which a mixed projection combining the usual Lagrange interpolation operator* Π_h *and Clement's [7]* L_2*-projection operator is used for the approximation of the Fourier Coefficients* $\mathbf{u}^e_n$*. In this case the additional smoothness requirement on* $\mathbf{f}$ *is not necessary to achieve the same order of convergence.*

3.5 Domains with reentrant edges and conical vertices; the singular function method

In this subsection, we show how the Fourier-finite-element method in combination with the singular function method (cf. [5, 14, 16]) can be used to construct an approximation algorithm with optimal convergence rate for the solution of problem (2.9) if Ω has reentrant edges.

We assume for the sake of definiteness that the domain Ω has one reentrant edge with interior $\frac{3\pi}{2} \leq \theta_1 < 2\pi$ and a conical vertex with interior angle $\theta^* < \theta_2 < \pi$, see Theorem 2.4 and Theorem 2.5, respectively, and that all the other edges of the domain are such that the expected regularity of the solution is not affected. By Theorem 2.4 and Theorem 2.5, the Fourier coefficients $\mathbf{u}^e_n$ $(n \in \mathbb{N}_0)$ of the solution $\mathbf{u} \in \mathcal{V}_0(\mathbf{curl}_{r\varphi z}, \mathrm{div}_{r\varphi z})$ of (2.9) can be split in the form $\mathbf{u}^e_n = \mathbf{w}^e_n + \mathbf{s}^e_n$

$(n \in \mathbb{N}_0)$ with a regular part $\mathbf{w}_n^e \in W^{2,2}_{1/2}(\Omega_a)^3$ $(n \in \mathbb{N}_0)$ and a singular part $\mathbf{s}_n^e = \mathbf{s}_{n1}^e + \mathbf{s}_{n2}^e + \mathbf{s}_{n3}^e$ for $n \in \mathbb{N}_0 \setminus \{1\}$ and $\mathbf{s}_1^e = \mathbf{s}_{11}^e + \mathbf{s}_{12}^e + \mathbf{s}_{13}^e + \mathbf{s}_{14}^e$, where

$$\begin{aligned}
\mathbf{s}_{n1}^e &= (\tilde{\eta}(R)\gamma_{n1}^e R^{\lambda_1 - 1}\sin\lambda_1\theta,\ \tilde{\eta}(R)\gamma_n^e \sin\lambda_1\theta,\ \tilde{\eta}(R)\gamma_{n1}^e R^{\lambda_1-1}\sin\lambda_1\theta)\\
\mathbf{s}_{n2}^e &= (\tilde{\eta}(R)\gamma_{n2}^e R^{\lambda_2 - 1}\sin\lambda_2\theta,\ 0,\ \tilde{\eta}(R)\gamma_{n2}^e R^{\lambda_2-1}\sin\lambda_2\theta)\\
\mathbf{s}_{n3}^e &= (\tilde{\eta}(R)\gamma_{n3}^e R^{\lambda_3 - 1}\sin\lambda_3\theta,\ 0,\ \tilde{\eta}(R)\gamma_{n3}^e R^{\lambda_3-1}\sin\lambda_3\theta), \quad \lambda_k := \frac{k\pi}{\theta_1}\\
\mathbf{s}_{14}^e &= (\tilde{\eta}(R)\gamma^e R^\alpha P_\alpha(\cos\theta),\ \tilde{\eta}(R)\gamma^e R^\alpha P_\alpha(\cos\theta),\ \tilde{\eta}(R)\gamma^e R^\alpha P_\alpha^{-1}(\cos\theta))
\end{aligned} \tag{3.33}$$

In (3.33), only the singular function $\mathbf{s}_{n1}^e$ does not belong to the space $W^{1,2}_{1/2}(\Omega_a)^3$ and consequently cannot be approximated by the nodal finite elements as described in Subsection 3.2. This problem is solved subsequently by adding to the finite element subspaces in (3.9) the function $\mathbf{s}_{n1}^e$ (the singular function method). The remainder part of the coefficients $\mathbf{u}_n^e$, namely

$$\tilde{\mathbf{w}}_n^e := \mathbf{w}_n^e + \mathbf{s}_{n2}^e + \mathbf{s}_{n3}^e, \quad n \in \mathbb{N}_0 \setminus \{1\};\ \tilde{\mathbf{w}}_1^e := \mathbf{w}_1^e + \mathbf{s}_{12}^e + \mathbf{s}_{13}^e + \mathbf{s}_{14}^e \tag{3.34}$$

