

# Donaldson invariants in Algebraic Geometry

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## Abstract

In these lectures I want to give an introduction to the relation of Donaldson invariants with algebraic geometry: Donaldson invariants are differentiable invariants of smooth compact 4-manifolds  $X$ , defined via moduli spaces of anti-self-dual connections. If  $X$  is an algebraic surface, then these moduli spaces can for a suitable choice of the metric be identified with moduli spaces of stable vector bundles on  $X$ . This can be used to compute Donaldson invariants via methods of algebraic geometry and has led to a lot of activity on moduli spaces of vector bundles and coherent sheaves on algebraic surfaces.

We will first recall the definition of the Donaldson invariants via gauge theory. Then we will show the relation between moduli spaces of anti-self-dual connections and moduli spaces of vector bundles on algebraic surfaces, and how this makes it possible to compute Donaldson invariants via algebraic geometry methods. Finally we concentrate on the case that the number  $b_+$  of positive eigenvalues of the intersection form on the second homology of the 4-manifold is 1. In this case the Donaldson invariants depend on the metric (or in the algebraic geometric case on the polarization) via a system of walls and chambers. We will study the change of the invariants under wall-crossing, and use this in particular to compute the Donaldson invariants of rational algebraic surfaces.

*Keywords:* Donaldson invariants, moduli spaces of sheaves.

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## 1 Introduction

Donaldson invariants [D1] have played an important role in the study and classification of compact differentiable 4-manifolds  $X$ . The most comprehensive introduction to Donaldson invariants is [D-Kr]. Discrete invariants of 4-manifolds are the fundamental group  $\pi_1(X)$  and the intersection form on  $H_2(X, \mathbb{Z})$ . If  $X$  is simply-connected, then the homotopy type of  $X$  is essentially determined by the intersection form. Friedman showed that in this case  $X$  is determined up to homeomorphism by its homotopy type.

In order to attempt to make a differentiable classification, one needs additional invariants. The Donaldson invariants are defined via gauge theory in terms of moduli spaces of anti-self-dual connections on differentiable bundles on  $X$ . If  $X$  is an algebraic surface, then these moduli spaces can be identified with moduli spaces of stable vector bundles on  $X$ . This makes it possible to apply methods of algebraic geometry to compute the Donaldson invariants. In fact, because of this, for a long time most of the calculations of Donaldson invariants have been carried out in the case of algebraic surfaces. On the other hand the Donaldson invariants have provided a lot of interest for the study of moduli spaces of vector bundles and coherent sheaves on algebraic surfaces.

Some results obtained with Donaldson invariants are:

1. Algebraic surfaces are essentially indecomposable: If a simply-connected algebraic surface  $X$  is the connected sum  $X = Y \# Z$  of two 4-manifolds, then either  $Y$  or  $Z$  must have a negative definite intersection form. An example where this happens is when  $X$  is the blow up of  $Y$  in a point.
2. The differentiable classification of elliptic surfaces.
3. The Kodaira dimension of an algebraic surface is a differentiable invariant.

Recently the Seiberg-Witten invariants have appeared, which are also defined via gauge theory, but are often easier to compute [W],[D2]. A number of conjectures from Donaldson theory were immediately proved, e.g.

1. The plurigenera of algebraic surfaces are differentiable invariants.
2. The generalized Thom conjecture: Let  $X$  be an algebraic surface, then each smooth algebraic curve in  $X$  minimizes the genus of embedded 2-manifolds in its homology class.

Conjecturally the Donaldson- and Seiberg-Witten invariants are very closely related and in particular the Donaldson invariants can be computed in terms of the Seiberg-Witten invariants.

Since the appearance of the Seiberg-Witten invariants the interest in the Donaldson invariants has become a bit less, but there is still a large number of interesting open questions.

## 2 Definition and properties of the Donaldson invariants

In this lecture we define the Donaldson invariants via gauge theory and state some of their most important properties. We prefer here to formulate everything in terms of vector bundles, which should be more familiar to the audience, instead of principal bundles, which would be more natural.

### 2.1 Moduli spaces of connections

Let  $X$  be a smooth simply-connected compact oriented 4-manifold. Let  $P$  be a principal  $SU(2)$ - or  $SO(3)$ -bundle on  $X$ . The Donaldson invariants are defined via intersection theory on a moduli space of anti-self-dual connections on  $P$ .

$SU(2)$ -bundles on  $X$  are classified by their second Chern class  $c_2(P)$ , and  $SO(3)$ -bundles on  $X$  are classified by their second Stiefel-Whitney class  $w_2(P) \in H^2(X, \mathbb{Z}/2)$  and their first Pontrjagin class  $p_1(P) \in H^4(X, \mathbb{Z})$ .

In the  $SU(2)$ -case the moduli space of anti-self-dual connections on  $P$  can be identified with the moduli space of anti-self-dual connections on the associated complex vector bundle on  $E$  with first Chern class  $c_1 = 0$ . In the  $SO(3)$ -case (after choosing a lift  $c_1 \in H^2(X, \mathbb{Z})$  of  $w_2(P)$ ) it corresponds to a moduli space of Hermitian Yang-Mills connections on the associated complex vector bundle with Chern classes  $c_1, c_2$  (with  $c_1^2 - 4c_2 = p_1(P)$ ). For simplicity, in the following we will concentrate on the  $SU(2)$ -case.

Let  $E$  be a rank 2 complex differentiable vector bundle on  $X$ . We fix a hermitian metric  $h$  on  $E$ . (That is for each  $x \in X$  we have a hermitian inner product on the fiber  $E(x)$ , varying smoothly with  $x$ .) We denote by  $\Omega^i(E)$  the space of  $\mathbb{C}^\infty$ -sections of  $E \otimes \Lambda^i T_X^*$ .

A hermitian connection on  $E$  is a connection  $D : \Omega^0(E) \rightarrow \Omega^1(E)$ , which is compatible with  $h$ . (That  $D$  is a connection means that it is a linear map

satisfying the Leibniz rule

$$D(f \cdot s) = d(f \otimes s) + f \cdot D(S)$$

and that  $D$  is compatible with the metric means furthermore that

$$d(h(s_1, s_2)) = h(D(s_1), s_2) + h(s_1, D(s_2)) .)$$

$D$  is called *reducible* if  $E$  is the direct sum  $L_1 \oplus L_2$  of two line bundles, and  $D = D_1 \oplus D_2$  with  $D_i$  a connection on  $L_i$ .

We write  $\mathcal{A}(E)$  for the space of hermitian connections on  $E$ , (which are trivial on  $\det(E)$  in the case  $c_1(E) = 0$  and equal to a fixed connection on  $\det(E)$  otherwise).  $\mathcal{A}^*(E) \subset \mathcal{A}(E)$  is the subspace of irreducible connections. The *gauge group*  $\mathcal{G}$  is the set of  $\mathbb{C}^\infty$ -automorphisms of  $E$  which are compatible with  $h$  and act as the identity on  $\det(E)$ .  $\mathcal{G}$  acts on  $\mathcal{A}(E)$  and  $\mathcal{A}^*(E)$  via

$$\begin{array}{ccc} \Omega^0(E) & \xrightarrow{D} & \Omega^1(E) \\ \downarrow \alpha & & \downarrow \alpha \\ \Omega^0(E) & \xrightarrow{\alpha(D)} & \Omega^1(E). \end{array}$$

Let  $\mathcal{B}(E) := \mathcal{A}(E)/\mathcal{G}$ ,  $\mathcal{B}^*(E) := \mathcal{A}^*(E)/\mathcal{G}$ .

## 2.2 ASD-connections

We assume in this part that  $c_1(E) = 0$ . Now fix a Riemannian metric  $g$  on  $X$ . It gives rise to a Hodge star operator

$$*_g : \Lambda^2 T_X^* \rightarrow \Lambda^2 T_X^*, \quad *_g^2 = 1.$$

We write  $\Lambda_+$  for the  $(+1)$ -eigenbundle and  $\Lambda_-$  for the  $(-1)$ -eigenbundle of  $*_g$ .

**Definition 2.1** For  $D \in \mathcal{A}^*(E)$ , let  $F(D) = D \circ D \in \Omega^2(\text{End}(E))$  be its curvature.  $F$  is called anti-self-dual (ASD), if

$$*_g F(D) = -F(D).$$

In other words, writing  $F(D) := F_-(D) + F_+(D)$ , with  $F_-(D)$  a section of  $\Lambda_-(\text{End}(E))$  (and similarly for  $F_+(D)$ ), the condition is  $F_+(D) = 0$ . We write  $N_g(E)$  for the moduli space

$$N_g(E) := \{\text{ASD-connections on } E\}/\mathcal{G} \subset \mathcal{B}^*(E).$$

In the case  $c_1(E) \neq 0$  we have instead to take the moduli space of Hermitian Yang-Mills connections on  $E$ , because only these correspond to the moduli space of ASD-connections on the corresponding principal bundle. The differentiable type of  $E$  is determined by its Chern classes  $c_1(E)$ , and  $c_2(E)$ . Therefore we also write  $N_g(c_1, c_2)$  for  $N_g(E)$ .

