FUNDAMENTS OF TRANSPORT EQUATION SPLITTING AND THE EIGENVALUE PROBLEM

Velimir Stančić, *Institute of Nuclear Sciences - "Vinča"*
Yugoslavia, Belgrade P.O. Box 522

Abstract - *In order to remove some singularities concerning the boundary conditions of one dimensional transport equation, a splitted form of transport equation describing the forward i.e.* $\mu \geq 0$, *and a backward* $\mu < 0$ *directed neutrons is being proposed here. The eigenvalue problem has also been considered here.*

I INTRODUCTION

The singular eigenfunction expansion method [1] has enabled the application of functional analysis methods in transport theory. However the users were discouraged in applications, since in most problems, like slab problems, there has occurred an extra problem to solve the Fredholm integral equation in order to determine the expansion coefficients. Several reasons are sources of this difficulty. One of them is the use of full range expansion technics even in the regions where the function is singular, e.g. the free boundary condition requires for the distribution being equal to zero. Moreover, at $\mu = 0$ the transport equation becomes integral one. Both reasons motivated us to redefine the transport equation in plane geometry. For this we define the flux distribution as a direct sum of forward i.e. $\mu \geq 0$, and a backward $\mu < 0$ directed neutrons. Further, in order to avoid sum ambiguity with respect to elementary solutions and eigenfunctions, and to suggest further development, a much modern approach to this theory is proposed [2,3]. Due to space constrained, only fundaments of this approach will be presented here.

II SPLITTED FORM OF TRANSPORT EQUATION

We consider the direct sum Hilbert space $\mathcal{H} = \mathcal{L}_{2-} \oplus \mathcal{L}_{2+}$ where $\mathcal{L}_{2-} = \mathcal{L}_2([-1,0])$, and $\mathcal{L}_{2+} = \mathcal{L}_2([0,+1])$. Let $e_\pm \in \mathcal{H}$ be unit functions on $\mathcal{L}_{2\pm}$ respectively and zero otherwise, then any vector $\Psi \in \mathcal{H}$ will be written in a form $\Psi = \Psi e_- + \Psi e_+$, with $\Psi_\pm = \Psi e_\pm \in \mathcal{L}_{2\pm}$. The scalar product is $(\Psi,\Phi)_{\mathcal{H}} = (\Psi_-,\Phi_-)_- + (\Psi_+,\Phi_+)_+$ where $(\Psi_\pm,\Phi_\pm)_\pm = \pm\int_0^{\pm}\Psi_\pm(\mu)\overline{\Phi}_\pm(\mu)d\mu$. In this analysis the space variable will be considered as parameter. The homogeneous stationary one-speed transport equation with isotropic scattering in infinite plane of thickness d

$$\mu\frac{d}{dx}\Psi(x,\mu) + \Psi(x,\mu) = \frac{c}{2}\int_{-1}^{+1}\Psi(x,\mu)d\mu \qquad (1)$$

with boundary conditions $\Psi_0(\mu) = \Psi(x = 0,\mu)$ and $\Psi_d(\mu) = \Psi(x = d,\mu)$ will be split into two equations, written in operator form

$$\begin{aligned} \mathbf{MD}\Psi_+ + \mathbf{B}_+\Psi_+ &= \tfrac{c}{2}\mathbf{P}_-\Psi_-, \ \mu \geq o \\ \mathbf{MD}\Psi_- + \mathbf{B}_-\Psi_- &= \tfrac{c}{2}\mathbf{P}_+\Psi_+, \ \mu < o \end{aligned} \qquad (2)$$

Here $\mathbf{M}\Psi = \mu$, $\mathbf{D} = d/dx$, $\mathbf{B}_\pm = \mathbf{I}_\pm - \frac{c}{2}\mathbf{P}_\pm$, $\mathbf{P}_\pm = \pm\int_0^{\pm 1}(\cdot)d\mu$ are projectors $\mathbf{P}_\pm : \mathcal{L}_{2\pm} \to \mathcal{H}$, and $\mathbf{P}\Psi = \frac{1}{2}\left[(\Psi,e_-)_- e_- + (\Psi,e_+)_+ e_+\right]$ is projector $\mathbf{P} : \mathcal{H} \to \mathcal{H}$. Eqs (2) can be written in space $\mathcal{H}$ in a form very similar to full range equation:

$$\mathbf{MD}\Psi + \mathbf{B}\Psi = 0 \qquad (3)$$

where $\mathbf{B} = \mathbf{I} - c\mathbf{P}$, and $\mathbf{I}, \mathbf{P}$ are identity and projection operators in $\mathcal{H}$:

$$\mathbf{I} = \begin{pmatrix} \mathbf{I}_- & 0 \\ 0 & \mathbf{I}_+ \end{pmatrix}, \ \mathbf{P} = \frac{1}{2}\begin{pmatrix} \mathbf{P}_- & \mathbf{P}_+ \\ \mathbf{P}_- & \mathbf{P}_+ \end{pmatrix} \qquad (4)$$