belong to the space $W^{1,2}_{1/2}(\Omega_a)^3$ and can be approximated optimally by the nodal finite element method as explained in the preceding subsections, where the grading parameter κ is to be chosen such that $\kappa < \min\{\alpha, \lambda_2 - 1\}$. Thus for the Fourier-Galerking approximation of the solution $\mathbf{u} \in \mathcal{V}_0(\mathbf{curl}_{r\varphi z}, \mathrm{div}_{r\varphi z})$ of (2.9), we introduce the spaces

$$\begin{aligned}
\tilde{V}_h^0 &:= V_h^0 \oplus \mathrm{span}(\mathbf{s}_{01}^e), \quad \tilde{V}_h^1 := V_h^1 \oplus \mathrm{span}(\mathbf{s}_{11}^e), \quad \tilde{V}_h^2 := V_h^2 \oplus \mathrm{span}(\mathbf{s}_{n1}^e),\ 2 \le n \le N\\
\tilde{V}_h^N &:= \{\mathbf{v}_{hN} : \mathbf{v}_{hN} = \sum_{n=0}^N (\mathbf{R}_n^s \mathbf{v}_{nh}^s + \mathbf{R}_n^a \mathbf{v}_{nh}^a), \mathbf{v}_{0h}^e \in \tilde{V}_h^0, \mathbf{v}_{1h}^e \in \tilde{V}_h^1, \mathbf{v}_{nh}^e \in \tilde{V}_h^2, 2 \le n \le N\}
\end{aligned}$$

where V_h^0, V_h^1, V_h^2 and $\mathbf{R}_n^e$ are defined in (3.9) and (3.7), respectively. Obviously, the relations $\tilde{V}_h^0 \subset \mathcal{U}_0^0(\Omega_a)$, $\tilde{V}_h^1 \subset \mathcal{U}_0^1(\Omega_a)$, $\tilde{V}_h^2 \subset \mathcal{U}_0^n(\Omega_a)$ $(n \ge 2)$ and $\tilde{V}_h^N \subset \mathcal{V}_0(\mathbf{curl}_{r\varphi z}, \mathrm{div}_{r\varphi z})$ are fulfilled.

The Fourier-finite-element approximation $\mathbf{u}_{hN}$ of the solution $\mathbf{u}$ of (2.9) is obtained by solving the Fourier-Galerkin equation:

Find $\mathbf{u}_{hN} \in \tilde{V}_h^N$ such that

$$b(\mathbf{u}_{hN}, \mathbf{v}) = h(\mathbf{v}) \quad \text{for all} \quad \mathbf{v} \in \tilde{V}_h^N \tag{3.35}$$

where the functionals $b(\cdot,\cdot)$ and $h(\cdot)$ are from (2.10) and (2.11), respectively. Again Cea's lemma infers that

$$\|\mathbf{u} - \mathbf{u}_{hN}\|_{\mathcal{V}_0(\mathbf{curl}_{r\varphi z}, \mathrm{div}_{r\varphi z})} \le C\|\mathbf{u} - \mathbf{v}\|_{\mathcal{V}_0(\mathbf{curl}_{r\varphi z}, \mathrm{div}_{r\varphi z})} \quad \text{for any} \quad \mathbf{v} \in \tilde{V}_h^N \tag{3.36}$$

The solution $\mathbf{u}_{hN} \in \tilde{V}_h^N$ of equation (3.35) can be expressed as a truncated Fourier series in the form

$$\mathbf{u}_{hN} = \sum_{n=0}^N (\mathbf{R}_n^s \mathbf{u}_{nh}^s + \mathbf{R}_n^a \mathbf{u}_{nh}^a)$$

where the Fourier coefficients $\mathbf{u}^e_{nh}$ $(0 \le n \le N)$ are the unique solutions of the Galerkin equations

$$\begin{aligned}
n = 0: &\quad \text{find} \quad \mathbf{u}^e_{0h} \in \tilde{V}^0_h : b_0(\mathbf{u}^e_{0h}, \mathbf{w}) &=& \quad h^e_0(\mathbf{w}) \quad \text{for} \quad \mathbf{w} \in \tilde{V}^0_h \\
n = 1: &\quad \text{find} \quad \mathbf{u}^e_{1h} \in \tilde{V}^1_h : b_1(\mathbf{u}^e_{1h}, \mathbf{w}) &=& \quad h^e_1(\mathbf{w}) \quad \text{for} \quad \mathbf{w} \in \tilde{V}^1_h \\
2 \le n \le N: &\quad \text{find} \quad \mathbf{u}^e_{nh} \in \tilde{V}^2_h : b_n(\mathbf{u}^e_{nh}, \mathbf{w}) &=& \quad h^e_n(\mathbf{w}) \quad \text{for} \quad \mathbf{w} \in \tilde{V}^2_h
\end{aligned}$$

where the functionals $b_n(\cdot,\cdot)$ and $h^e_n(\cdot)$ are from (2.28) and (2.29), respectively.