If  $D$  is an ASD-connection (or Hermitian Yang-Mills in case  $c_1(E) \neq 0$ ) on  $E$ , then by Chern-Weil theory

$$4c_2(E) - c_1^2(E) = -p_1(E) = \frac{1}{4\pi^2} \int_X \text{tr}(F(D) \wedge F(D)) = \frac{1}{4\pi^2} \int_X \|F_-(D)\|^2 > 0.$$

Let  $b_+(X)$  be the number of positive eigenvalues of the intersection form on  $H_2(X, \mathbb{R})$ . We write

$$k := (c_2 - c_1^2/4)(E).$$

Then we have the following generic smoothness result:

**Theorem 2.2** *If  $g$  is generic, then  $N_g(E)$  is a smooth manifold of dimension*

$$2d = 8k - 3(1 + b_+(X)).$$

*For a generic path  $g_t$  of metrics, the corresponding parameterized moduli space is smooth.*

*Furthermore  $N_E(g)$  is orientable. The orientation depends on the choice of an orientation of a maximal-dimensional subspace  $H^2(X, \mathbb{R})^+$  of  $H^2(X, \mathbb{R})$  on which the intersection form is positive definite.*

### 2.3 Relations to holomorphic vector bundles

Assume here, that  $c_1(E) = 0$ . Let  $X$  be a projective algebraic surface. Let  $H$  be an ample divisor on  $X$ . Let  $g(H)$  be the corresponding Hodge metric and  $\omega$  the Kähler form. We write  $\Lambda^{p,q}$  for the bundle of  $(p, q)$  forms. Then we get

$$\begin{aligned} \Lambda_+ &= \Re(\Lambda^{2,0} \oplus \Lambda^{0,2}) \oplus \mathbb{R}\omega \\ \Lambda_- &= \omega^\perp \quad \text{in } \Lambda^{1,1}. \end{aligned}$$

We can write  $D := \partial_D + \bar{\partial}_D$ , where  $\partial_D : \Omega^0(E) \rightarrow \Omega^{1,0}(E)$  and  $\bar{\partial}_D : \Omega^0(E) \rightarrow \Omega^{0,1}(E)$ . So we get

$$F(D) = \partial_D^2 + (\partial_D \bar{\partial}_D + \bar{\partial}_D \partial_D) + \bar{\partial}_D^2.$$

$D$  is ASD if

1.  $\bar{\partial}_D^2 = 0$ .
2.  $\partial_D^2 = 0$  and  $F(D) \wedge \omega = 0$ .

1. means that  $\bar{\partial}_D$  defines a holomorphic structure on  $E$ . 2. implies after some work that  $(E, \bar{\partial}_D)$  is  $\mu$ -stable with respect to  $H$ .

Recall that a vector bundle  $E$  of rank 2 on an algebraic surface  $X$  is called  $\mu$ -stable (slope stable) with respect to an ample divisor  $H$ , if

$$Hc_1(L) < \frac{Hc_1(E)}{2}$$

for all locally free subsheaves  $L$  of rank 1 of  $E$ . We denote by  $M_H^X(c_1, c_2)^\mu$  the moduli space of  $\mu$ -stable rank 2 bundles on  $X$  with Chern classes  $c_1$  and  $c_2$ . We have motivated (at least in the case  $c_1 = 0$ ) that there is a map

$$\Psi : N_{g(H)}(c_1, c_2) \rightarrow M_H^X(c_1, c_2)^\mu.$$

In fact this map exists for any  $c_1$ , and furthermore we get:

**Theorem 2.3** (Donaldson)  $\Psi$  is a homeomorphism.

This will give a relation between the Donaldson invariants (which we will define via moduli spaces of ASD-connections) and moduli of vector bundles.

## 2.4 Uhlenbeck compactification

We want to define the Donaldson invariants as intersection numbers on  $N_g(E)$  which is usually not compact. We have therefore to compactify.

**Theorem 2.4** Let  $(A_i)_i$  be a sequence in  $N_g(E)$ . After passing to a subsequence we obtain: There is a finite collection of points  $p_1, \dots, p_l \in X$  with multiplicities  $n_1, \dots, n_l > 0$ , such that up to gauge transformation  $A_i|_{X \setminus \{p_1, \dots, p_l\}}$  converges to an ASD-connection  $A_\infty$ .  $A_\infty$  can be extended to an ASD-connection on a vector bundle  $E'$  with

$$\det(E') = \det(E), \quad c_2(E) = c_2(E') + \sum_{i=1}^l n_i.$$

This leads to the Uhlenbeck compactification:



**Theorem 2.5** *There exists a topology on*

$$\coprod_{n \geq 0} N_g(c_1, c_2 - n) \times X^{(n)}$$

*such that the closure  $\overline{N}_g(c_1, c_2)$  is compact.*

Here  $X^{(n)} = X^n / \mathfrak{S}(n)$  denotes the  $n$ -th symmetric power of  $X$ , the quotient of the  $n$ -th power of  $X$  by the action of the symmetric group  $\mathfrak{S}(n)$  via permuting the factors. It parameterizes effective 0-cycles on  $X$  of degree  $n$ .

## 2.5 Definition of the invariants

We write  $H^*(X) := H^*(X, \mathbb{Q})$  and  $H_*(X) = H_*(X, \mathbb{Q})$ . If on  $X \times N_g(E)$  there exists a universal bundle  $\mathcal{E}$  with a universal connection  $\mathcal{D}$  with  $\mathcal{D}|_{X \times \{D\}} = D$ , then we can define the  $\mu$ -map as follows.

$$\mu : H_*(X) \rightarrow H^*(N_g(E)); \quad \mu(\alpha) = -\frac{1}{4}p_1(\mathcal{E})/\alpha.$$

Here  $-\frac{1}{4}p_1(\mathcal{E}) = (c_2(\mathcal{E}) - c_1(\mathcal{E})^2)/4$ , and the slant product  $-\frac{1}{4}p_1(\mathcal{E})/\alpha$  means: write

$$-\frac{1}{4}p_1(\mathcal{E}) = \sum_i \beta_i \otimes \gamma_i, \quad \beta_i \in H^*(X), \quad \gamma_i \in H^*(N_g(E)).$$

Then

$$-\frac{1}{4}p_1(\mathcal{E})/\alpha = \sum_i \langle \beta_i, \alpha \rangle \gamma_i.$$

If the universal bundle does not exist, its endomorphism bundle  $End(\mathcal{E})$  will still exist, and we can define  $\mu$  by replacing  $(c_2(\mathcal{E}) - c_1^2(\mathcal{E}))/4$  with  $-c_2(End(\mathcal{E}))/4$ .

It can be shown that  $\mu(\alpha)$  extends over the Uhlenbeck compactification  $\overline{N}_g(E)$ . For generic metric  $g$ ,  $\overline{N}_g(E)$  is a stratified space with smooth strata, and the submaximal stratum has codimension at least 4. Therefore  $\overline{N}_g(E)$  has a fundamental class.

Now let

$$d := 4c_2 - c_1^2 - \frac{3}{2}(1 + b_+(X))$$

and write  $d = l + 2m$ . Let  $\alpha_1, \dots, \alpha_l \in H_2(X)$  and let  $p \in H_0(X)$  be the class of a point. Then we define the *Donaldson invariant*

$$\Phi_{c_1, d}^{X, g}(\alpha_1 \cdots \alpha_l \cdot p^m) := \int_{[\overline{N}_g(E)]} \mu(\alpha_1) \cup \dots \cup \mu(\alpha_l) \cup \mu(p)^m.$$

More generally let

$$A_*(X) := \text{Sym}^*(H_2(X) \oplus H_0(X)).$$

This is graded by giving degree  $(2 - i/2)$  to elements in  $H_i(X)$ . We denote by  $A_d(X)$  the part of degree  $d$ . By linear extension we get a map  $\Phi_{c_1, d}^{X, g} : A_d(X) \rightarrow \mathbb{Q}$  and

$$\Phi_{c_1}^{X, g} := \sum_{d \geq 0} \Phi_{c_1, d}^{X, g} : A_*(X) \rightarrow \mathbb{Q}.$$

By definition the Donaldson invariants depend on the choice of the metric  $g$ , because the ASD-equation uses the Hodge  $*$  operator, which depends on  $g$ . We have however

- Theorem 2.6**
1. If  $b_+(X) > 1$ , then  $\Phi_{C, d}^{X, g}$  is independent of the generic metric  $g$ .
  2. If  $b_+(X) = 1$ , then  $\Phi_{C, d}^{X, g}$  depends only on the chamber of the period point of  $g$ .

We will discuss walls and chambers later. The result means that the Donaldson invariants are really invariants of the differentiable structure of  $X$ . In the case  $b_+(X) > 1$ , we can therefore drop the  $g$  from our notation.

The argument for showing the theorem is that one connects two generic metrics by a generic path in order to make a cobordism. Reducible connections occur in codimension  $b_+(X)$ , so they make no problem for  $b_+(X) > 1$ , but can disconnect the path for  $b_+(X) = 1$ .

## 2.6 Structure theorems

It is often useful to look at generating functions for the Donaldson invariants. For  $a \in H_2(X)$  and  $\alpha \in A_*(X)$  and a variable  $z$  we write

$$\Phi_C^X(\alpha e^{az}) := \sum_{n \geq 0} \Phi_C^X(\alpha a^n / n!) z^n.$$

**Definition 2.7** A 4-manifold  $X$  is of *simple type* if

$$\Phi_C^X(\alpha(p^2 - 4)) = 0$$

for all  $\alpha \in A_*(X)$  and all  $C \in H^2(X, \mathbb{Z})$ .

