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SPLIT OCTONION ELECTRODYNAMICS AND UNIFIED FIELDS OF DYONS

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ABSTRACT

Split octonion electrodynamics has been developed in terms of Zorn's vector matrix realization by reformulating electromagnetic potential, current, field tensor and other dynamical quantities. Corresponding field equation (Unified Maxwell's equations) and equation of motion have been reformulated by means of split octonion and its Zorn vector realization in unique, simpler and consistent manner. It has been shown that this theory reproduces the dyon field equations in the absence of gravito-dyons and vice versa.

Introduction

The question of existence of monopole [1-3] has become a challenging new frontier and the object of more interest in connection with quark confinement problem of quantum chromodynamics. The eight decades of this century witnessed a rapid development of the group theory and gauge field theory to establish the theoretical existence of monopoles and to explain their group properties and symmetries. Keeping in mind t'Hooft's solutions [4-5] and the fact that despite the potential importance of monopoles, the formalism necessary to describe them has been clumsy and not manifestly covariant, Rajput et. al. [6-7] developed the self consistent quantum field theory of generalised electromagnetic fields associated with dyons (particles carying electric and magnetic charges). The analogy between linear gravitational and electromagnetic fields leads the asymmetry in Einstein's linear equation of gravity and suggests the existence of gravitational analogue of magnetic monopole [8-9]. Like magnetic field, Cantani [10] introduced a new field (i.e. Heavisidian field) depending upon the velocities of gravitational charges (masses) and derived the covariant equations (Maxwell's equations) of linear gravity. Avoiding the use of arbitrary string variables [1], we have formulated manifestly covariant theory of gravito-dyons [11-12] in terms of two four-potentials and maintained the structural symmetry between generalised electromagnetic fields of dyons [13-14] and generalised gravito-Heavisidian fields of gravito-dyons.

Octonions, or Cayley numbers, have been little used in physics up to date [15-16]. Since octonions share with complex numbers and quaternions have many attractive mathematical properties, one might except that they would be equally usefl. However, they may occur as alternate mathematical structures, which is useful and more transparent. Gunaydin et. al. [17-19] has formulated the quark models and colour gauge theory in terms of split octonion algebra and the related groups SO(8), SO(7), G_2 and SU(3). Representation of the poincare group in split octonion Hilbert space, they have constructed octonion Hilbert space as the finite space of internal symmetries, the SU(3) group appears as the automorphism group of octonion representation leaving the complex subspace and the scalar product invariant. The approach of Gunaydin et. al. [17-19] has been followed by Domokos et. al. [20-21] and Morita [22[to the algebraic colour gauge theory and quark confinement problem. Use of Cayley octonions was made by Buoncristiani [23] in writing Yang-Mill's (and the Maxwell) field equation in a simple form and showed that octonion algebra accommodates both space-time symmetry (Lorentz invariance) and internal symmetry. Morques and Oliveria [24] have described the extension of quaternionic matrices to octonions as interpreting non-Riemannian geometry.

In this paper I have undertaken the study of reformulation of classical electrodynamics in terms of split octonions. Split octonion electrodynamics has been developed in terms of Zorn's vector matrix realization by reformulating electrodynamics potential, current, field tensor and other dynamical quantities. Corresponding field equation (Maxwell's equations) and equation of motion have been reformulated by means of split octonion and its Zorn vectors realization in unique, simples and consistent manner. Finally in this paper the technique of split octonion formalism and its isomorphic Zorn's vector matrix realization is applied to develop the unified theory of generalised electromagnetic fields associated with dyons and generalised gravito-Heavisidian fields associated with gravito-dyons. It has been shown that this theory reproduces the dyon field equations in the absence of gravito-dyons and vice-versa. Moreover, it leads to the usual dynamics of electric (or gravitational) charge (mass) in the absence of magnetic (or Heavisidian) charge (or mass) on dyons (or gravito-dyons) and vice-versa.

(2) Octonions: -

A composition algebra is defined as an algebra A with identity and with non degenerate quadratic form N defined over it such that N permits composition i.e. for x, y ε A,

$$N(xy) = N(x)N(y) \tag{1}$$

According to celebrated Hurwitz theorem [293], there exits only four different composition algebra over the real (or complex number) field. These are the real numbers R of dimension 1, complex number C of dimension 2, Hamilton's quaternions H of dimensions 4 and octonions (Cayley – Grave's numbers) of dimension 8. As such we have only four type of hyper complex number system, which yield the properties of division algebra [25]. Among these algebras, real and complex numbers follow commutativity and associativity. Quaternions defined in previous section obey associativity but not commutativity while octonions are neither commutative nor associative but alternative [25]. Composition algebra is said to be division algebra if quadratic form N is anistropic i.e.

If $N(x) = 0$ implies that $x = 0$. (2)

Otherwise the algebra is called split [26-27].

