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SYNCHRONIZATION OF IDENTICAL CHAOTIC SYSTEMS THROUGH EXTERNAL CHAOTIC DRIVING

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Abstract

In recent years, the study of synchronization of identical chaotic systems subjected to a common fluctuating random driving signal has drawn considerable interest. In this communication, we report that it is possible to achieve synchronization between two identical chaotic systems, which are not coupled directly but subjected to an external chaotic signal. The external chaotic signal may be obtained from any chaotic system identical or nonidentical to both identical chaotic systems. Results of numerical simulations on well known Rössler and jerk dynamical systems have been presented.

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1. Introduction

Chaotic systems are dynamical systems that defy synchronization due to their essential feature of displaying high sensitivity to initial conditions. As a result, two identical chaotic systems starting at nearly the same initial points in phase space develop onto trajectories which become uncorrelated in the course of time. Nevertheless, it has been shown that it is possible to synchronize these kinds of systems, to make them evolve on the same chaotic trajectory by establishing appropriate coupling between them [1]-[4]. When we deal with coupled identical systems, synchronization appears as the equality of the state variables while evolving in time. We refer to this type of synchronization as identical synchronization (IS). The appearance of this kind of synchronization has been established by means of several different coupling mechanisms, such as the Pecora and Carroll method [3]-[7], the negative feedback [7]-[9], the sporadic driving [10], the active-passive decomposition [11], [12], a hybrid method of partial replacement [13] etc.

Besides the work on synchronization of identical chaotic systems by establishing direct coupling between them, in recent years there has been growing interest on the topic of synchronization of identical chaotic systems, which are not directly coupled but subjected to a common fluctuating random driving signal. Initially, Maritan and Banavar [14] showed that two independent identical chaotic systems can be synchronized by using a common sequence of random numbers. These results were heavily criticized by some authors [15]-[16], who showed that the largest Lyapunov exponent of the noisy systems is always positive and hence in contradiction with the criterion of conditional Lyapunov exponent, which states that the largest conditional Lyapunov exponent should be negative for possessing the stable synchronization. They also speculated that the observed synchronization may be due to the artifact of limited precision in numerical calculations. After several numerical [17]-[20] as well as experimental [21], [22] efforts, a widespread belief has emerged (except for a few contradictory results [23]-[25]) that it is not possible to synchronize two chaotic systems by injecting the same noisy unbiased zero-mean signal to both of them. However, a non-zero mean noisy signal leads to synchronization. Later, Lai and Zhou [23] and Rim et al. [26] explained the actual mechanism of synchronization in such noisy cases, according to which the presence of noise allows the system to spend more time in the convergence region where the local Lyapunov exponent is negative hence yielding a global negative Lyapunov exponent, which fulfills the essential requirement for synchronizing two chaotic systems.

In this paper, we discuss the possibility of the synchronization of two identical chaotic systems, which are not coupled directly but through an external chaotic driving signal i.e. we are considering a chaotic signal instead of noise as considered in the past by several authors [14]-[26]. In the next section, we give the formulation and stability of synchronization for such a coupling mechanism.

2. Formulation

Consider an n-dimensional dynamical system, which is chaotic, as

$$\dot{X} = F(X), \tag{1}$$

where $X = (x_1, x_2, x_3, ..., x_n)^T$ and $F(X) = (f_1(X), f_2(X), .., f_n(X))^T$. Consider another chaotic system identical to (1) but starting from different initial conditions (i.e. with different variables), as

$$\dot{X}' = F(X'), \qquad (2)$$

where $X' = (x_1^{'}, x_2^{'}, ..., x_n^{'})^T$ and $F(X') = (f_1(X'), f_2(X'), ..., f_n(X'))^T$. Now our aim is to synchronize both identical chaotic systems without establishing direct coupling between them as done in the case of Pecora-Carroll [3]-[7], feedback [7]-[9] and active-passive decomposition [11], [12] techniques. For this purpose we consider another chaotic system, named as external chaotic system, which may be structurally identical or non-identical to the systems (1) and (2) as