Many 4-manifolds like  $K3$  surfaces and complete intersections are known to be of simple type, and it is possible that all simply-connected 4-manifolds are of simple type. The famous structure theorem of Kronheimer and Mrowka [Kr-Mr] says that all the Donaldson invariants of a manifold of simple type organize themselves in a nice generating function, which depends only on a finite amount of data: a finite number of cohomology classes in  $H^2(X, \mathbb{Z})$  (the *basic classes*) and rational multiplicities associated to these numbers.

**Theorem 2.8** *Let  $X$  be a simply-connected 4-manifold of simple type. Then there exist so-called basic classes  $K_1, \dots, K_l \in H^2(X, \mathbb{Z})$  and rational numbers  $\alpha_1(C), \dots, \alpha_l(C)$ , such that for all  $a \in H_2(X)$*

$$\Phi_C^X(e^{at}(1 + p/2)) = e^{(a \cdot a)t^2/2} \sum_{i=1}^l \alpha_i(C) e^{\langle K_i, a \rangle t}.$$

(Here  $(a \cdot a)$  denotes the intersection form on  $H_2(X)$ , and  $\langle K_i, a \rangle$  the dual pairing between cohomology and homology.)

### 3 Algebro-geometric definition of Donaldson invariants

Let  $X$  be a simply-connected algebraic surface, and let  $H$  be an ample divisor on  $X$ . For a sheaf  $F$  and a line bundle  $L$  on  $X$  we denote  $F(nL) := F \otimes L^{\otimes n}$ . Let  $\chi(F) = \sum_i (-1)^i \dim H^i(X, F)$  be the holomorphic Euler characteristic of  $F$ . Recall that a torsion-free coherent sheaf  $F$  on  $X$  is  $\mu$ -stable with respect to  $H$  if  $(c_1(G) \cdot H)/rk(G) < (c_1(F) \cdot H)/rk(F)$  for all non-zero strict subsheaves of  $F$ .  $F$  is called (Gieseker)  $H$ -semistable if  $\chi(G(nH)) \leq \chi(F(nH))$  for all nonzero strict subsheaves  $G$  of  $F$ .

We denote by  $M := M_H^X(C, c_2)$  the moduli space of (Gieseker)  $H$ -semistable rank 2 torsion-free coherent sheaves  $F$  on  $X$  with  $c_1(F) = C$  and  $c_2(F) = c_2$ . We want to relate  $M$  to the Uhlenbeck compactification  $N := N_{g(H)}(C, c_2)$ . Here  $g(H)$  is the Fubini-Study metric associated to  $H$ . As the Donaldson invariants are defined in terms of the Uhlenbeck compactification, this allows us to compute them on the moduli space  $M$  of sheaves.

The steps of the argument are as follows:

1. Introduce the determinant bundles  $L_1(nH)$  on  $M$  for  $n \gg 0$ .

2. Construct sections of  $L_1(nH)^{\otimes m}$  for  $n, m \gg 0$  and show that the corresponding linear system is base-point free, thus giving a morphism

$$\Psi : M \rightarrow \mathbb{P}(H^0(M, L_1(nH)^{\otimes m})^\vee).$$

3. Show that  $Im(\Psi)$  is homeomorphic to  $N$ .
4. Apply this to the computation of the Donaldson invariants.

### 3.1 Determinant line bundles

We will assume for simplicity that there is a universal sheaf  $\mathcal{E}$  over  $X \times M$ . For instance, this is the case if  $H$  is general and either  $C$  is not divisible by 2 or  $c_2 - C^2/4$  is odd.

For a coherent sheaf  $\mathcal{F}$  on  $X \times M$ , let

$$0 \rightarrow G_l \rightarrow \dots \rightarrow G_s \rightarrow 0$$

be a finite complex of locally free sheaves which is quasi-isomorphic to  $Rp_{2*}(\mathcal{F})$ . Then we put

$$\det(p_{2!}(\mathcal{F})) := \bigotimes \det(G_j)^{(-1)^j} \in Pic(M).$$

**Definition 3.1** Let  $D \in |nH|$  be a smooth curve. For a general  $E \in M$  let  $\chi_1 := \chi(E|_D)$ . Let  $a \in X$  be a point. Then we put

$$L_1(nH) := \det(p_{2!}(\mathcal{E}|_{D \times M}))^{\otimes -2} \otimes \det(\mathcal{E}|_{\{a\} \times M})^{\otimes \chi_1}.$$

Let  $M_D$  be the moduli space of semistable rank 2 vector bundles on  $D$  of degree  $D \cdot C$ . Assume for simplicity that also on  $D \times M_D$  there is a universal sheaf  $\mathcal{G}$ . Let  $G$  be any element in  $M_D$ . Then we define

$$L_0 := \det(p_{2!}\mathcal{G})^{\otimes -2} \otimes \det(\mathcal{G}|_{\{a\} \times M_D})^{\otimes \chi(G)}.$$

**Remark 3.2**  $L_1(nH)$  is independent of the choice of  $\mathcal{E}$  (and also of  $D$  and  $a$ ). Any other choice of a universal sheaf  $\mathcal{F}$  can be written as  $\mathcal{F} = \mathcal{E} \otimes p_2^* \lambda$  for  $\lambda$  a line bundle on  $M$ . Then the projection formula implies that  $Rp_{2*}(\mathcal{E} \otimes p_2^* \lambda) = Rp_{2*}(\mathcal{E}) \otimes \lambda$ , and therefore

$$\det(p_{2!}(\mathcal{E} \otimes p_2^* \lambda)) = \lambda^{\otimes \chi(E)} \otimes \det(p_{2!}(\mathcal{E})).$$

So  $L_1(nH)$  stays unchanged if we replace  $\mathcal{E}$  by  $\mathcal{E} \otimes \lambda$ . In fact we do not need the existence of  $\mathcal{E}$  in order to define  $L_1(nH)$ . The definition is part of a more general formalism of determinant sheaves as was explained in the lectures of Huybrechts and Lehn (see [LP], [H-L] where these line bundles are defined via descent from the corresponding Quot scheme).

In the same way we see that  $L_0$  is independent of the choice of  $\mathcal{F}$  and indeed we do not need the existence of  $\mathcal{G}$  to define  $L_0$ .

### 3.2 Construction of sections of $L_1(nH)$

We have the following theorem

**Theorem 3.3** [D-N]  $L_0$  is ample on  $M_D$ .

Let  $U(D) \subset M$  be the open subset of all sheaves  $E$  such that  $E|_D$  is semistable. Thus for  $E \in U(D)$ , we get that  $E|_D \in M_D$ . We obtain therefore a rational map

$$j : M \rightarrow M_D,$$

which is defined on  $U(D)$ . By definition we see that

$$j^*(L_0) = L_1(nH) \quad \text{on } U(D).$$

Fix an integer  $m \gg 0$ . As  $L_0$  is ample,  $L_0^{\otimes m}$  will have many sections. So we want to extend the pullbacks  $j^*(s)$  of sections  $s \in H^0(M_D, L_0^{\otimes m})$  to sections  $\tilde{s} \in H^0(M, L_1(nH))^{\otimes m}$ . By Bogomolov's theorem ([H-L] p. 174) we have the following: For  $n \gg 0$  and all  $E \in M$  the restriction  $E|_D$  is semistable, unless  $E$  is not locally free over  $D$ . For  $c_2 \gg 0$  the general element in  $M$  is locally free. If  $E \in M$  is not locally free, then its singularities occur in codimension 2. Therefore the condition that  $E|_D$  is not locally free has codimension 1 in the locus of not locally free sheaves. So, putting things together, we see that the complement  $M \setminus U(D)$  has codimension  $\geq 2$  in  $M$ . Furthermore  $M$  is normal. Therefore every  $j^*(s)$  for  $s \in H^0(M_D, L_0^{\otimes m})$  extends to  $\tilde{s} \in H^0(M, L_1(nH))^{\otimes m}$ .

More precisely one can show the following ([Li], Prop. 2.5).

**Lemma 3.4** For every  $s \in H^0(M_D, L_0^{\otimes m})$  the pullback  $j^*(s)$  extends to  $\tilde{s} \in H^0(M, L_1(nH))^{\otimes m}$ . Furthermore the vanishing locus of  $\tilde{s}$  is

$$Z(\tilde{s}) = \{E \in M \mid E|_D \text{ is semistable and } s(E|_D) = 0 \\ \text{or } E|_D \text{ is not semistable}\}.$$

Now choose  $m, n \gg 0$ .

**Proposition 3.5**  $H^0(M, L_1(nH)^{\otimes m})$  is base-point free.

*Proof.* Let  $E \in M$ . By the theorem of Mehta and Ramanathan (see [H-L] Theorem. 7.2.1), we can find a smooth curve  $D \in |nH|$  such that  $E|_D$  is semistable. Choose  $s \in H^0(M_D, L_0^{\otimes m})$ , such that  $s(E|_D) \neq 0$ . Then  $\tilde{s}(E) \neq 0$ .  $\square$

### 3.3 Uhlenbeck compactification

$L_1(nH)^{\otimes m}$  defines a morphism

$$\Psi : M \rightarrow \mathbb{P}(H^0(M, L_1(nH)^{\otimes m})^\vee).$$

**Theorem 3.6**  $\Psi(M)$  is homeomorphic to the Uhlenbeck compactification  $N$ .

We want to give a brief sketch of the proof of this theorem.

