Holomorphic bundles over elliptic manifolds

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Contents

Introduction:

In this series of lectures we shall examine holomorphic bundles over compact elliptically fibered manifolds. We shall examine constructions of such bundles as well as (duality) relations between such bundles and other geometric objects, namely $K3$-surfaces and del Pezzo surfaces.

We shall be dealing throughout with holomorphic principal bundles with structure group $G_{\mathbf{C}}$ where G is a compact, simple (usually simply connected) Lie group and $G_{\mathbf{C}}$ is the associated complex simple algebraic group. Of course, in the special case $G = SU(n)$ and hence $G_{\mathbf{C}} = SL_n(\mathbf{C})$, we are considering holomorphic vector bundles with trivial determinant. In the other cases of classical groups, $G = SO(n)$ or $G = \mathrm{Sympl}(2n)$ we are considering holomorphic vector bundles with trivial determinant equipped with a non-degenerate symmetric, or skew symmetric pairing. In addition to these classical cases there are the finite number of exceptional groups. Amazingly enough, motivated by questions in physics, much interest centers around the group E_8 and its subgroups. For these applications it does not suffice to consider only the classical groups. Thus, while often first doing the case of $SU(n)$ or more generally of the classical groups, we shall extend our discussions to the general semi-simple group. Also, we shall spend a good deal of time considering elliptically fibered manifolds of the simplest type – namely, elliptic curves.

The basic references for the material covered in these lectures are:

1. M. Atiyah, *Vectors bundles over an elliptic curve,* Proc. London Math. Soc. **7** (1967) 414-452.

2. R. Friedman, J. Morgan, E. Witten, *Vector bundles and F-theory,* Commun. Math. Phys. **187** (1997) 679-743.

3. ——, *Principal G-bundles over elliptic curves,* Math. Research Letters **5** (1998) 97-118.

4. ——, *Vector bundles over elliptic fibrations,* J. Alg. Geom. **8** (1999) 279-401.

1 Lie Groups and Holomorphic Principal $G_{\mathbf{C}}$-bundles

In this lecture we review the classification of compact simple groups or equivalently of complex linear simple groups. Then we turn to a review of the basics of holomorphic principal bundles over complex manifolds. We finish the section with a discussion of isomorphism classes of G-bundles (G compact) over the circle.

1.1 Generalities on roots, the Weyl group, etc.

A good general reference for Root Systems, Weyl groups, etc. is [3]. Let G be a compact group and T a maximal torus for G. Of course, T is unique up to conjugation in G. The rank of G is by definition the dimension of T. We denote by W the Weyl group of T in G, i.e., the quotient of the subgroup of G conjugating T to itself modulo the normal subgroup of elements commuting with T (which is T itself). This is a finite group. We denote by $\mathfrak{g}$ the Lie algebra of G and by $\mathfrak{t} \subset \mathfrak{g}$ the Lie algebra of T. The group G acts on its Lie algebra $\mathfrak{g}$ by the adjoint representation. The exponential mapping $\exp\colon \mathfrak{t} \to T$ is a covering projection with kernel $\Lambda \subset \mathfrak{t}$, where Λ is the fundamental group of T. The adjoint action of W on $\mathfrak{t}$ covers the conjugation action of W on T.

The complexification $\mathfrak{g}_{\mathbf{C}}$ decomposes into the direct summand of subspaces invariant under the conjugation action of the maximal torus T. The subspace on which the action is trivial is the complexification of the Lie algebra $\mathfrak{t}$ of the maximal torus. All other subspaces are one dimensional and are called the *root spaces*. The non-trivial character by which the torus acts on a root space is called a *root* of G (with respect to T) for this subspace. By definition the roots of G are non-zero elements of the character group (dual group) of T. This character group is a free abelian group of dimension equal to the rank of G. In the case when G is semi-simple, the roots of G span a subgroup of finite index inside entire character group. When G is not semi-simple the center of G is positive dimensional, and the roots span a sublattice of the group of characters of codimension equal to the dimension of the center of G.

Equivalently, the roots can be viewed as elements of the dual space $\mathfrak{t}^*$ of the Lie algebra of T, taking integral values on Λ. Since all the root spaces are one-dimensional, we see that the dimension of G as a group is equal to the rank of G plus the number of roots of G.