An octonion is defined as,

$$X = X_0 e_0 + X_i e_i, \quad X_0, X_i \varepsilon R \tag{3}$$

where i =1,2,....7, e_i are octonion unit elements satisfying following multiplication rules [28],

$$\begin{aligned} e_j e_k &= -\delta_{jk} e_0 + f_{jkl} e_l; \\ e_i e_0 &= e_0 e_i = e_i; \\ e_0 e_0 &= e_0; \end{aligned} \tag{4}$$

where δ_{jk} is the usual Kronecker delta symbol and f_{jkl} Levi-Civita tensor for quaternions, is fully anti symmetric tensor with

$$f_{jkl} = +1 \text{ for jkl} = 123, 516, 624, 435, 471, 673, 672. \tag{5}$$

for octonion. The cyclic symmetry is obtained by ordering seven points clockwise on circle with the numbering (1243657) (fig.1). Then a triangle is obtained from (123) by six successive rotations of angle 2π/7

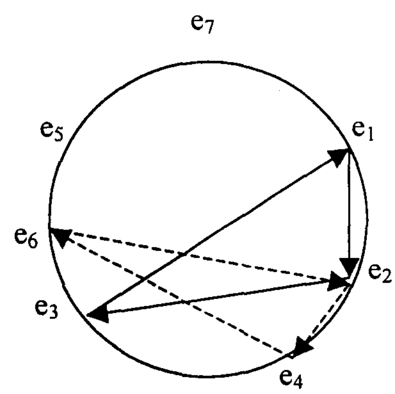

fig. (1)

Another convenient way of representing the multiplication table by singling out one of the elements is provided by the triangular diagram (fig. 2.), where arrows show the directions along which the multiplication has a positive sign, e.g. i.e. $e_6 e_7 = e_3$.

$$e_1 e_2 = e_3 \; ; \; e_6 e_4 = e_2 \; ; \; e_4 e_7 = e_2 \; ;$$
$$e_5 e_1 = e_6 \; ; \; e_4 e_3 = e_5 \; ; \; e_7 e_2 = e_5 \; ; \qquad (6)$$
$$e_6 e_7 = e_3$$

so on and $e_a e_b + e_b e_a = -2\delta_{ab} \; (a, b = 1, 2, \ldots .7)$.

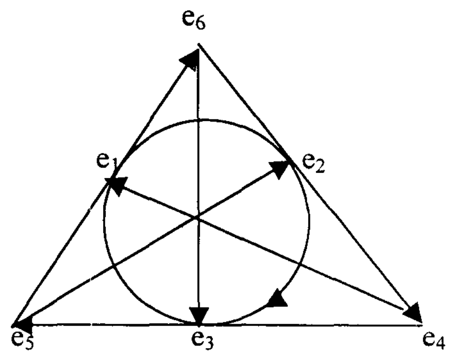

fig.2

For convenience let us write the multiplication table explicity.

Table 1:

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_7	$-e_6$	e_5	$-e_4$
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	$-e_5$	e_4	e_7	$-e_6$
e_4	e_4	$-e_7$	$-e_6$	e_5	$-e_0$	$-e_3$	e_2	e_1
e_5	e_5	e_6	$-e_7$	$-e_4$	e_3	$-e_0$	$-e_1$	e_2
e_6	e_6	$-e_5$	e_4	$-e_7$	$-e_2$	e_1	$-e_0$	e_3
e_7	e_7	e_4	e_5	e_6	$-e_1$	$-e_2$	$-e_3$	$-e_0$

The above multiplication table directly follows that the algebra O is not associative i.e.

$$e_j(e_k e_l) \neq (e_j e_k)e_l. \tag{7}$$

The commutation rules for octonion basis elements are given by,

$$[e_j, e_k] = 2f_{jkl}e_l$$
$$\{e_j, e_k\} = -2\delta_{jk}e_{0.} \tag{8}$$

and the associator ,

$$\{e_j, e_k, e_l\} = (e_j e_k)e_l - e_j(e_k e_l)$$
$$= -\delta_{jk}e_l + \delta_{kl}e_j + (\varepsilon_{jkl}\varepsilon_{m\ln} - \varepsilon_{klp}\varepsilon_{jpn})e_n. \tag{9}$$

Though the quaternion units e_1, e_2, e_3 given in previous section are anti commutative and associative, the octonion units $e_1, e_2, ...e_7$ are anti commutative and anti associative. The later property gives,

$$(e_j e_k)e_l = -e_j(e_k e_l) \tag{10a}$$

$$\{e_j, e_k, e_l\} = (e_j e_k)e_l + e_j(e_k e_l) = 0. \tag{10b}$$

Thus the algebra of octonions is alternative i.e. the associator of three elements X, Y, Z ε O is alternating function of arguments [29]

$$[X,Y,Z] = (XY)Z - X(YZ)$$
$$= [Z,X,Y] = [Y,Z,X] = -[Y,X,Z]. \tag{11}$$

Real, complex and quaternion algebras are sub algebras of octonions algebra O.

Octonion conjugate is defined as,

$$\overline{X} = X_0 e_0 - X_i e_i \ , \qquad (i = 1,2.....7) \tag{12}$$

and

$$\overline{\overline{X}} = X\ ; \ \overline{XY} = \overline{Y}\,\overline{X} \tag{13}$$

The norm N of the octonion is defined as,

$$N(X) = X\overline{X} = \overline{X}X = (X_0^2 + X_i^2)e_0 \tag{14}$$

while the inverse is defined as,

$$X^{-1} = \frac{\overline{X}}{N(X)};$$
$$X^{-1}X = XX^{-1} = 1.e_{0.} \tag{15}$$