For  $E \in M$ , we introduce the pair  $(A(E), Z(E))$ , where

1. If  $E$  is  $\mu$ -stable, then

$$A(E) = E^{\vee\vee}, \quad Z(E) = \sum_{p \in X} l(E^{\vee\vee}/E)_p \cdot p.$$

$l(E^{\vee\vee}/E)_p$  is the length of the sheaf  $E^{\vee\vee}/E$  at  $p$ .  $Z(E)$  is an effective 0-cycle of length  $k := c_2(E) - c_2(E^{\vee\vee})$  on  $X$ , i.e. a point in the symmetric power  $X^{(k)}$ .

2. If  $E$  is not  $\mu$ -stable, we have the Harder-Narasimhan filtration

$$0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0,$$

where  $F$  and  $G$  are rank 1 sheaves with  $\deg_H(F) = \deg_H(G)$ . We put

$$A(E) = F^{\vee\vee} \oplus G^{\vee\vee}, \quad Z(E) = \sum_{p \in X} l((F^{\vee\vee} \oplus G^{\vee\vee})/(F \oplus G))_p \cdot p \in X^{(k)}.$$

**Claim:** For  $E_1, E_2 \in M$  we have  $\Psi(E_1) = \Psi(E_2)$  if and only if  $(A(E_1), Z(E_1)) = (A(E_2), Z(E_2))$ . In other words: the sets  $\Psi(M)$  and  $N$  are equal.

We want to check the claim in a special case. Assume  $\Psi(E_1) = \Psi(E_2)$ , where  $E_1$  and  $E_2$  are  $\mu$ -stable. Take  $D \in |nH|$  general, then  $E_1|_D = E_2|_D$  (otherwise, as  $L_0^{\otimes m}$  is very ample on  $M_D$ , we can find a section  $s \in H^0(M_D, L_0^{\otimes m})$ , such that  $0 = s(E_1|_D) \neq s(E_2|_D)$ . Then  $\tilde{s}(E_1) = 0, \tilde{s}(E_2) \neq 0$ ). The exact sequence

$$Hom(E_1^{\vee\vee}, E_2^{\vee\vee}) \rightarrow Hom(E_1|_D, E_2|_D) \rightarrow H^1(Hom(E_1^{\vee\vee}, E_2^{\vee\vee}(-nH))) = 0$$

implies that  $Hom(E_1^{\vee\vee}, E_2^{\vee\vee}) \neq 0$ . The  $\mu$ -stability of  $E_1, E_2$  and therefore also of  $E_1^{\vee\vee}, E_2^{\vee\vee}$  then implies  $E_1^{\vee\vee} = E_2^{\vee\vee}$ . Now assume  $p \in Z(E_1)$  but  $p \notin Z(E_2)$ . Then we choose  $D \in |nH|$  such that  $p \in D$  and  $E_2|_D$  is semistable. Then  $E_1|_D$  is not semistable and therefore we can find  $s \in H^0(M_D, L_0^{\otimes m})$ , such that  $\tilde{s}(E_1) = 0$  and  $\tilde{s}(E_2) \neq 0$ .

### 3.4 Donaldson invariants via algebraic geometry

Let again  $M := M_H^X(C, c_2)$  be the moduli space of Gieseker  $H$ -semistable sheaves with Chern classes  $C$  and  $c_2$ . Assume that there is a universal sheaf  $\mathcal{E}$  over  $X \times M$ . Write

$$d := 4c_2 - C^2 - 3(1 + p_g(X)).$$

Let  $\nu : H_*(X) \rightarrow H^{4-*}(M)$  be defined by

$$\nu(a) := c_2(\mathcal{E}) - \frac{1}{4}c_1(\mathcal{E})^2/a,$$

(i.e. we write

$$c_2(\mathcal{E}) - \frac{1}{4}c_1(\mathcal{E})^2 := \sum_i \beta_i \otimes \gamma_i, \quad \beta_i \in H^*(X, \mathbb{Q}), \quad \gamma_i \in H^*(M, \mathbb{Q}),$$

then

$$\nu(\alpha) = \sum_i \langle \beta_i, a \rangle \gamma_i.$$

Again  $\nu$  is independent of the choice of a universal sheaf, and, if no universal sheaf exists,  $\nu$  can be defined without using it. We denote again by  $A_d(X)$  the set of elements of degree  $d$  in  $Sym^*(H_0(X) \oplus H_2(X))$ , where the class  $p$  of a point in  $H_0(X)$  is given weight 2 and a class in  $H_2(X)$  is given weight 1. For  $\alpha := a_1 \cdot \dots \cdot a_k \in A_d(X)$ , we define

$$\nu(\alpha) := \nu(a_1) \cup \dots \cup \nu(a_k) \in H^{2d}(M)$$

and

$$\gamma_{C,d}^{X,H}(\alpha) := \int_M \nu(\alpha).$$

**Theorem 3.7** [M],[Li] *Under the conditions specified below we have*

$$\Phi_{C,d}^{X,g(H)} = (-1)^{(C^2 + K_X C)/2} \gamma_{C,d}^{X,H}.$$

**Conditions:**

1. Locally-free  $\mu$ -stable sheaves are dense in  $M$  (otherwise replace  $M$  by the closure of the locus of locally free sheaves).
2. Every  $L$  in  $Pic(S) \setminus \{0\}$  with  $L \equiv C \pmod{2}$  and  $LH = 0$  satisfies  $L^2 < -(4c_2 - C^2)$  (this means that  $H$  does not lie on a wall, see below).
3.  $M_H^X(C, c_2)$  has dimension  $4c_2 - c_1^2 - 3\chi(\mathcal{O}_X)$  and

$$\dim(M_H^X(C, n)) + 2(c_2 - n) < \dim(M_H^X(C, c_2)), \quad \text{for all } n < c_2.$$

4. If  $C \in 2H^2(X, \mathbb{Z})$  there is an extra condition; e.g. for  $\alpha \in Sym^d(H_2(X))$ , the condition is  $d > 2c_2 - C^2/2$ .

The point is that the classes  $\mu(\alpha)$  and  $\nu(\alpha)$  are related by  $\Psi^*$ :  $\Psi^*(\mu(\alpha)) = \nu(\alpha)$ . Furthermore the fundamental classes of  $M$  and  $N$  are related by  $\Psi_*$ : up to different sign convention  $\Psi_*([M]) = [N]$ . Then the theorem follows from the projection formula.

## 4 Flips of moduli spaces and wall-crossing for Donaldson invariants

In this and the next lecture we want to determine the dependence of the Donaldson invariants on the metric in the case  $b_+ = 1$  when they indeed depend on the metric. In this lecture we will restrict to the case of algebraic surfaces. In this case the change of metric corresponds to a change of ample divisor  $H$ . So we study how the moduli spaces  $M_H^X(C, c_2)$  vary with  $H$ . We will find out that under suitable assumptions, the variation is described by an explicit series of blow ups and blow downs, with centers projective bundles over Hilbert schemes of points. We use this to determine the change



of the Donaldson invariants as an explicit intersection number on a suitable Hilbert scheme of points on  $X$ . Finally one can compute the leading terms of this intersection number. We will follow mostly [E-G1]. A similar approach can be found in [Fr-Q].

#### 4.1 Walls and chambers

We start by reviewing general results about the dependence on the metric. Let  $X$  be a compact simply-connected differentiable 4-manifold.

In the case  $b_+(X) > 1$  the Donaldson invariants  $\Phi_{C,d}^{X,g}$  are independent of the metric  $g$  (as long as it is generic). Now assume  $b_+(X) = 1$ . In this case the Donaldson invariants will indeed depend on the metric  $g$ . Let  $H^2(X, \mathbb{R})^+$  be the set of all  $\alpha \in H^2(X, \mathbb{R})$  with  $\alpha^2 > 0$ . In fact the Donaldson invariants depend on  $g$  via a system of walls and chambers in  $H^2(X, \mathbb{R})^+$ .

We fix  $C \in H^2(X, \mathbb{Z})$  and  $d \in \mathbb{Z}_{\geq 0}$ . The positive cone  $H^2(X, \mathbb{R})^+ / \mathbb{R}^+$  has two connected components  $\Omega^+$  and  $\Omega^-$ . A homology orientation (i.e. the choice of an orientation on a maximal-dimensional linear subspace of  $H^2(X, \mathbb{R})$  on which the intersection form is positive definite), which is needed to define an orientation on the moduli space of ASD-connections, is equivalent to the choice of one of them, say  $\Omega^+$ .