$$\dot{Z} = G(Z), \qquad (3)$$

where $Z = (z_1, z_2, z_3, ..., z_n)^T$ and $G(Z) = (g_1(Z), g_2(Z), ..., g_n(Z))^T$. Now we choose a drive variable from external chaotic system e.g. z_i $(1 \le i \le n)$ and name it external chaotic drive variable. Now a feedback control, which is proportional to the difference of external chaotic drive variable (z_i) and one of the variables e.g. x_i in the first identical chaotic system, is applied to the first chaotic system. The similar feedback control, which is proportional to the difference of external chaotic drive variable and counterpart of x_i in the second identical chaotic system i.e. $x_i^{'}$, is applied to the second chaotic system. In this way both identical chaotic systems are not coupled directly, however they are coupled through an external chaotic system. After establishing the proposed coupling, both identical systems (1) and (2) will look as:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, ..., x_n) \\
&. \\
\dot{x}_i &= f_i(x_1, x_2, ..., x_n) - c(x_i - z_i), \qquad (4) \\
&. \\
\dot{x}_n &= f_n(x_1, x_2, ..., x_n)
\end{aligned}$$

and

$$\begin{aligned}
\dot{x}_1^{'} &= f_1(x_1^{'}, x_2^{'}, ..., x_n^{'}) \\
&. \\
\dot{x}_i^{'} &= f_i(x_1^{'}, x_2^{'}, ..., x_n^{'}) - c(x_i^{'} - z_i), \qquad (5) \\
&. \\
\dot{x}_n^{'} &= f_n(x_1^{'}, x_2^{'}, ..., x_n^{'})
\end{aligned}$$

where c is a constant and termed as feedback constant or coupling strength. The pair of identical chaotic systems (4) and (5) synchronizes if the dynamical system describing the evolution of the difference,

$$\begin{bmatrix} \dot{e}_1 \\ . \\ \dot{e}_i \\ . \\ \dot{e}_n \end{bmatrix} = \begin{bmatrix} \dot{x}_1^{'} - \dot{x}_1 \\ . \\ \dot{x}_i^{'} - \dot{x}_i \\ . \\ \dot{x}_n^{'} - \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1(X^{'}) - f_1(X) \\ . \\ f_i(X^{'}) - f_i(X) - c(x_i^{'} - x_i) \\ . \\ f_n(X^{'}) - f_n(X) \end{bmatrix}, \quad (6)$$

possesses a stable fixed point at the origin $E = 0$, where $E = (e_1,\ e_2,\ ...,\ e_n)^T$ and $e_i = x_i^{'} - x_i$ for $i = 1$ *to* n. The dynamics of the error system can be understood by studying the following linearized system for small *E*:

$$\dot{E} = DF \cdot E, \quad (7)$$

where *DF* is the Jacobian matrix of derivatives with respect to *E* i.e.

$$DF = \begin{bmatrix} \frac{\partial f_1}{\partial e_1} & . & \frac{\partial f_1}{\partial e_i} & . & . & \frac{\partial f_1}{\partial e_n} \\ . & . & . & . & . & . \\ \frac{\partial f_i}{\partial e_1} & . & \frac{\partial f_i}{\partial e_i} - c & . & . & \frac{\partial f_i}{\partial e_n} \\ . & . & . & . & . & . \\ \frac{\partial f_n}{\partial e_1} & . & \frac{\partial f_n}{\partial e_i} & . & . & \frac{\partial f_n}{\partial e_n} \end{bmatrix}. \quad (8)$$

All the derivatives in the Jacobian matrix are evaluated at $E = 0$ i.e., on the synchronization manifold. To study the stability of identical synchronization in this case, one can use the three basic criteria: (i) largest eigenvalue of the Jacobian matrix *DF* must be negative, (ii) existence of suitable Lyapunov function for the difference system (6), (iii) all the conditional Lyapunov exponents (CLE's) must be negative, out of which the criterion of conditional Lyapunov exponent is of great importance [5], [27]. At first sight one would be tempted to think that the evolution equation of difference system is similar to the case of feedback technique [6], [7]. However it is important to note that the conditional Lyapunov exponents, that should be negative in order to obtain synchronization, correspond here to the behaviour of the difference between equally driven identical chaotic systems, which are not directly coupled to each other. This means that the reference state that one has to use to obtain these exponents is that of an equally driven identical chaotic system and not the external chaotic driver system. If one considers the case in which the external driving system is not identical to both systems then the conditional Lyapunov exponents and thus the statistics of trajectory separation will depend on the external chaotic driving signal as it alters the behaviour of both identical chaotic systems [28].