The collection of all roots forms an algebraic object inside $\mathfrak{t}^*$ called a *root system*. By definition a root system on a vector space V is a finite subset $\Phi \subset V^*$ of roots such that for each root $a \in /\Phi$ there is a dual coroot $/a^\vee \in V$ such that the 'reflection' $r_a\colon V^* \to V^*$ defined by $r_a(b) = b - \langle b, a^\vee \rangle a$

normalizes the set Φ. It is easy to see that if such $a^\vee$ exists then it is unique and furthermore that the $\{r_a\}_{a\in\Phi}$ generate a finite group. The element r_a is called the *reflection in the wall perpendicular to a*. The wall, denoted W_a^*, fixed by r_a is simply the kernel of the linear map $a^\vee$ on V^*. The group W generated by these reflections is the Weyl group. Dually we can consider the wall $W_a \subset V$ determined by $a = 0$. There is the dual reflection $r_a: V \to V$ given by the formula $r_a(v) = v - \langle a, v\rangle a^\vee$. These reflections generate the adjoint action of W on V. The set $\Phi \subset V^*$ generates a lattice in V^* called the *root lattice* and denoted Λ^{root}. Dually the coroots span a lattice in V called the coroot lattice and denoted Λ. Back to the case of a root system of a Lie group G with maximal torus T, since any root of G must take integral values on $\pi_1(T)$ we see that the lattice $\pi_1(T) \subset \mathfrak{t}$ is contained in the integral dual of the root lattice. Furthermore, one can show that the coroot lattice Λ is contained in $\pi_1(T)$.

The walls $\{W_a\}$ divide V into regions called Weyl chambers. Each chamber has a set of walls. We say that a set of roots is a set of *simple roots* if (i) the walls associated with the roots are exactly the walls of some chamber C, and (ii) the roots are non-negative on this chamber. It turns out that a set of simple roots is in fact an integral basis for the root lattice. Also, every root is either a non-negative or a non-positive linear combination of the simple roots, and roots are called positive or negative roots depending on the sign of the coefficients when they are expressed as a linear combination of the simple roots. (Of course, these notions are relative to the choice of simple roots.)

Since the Weyl group action on V^* is finite, there is a Weyl-invariant inner product on V^*. This allows us to identify V and V^* in a Weyl-invariant fashion and consider the roots and coroots as lying in the same space. When we do this the relative lengths of the simple roots and the angles between their walls are recorded in a Dynkin diagram which completely classifies the root system and also the Lie algebra up to isomorphism. If the group is simple, then this inner product is unique up to a positive scalar factor. In general, we can use this inner product to identify $\mathfrak{t}$ and $\mathfrak{t}^*$. When we do we have $\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$.

There is one set of simple roots for each Weyl chamber. It is a simple geometric exercise to show that the group generated by the reflections in the walls acts simply transitively on the set of Weyl chambers, so that all sets of simple roots are conjugate under the Weyl group, and the stabilizer in the Weyl group of a set of simple roots is trivial. Thus, the quotient $\mathfrak{t}/W$ is identified with any Weyl chamber.

A Lie algebra is said to be *simple* if it is not one-dimensional and has no non-trivial normal subalgebras. A Lie algebra that is a direct sum of simple algebras is said to be *semi-simple*. A Lie algebra is semi-simple if and only if it has no non-trivial normal abelian subalgebras. A compact Lie group is said to be simple, resp., semi-simple, if and only if its Lie algebra is simple, resp., semi-simple. Notice that a simple Lie group is not necessarily simple as a group. It can have a non-trivial (finite) center, when then produces finite normal subgroups of G. These are the only normal subgroups of G if G is simple as a Lie group.

A simply connected compact semi-simple group is a product of simple groups. In general a compact semi-simple group is finitely covered by a product of simple groups.

There are a finite number of simple Lie groups with a given Lie algebra $\mathfrak{g}$. All are obtained in the following fashion. There is exactly one simply connected group G with $\mathfrak{g}$ as Lie algebra. For this group the fundamental group of the maximal torus is identified with the coroot lattice Λ of the group. This group has a finite center $\mathcal{C}G$ which is in fact identified with the dual to the root lattice modulo the coroot lattice. Any other group with the same Lie algebra is of the form G/C where $C \subset \mathcal{C}G$ is a subgroup. Thus, the fundamental group of the maximal torus of G/C is a lattice in $\mathfrak{t}$ containing the coroot lattice and contained in the dual to the root lattice. Its quotient by the coroot lattice is C.

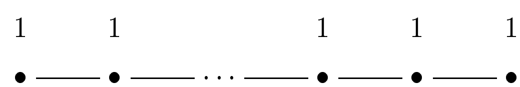
Each compact simple group G embeds as the maximal compact subgroup of a simple complex linear group $G_{\mathbf{C}}$. The Lie algebra of $G_{\mathbf{C}}$ is the complexification of the Lie algebra of G and there are maximal complex tori of $G_{\mathbf{C}}$ containing maximal tori of G as maximal compact subgroups. For example, the complexification of $SU(n)$ is $SL_n(C)$ whereas the complexification of $SO(n)$ is the complex special orthogonal group $SO(n, \mathbf{C})$. We can recover the compact group from the semi-simple complex group by taking a maximal compact subgroup (all such are conjugate in the complex group and hence are isomorphic). Any such maximal compact subgroup is called the *compact form* of the group. The fundamental group of a complex semi-simple group is the same as the fundamental group of its compact form.