The norm given by equation (14) is non-degenerate and positively defined (over R) and therefore every element X ε O has the unique inverse element $X^{-1}\varepsilon$ O. Left and right quotient B/A of A divided by B "from the left" is the solution of the equation

$$BX = A \tag{16}$$

and the right quotient A/B of A divided by B "from the right" is the solution of the equation

$$YB = A \tag{17}$$

and hence [30],

$$X = B / A = \overline{B}A / N(B) \tag{18}$$

$$Y = A / B = \overline{A}B / N(B). \tag{19}$$

Real octonion algebra can also be obtained by the extension of the quaternions. For this we decompose an element X ε O as (taking $e_0 = 1$)

$$\begin{aligned} X &= (X_0 + X_1 e_1 + X_2 e_2 + X_3 e_3) + e_7 (X_7 + X_4 e_1 + X_5 e_2 + X_6 e_3) \\ &= Q_1 + e_7 Q_2 \end{aligned} \tag{20}$$

where Q_1, Q_2 belong to quaternion sub algebra H generated by $e_j\,(j = 1,2,3)$. In this form we can write the octonion multiplication as [29],

$$\begin{aligned} XY &= (Q_1 + e_7 Q_2)(R_1 + e_7 R_2) \\ &= (Q_1 R_1 - R_2 Q_2) + e_7 (Q_1 R_2 + R_1 Q_2) \end{aligned} \tag{21}$$

where (~) denotes the quaternion conjugate $e_j \to -e_j$ $(j = 1,2,3)$. Thus we can write quaternion conjugate (~) on an octonion as,

$$\widetilde{X} = \widetilde{Q}_1 + e_7 \widetilde{Q}_2 \tag{22}$$

Equation (4) can also be written as a "quaternion extension" of complex numbers i.e.

$$\begin{aligned} X &= X_0 + \sum_{A=1}^{7} X_A e_A \\ &= (X_0 + e_7 X_7) + (X_1 + e_7 X_4) e_1 + (X_2 + e_7 X_5) e_2 + (X_3 + e_7 X_6) e_3 \\ &= X_0 + X_1 + X_2 + X_3 \end{aligned} \tag{23}$$

where $X_0 = X_0 + e_7 X_7$ and $X_j = X_j + e_7 X_{j+3}\,(j = 1,2,3)$. Then the octonion multiplication (21) reduces to,

$$XY = (X_0 + \sum_{j=1}^{3} X_j e_j)(Y_0 + \sum_{j=1}^{3} Y_j e_j)$$
$$= (X_0 Y_0 - \sum_{j=1}^{3} X_j Y_j *) + \sum_{l=1}^{3}(X_0 Y_l + Y_0 * X_l + \sum_{j,k=1}^{3} \varepsilon_{jkl} X_j * Y_j * e_k) \qquad (24)$$

where * operation denotes complex conjugate with respect to e_7 (*i.e.* $e_7 \rightarrow -e_7$).

It is known that the group of automorphism of octonion algebra is the exceptional Lie group G_2 [29,31] and that when any one of the seven imaginary units $e_1, e_2, \ldots\ldots e_7$ is held fixed the group of automorphism is reduced to SU (3) which is the subgroup of G_2. Since G_2 has only real representation [32]. The group G_2 also has an SU (2) $\times$ SU (2) subgroup arises from the fact that octonions are constructed from two quaternions given by equation (20-23). The SU (3) subgroup can be imbedded in G_2 in seven different ways and for each imbedding of SU (3) there are three different imbeddings of SU (2)$\times$ SU (2) involving I, U, V spin subgroup of SU (3).

There are four possible "bilinear forms" [29] that can be described over the real octonion algebra that induces the usual octonionic norm and satisfy the composition law. These bilinear forms are

(i) The real bilinear form $(X,Y)_R$ i.e.

$$(X,Y)_R = \tfrac{1}{2}(\bar{X}Y + \bar{Y}X) = \sum_{a=1}^{7} X_a Y_a \qquad (25)$$

Over bar (-) denotes the octonion conjugation with respect to seven octonion units ($e_1, e_2, \ldots\ldots e_7$). This bilinear product has the invariance group SO (8) which is also invariance group for octonionic norm given by equation (14).

(ii) The complex bilinear product$(X,Y)_C$ i.e.

$$(X,Y)_C = \tfrac{1}{2}(\bar{X}Y + \tilde{\bar{Y}}X) \qquad (26)$$

where (~) denotes the quaternion conjugation i.e. change of sign in quaternion basis elements. Equation (26) reduces to the following form in terms if real and imaginary components;

$$(X,Y) = \sum_{a=1}^{7} X_a Y_a + e_7(X_0 Y_7 - X_7 Y_0 + X_4 Y_1 - X_1 Y_4$$
$$+ X_5 Y_2 - X_2 Y_5 + X_6 Y_3 - X_3 Y_6). \qquad (27)$$

Real part of this equation is same as (25) and invariant under SO (8) transformations while the imaginary part has the invariance group SP (8). Thus the invariance group of equation (27) is SO (8) $\cap$ SP (8)$\approx$U (4). From equation (24) we draw the conclusion that the invariance group of complex bilinear product of real octonion is U (4). In view of "quaternion extension" of complex numbers to octonions.

(i) Quaternion bilinear product $(X,Y)_H$ i.e.

$$\begin{aligned}(X,Y)_H &= \tfrac{1}{2}\{(\bar{X}Y)+(\bar{Y}X)^*\} \\ &= \tfrac{1}{2}(\bar{X}Y+\bar{X}^*Y^*)\end{aligned} \tag{28}$$

where (*) denotes the complex conjugation (i.e. $e_7 \to -e_7$). This product has the invariance group SO(3)$\times$SO(3).

(iv) Octonion bilinear product

$$(X,Y)_O = XY. \tag{29}$$

The invariance group of this form is the trivial multiplication by ± 1 [27].

Due to non-associativity of octonions it is not possible to write them in matrix form like quaternions. Even it is not possible to write them in 2×2 quaternion valued matrices while they are derived direct extension of quaternions. The octonion unit e_7, which links two quaternions to octonion, does not commute with quaternion units e_1, e_2, e_3 and hence can not be taken as invariant quantity. On the other hand due to alternativity and other properties, one can split octonions, where e_7 is replaced by imaginary quantity, in terms of 2×2 Zorn's vector matrices. This type of algebra of octonion is known as aplit octonion algebra [27].