3. Numerical Results

In this section, we present the results of our numerical simulations to demonstrate the synchronization of identical chaotic systems using external chaotic driving. For this purpose, we first consider the well-known Rössler system [29] as:

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c) \end{aligned} \quad , \qquad (9)$$

which is known to exhibit chaotic behaviour at the parameter values $a = b = 0.2$ and $c = 5.7$. As we have stated in the last subsection, the external chaotic system may be structurally identical or nonidentical to the two identical chaotic systems, which are to be made synchronized. First of all we consider the simplest case, when two identical chaotic Rössler systems are driven by another identical Rössler system as:

$$\left.\begin{aligned} \dot{z}_1 &= -z_2 - z_3 \\ \dot{z}_2 &= z_1 + 0.2z_2 \\ \dot{z}_3 &= 0.2 + z_3(z_1 - 5.7) \end{aligned}\right\} \text{External chaotic driver} \qquad (10)$$

$$\left.\begin{aligned} \dot{x}_1 &= -x_2 - x_3 - c(x_1 - z_1) \\ \dot{x}_2 &= x_1 + 0.2x_2 \\ \dot{x}_3 &= 0.2 + x_3(x_1 - 5.7) \end{aligned}\right\} \text{Identical chaotic system A} \qquad (11)$$

$$\left.\begin{aligned} \dot{x}_1' &= -x_2' - x_3' - c(x_1' - z_1) \\ \dot{x}_2' &= x_1' + 0.2x_2' \\ \dot{x}_3' &= 0.2 + x_3'(x_1' - 5.7) \end{aligned}\right\} \text{Identical chaotic system B} \qquad (12)$$

In Figure 1, we have shown the Euclidean error between the identical system A and identical system B ($|e| = \left\{(x'-x)^2 + (y'-y)^2 + (z'-z)^2\right\}^{\frac{1}{2}}$) as a function of time for different values of coupling strength (c). It is clear from Figure 1 that for appropriate choice of coupling strength the two identical Rössler systems (which are not directly coupled with each other, however they are coupled through an external chaotic signal generated from another identical Rössler system) are synchronized. The results observed in Figure 1 are obvious as we know from the knowledge of feedback technique [6], [7] that two identical chaotic systems (named as drive and response systems) can be synchronized by applying the feedback control (which is proportional to the difference of drive variable and its counterpart in the response system) to one of them (i.e. the response system). Here (in Figure 1) the external chaotic Rössler system (10) and the identical chaotic Rössler system A (11) are coupled through the so-called feedback control and hence for appropriate choice of coupling strength systems (10) and (11) will possess the identical synchronization i.e. $Z = X$. Similarly the external chaotic Rössler system (10) and identical chaotic Rössler system B (12) will also possess identical synchronization i.e. $Z = X'$ and hence $X = X'$ i.e. both identical chaotic systems A and B are in identical synchronization though both are not coupled directly.