We remark the operator $\mathbf{B}$ is still positive definite in $\mathcal{H}$ in the sense $\langle\Psi|\mathbf{B}\Psi\rangle_{\mathcal{H}} > (1-c)\langle\Psi|\Psi\rangle_{\mathcal{H}}$. Moreover the operator $\mathbf{T} = \mathbf{M}^{-1}\mathbf{B}$ can be transformed to a selfadjoint one [2] on $\mathcal{H}$, or represented as selfadjoint on $\mathcal{K}$ of $\mathcal{H}$ supplied with the inner product $(\Psi,\Phi)_{\mathcal{K}} = (\mathbf{B}^{-1}\Psi,\Phi)_{\mathcal{H}}$.

Formal solution of the Eq (3) is

$$\Psi(x,\mu) = exp(-x\mathbf{T})\Psi_0(\mu) \qquad (5)$$

We investigate the eigenfunction problem of the operator T in order to find the functional form of (5).

III EIGENFUNCTIONS AND COMPLETENESS

The eigenvalue problem may be defined by

$$\mathbf{M}\Phi_\nu = \nu\mathbf{B}\Phi_\nu \tag{6}$$

Here is a diagonal matrix with elements ν_+, ν_- respectively. Obviously the continuous parts of the spectra are $\nu_{c+} \in [0,1]$, $\nu_{c-} \in [-1,0]$. The discrete eigenfunctions are solutions of the equation $\nu_{d\pm} = \pm\nu_0$

$$-\frac{c\nu_0}{2} ln\left(\frac{\nu_0}{\nu_0 - 1}\right) = 0 \tag{7}$$

Solutions of this equation compared to the full range eigenvalues are given in the Table I.

TABLE I

Full Range (FR) and Splitted Form (SF) Eigenvalues ν_0 as a Function of **c**

c	$1/\nu_0$ FR	$1/\nu_0$ SF
0.0	1.00000	1.00000
0.1	1.00000	1.00000
0.2	0.99991	0.99995
0.3	0.99741	0.99872
0.4	0.98562	0.99302
0.5	0.95750	0.98017
0.6	0.90733	0.95912
0.7	0.82863	0.92981
0.8	0.71041	0.89264
0.9	0.52543	0.84813
1.0	0.00000	0.79681

The spectral family Λ of the operator T is $\Lambda = \Lambda_- \cup \Lambda_+$, where $\Lambda_\pm = (0, \pm 1] \cup \{\pm\nu_0\}$. The discrete and continuous eigenfunctions are of the same form as for the full range [1]. The eigenfunction sets are $\{\phi_{\pm\nu}, \nu \in \Lambda_\pm\}$. To make these sets orthogonal, one can find the weight functions $W_\mp(\mu)$ and spectral measures $d\rho_\pm(\nu), \nu \in \Lambda_\pm$, applying the same procedure as in the half range example [1]. Since the operator $\mathcal{B}$ constructed here conserves the algebraic properties as in full range [2], the eigenfunction completeness may be accomplished by using the rigged Hilbert space method [2].

Any function $\Psi \in \mathcal{H}$ of a form $\Psi = \Psi e_- + \Psi e_+$, with $\Psi_\pm = \Psi e_\pm \in L_{2\pm}$ can be expanded in terms of these eigenfunction sets $\Psi_\pm = \int_{\Lambda_\pm} d\rho_\pm(\nu)\left(W_\mp \Psi_\pm, \phi_\pm\right)_\pm \phi_\pm$. Using this expansion one find the Green function [1] and a solution of the Eq (1).

However, some effort is required to find the explicit form of expression (5). Thus, if one uses the eigenfunction expansion on $\mathcal{H}$, then the identity resolution in $\mathcal{H}$ should be derived. On the other side, if one uses the functional calculus of the transport operator T, then the isomorphisms of the generated algebra must be found. Such an analysis would be out of the scope of this paper.

IV CONCLUSIONS

There is a significant descrepancy between the FR and SF eigenvalues for dominantly scattering systems (c>0.5). The method proposed offers a variety of applications as well as further theoretical considerations.

REFERENCES

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OSNOVI RAZDVAJANJA TRANSPORTNE JEDNAČINE I ZADATAK SVOJSTVENIH VREDNOSTI

Velimir Stančić

Sadržaj - *Sa ciljem da se otklone neki sungulariteti u vezi graničnih uslova jednodimenzione transportne jednačine, predložena je transportna jednačina u razdvojenom obliku koja opisuje napred* $\mu \geq 0$ *i nazad* $\mu < 0$ *rasejane neutrone. Takođe je razmatran zadatak svojstvenih vrednosti.*