Thus in the light of celebrated Hurwitz theorem we have only four division algebra following the properties of division ring. That is, if a and b are given elements in the field, the equation [33];

$$ax+b=0 \tag{30}$$

has a unique solution for x when the field is a division ring. To go beyond octonion, we loose alternativity as well as the property of being a division ring.

(3) <u>Split octonions</u>:-

It is almost clear that due to non-associativity of octonions a matrix realization for octonion units is not possible in their usual form. However, one can relate split octonion units with Zorn's vector matrix [34]. For split octonions one deal with the algebra of Cayley-Dickson's process with O (1,1-1) for $\mu_1 = 1, \mu_2 = 1 \, and \, \mu_3 = -1$.

The division and split octonion algebra's have some differences which may characteristically be expressed in terms of idempotents (by definition, an element u=0 is idempotent if $u^2 = u$). Any finite-dimensional power associative algebra which is not a nil algebra contains an idempotent u = 0.

Definitions:- The idempotents $u_1, u_2, ...u_n \varepsilon A$ (for arbitary algebra) are called pair wise orthogonal in case $u_i u_j = 0 \, for \, i \neq ji, j = 1,2,....n$. An idempotent u$\varepsilon A$ is called primitive if there do not exit orthogonal idempotents u,w (uw = wv = 0) so that u = v + w. In a finite dimensional algebra A, any idempotent may be written as a sum of pair wise orthogonal primitive idempotents.

For the split octonion algebra the following new basis is considered [35-39] on the complex field (instead of real field) i.e.

$$
\begin{aligned}
u_1 &= \frac{1}{2}(e_1 + ie_4), \quad u_1^* = \frac{1}{2}(e_1 - ie_4), \\
u_2 &= \frac{1}{2}(e_2 + ie_5), \quad u_2^* = \frac{1}{2}(e_2 - ie_5), \\
u_3 &= \frac{1}{2}(e_3 + ie_6), \quad u_3^* = \frac{1}{2}(e_3 - ie_6), \\
u_0 &= \frac{1}{2}(1 + ie_7), \quad u_0^* = \frac{1}{2}(1 - ie_7),
\end{aligned}
\tag{31}
$$

where i $=\sqrt{-1}$ and is assumed to commute with e_A (A = 1,2,...7) octonion units,

$$
\begin{aligned}
& u_i u_j = \varepsilon_{ijk} u_k^* \\
& u_i^* u_j^* = -\varepsilon_{ijk} u_k \, (i,j,k = 1,2,3) \\
& u_i u_j^* = -\delta_{ij} u_0; \quad u_i u_o = 0; \quad u_i^* u_0 = u_i^* \\
& u_i^* u_j = -\delta_{ij} u_0; \quad u_i u_0^* = u_0; \quad u_i^* u_0^* = 0 \\
& u_0 u_i = u_i; \quad u_0^* u_i = 0; \quad u_0 u_i^* = 0 \\
& u_0^* u_i^* = u_i^* \\
& u_0^2 = u_0; \quad u_0^{*2} = u_0^* \\
& u_0 u_0^* = u_0^* u_0 = 0.
\end{aligned}
\tag{32}
$$

Gunaydin and Gursey [35-39] pointed out that the automorphism group of octonion is G_2 and its subgroup which leaves imaginary octonion unit e_7 invariant (or equivalently the idempotents u_0 and $u*_0$) is SU (3) where the units u_i and $u*_i$ (i = 1,2,3) transform respectively like a triplet and anti triplet and are associated with colour and anti colour triplet of SU (3) group. Let us introduce a convenient realization for the basis elements (u_0, u_i, u_0^*, u_i^*) through the use of Pauli matrices. Identifying can do this

$$u_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad u_0^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad u_i = \begin{pmatrix} 0 & 0 \\ e_i & 0 \end{pmatrix}, \quad u_i^* = \begin{pmatrix} 0 & -e_i \\ 0 & 0 \end{pmatrix} \quad (i = 1,2,3) \tag{33}$$

where $1, e_1, e_2, e_3$ are quaternion units satisfying the multiplication rule $e_i e_j = -\delta_{ij} + \varepsilon_{ijk} e_k$. As such for an arbitrary split octonion A we have [40],

$$A = a u_0^* + b u_0 + x_i u_i^* + y_i u_i = \begin{pmatrix} a & -\vec{x} \\ \vec{y} & b \end{pmatrix} \tag{34}$$

which is a realization via the 2×2 Zorn's vector matrices,

$$\begin{pmatrix} a & \vec{x} \\ \vec{y} & b \end{pmatrix}$$

where a and b are scalars and $\vec{x}$ and $\vec{y}$ are three vectors with product,

$$\begin{pmatrix} a & \vec{x} \\ \vec{y} & b \end{pmatrix}\begin{pmatrix} c & \vec{u} \\ \vec{v} & d \end{pmatrix} = \begin{pmatrix} ac + \vec{x}.\vec{v} & a\vec{u} + d\vec{x} - \vec{y} \times \vec{v} \\ c\vec{y} + b\vec{v} + \vec{x} \times \vec{u} & \vec{y}.\vec{u} + bd \end{pmatrix} \tag{35}$$

and $\times$ denotes the usual vector product, e_i (i = 1,2,3) with $e_i \times e_j = \varepsilon_{ijk} e_k$ and $e_i e_j = \delta_{ij}$ then we can relate the split octonions to the vector matrices given by Eq. (33). Octonion conjugation of equation (34) is defined as,

$$\overline{A} = bu_0^* + au_0 - x_i u_i^* - y_i u_i = \begin{pmatrix} b & \vec{x} \\ -\vec{y} & a \end{pmatrix} \tag{36}$$

The norm of A is then defined as,

$$\overline{A}\,A = A\,\overline{A} = (ab + \vec{x}.\vec{y}\ ).1 \tag{37}$$

where 1 is the identity element of the algebra given by $1 = 1u_0^* + 1u_0$.