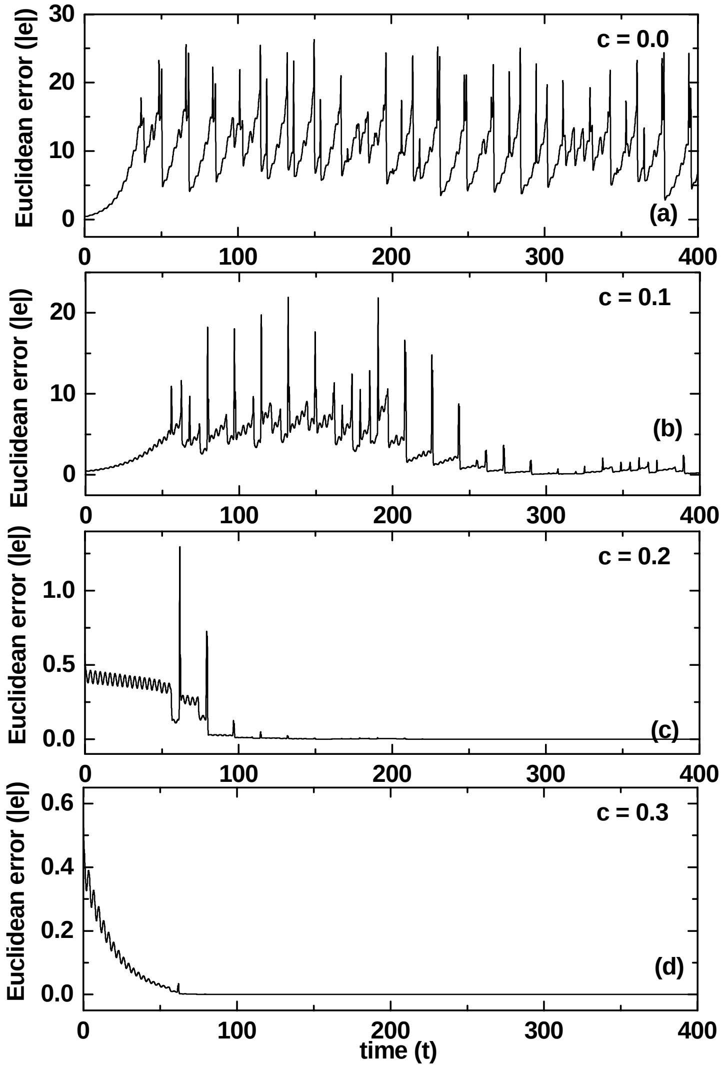

Figure 1: Euclidean error (i.e. $|e| = \{(x'-x)^2 + (y'-y)^2 + (z'-z)^2\}^{\frac{1}{2}}$) between the two identical Rössler systems driven by another identical Rössler system (Eqs. (10)-(12)) as a function of time (t) for different values of coupling strength (c): (a) $c = 0.0$, (b) $c = 0.1$, (c) $c = 0.2$ and (d) $c = 0.3$.

Now we consider a rather complicated case, when the external chaotic system is structurally non-identical to the identical chaotic systems to be synchronized. In this case the previous argument does not hold i.e. the external chaotic system does not separately possess identical synchronization with both the identical chaotic systems even though it is coupled with these systems with the feedback control. Now we explore this situation numerically and see whether the identical chaotic systems, which are coupled through an external chaotic signal generated from a non-identical chaotic system, can be synchronized or not?