**Definition 4.1** Let  $g$  be a Riemannian metric on  $X$ . The *period point*  $\omega(g)$  is the point in  $\Omega^+$  defined by the one-dimensional subspace of  $g$ -self-dual  $g$ -harmonic 2-forms. I.e these are the harmonic two forms  $\eta \in \Omega^2(X)$  with  $*_g \eta = \eta$ . By the Hodge theorem this is a 1-dimensional subspace of  $H^2(X, \mathbb{R})$ . An element  $\xi \in H^2(X, \mathbb{Z}) + C/2$  is called of *type*  $(C, d)$  if

$$(d + 3)/4 + \xi^2 \in \mathbb{Z}_{\geq 0}.$$

In this case

$$W^\xi := \{L \in \Omega^+ \mid \xi \cdot L = 0\}$$

is called the corresponding wall of type  $(C, d)$ . The *chambers of type*  $(C, d)$  are the connected components of complement of the walls of type  $(C, d)$  in  $\Omega^+$ .

It turns out that the Donaldson invariants with respect to the Fubini-Study metric corresponding to  $H$  depend only on the chamber of the period point of  $H$ .

**Theorem 4.2** [K-M]

1.  $\Phi_{C,d}^{X,g}$  depends only on the chamber (of type  $(C, d)$ ) of  $\omega(g)$ .
2. For all  $\xi$  of type  $(C, d)$  there exists a linear map  $\delta_{\xi,d}^X : A_d(X) \rightarrow \mathbb{C}$  such that

$$\Phi_{C,d}^{X,g_1} - \Phi_{C,d}^{X,g_2} = \sum_{\xi\omega(g_2) < 0 < \xi\omega(g_1)} (-1)^{(\xi-C/4)C} \delta_{\xi,d}^X.$$

**4.2 Interpretation of the walls in algebraic geometry**

Now let  $X$  be a simply-connected algebraic surface with geometric genus  $p_g = 0$  (this is equivalent to  $b_+(X) = (1 + 2p_g(X)) = 1$ ). Let  $H$  be an ample divisor on  $X$ . Let  $\mathcal{C}$  be the ample cone of  $X$ . We choose  $\Omega^+$  as the connected component of  $H^2(X, \mathbb{R})^+ / \mathbb{R}^+$ , which contains  $\mathcal{C} / \mathbb{R}^+$ . Then the period point of the Fubini-Study metric  $g(H)$  is  $\omega(g(H)) = \mathbb{R}^+ H \in \mathcal{C} / \mathbb{R}^+$ . Fix  $C \in H^2(X, \mathbb{Z})$  and  $c_2 \in \mathbb{Z}$ , such that  $d := 4c_2 - C^2 - 3$  is a nonnegative integer. By Section 3 we can compute  $\Phi_{C,d}^{X,g(H)}$  on  $M_H^X(C, c_2)$ . So we now need to know how  $M_H^X(C, c_2)$  depends on  $H$ .

Let  $E$  be a torsion-free rank 2 sheaf on  $X$  with Chern classes  $C$  and  $c_2$ . Let  $H_+$  and  $H_-$  be two ample line bundles on  $X$ , and assume that  $E$  is Gieseker stable with respect to  $H_-$ , but Gieseker unstable with respect to  $H_+$ . Then the Harder-Narasimhan filtration of  $E$  with respect to  $H_+$  gives an exact sequence

$$0 \rightarrow \mathcal{I}_{Z_1}(F) \rightarrow E \rightarrow \mathcal{I}_{Z_2}(G) \rightarrow 0,$$

where

1. The class  $\xi := (F - G)/2$  satisfies

$$\xi H_- < 0 < \xi H_+.$$

2.  $\mathcal{I}_{Z_1}$  and  $\mathcal{I}_{Z_2}$  are the ideal sheaves of 0-dimensional subschemes  $Z_1 \in X^{[n]}$  and  $Z_2 \in X^{[m]}$  and  $c_2(E) = FG + n + m$  or equivalently

$$c_2 - C^2/4 + \xi^2 = n + m \geq 0.$$

This means that  $\xi$  is a class of type  $(C, d)$  and there exists an ample line bundle  $H$  between  $H_+$  and  $H_-$  with  $\xi H = 0$ . In other words  $\xi$  defines a wall of type  $(C, d)$ , such that  $W^\xi$  intersects  $\mathcal{C}$ .

**Definition 4.3** Let  $E_\xi^{n,m}$  be the set of all sheaves  $E$  lying in extensions

$$0 \rightarrow \mathcal{I}_{Z_1}(F) \rightarrow E \rightarrow \mathcal{I}_{Z_2}(G) \rightarrow 0,$$

with  $\xi := (F - G)/2$ ,  $Z_1 \in X^{[n]}$ ,  $Z_2 \in X^{[m]}$ .

Then we conclude

1.  $M_H^X(C, c_2)$  depends only on the chamber of type  $(C, d)$  of  $H$ . In particular the Donaldson invariants are constant on each chamber.
- 2.

$$M_{H_-}^X(C, c_2) \setminus M_{H_+}^X(C, c_2) \subset \coprod_{\xi} \coprod_{n,m} E_\xi^{n,m}.$$

Here the sums run over all  $\xi$  of type  $(C, d)$  such that  $H_- < 0 < \xi H_+$ , and over all  $n, m \in \mathbb{Z}_{\geq 0}$  with  $n + m = c_2 - C^2/4 + \xi^2$ .

We would like to say that  $M_{H_+}^X(C, c_2)$  is obtained from  $M_{H_-}^X(C, c_2)$ , by removing the  $E_\xi^{n,m}$  and replacing them by the  $E_{-\xi}^{m,n}$  (for  $\xi$  classes of type  $(C, d)$  with  $\xi H_- < 0 < \xi H_+$ ). This however is not quite true. The problem is that  $E_\xi^{n,m}$  and  $E_{-\xi}^{l,r}$  can intersect, i.e. we can have a diagram

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & & & \mathcal{I}_{W_1}(G) & & & \\
 & & & \downarrow & \searrow & & \\
 0 & \longrightarrow & \mathcal{I}_{Z_1}(F) & \longrightarrow & E & \longrightarrow & \mathcal{I}_{Z_2}(G) \longrightarrow 0 \\
 & & & \searrow & \downarrow & & \\
 & & & & \mathcal{I}_{W_2}(F) & & \\
 & & & & \downarrow & & \\
 & & & & 0 & & 
 \end{array}$$

To deal with this, we need a finer notion of stability. We use: Gieseker stability is not invariant under tensorizing by a line bundle.

Assume  $H_-$  and  $H_+$  are separated by a unique wall  $W^\xi$  with  $\xi H_- < 0 < \xi H_+$ . Let  $H$  lie between  $H_-$  and  $H_+$  with  $H\xi = 0$ . If  $E$  is a torsion-free  $H$ -semistable sheaf of rank 2 with Chern classes  $C$  and  $c_2$ , then

1.  $E$  is  $H_-$ -semistable, if and only if  $E(l(H_- - H_+))$  is  $H$ -semistable for  $l \gg 0$ .

2.  $E$  is  $H_+$ -semistable, if and only if  $E(l(H_+ - H_-))$  is  $H$ -semistable for  $l \gg 0$ .

This gives us a finer notion of stability. By using a parabolic structure of length 1 (which essentially amounts to tensorizing with a fractional power of  $H_- - H_+$ ), we get moduli spaces

$$M_a, \quad a \in [-1, 1], \quad M_{-1} = M_{H_-}^X(C, c_2), \quad M_1 = M_{H_+}^X(C, c_2).$$

There are *miniwalls*  $a_i \in [-1, 1]$  such that for all  $i$  and  $0 < \epsilon \ll 1$

$$M_{a_i+\epsilon} = (M_{a_i-\epsilon} \setminus E_\xi^{n,m}) \amalg E_{-\xi}^{m,n}$$

for suitable  $n, m$  (for more details look at [E-G1]).

### 4.3 Flip construction

Now we want to see more precisely what happens to  $M_H^X(C, c_2)$  when  $H$  crosses a wall. We want to see that its change can be described by a sequence of blow ups and blow downs. For this we need an additional assumption which essentially guarantees that all the involved moduli spaces will be smooth near the locus where they change.

**Definition 4.4** A class  $\xi$  of type  $(C, d)$  defines a *good wall* if  $W^\xi$  contains ample divisors and  $2\xi + K_X$  and  $-2\xi + K_X$  are not effective. In particular if  $-K_X$  is effective, then all walls in the ample cone are good.

We want to describe the wall-crossing through a good wall defined by  $\xi$ . Let  $b$  be a miniwall, as above, and let

$$M_- := M_{b-\epsilon}, \quad M_+ := M_{b+\epsilon} = (M_- \setminus E_\xi^{n,m}) \amalg E_{-\xi}^{m,n}.$$

We write

$$E_- := E_\xi^{n,m}, \quad E_+ := E_{-\xi}^{m,n}.$$

Let  $T := X^{[n]} \times X^{[m]}$ , let  $\mathbb{Z}_1, \mathbb{Z}_2 \subset S \times T$  be the two universal families i.e.

$$\begin{aligned} \mathbb{Z}_1 &:= \{(x, Z, W) \in X \times T \mid x \in Z\} \\ \mathbb{Z}_2 &:= \{(x, Z, W) \in X \times T \mid x \in W\}. \end{aligned}$$

Let  $q$  be the projection  $S \times T \rightarrow T$ . We write  $C := F + G$ ,  $\xi := F - G/2$  and write  $\mathcal{F} := \mathcal{I}_{\mathbb{Z}_1}(F)$ ,  $\mathcal{G} := \mathcal{I}_{\mathbb{Z}_2}(G)$ , (these are sheaves on  $X \times T$  and we suppress the various pullbacks in the notation). Finally we write

$$\mathcal{A}_- := \text{Ext}_q^1(\mathcal{F}, \mathcal{G}), \quad \mathcal{A}_+ := \text{Ext}_q^1(\mathcal{G}, \mathcal{F}).$$

These are the relative Ext sheaves. Under our assumptions these sheaves are locally free and the fibers over a point  $(Z, W) \in T$  are

$$\mathcal{A}_-(Z, W) := \text{Ext}^1(\mathcal{I}_Z(F), \mathcal{I}_W(G)), \quad \mathcal{A}_+(Z, W) := \text{Ext}^1(\mathcal{I}_W(G), \mathcal{I}_Z(F)).$$

We denote the tautological sub-line bundles on  $\mathbb{P}(\mathcal{A}_-)$  and  $\mathbb{P}(\mathcal{A}_+)$  by  $\tau_-$  and  $\tau_+$ .