3.6 Error estimates in $H^1(\hat{\Omega})^3$

For estimating the error in (3.36), we define again a projection $\mathbf{r}_{hN}\mathbf{u} : \mathcal{V}_0(\mathbf{curl}_{r\varphi z}, \mathrm{div}_{r\varphi z}) \to \tilde{V}^N_h$ such that for $\mathbf{u} \in \mathcal{V}_0(\mathbf{curl}_{r\varphi z}, \mathrm{div}_{r\varphi z})$

$$\mathbf{r}_{hN}\mathbf{u} = \sum_{n=0}^{N} (\mathbf{R}^s_n \mathbf{u}^s_{nh} + \mathbf{R}^a_n \mathbf{u}^a_{nh}) \quad \text{with} \quad \mathbf{u}^e_{nh} = \mathbf{s}^e_{n1} + \mathbf{\Pi}_h \tilde{\mathbf{w}}^e_n, \quad n = 0, 1, \cdots, N \tag{3.37}$$

where $\mathbf{s}^e_{n1}$ and $\tilde{\mathbf{w}}^e_n$ are from (3.33) and (3.34), respectively.

Owing to the fact that the solution $\mathbf{u} \in \mathcal{V}_0(\mathbf{curl}_{r\varphi z}, \mathrm{div}_{r\varphi z})$ does not belong to the space $V_N(\Omega)$, the fine a priori estimate (2.33) of the Fourier coefficients $\mathbf{u}^e_n$ $(n \in \mathbb{N})$ can no longer be proved. Consequently, for the proof of convergence $\mathbf{u}_N \to \mathbf{u}$ in the following theorem we require additionally the condition $\frac{\partial \mathbf{f}}{\partial \varphi} \in X^0_{1/2}(\Omega)^3$.

Theorem 3.3. *Let* $\mathbf{u} \in \mathcal{V}_0(\mathbf{curl}_{r\varphi z}, div_{r\varphi z})$ *be the solution of* (2.9) *with* $\mathbf{f} \in X^0_{1/2}(\Omega)^3$ *and let* $\mathbf{u}_N$ *be its Fourier approximation defined by* (3.8)*. Suppose that* $\frac{\partial \mathbf{f}}{\partial \varphi} \in X^0_{1/2}(\Omega)^3$*, then there exists a constant* $C > 0$ *independent of* $N > 0$*,* $\mathbf{u}$ *and* $\mathbf{f}$ *such that*

$$\|\mathbf{u} - \mathbf{u}_N\|_{V_N(\Omega)} \le C N^{-1} \Big\| \frac{\partial \mathbf{f}}{\partial \varphi} \Big\|_{X^0_{1/2}(\Omega)^3} \tag{3.38}$$

Proof. It follows from the completeness relation (2.22), the definition of the norm $\|\cdot\|_{\mathcal{U}^n(\Omega_a)}$ (cf. (2.17)), the a priori estimates (2.31) and Remark 2.3 the inequalities

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_N\|^2_{\mathcal{V}_0(\mathbf{curl}_{r\varphi z}, \mathrm{div}_{r\varphi z})} &\le C \sum_{n=N+1}^{\infty} \sum_{e \in \{s,a\}} \|\mathbf{u}^e_n\|^2_{\mathcal{U}^n(\Omega_a)} \\
&\le C N^{-2} \sum_{n=N+1}^{\infty} \sum_{e \in \{s,a\}} n^2 \|\mathbf{u}^e_n\|^2_{\mathcal{U}^n(\Omega_a)} \le C N^{-2} \| \frac{\partial \mathbf{f}}{\partial \varphi} \|_{X^0_{1/2}(\Omega)^3} \blacksquare
\end{aligned}$$

Theorem 3.4. *For* $\mathbf{f} \in X^0_{1/2}(\Omega)^3$*, let* $\mathbf{u} \in \mathcal{V}_0(\mathbf{curl}_{r\varphi z}, div_{r\varphi z})$ *be the solution of the boundary value problem* (2.9) *and* $\mathbf{u}_{hN}$ *its Fourier-Galerkin approximation defined according to* (3.35)*. Suppose that* $\partial \mathbf{f}/\partial \varphi \in X^0_{1/2}(\Omega)^3$ *and that the triangulation* $\mathcal{T}_h$ *satisfies Assumption 3.1. Then there exists a constant* $C > 0$ *independent of* h*,* $N > 0$ *and* $\mathbf{f}$ *such that*