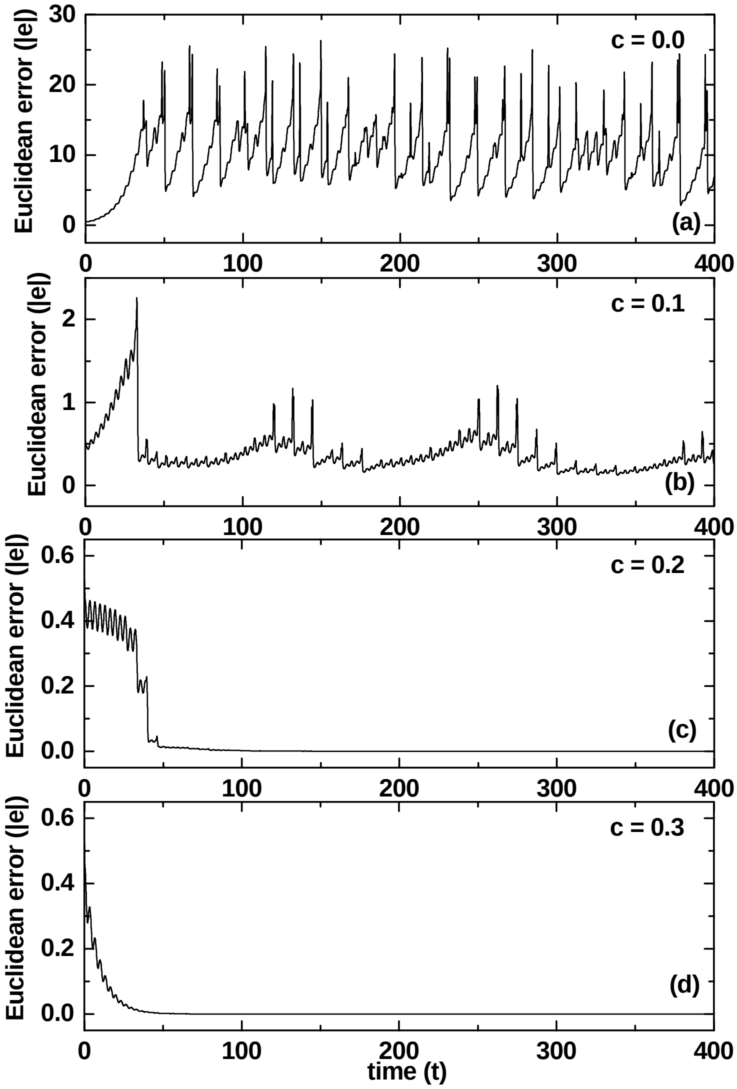

Figure 2: Euclidean error (i.e. $|e| = \{(x'-x)^2 + (y'-y)^2 + (z'-z)^2\}^{\frac{1}{2}}$) between the two identical Rössler systems driven by a jerk dynamical system (Eqs. (13)-(15)) as a function of time (t) for $G(z_1) = |z_1| - 2$ and different values of coupling strength (c):(a) *c = 0.0*, (b) *c = 0.1*, (c) *c = 0.2* and (d) *c = 0.3*.

For this purpose we consider a jerk dynamical system

$$\left.\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= z_3\\ \dot{z}_3 &= -0.6z_3 - z_2 + G(z_1)\end{aligned}\right\}\text{External chaotic driver} \qquad (13)$$

as an external chaotic driver. Equation (13) exhibits chaotic behaviour for $G(z_1) = |z_1| - 2$ and $G(z_1) = 0.58(z_1^2 - 1)$ [6], [30]-[32]. Now we consider two identical Rössler systems driven by an external jerk dynamical system (13) as:

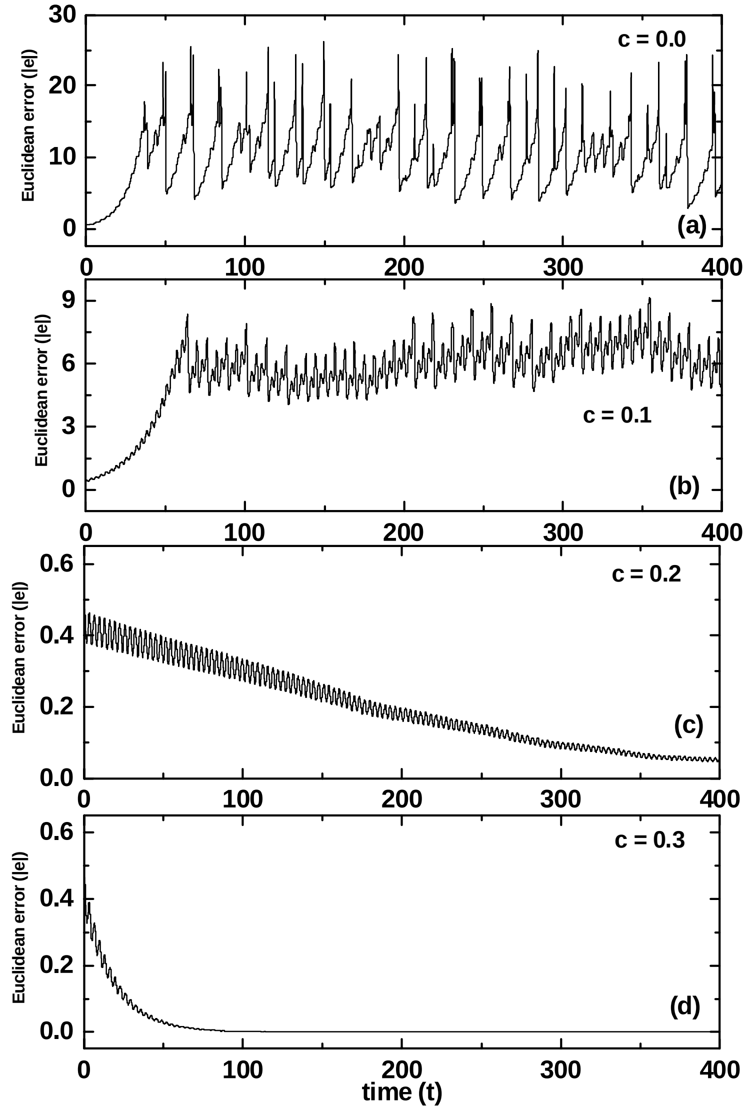

Figure 3: Euclidean error (i.e. $|e| = \{(x'-x)^2 + (y'-y)^2 + (z'-z)^2\}^{\frac{1}{2}}$) between the two identical Rössler systems driven by a jerk dynamical system (Eqs. (13)-(15)) as a function of time (t) for $G(z_1) = 0.58(z_1^2 - 1)$ and different values of coupling strength (c):(a) *c = 0.0*, (b) *c = 0.1*, (c) *c = 0.2* and (d) *c = 0.3*.

$$\left.\begin{aligned} \dot{x}_1 &= -x_2 - x_3 - c(x_1 - z_1) \\ \dot{x}_2 &= x_1 + 0.2x_2 \\ \dot{x}_3 &= 0.2 + x_3(x_1 - 5.7) \end{aligned}\right\} \text{Identical chaotic system A} \qquad (14)$$

$$\left.\begin{aligned} \dot{x}_1' &= -x_2' - x_3' - c(x_1' - z_1) \\ \dot{x}_2' &= x_1' + 0.2x_2' \\ \dot{x}_3' &= 0.2 + x_3'(x_1' - 5.7) \end{aligned}\right\} \text{Identical chaotic system B} \qquad (15)$$

We have numerically solved the coupled system described by equations (13)-(15).