**Lemma 4.5** 1.  $\mathcal{A}_-$  is locally free of rank  $-\xi(2\xi - K_S) + n + m - 1$ .

2. The tautological extension

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{G}(\tau_-) \rightarrow 0$$

gives an isomorphism  $\mathbb{P}(\mathcal{A}_-) \simeq E_-$ .

3.  $N_{E_-/M_-} = \mathcal{A}_+(\tau_-)$ .

*Proof.* 1. follows from Riemann-Roch, because the condition of a good wall implies  $\text{Hom}_q(\mathcal{F}, \mathcal{G}) = 0$  and  $\text{Ext}_q^2(\mathcal{F}, \mathcal{G}) = 0$ .

2. is then easy.

For 3. we use that  $T_{M_-} = \text{Ext}_q^1(\mathcal{E}, \mathcal{E})$ , where  $\mathcal{E}$  is the universal sheaf on  $X \times M_-$ . Then one has to do some diagram-chasing.  $\square$

Part 3. of this lemma lets us hope that the blow up of  $M_-$  along  $E_-$  and the blow up of  $M_+$  along  $E_+$  might be the same. So let  $\widetilde{M}_-$  be the blow up of  $M_-$  along  $E_-$  and  $\widetilde{M}_+$  the blow up of  $M_+$  along  $E_+$ . Let  $D \simeq \mathbb{P}(\mathcal{A}_-) \times_T \mathbb{P}(\mathcal{A}_+)$  be the exceptional divisor. We see that  $\mathcal{O}(D)|_D = \mathcal{O}_D(\tau_- + \tau_+)$ .

**Theorem 4.6**  $\widetilde{M}_- = \widetilde{M}_+$ .

*Proof.* Let  $\mathcal{E}_-$  be the universal family on  $X \times M_-$  (and denote by the same symbol its pullback to  $\widetilde{M}_-$ ). Let  $\mathcal{E}_+$  be the kernel of the composition  $\mathcal{E}_- \rightarrow \mathcal{E}_-|_D \rightarrow \mathcal{G}(\tau_-)_D$ , where  $\mathcal{G}(\tau_-)_D$  is the pullback of  $\mathcal{G}(\tau_-)$  from  $S \times T$  to  $S \times D$ . So we define  $\mathcal{E}_+$  via elementary transform along the exceptional divisor  $D$ . Then we check that  $\mathcal{E}_+$  is a  $b + \epsilon$  stable family for  $1 \gg \epsilon > 0$ .

Thus  $\mathcal{E}_+$  defines a morphism  $\widetilde{M}_- \rightarrow M_+$ . It is not difficult to check that it is the blow up along  $E_+$ .  $\square$

So we see that  $M_{H_+}^X(C, c_2) = M_1$  is obtained from  $M_{H_-}^X(C, c_2) = M_{-1}$  via a sequence of blow ups along smooth subvarieties of the form  $E_\xi^{n,m}$  followed by a blow up of the exceptional divisor in another direction to  $E_{-\xi}^{m,n}$ .

#### 4.4 Computation of the wall-crossing

Now we want to compute the wall-crossing terms  $\delta_{\xi,d}^X$ . For simplicity we restrict to  $Sym^d(H_2(X))$ . Let  $a \in H_2(X)$ . Let  $b$  run through the miniwalls corresponding to  $\xi$  and write  $\widetilde{M}_b$  for the blow up  $\widetilde{M}_{b-\epsilon} = \widetilde{M}_{b+\epsilon}$ . From the definitions we see that

$$\delta_{\xi,d}^X(a^d) = \pm \left( \int_{M_{H_+}^X(C, c_2)} \nu(a)^d - \int_{M_{H_-}^X(C, c_2)} \nu(a)^d \right) = \sum_b \int_{\widetilde{M}_b} (\nu_+(a)^d - \nu_-(a)^d).$$

Here  $\nu_-(a) := (c_2(\mathcal{E}_-) - c_1(\mathcal{E}_-)^2/4)/a$ , and similarly for  $\nu_+$ . (The  $\mathcal{E}_-$  and  $\mathcal{E}_+$  and therefore the  $\nu_-$  and  $\nu_+$  also depend on  $b$ .)

Let us again put ourselves in the situation of the previous section:  $\widetilde{M}_-$  is the blow up of  $M_-$  along  $E_-$  and  $\widetilde{M}_+$  the blow up of  $M_+$  along  $E_+$ , and  $D$  is the exceptional divisor.

**Lemma 4.7** 1.  $\nu_+(a) - \nu_-(a) = -\langle \xi, a \rangle D$ .

2.  $\int_{\widetilde{M}_-} (\nu_+(a)^d - \nu_-(a)^d)$  is the evaluation of a suitable (explicitly computable) cohomology class on  $X^{[n]} \times X^{[m]}$ .

*Proof.* 1. Is an easy application of Riemann-Roch without denominators see [Fu] (which tells how to compute the Chern classes of sheaves supported on subvarieties).

2.

$$\nu_+(a)^d - \nu_-(a)^d = (\nu_+(a) - \nu_-(a))(\nu_+(a)^{d-1} + \dots + \nu_-(a)^{d-1})$$

is by 1. divisible by  $D$ , so we can view it as a class on  $D$ . We can then push the class from  $D$  down to  $T$ .  $\square$

Putting all this together and summing over all the miniwalls corresponding to a given wall  $\xi$  we obtain the following: Note that the various  $X^{[n]} \times X^{[m]}$  with  $m + n = l$  can be collected to  $(X \sqcup X)^{[l]} = \coprod_{n+m=l} X^{[n]} \times X^{[m]}$ .

**Theorem 4.8**

$$\delta_{\xi,d}^X(a^d) = \pm \sum_{b=0}^d 2^b \binom{d}{b} \langle \xi, a \rangle^{d-b} \cdot \int_{(X \sqcup X)^{[l]}} \alpha^b s_{2l-b}(\text{Ext}_q^1(\mathcal{I}_{\mathbb{Z}_1}, \mathcal{I}_{\mathbb{Z}_2} \otimes (\mathcal{O}(-2\xi) \oplus \mathcal{O}(-2\xi + K_X))))).$$

Here  $p$  and  $q$  are the projections of  $X \times (X \sqcup X)^{[l]}$  to  $X$  and  $(X \sqcup X)^{[n]}$  respectively and  $l := c_2 - C^2/4 + \xi^2$ .  $\mathbb{Z}_1$  and  $\mathbb{Z}_2 \subset X \times (X \sqcup X)^{[n]}$  are the universal families

$$\begin{aligned} \mathbb{Z}_1 &:= \{(x, Z, W) \in X \times (X \sqcup X)^{[l]} \mid x \in Z\}, \\ \mathbb{Z}_2 &:= \{(x, Z, W) \in X \times (X \sqcup X)^{[l]} \mid x \in W\}. \end{aligned}$$

$s_i(E)$  denotes the  $i$ -th Segre class, defined by

$$1 + s_1(E) + s_2(E) + \dots = 1/(1 + c_1(E) + c_2(E) + \dots)$$

and  $\alpha := q_*(p^* \alpha \cdot ([\mathbb{Z}_1] + [\mathbb{Z}_2]))$ . So we are reduced to a (very complicated) intersection computation on the Hilbert scheme of points on  $X$ . The intersection theory of  $X^{[n]}$  is in general not understood. It gets harder very fast as  $n$  grows. So in our case the difficulty of the computation depends on the number  $l := c_2 - C^2/4 + \xi^2$ . The intersection number above can be computed for  $l$  not too large, say  $l \leq 3$ . For  $l = 0$  we get for instance

$$\delta_{\xi,d}^X(a^d) = \pm \langle \xi, a \rangle^d.$$

There is an alternative way of carrying out the final step of the computation, i.e. the computation in the cohomology ring of the Hilbert scheme of points. Assume  $X$  is a blow up of  $\mathbb{P}_2$ . On  $\mathbb{P}_2$  we have actions of  $\mathbb{C}^*$  with finitely many fixpoints. We can do the blow up in such a way that  $X$  still carries an action of  $\mathbb{C}^*$  with finitely many fixpoints (at each step it is enough to only blow up fixpoints). This action lifts to an action on  $X^{[n]}$ , which still has only finitely many fixpoints. All the intersection numbers we have to compute for the wall-crossing are indeed intersection numbers of Chern classes of equivariant bundles for this action.