$$\|\mathbf{u} - \mathbf{u}_{hN}\|_{\mathcal{V}_0(\mathbf{curl}_{r\varphi z}, div_{r\varphi z})} \le C(N^{-1} + \beta(h,\kappa)) \Big(\|\mathbf{f}\|_{X^0_{1/2}(\Omega)^3} + \Big\| \frac{\partial \mathbf{f}}{\partial \varphi} \Big\|_{X^0_{1/2}(\Omega)^3} \Big) \tag{3.39}$$

where $\beta(h,\kappa)$ *is taken from* (3.19)*.*

Proof. Starting from (3.36) and using the triangle inequality we get the inequality

$$\|\mathbf{u}-\mathbf{u}_{hN}\|_{\mathcal{V}_0(\mathbf{curl}_{r\varphi z},\mathrm{div}_{r\varphi z})} \leq \|\mathbf{u}-\mathbf{u}_N\|_{\mathcal{V}_0(\mathbf{curl}_{r\varphi z},\mathrm{div}_{r\varphi z})} + \|\mathbf{u}_N-\mathbf{r}_{hN}\mathbf{u}\|_{\mathcal{V}_0(\mathbf{curl}_{r\varphi z},\mathrm{div}_{r\varphi z})} \quad (3.40)$$

Owing to the completeness relationship (2.22), the definition of the norm $\|\mathbf{u}_n^e\|_{\mathcal{U}^n}$ (cf. (2.17)), the relations (3.37), (3.34), (3.33), Lemma 3.2, Lemma 3.3 and Remark 2.3, we have

$$\begin{aligned}
\|\mathbf{u}_N - \mathbf{r}_{hN}\mathbf{u}\|^2_{\mathcal{V}_0(\mathbf{curl}_{r\varphi z},\mathrm{div}_{r\varphi z})} &\leq C\sum_{n=0}^{N}\sum_{e\in\{s,a\}} \|\mathbf{u}_n^e - \mathbf{u}_{nh}^e\|^2_{\mathcal{U}^n(\Omega_a)} \\
\leq\quad & C\sum_{n=0}^{N}\sum_{e\in\{s,a\}} \|\tilde{\mathbf{w}}_n^e - \Pi_h\tilde{\mathbf{w}}_n^e\|^2_{\mathcal{U}^n(\Omega_a)} \\
\leq\quad & C\Big(\sum_{n=1}^{N}\sum_{e\in\{s,a\}} \Big\{\|\mathbf{w}_n^e - \Pi_h\mathbf{w}_n^e\|^2_{W^{1,2}_{1/2}(\Omega_a)^3} + n^2\Big\|\frac{1}{r}(\mathbf{w}_n^e - \Pi_h\mathbf{w}_n^e)\Big\|^2_{X(\Omega_a)^3}\Big\} \\
+\quad & \sum_{n=1}^{N}\sum_{e\in\{s,a\}} \Big\{\|\mathbf{s}_{n2}^e - \Pi_h\mathbf{s}_{n2}^e\|^2_{H^1(\Omega_a)^3} + n^2\Big\|\mathbf{s}_{n2}^e - \Pi_h\mathbf{s}_{n2}^e\Big\|^2_{L_2(\Omega_a)^3}\Big\} \\
+\quad & \sum_{n=1}^{N}\sum_{e\in\{s,a\}} \Big\{\|\mathbf{s}_{n3}^e - \Pi_h\mathbf{s}_{n3}^e\|^2_{H^1(\Omega_a)^3} + n^2\Big\|\mathbf{s}_{n3}^e - \Pi_h\mathbf{s}_{n3}^e\Big\|^2_{L_2(\Omega_a)^3}\Big\}\Big) \\
\leq\quad & C\beta^2(h,\kappa)\sum_{n=0}^{\infty}\sum_{e\in\{s,a\}} (1+n^2)\|\mathbf{f}_n^e\|^2_{X(\Omega_a)^3} \\
\leq\quad & C\beta^2(h,\kappa)\Big(\|\mathbf{f}\|^2_{X^0_{1/2}(\Omega)^3} + \Big\|\frac{\partial\mathbf{f}}{\partial\varphi}\Big\|^2_{X^0_{1/2}(\Omega)^3}\Big) \qquad (3.41)
\end{aligned}$$

where we have also used the fact that the norms $\|\tilde{\mathbf{w}}_n^e\|^2_{\mathcal{U}^n(\Omega_a)}$ and $\|\tilde{\mathbf{w}}_n^e\|^2_{V^n(\Omega_a)}$ are equivalent, since $\tilde{\mathbf{w}}_n^e \in V^n(\Omega)$. Finally, Theorem 3.4 follows from Theorem 3.3 and relations (3.40) and (3.41). ■

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