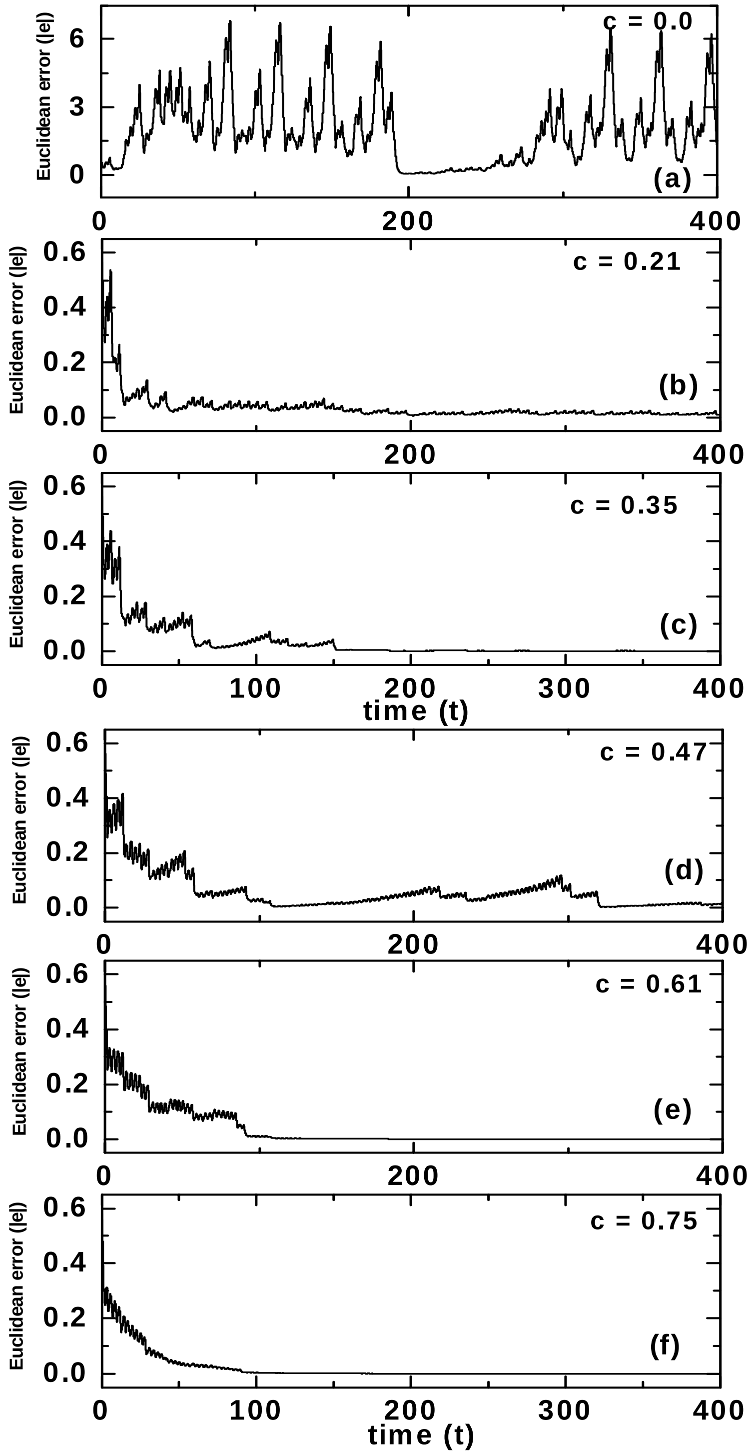

Figure 4: Euclidean error (i.e. $|e| = \{(x'-x)^2 + (y'-y)^2 + (z'-z)^2\}^{\frac{1}{2}}$) between the two identical jerk dynamical systems having absolute nonlinearity and driven by a jerk dynamical system having quadratic nonlinearity (Eqs. (16)-(18)) as a function of time (t) for different values of coupling strength (c): (a) *c = 0.0*, (b) *c = 0.21*, (c) *c = 0.35*, (d) *c = 0.47*, (e) *c = 0.61* and (f) *c = 0.75*.

In Figures 2 and 3 (respectively for $G(z_1) = |z_1| - 2$ and $G(z_1) = 0.58(z_1^2 - 1)$) we have depicted the results for different values of coupling strength (c). In this complicated situation also we observe that for appropriate choice of coupling strength both identical Rössler systems synchronize though they are not coupled directly but through an external chaotic signal generated from a structurally non-identical system.

Finally, in Figure 4, we have depicted the results corresponding to the case where two identical chaotic jerk dynamical systems (having absolute nonlinearity) are driven by the jerk dynamical system having quadratic nonlinearity as:

$$\left.\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= z_3\\ \dot{z}_3 &= -0.6z_3 - z_2 + 0.58(z_1^2 - 1)\end{aligned}\right\}\text{External chaotic driver} \qquad (16)$$

$$\left.\begin{aligned}\dot{x}_1 &= x_2 - c(x_1 - z_1)\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= -0.6x_3 - x_2 + |x_1| - 2\end{aligned}\right\}\text{Identical chaotic system A} \qquad (17)$$

$$\left.\begin{aligned}\dot{x}_1' &= x_2' - c(x_1' - z_1)\\ \dot{x}_2' &= x_3'\\ \dot{x}_3' &= -0.6x_3' - x_2' + |x_1'| - 2\end{aligned}\right\}\text{Identical chaotic system A} \qquad (18)$$

In this case also, we observe the synchronization between identical chaotic jerk systems driven by another non-identical chaotic jerk system.

4. Conclusions

In this communication, we have numerically analyzed the possibility of the synchronization of two identical chaotic systems, which are not directly coupled. However, they are coupled through an external chaotic driving signal, which is generated from any chaotic system. With these numerical simulations, we have found that it is possible to achieve synchronization between two identical chaotic systems without coupling them directly but using external chaotic driving. We have not yet analyzed the stability of synchronization for this case using the conditional Lyapunov exponent criterion (which suggests us the appropriate value of coupling strength for the stable synchronization). Since it is not straightforward to calculate the conditional Lyapunov exponents in the present case of external chaotic driving (unlike to the cases of Pecorra-Carroll, feedback and active-passive decomposition techniques [3]-[8], [11]-[13]), as the conditional Lyapunov exponents here correspond to the behaviour of the difference between equally driven identical chaotic systems, which are not directly coupled to each other. Hence the stability of synchronization in this case depends on the external chaotic signal as it affects the behaviour of both identical systems. The calculation of the conditional Lyapunov exponents (CLE's) for this case is under consideration and will be reported soon.

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