We can therefore apply the Bott residue formula. This allows us to compute the intersection numbers by looking at the weights of the action on the fibers of the equivariant bundles over the fixpoints. This gives an algorithm

for computing the wall-crossing for rational surfaces. We used this in [E-G2] to compute the Donaldson invariants of  $\mathbb{P}_2$  of degree  $\leq 50$ .

The fact that we can compute the Donaldson invariants of  $\mathbb{P}_2$ , where there are no walls might seem surprising. We use the blow up formulas (see the next lecture) to relate the Donaldson invariants of  $\mathbb{P}_2$  to those of the blow up of  $\mathbb{P}_2$  in a point. On this blow up we can then apply the wall-crossing in order to do the computation.

## 5 Wall-crossing and modular forms

Let  $X$  be a simply-connected 4-manifold with  $b_+(X) = 1$ . In this case the Donaldson invariants were first studied in [K]. In this lecture I want to give a generating function for the wall-crossing terms  $\delta_{\xi,d}^X$ . We will see that such a generating function can be found in terms of modular forms. This is the contents of the paper [G]. The strategy will be to compare the wall-crossing on  $X$  and on the connected sum of  $X$  with  $\mathbb{P}_2$  with the opposite orientation. This will give us recursion formulas for the  $\delta_{\xi,d}^X$ .

### 5.1 Ingredients

There are several ingredients which have to be put together in order to compute the generating function.

**(1) Kotschik-Morgan conjecture.** In their paper [K-M], where they show that the Donaldson invariants  $\Phi_{C,d}^{X,g}$  depend only on the chamber of the period point of the metric  $g$ , Kotschik and Morgan also make a conjecture about the structure of the wall-crossing terms  $\delta_{\xi,d}^X$ . For a class  $\xi \in H^2(X)$  and  $a \in H_2(X)$ , we denote by  $\langle \xi, a \rangle$  the pairing of  $H^2(X)$  with  $H_2(X)$  and by  $(a \cdot a)$  the intersection form on the middle homology  $H_2(X)$ .

**Conjecture 5.1** [K-M]  $\delta_{\xi,d}^X(a^d)$  is for  $a \in H_2(X)$  a polynomial in  $\langle \xi, a \rangle$  and  $(a \cdot a)$ , whose coefficients depend only on  $\xi^2$ ,  $d$  and the homotopy type of  $X$ .

In a series of papers Fehar and Leness are working on a proof of this conjecture [Fe-Le1],[Fe-Le2-4].

**(2) Blow up formulas.** The blow up formulas relate the Donaldson invariants of a 4-manifold  $X$  with those of the connected sum  $\widehat{X} := X \# \overline{\mathbb{P}}_2$  of  $X$  with  $\mathbb{P}_2$  with the opposite orientation. In the case that  $X$  is an algebraic



surface, we can take  $\widehat{X}$  to be the blow up of  $X$  in a point. In the case  $b_+(X) = 1$ , when the Donaldson invariants depend on the choice of a metric, we need to choose the metric on  $\widehat{X}$  to be very close to the pullback of a metric on  $X$ , in order to make the blow up formulas applicable. Let  $E$  be the class of the exceptional divisor, then  $H^2(\widehat{X}, \mathbb{R}) = H^2(X, \mathbb{R}) \oplus \mathbb{R}E$ . We will identify  $H^2(X, \mathbb{R})$  with the classes in  $H^2(\widehat{X}, \mathbb{R})$  orthogonal to  $E$ .

If  $L \in H^2(X, \mathbb{R})^+$  is (a representative of) the period point of the metric  $g$ , we write

$$\Phi_{C,d}^{X,L} := \Phi_{C,d}^{X,g}$$

For  $C \in H^2(\widehat{X}, \mathbb{Z})$ ,  $H \in H^2(X, \mathbb{R})^+$ , we write

$$\Phi_{C,d}^{\widehat{X},H} := \Phi_{C,d}^{\widehat{X},H-\epsilon E}$$

for  $0 < \epsilon \ll 1$ . (This will be independent of  $\epsilon$  for sufficiently small  $\epsilon > 0$ .)

**Theorem 5.2** *Let  $C \in H^2(X, \mathbb{Z})$ ,  $a \in H_2(X)$ ,  $H \in H^2(X, \mathbb{R})^+$ . We write  $e \in H_2(\widehat{X}, \mathbb{Z})$  for the Poincaré dual of  $E$ . Then*

1.  $\Phi_{C,d}^{\widehat{X},H}(a^d) = \Phi_{C,d}^{X,H}(a^d)$ .
2.  $\Phi_{C+E,d+1}^{\widehat{X},H}(e a^d) = \Phi_{C,d}^{X,H}(a^d)$ .
3.  $\Phi_{C,d}^{\widehat{X},H}(e^2 a^{d-2}) = 0$ .

This result holds also if  $b_+ > 1$ . In fact this is the case in which it was originally proved. In the case  $b_+ > 1$  the Donaldson invariants are independent of the metric, so one does not have to worry about which metric to choose on  $\widehat{X}$  for a given metric on  $X$ .

More generally Fintushel and Stern [F-S] found generating functions for the blow up formulas: Let  $p \in H_0(X)$  be the class of a point. Then there are power series

$$B(x, t) = \sum_k B_k(x) t^k,$$

$$S(x, t) = \sum_k S_k(x) t^k,$$

such that

$$\Phi_C^{\widehat{X},H}(a^d e^k) = \Phi_C^{X,H}(a^d B_k(p)),$$

$$\Phi_{C+E}^{\widehat{X},H}(a^d e^k) = \Phi_C^{X,H}(a^d S_k(p)).$$

$B(x, t)$  and  $S(x, t)$  can be expressed in terms of elliptic functions, e.g.  $S(x, t) = e^{-t^2 x/6} \sigma(t)$ , where  $\sigma$  is the Weierstrass  $\sigma$  function.

**(3) Vanishing results.** The last ingredient is that in certain cases (for rational ruled surfaces) the Donaldson invariants vanish. This will give a starting point for the calculations.

**Lemma 5.3** *Let  $X$  be a rational ruled surface. Let  $F$  be the class of a fiber and assume  $CF = 1$ . Let  $H$  be an ample divisor on  $X$ . Then  $M_{F+\epsilon H}^X(C, c_2) = \emptyset$  for  $0 < \epsilon \ll 1$ . In particular  $\Phi_{C,d}^{X, F+\epsilon H} = 0$  for  $0 < \epsilon \ll 1$ .*

More generally the following holds: Let  $f : X \rightarrow C$  be a surjective morphism of an algebraic surface to a curve. Let  $F$  be the class of a fiber and let  $H$  be ample on  $X$ . Then a vector bundle  $E$  over  $X$  is semistable with respect to  $F + \epsilon H$  if and only if the restriction of  $E$  to the generic fiber of  $f$  is semistable. This fact is also e.g. used by Friedman to study the Donaldson invariants of elliptic surfaces.

## 5.2 The result

Our aim is to show:

**Theorem 5.4** *Let  $a \in H_2(X)$  and let  $t$  be a variable. Then*

$$\delta_\xi^X(\exp(at)) = \text{Coeff}_{q^0} \left[ f(\tau) R(\tau) \theta(\tau)^{\sigma(X)} q^{-\xi^2/2} \exp \left( -\frac{\langle \xi, a \rangle t}{f(\tau)} - \frac{(a \cdot a) G(\tau) t^2}{f(\tau)^2} \right) \right].$$

Here  $\sigma(X)$  is the signature of  $X$ . For the rest of the notations I briefly review modular forms.

**Review of modular forms:** Let  $\mathbb{H} := \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$  be the complex upper half plane. For  $\tau$  in  $\mathbb{H}$  we denote  $q := e^{2\pi i \tau}$ . The group  $SL_2(\mathbb{Z})$  acts on  $\mathbb{H}$  by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

A function  $g : \mathbb{H} \rightarrow \mathbb{C}$  is called a modular form of weight  $k$  for  $SL_2(\mathbb{Z})$ , if

$$g\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),$$

and furthermore  $g$  has a  $q$ -development

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n.$$

One can associate an elliptic curve  $E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$  to  $\tau \in \mathbb{H}$ , and  $E_\tau$  and  $E_{\tau'}$  are isomorphic if and only if  $\tau$  and  $\tau'$  are related by an element of  $SL_2(\mathbb{Z})$ . Therefore modular forms are related to moduli of elliptic curves.

One can also talk about modular forms for subgroups  $\Gamma$  of finite index of  $SL_2(\mathbb{Z})$ . In this case one requires the transformation behavior only for the elements in  $\Gamma$  and the requirement on the  $q$ -development has to be modified.