(4) Split octonion Electrodynamics:-

Split octonions (i=1,2,3) are the bivalued (or bidimensional) representations of quaternion units e_0, e_1, e_2, e_3 constituting an Euclididean four-space of relativistic space-time world. As such a split octonion may be expressed as

$$\begin{aligned} A &= x_\mu u_\mu + y_\mu u_\mu^* \\ &= u_0 x_4 + u_i x_i + u_0^* y_0 + u_i^* y_i \end{aligned} \tag{38}$$

where $x_\mu = \{x, x_4\}$ and $y_\mu = \{y, y_4\}$. Any four-vector A_μ (complex or real) can equivalently be written in the following Zorn's matrix realization

$$Z(A) = \begin{pmatrix} x_4 & -\vec{x} \\ \vec{y} & y_4 \end{pmatrix} \tag{39a}$$

and

$$Z(\bar{A}) = \begin{pmatrix} x_4 & \vec{y} \\ -\vec{x} & y_4 \end{pmatrix} \tag{39b}$$

where (-) sign is taken fron the definition of u_i^* in equation (33). For x=y we have

$$Z(A) = \begin{pmatrix} x_4 & -\vec{x} \\ \vec{x} & x_4 \end{pmatrix} \tag{40}$$

which is the equivalent matrix realization for the bivalued 4-dimensional Euclidean space-time. The norm of a space-time four-time is then expressed as

$$Z(A)Z(\bar{A}) = \begin{pmatrix} x_4 & -\vec{x} \\ \vec{x} & x_4 \end{pmatrix} \begin{pmatrix} x_4 & \vec{x} \\ -\vec{x} & x_4 \end{pmatrix}$$
$$= \begin{pmatrix} x_4^2 + |x^2| & 0 \\ 0 & x_4^2 + |x^2| \end{pmatrix} \tag{41}$$
$$= \{x_4^2 + x_1^2 + x_2^2 + x_3^2\}.1(x_4 = ict)$$

which is the norm of a Euclidean space-time four-vector and 1 is the unit elements (i.e. 2×2 unit matrix). The space –time four-differential operators are written as,

$$D = u_\mu \partial_\mu + u_\mu^* \partial_\mu = u_0 \partial_4 + u_i \partial_i + u_0^* \partial_4 + u_i^* \partial_i$$
$$or\, equivalent\, Z(D) = \begin{pmatrix} \partial_4 & -\vec{\partial} \\ \vec{\partial} & \partial_4 \end{pmatrix} \tag{42}$$

where $\partial_\mu = \partial / \partial x_\mu$ $(\mu = 1,2,3,4)$.

For brevity we omit Z and simply write a Zorn's vector matrix notation. Thus for Z(D), we write

$$Z(\bar{D}) = \begin{pmatrix} \partial_4 & \vec{\partial} \\ -\vec{\partial} & \partial_4 \end{pmatrix} \tag{43}$$

and so on. The norm of four-differential operator is defined by

$$\overline{D}D = D\overline{D} = (\partial_4^2 + \nabla^2).1 = [\,].1 = \partial_\mu \partial_\mu$$
$$= -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{44}$$

which is the form of D'Alembertian operator. Here our aim is to reformulate usual electrodynamics by means of split octonions. Maintaining the usual form of Euclidean 4 space here we have made an attempt to reformulate the classical electrodynamics in terms of split octonions and their Zorn's matrix realization.

Let us define the electromagnetic four-potential in terms of split octonion vector realization as

$$A = \begin{pmatrix} A_4 & -\vec{A} \\ \vec{A} & A_4 \end{pmatrix} \tag{45}$$

with $A_4 = i\phi_e$ (e stands for electric charge) and we take the following usual definition of electric and magnetic fields from the compounds of electromagnetic four-potential $A_\mu = \{\vec{A}, i\phi_e\}$ i.e.

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi_e \qquad (46)$$
$$\vec{H} = -\vec{\nabla} \times \vec{A}.$$

Now operating equation (43) to equation (45) and using relation (35), we get

$$\bar{D} A = \begin{pmatrix} \partial_4 & \vec{\partial} \\ -\vec{\partial} & \partial_4 \end{pmatrix} \begin{pmatrix} A_4 & -\vec{A} \\ \vec{A} & A_4 \end{pmatrix}$$
$$= \begin{pmatrix} \partial_4 A_4 + \vec{\nabla}.\vec{A} & -\partial_4 \vec{A} + \vec{\nabla} A_4 + \vec{\nabla} \times \vec{A} \\ \partial_4 \vec{A} - \vec{\nabla} A_4 - \vec{\nabla} \times \vec{A} & \partial_4 A_4 + \vec{\nabla}.\vec{A} \end{pmatrix} \qquad (47)$$

Now using the definition of electric and magnetic fields given by equation (46) and imposing Lorentz gauge condition

$$\partial_4 A_4 + \vec{\nabla}.\vec{A} = \partial\phi_0 / \partial t + \vec{\nabla}.\vec{A} = 0$$

$$-\partial_4 \vec{A} + \vec{\nabla} A_4 + \vec{\nabla} \times \vec{A} = -iE + H = -i\psi^*$$

$$\partial_4 \vec{A} - \vec{\nabla} A_4 - \vec{\nabla} \times \vec{A} = iE - H = i\psi^* \qquad (48)$$

Where $\psi = E - iH \;\; (i = \sqrt{-1})$.