All the functions appearing in the theorem are (related to) modular forms.

$$\Delta(\tau) := q \prod_{k=1}^{\infty} (1 - q^k)^{24}$$

is the discriminant, a modular form for  $SL_2(\mathbb{Z})$ .

$$\eta(\tau) = \Delta(\tau)^{1/24}$$

is the Dirichlet eta-function.

$$\theta(\tau) := \sum_{n \in \mathbb{Z}} q^{n^2/2}$$

is the theta function for  $\mathbb{Z}$ .

$$G_2(\tau) := -\frac{1}{24} + \sum_{n=1}^{\infty} \left( \sum_{d|n} d \right) q^n$$

is an Eisenstein series and

$$e_3(\tau) := \frac{1}{12} + \sum_{n=1}^{\infty} \left( \sum_{d|n, d \text{ odd}} d \right) q^n,$$

the value of the Weierstrass  $\wp$ -function at one of the two-division points. We put

$$\begin{aligned} R(\tau) &:= \frac{\Delta(\tau)^2}{\Delta(\tau/2)\Delta(2\tau)}, \\ f(\tau) &:= e^{-\pi i/4} \frac{\eta(\tau)^3}{\theta(\tau)}, \\ G(\tau) &:= G_2(\tau) + e_3(\tau)/2. \end{aligned}$$

As a corollary to this result we can compute all the Donaldson invariants of the projective plane  $\mathbb{P}_2$ . The projective plane is in some respects the simplest algebraic surface. Therefore, if one wants to understand the Donaldson invariants of algebraic surfaces, one should at least be able to compute them for  $\mathbb{P}_2$ .

Let  $H \in H^2(\mathbb{P}_2, \mathbb{Z})$  be the hyperplane class, and let  $h$  be its Poincaré dual.

### Corollary 5.5

$$\Phi_H^{\mathbb{P}_2, H}(\exp(ht)) = \text{Coeff}_{q^0} \left[ f(\tau)R(\tau) \sum_{a \geq n > 0} (-1)^{n+\frac{1}{4}} q^{\frac{1}{2}((a \cdot a) - (n - \frac{1}{2})^2)} \exp\left(-\frac{(n + \frac{1}{2})t}{f(\tau)} - \frac{G(\tau)t^2}{f(\tau)^2}\right) \right].$$

There is a similar formula for  $\Phi_0^{\mathbb{P}_2, H}$ .

*Proof.* The blow up  $X$  of  $\mathbb{P}_2$  at a point is a ruled surface, the class of the fiber is  $F = H - E$ . So we get  $\Phi_H^{X, F+\epsilon H} = 0$  by the vanishing result above. On the other hand the blow up formulas give that  $\Phi_H^{\mathbb{P}_2, H}(h^d) = \Phi_H^{X, H}(h^d)$ , and the last can be computed by adding all the wall-crossing terms  $\delta_{\xi, d}^X$  for all classes  $\xi$  of type  $(H, d)$  with  $\xi H > 0 > \xi F$ .  $\square$

### 5.3 Proof of the theorem

Now I want to sketch the proof of the theorem. The idea is as follows: We want to relate the wall-crossing on  $X$  and its blow up  $\widehat{X}$ . So fix  $C \in H^2(X, \mathbb{Z})$  and let  $\xi$  define the only wall of type  $(C, d)$  on  $X$  between  $H_-$  and  $H_+$ . Instead of directly applying the wall-crossing formula for the wall  $W^\xi$ , we can also first apply the blow up formulas, then cross all the walls between  $H_-$  and  $H_+$  on  $\widehat{X}$  and then apply the blow up formula again to get back to  $X$ . This gives us two different ways to compute the wall-crossing term  $\delta_\xi^X$ , which will give us recursion formulas.

By definition we see that the classes  $\eta$  of type  $(C, d)$  on  $\widehat{X}$  with  $H_- \eta < 0 < H_+ \eta$  are precisely the

$$\eta = \xi + nE, \quad n \in \mathbb{Z}, \quad n^2 \leq (d+3)/4 + \xi^2,$$

and the classes of type  $(C + E, d + 1)$  are precisely the

$$\eta = \xi + (n + 1/2)E, \quad n \in \mathbb{Z}, \quad (n + 1/2)^2 \leq (d + 4)/4 + \xi^2.$$

We write

$$\delta_\xi^X := \sum_{d \geq 0} \delta_{\xi, d}^X.$$

Then together with the above discussion the blow up formulas give:

$$\delta_\xi^X(a^d) = \sum_{n \in \mathbb{Z}} \delta_{\xi+nE}^{\widehat{X}}(a^d), \quad (5.0.1)$$

$$\delta_\xi^X(a^d) = \sum_{n \in \mathbb{Z}} (-1)^{n-1} \delta_{\xi+(n+1/2)E}^{\widehat{X}}(e a^d), \quad (5.0.2)$$

$$0 = \sum_{n \in \mathbb{Z}} \delta_{\xi+nE}^{\widehat{X}}(e^2 a^{d-2}). \quad (5.0.3)$$

Now we use the Kotschick-Morgan conjecture. Let  $X(b)$  be the blow up of  $X$  in  $b$  points. The Kotschick-Morgan conjecture allows us to write

$$\delta_\xi^{X(b)}(a^d/d!) = \sum_{l+2k=d} \frac{\langle \xi, a \rangle^l}{l!} \frac{(a \cdot a)^k}{k!} P(l, k, b, \xi^2),$$

for universal constants  $P(l, k, b, w)$  for  $l, k, b \in \mathbb{Z}$ ,  $w \in \mathbb{Z}/4$ . Then the relations (5.0.1), (5.0.2), (5.0.3) imply in turn

$$P(l, k, b, w) = \sum_{n \in \mathbb{Z}} P(l, k, b + 1, w - n^2), \quad (5.0.4)$$

$$P(l, k, b, w) = \sum_{n \in \mathbb{Z}} (-1)^n (n + 1/2) P(l, k, b + 1, w - (n + 1/2)^2), \quad (5.0.5)$$

$$\sum_{n \in \mathbb{Z}} n^2 P(l, k, b, w - n^2) = 2 \sum_{n \in \mathbb{Z}} P(l, k, b, w - n^2). \quad (5.0.6)$$

Now we put

$$\Lambda_X := \sum_{l, k, b, w} P(l, k, b, w) q^{w/2} \frac{L^l Q^k t^b}{l! k! b!},$$

for variables  $q, L, Q, t$ . We see that we have encoded all the information about the wall-crossing formulas into the generating function  $\Lambda_X$ . So our task is to determine  $\Lambda_X$  explicitly.

The formulas (5.0.4), (5.0.5), (5.0.6) for the  $P(l, k, b, w)$  translate into the following differential equations for  $\Lambda_X$ .

$$\begin{aligned}\theta(\tau) \frac{\partial}{\partial t} \Lambda_X &= \Lambda_X, \\ \eta(\tau)^3 \frac{\partial}{\partial L} \frac{\partial}{\partial t} \Lambda_X &= \Lambda_X, \\ 2\theta(\tau) \frac{\partial}{\partial Q} \Lambda_X &= \left( q \frac{\partial}{\partial q} \theta(\tau) \right) \frac{\partial^2}{\partial L^2} \Lambda_X.\end{aligned}$$

These differential equations are trivial to solve: Writing

$$\lambda_X(q) := \Lambda_X(0, 0, 0, q)$$

we get

$$\Lambda_X = \exp\left(-\frac{L}{f(\tau)} - \frac{Q G(\tau)}{f(\tau)^2} + \frac{t}{\theta}\right) \lambda_X(q).$$

Finally we need to determine  $\lambda_X(q)$ . It is enough to do this in the case  $X = \mathbb{P}_1 \times \mathbb{P}_1$ : For every simply-connected 4-manifold with  $b_+ = 1$ , the blow up  $Y$  of  $X$  in two points is homotopy-equivalent to the blow up of  $\mathbb{P}_1 \times \mathbb{P}_1$  in a number of points. The Kotschick-Morgan conjecture says that the wall-crossing terms only depend on the homotopy type of  $X$ .

Let  $F$  and  $G$  be the fibers of the two projections of  $\mathbb{P}_1 \times \mathbb{P}_1$  onto its factors. By the vanishing result we get

$$\Phi_{F+G, d}^{\mathbb{P}_1 \times \mathbb{P}_1, F+\epsilon G} = \Phi_{F+G, d}^{\mathbb{P}_1 \times \mathbb{P}_1, G+\epsilon F} = 0.$$

Therefore the sum of all the wall-crossing terms for all the walls between  $F$  and  $G$  has to vanish. This is enough to determine all the coefficients of  $\Lambda_{\mathbb{P}_1 \times \mathbb{P}_1}(q)$ . This part of the calculation is slightly more difficult and involves some tricks with modular forms.

## 5.4 Further results

This result has later been used in [G-Z] to prove structure theorems (like those of Kronheimer and Mrowka in the  $b_+ > 1$  case) also for manifolds with  $b_+ = 1$ . These results work when one takes the limit of the Donaldson invariants  $\Phi_C^{X, H}$  as  $H$  tends to a class  $F$  with  $F^2 = 0$ . We write  $\Phi_C^{X, F}$  for this limit.

We get for instance the following: Let  $X$  be a rational elliptic surface (i.e. the blow up of  $\mathbb{P}_2$  in the 9 points of intersection of two smooth cubics). Let  $F$  be the class of a fiber. Then for all  $a \in H_2(X)$  we get

$$\Phi_H^{X,F}(e^{at}(1 + p/2)) = -\frac{e^{(a \cdot a)t^2/2}}{\cosh(\langle F, a \rangle t)}.$$

To prove such results, one has to sum over all walls between two classes  $F, G$  with  $F^2 = G^2 = 0$ . These sums organize themselves into theta functions, and somewhat complicated arguments with modular forms will give the result.

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