Thus equation (47) reduces to

$$\bar{D} A = F \qquad (49)$$

where

$$F = \begin{pmatrix} 0 & -F \\ F & 0 \end{pmatrix} and\, F = i\psi^*. \qquad (50)$$

Equation (49) is the split octonion form of field tensor $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$, components of which describe the electric and magnetic fields. Off-diagonal vector components of split octonions (40-49) are the quaternion conjugate to each other, while the scalar components along principal diagonal are associated with quaternion scalar (i.e. the real quaternion units 1 (e_0). Rank of a tensor (in special relativity) can be increased or decreased by associating two similar or conjugate operators with each other. Consequently, we get

$$DA = F^T \tag{51}$$

$$\text{where } F^T = \begin{pmatrix} 0 & -F^* \\ F^* & 0 \end{pmatrix} \text{and } F^* = -i\psi. \tag{52}$$

From equation (49) and (51), we get

$$\tfrac{1}{2}(\overline{D}A + D\overline{A}) = \begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix} = H \tag{53a}$$

$$\tfrac{1}{2}i(\overline{D}A - D\overline{A}) = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} = E \tag{53b}$$

where E and h are the electric and magnetic fields given by equation (46). Here the situation is different from quaternion electrodynamics, where $\frac{1}{2}(\overline{D}A + D\overline{A})$ represents Lorentz condition (i.e. the scalar part of quaternion) and $\frac{1}{2}i(\overline{D}A - D\overline{A})$ represents vector part (pure quaternion) of a quaternion. This is not surprising, because for the case of quaternion we have single valued representation, while for split octonion it is bivalued quaternion representation. Former case deals with quaternion conjugation only, while the later has two-fold degeneracy of quaternions and as such quaternionic and octonionic conjugation play different manifestation of space-time world.

Now we are led to write the Maxwell's equation from split octonionic representation described above. We may write the following form of field equation from split octonion analysis, i.e.

DF=J (54a)

To see that this equation is the octonionic form of Maxwell's equation $F_{\mu\nu,\nu} = J_\mu$, we asociated D with equation (42) and F with equation (49) and use the multiplication rule (35) allowed for Zorn's matrices. As such we have

$$DF = J = \begin{pmatrix} J_4 & -\vec{J} \\ \vec{J} & J_4 \end{pmatrix} \tag{54b}$$

where we have used the following forms of Maxwell's equations

$$\begin{aligned} &\vec{\nabla}.\vec{E} = -\rho_0 ; \vec{\nabla}.\vec{H} = 0 \\ &\vec{\nabla} \times \vec{E} = -\partial\vec{H}/\partial t ; \vec{\nabla} \times \vec{H} = -\vec{J} + \partial\vec{E}/\partial t \end{aligned} \tag{55}$$

Thus equation (53) represents the split octonion forms of four-current associated with Maxwell's equation. This equation can also be written as

$$DF = D(\overline{D}A) = (D\overline{D})A = []A = J \tag{56}$$

which is the equivalent split octonion form of covariant field equation, $[]A_\mu = J_\mu$ of classical electrodynamics. Lorentz force equation of motion can be written as

$$f = e\overline{F}u, \tag{57}$$

where

$$f = \begin{pmatrix} f_4 & -\vec{f} \\ \vec{f} & f_4 \end{pmatrix}$$

with $f_4 = -ie\vec{E}.\vec{u}, \vec{f} = e[\vec{E} + \vec{u} \times \vec{H}], u_4 = i\,(c = \hbar = 1)$

$$u = \begin{pmatrix} u_4 & -\vec{u} \\ \vec{u} & u_4 \end{pmatrix} \tag{58a}$$

$$F = \begin{pmatrix} 0 & \vec{F} \\ -\vec{F} & 0 \end{pmatrix} \tag{58b}$$

(5) Unified split octonion fields of dyons and gravito-dyons:-

We write the complex generalised four-potential (V_μ) associated with dyons (or gravito-dyons)[41] in the following form

$$V = u_0 V_4 + u_i V_i + u_0^* V_0 + u_i^* V_i$$

$$= \begin{pmatrix} V_4 & -\vec{V} \\ \vec{V} & V_4 \end{pmatrix} \tag{59}$$

Where V_4 and $\vec{V}$ are the temporal and spatial components of generalised four-potentials of dyons(gravito-dyons).These are complex quantities and their real and imaginary components are electric (or gravitational) and magnetic (or Heavisidian) constituents. Operating split octonion conjugate of four-differential operator given by equation (42) on equation (59) and using the commutation relations of split octonion units,[42] we get

$$\bar{D} V = \begin{pmatrix} \partial_4 & \vec{\partial} \\ -\vec{\partial} & \partial_4 \end{pmatrix} \begin{pmatrix} V_4 & -\vec{V} \\ \vec{V} & V_4 \end{pmatrix}$$

$$= \begin{pmatrix} \partial_4 V_4 + \vec{\nabla}.\vec{V} & -\partial_4 \vec{V} + \vec{\nabla} V_4 + \vec{\nabla} \times \vec{V} \\ \partial_4 \vec{V} - \vec{\nabla} V_4 - \vec{\nabla} \times \vec{V} & \partial_4 V_4 + \vec{\nabla}.\vec{V} \end{pmatrix} \tag{60}$$

where

$$\partial_4 V_4 + \vec{\nabla}.\vec{V} = (\partial_\mu V_\mu)(u_0 + u_0^*) = 0.1 = 0$$

and (61)

$$-\partial_4 \vec{V} + \vec{\nabla} V_4 + \vec{\nabla} \times \vec{V} = -i\vec{E} + \vec{H} = -i\alpha \vec{\psi}^*$$

where α as a constant parameter [43]. Its values are given by

α=+1 (for generalised electromagnetic fields of dyons) (62a)

α=-1 (for generalised gravito-Heavisidian fields of gravito-dyons). (62b)

Thus, when α=+1, V is generalised split octonion potential of dyons while for α=-1, V is generalised split octonion potential of gravito-dyons. Thus

ψ = E-iH when α=+1, electromagnetic case (63a)

= G-iH when α=-1, gravito-Heavisidian case (63b) where ψ is the complex vector field, E and H are electromagnetic fields of dyons and G and h are respectively, the gravitational and Heavisidian fields of gravito-dyons. Equation (60) leads the following split octonion expression for unified fields of dyons and gravito dyons, i.e.

$$\bar{D}V = G \tag{64a}$$

where

$$G = \begin{pmatrix} 0 & -\vec{G} \\ \vec{G} & 0 \end{pmatrix} and\, \vec{G} = i\psi^{*}. \tag{64b}$$

Consequently, one obtains

$$D(\overline{D}V) = D(\alpha G) = \alpha J \tag{65}$$

where

$$\begin{aligned} J &= J_4 u_0^{*} + J_i u_i^{*} + J_4 u_0 + J_i u_i \\ &= \begin{pmatrix} J_4 & -\vec{J} \\ \vec{J} & J_4 \end{pmatrix} \end{aligned} \tag{66}$$

is the split octonion form of generalised four-current associated with dyons (orgravito-dyons). Using equation (63)-(65), we get

$$D(\overline{D}V) = (D\overline{D})V = D(\alpha G) = \alpha(DG) = \alpha J \tag{67a}$$

or equivalently

$$[]V = \alpha J \tag{67b}$$

To derive the Lorentz force equation of motion for unified electromagnetic and gravito-Heavisidian fields of dyons and gravito-dyons, one has to take the account the effective mass equation of Rajput [43], i.e.

$$M = M_{eff} = m - {(\alpha - 1)}\big/{2h} \tag{68}$$

which yields

M=m+h for gravito-Heavisidian fields (69a)

M=m for electromagnetic case (69b)

Where m and h are gravitational and Heavisidian charges (masses). The unified Lorentz force equation of motion in split octonion representation may be written as

$$\mathrm{Re}[Q\bar{G}]u = \begin{pmatrix} f_4 & -\vec{f} \\ \vec{f} & f_4 \end{pmatrix} = f \qquad (70)$$

where

$f = f_4 u_0^* + f_i u_i^* + f_4 u_0 + f_i u_i$

We have

$$f_4 = -i[e\vec{E}.\vec{u} + g\vec{H}.\vec{u}] \text{ (for electromagnetic case)} \qquad (71a)$$

and

$$f_4 = i[m\vec{G}.\vec{u} + h\vec{H}.\vec{u}] \text{ (for gravito-Heavisidian case)} \qquad (71b)$$

while the spatial components of unified Lorentz force are given by

$$\vec{f} = M\ddot{x} = \mathrm{Re}[q\psi^* - i\vec{u} \times \psi^*] \qquad (72)$$

Equation (72) yields

$$\vec{f} = m\ddot{x} = e\{\vec{E} + \vec{u} \times \vec{H}] + g[\vec{H} - \vec{u} \times \vec{E}] \text{ (for electromagnetic case)} \qquad (73a)$$

and

$$\vec{f} = (m+h)\ddot{x} = m[\vec{G} + \vec{u} \times \vec{H}] + h[\vec{H} - \vec{u} \times \vec{G}] \text{ (for gravito-Heavisidian case)} \qquad (73b)$$

In equation (70), Q is the generalised charge on dyons is defined as

Q=e-ig (for electromagnetic case) (74a)

Q-m-ih (for gravito-Heavisidian case) (74b)

Where e, g, m and h are respectively electric, magnetic, gravitational and Heavisidian charges.

Conclusion

The lack of associativity in octonion formulation of dyons forbids their grup theoretical study in terms of abelian and non-Abelian gauge structures. However, split octonion basis of octonions presented in Section 3, gives rise to their isomorphic matrix representation associated with 2×2 Zorn's vector matrices [18]. As such, any four-dimensional relativistic four vector may be reproduced in terms of split octonions as its bivalued representation of Zorn's vector matrices by taking scalar component along principal diagonal and vector component as off diagonal elements. Split basis of octonions is related to Pauli-spin matrices (or quaternions) by equation (33). Split octonion conjugate is defined by equation (36), while the norm of split octonion is given by equation (37).

Equation (38) shows some other properties like anti-automorphism, and associativity of trace acting on Zorn's vector matrices.

Eqaution (39) is the representation of Euclidean four-dimensional space-time vector in terms of split octonion or Zorn's vector realization. Equation (41) shows the norm of space-time four-vector, equation (42) and (43) are the split octonion equivalents of four differential operator and its octonion conjugate, respectively, giving rise to invariant D'Alembertain operator. Starting from split octonion form of electromagnetic four potential and keeping in view the definition of electric and magnetic field, I have derived equation (45), which is the split octonion equivalent form of electromagnetic field tensor, $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$. Its another counterpart is given by equation (51). It has been shown that electric and magnetic fields in terms of Zorn's matrix realization of split octonions can separately be obtained by combining equation (49) and (51) and are derived in equation (53). Equation (54a) is the split octonion field equations, whose four-current reproduced by equation (54b). Equation (54b) reduces to equation (56), which is the equivalent split octonion form of covariant field equation $[]A_{\mu} = J_{\mu}$ (in terms of Lorentz gauge). Lorentz force equation of motion has been reformulated by split octonions in equation (57).

Equation (65) and (67) are the split octonion equivalents of unified field (Maxwell-Dirac) equations of dyons and gravito-dyons. These equations thus reproduce the generalised field equation (Macwell-Dirac equation) of dyons for $\alpha = +1$. On the other hand, for $\alpha = -1$ its give the dynamics of gravito-Heavisidian fields of gravito-dyons. Moreover, equations (65) and (67) reproduces the dynamics of electric (or gravitational) charge (or mass) on dyons (or gravito-dyons) for $\alpha = +1$ (or $\alpha = -1$). Equation (67) is the split octonion equivalent of unified covariant field equation $[]V_{\mu} = \alpha J_{\mu}$ of generalised fields of dyons and gravito-dyons derived earlier by Rajput [43]. Split octonion electrodynamics leads to bivalued representation of the two basis elements u_0, u_1, u_2, u_3 and $u_0^*, u_1^*, u_2^*, u_3^*$ of split octonions, which can be interpreted as the dual basis or conjugate basis to each other.

REFERENCES

[1] P. A. M. Dirac, Proc. R. Soc. London A133 (1931) 60.

[2] P. A. M. Dirac, Phys. Rev.74(1948)817.

[3] J. Schwinger, Science 165(1978),757.

[4] G. t'Hooft, Nucl. Phys.B79 (1974), 276.

[5] G. t'Hooft, Nucl. Phys.B138(1978)1.

[6] B. S. Rajput and D. C. Joshi, Hadronic J.4 (1981) 1805.

[7] B S. Rajput and Om Prakash, Indian J. Phys.A53 (1979) 274.

[8] J. S. Dowker and J. A. Roche, Proc. R. Soc.92 (1967) 1.

[9] A. Singh, Lett. Nuovo Cimento32 (1981) 232.

[10] D. D. Cantani, Nuovo Cimento B60 (1980) 67.

[11] P. S. Bisht, O. P. S. Negi and B. S. Rajput,Indian J. Pure and Appl. Phys. 28 (1990) 157.

[12] P. S. Bisht, O. P. S. Negi and B. S. Rajput, Prog. Theor. Phys85 (1991) 157.

[13] D. S. Bhakuni, O. P. S. Negi and B. S. Rajput, Lett. Nuovo Cimento 36 (1983) 499.

[14] B. S. Rajput, S. R. Kumar and O. P. S. Negi, Lett. Nuovo Cimento 36 (1983) 75.

[15] A. Pais, Phys. Rev. Lett.7 (1964) 291.

[16] A. Gamba, High Energy Physics and Elementry Particles (International Atomic Agency, Vienna, 1965) 641.

[17] M. Gunaydin and F. Gursey, J. Math. Phys.14 (1973)1651.

[18] M. Gunaydin and F. Gursey, Phys. Rev.D9 (1974) 3387.

[19] M. Gunaydin and F. Gursey, Lett. Nuovo Cimento6 (1973) 401.

[20] G. Domokos and S. Kovesi Domokos, J. Math. Phys.19 (1978) 1477.

[21] G. Domokos and S. Kovesi Domokos, Phys. Rev.D19 (1979) 3984.

[22] K. Morita, Prog. Theor. Phys.65 (1981) 2027.

[23] A. M. Buoncristiani, J. Math. Phys.14 (1973) 849.

[24] S. Morques and C. G. Oliveria, J. Math. Phys.26 (1985) 3131.

[25] D. E. Littlewood, "A unitrary algebra", Heinrman (1956).

[26] L. Dickson, Ann. Maths.20 (1918) 155.

[27] I. L. Kantor and Solodovnikov, "Hyper complex numbers-An elementry introduction to Algebra"-Springer Verlag, New York, (1989), Chs. 6 and 7.

[28] H. Freudenthal, "Oktaven, Anonahamingruppen and Oktavenglometric", Utretch (1951).

[29] M. Gunaydin, J. Math. Phys.17 (1976) 1875.

[30] B. S. Rajput and Om Prakash, Ind. J. Pure and Appl. Phys.16 (1978) 593.

[31] P. Zenczykowski, Int. J. Theor. Phys.29 (1990) 835.

[32] M. L. Mehta, J. Math. Phys. 7 (1966) 1824.

[33] R. Gilmore, "Lie Groups, Lie algebras and some other applications", Wiley Int., (1974).

[34] M. Zorn, Proc. Acad. Sc.21 (1935) 355.

[35] F. Gursey,"In Johns Hopkins Workshop on current problems in High Energy Physics", John Hopkins Univ (1974) 15.

[36] F. Gursey,"Lecture notes in Physics Sequ of VII International Colliquium and Integrative Conference on Group theory and Mathematical Physics", Edited by W. Beiglbock et. Al. (1978) 508.

[37] M. Gunaydin and F. Gursey, J. Math. Phys.14 (1973) 1651.

[38] M. Gunaydin and F. Gursey, Lett. Nuovo Cimento 6(1973) 401.

[39] M. Gunaydin and F. Gursey, Phys. Rev.D9 (1974) 3387.

[40] P. S. Bisht and O. P. S. Negi, Ind. J. Pure and Appl. Phys32 (1994) 193.

[41] P. S. Bisht, O. P. S. Negi and B. S. Rajput, Int. J. Theoretical Physics, 32 (1993) 2099.

[42] P. S. Bisht and O. P. S. Negi, Ind. J. Pure and Appl. Phys31 (1993) 292.

[43] B. S. Rajput, J. Math. Phys.25 (1984) 351.