1.2 Classification of simple groups

Let us look at the classification of such objects.

1.2.1 Groups of A_n-type

The first series of groups is the series $SU(n+1)$, $n \geq 1$. These are the groups of A_n-type. The maximal torus of $SU(n+1)$ is usually taken to be the group of all diagonal matrices with entries in S^1 and the product of the entries being one. We identify this in the obvious way with $\{(\lambda_1, \dots, \lambda_{n+1}) | \prod_{i=1}^{n+1} \lambda_i = 1\}$. The rank of $SU(n+1)$ is n. The Lie algebra $\mathfrak{su}(n)$ is the space of matrices of trace zero, and the root space $\mathfrak{g}^{ij}; 1 \leq i, j \leq n, i \neq j$ consists of matrices with non-trivial entry only in the ij position. The root associated to this root space is denoted α_{ij} and is given by $\alpha_{ij}(\lambda_1, \dots, \lambda_{n+1}) = \lambda_i \lambda_j^{-1}$. Writing things additively, we identify $\mathfrak{t}$ with $\{(z_1, \dots, z_{n+1}) | \sum_i z_i = 0\}$ and then $\alpha_{ij} = e_i - e_j$ where e_i is the linear map which is projection onto the i^{th} coordinate. The usual choice of simple roots are $\alpha_{12}, \alpha_{23}, \dots, \alpha_{nn+1}$. With this choice the positive roots are the α_{ij} where $i < j$. When $i < j$ we have $\alpha_{ij} = \alpha_{i(i+1)} + \cdots + \alpha_{(j-1)j}$. The Weyl group is the symmetric group on $n+1$ letters. This group acts in the obvious way on $\mathbf{R}^{n+1}$ and leaves invariant the subspace we have identified with $\mathfrak{t}$. This is the Weyl group action on $\mathfrak{t}$. In particular, the restriction of the standard inner product on $\mathbf{R}^{n+1}$ to $\mathfrak{t}$ is Weyl invariant. Also, it is easy to see that all the roots are conjugate under the Weyl action. Notice that each simple root has length 2 and meets the previous simple root and the succeeding simple root (in the obvious ordering) in -1, and is perpendicular to all other simple roots. All this information is recorded in the Dynkin diagram for A_n. Since each root has length two, under the induced identification of $\mathfrak{t}$ with $\mathfrak{t}^*$ every root α is identified with its coroot $\alpha^\vee$. Thus, in this case, and as we shall see, in all other simply laced cases, one can identify the roots and coroots and hence their lattices.



The center $\mathcal{C}(SU(n+1))$ is the cyclic group of order $n+1$ consisting of diagonal matrices with diagonal entry ζ an $n+1$ root of unity. Thus for each cyclic subgroup of $\mathbf{Z}/(n+1)\mathbf{Z}$ there is a form of $SU(n)$ with fundamental group this cyclic subgroup. The full quotient $SU(n+1)/\mathcal{C}SU(n+1)$ is often called $PU(n)$. The intermediate quotients are not given names.

1.2.2 Groups of B_n-type

For any $n \geq 3$, the group $Spin(2n+1)$ is a simple group of type B_n. The standard maximal torus of this group is the subgroup of matrices that project into $SO(2n+1)$ to block diagonal matrices $SO(2) \times SO(2) \times \cdots \times SO(2) \times \{1\}$. The Lie algebra of this torus is naturally a product of n copies of $\mathbf{R}$, one for each $SO(2)$, and we let e_i be the projection of $\mathfrak{t}$ onto the i^{th}-factor. The Lie algebra is all skew symmetric matrices. For any $1 \leq i, j \leq n$ let $\mathfrak{g}^{ij}, 1 \leq i < j \leq n, i \neq j$ be the subspace of skew symmetric matrices with non-zero entries only in places $(2i-1, 2j-1), (2i-1, 2j), (2i, 2j-1), (2i, 2j)$ and the symmetric lower diagonal positions. This is a four-dimensional subspace of the Lie algebra of $SPin(2n+1)$. Thus, there are four roots associated with this space, they are $\pm e_i \pm e_j$. There are also subspaces $\mathfrak{g}^i; 1 \leq i \leq n$, where $\mathfrak{g}^i$ has non-zero entries only in positions $(2i-1, 2n+1)$ and $(2i, 2n+1)$ as well as the symmetric lower diagonal positions. The two roots associated to $\mathfrak{g}^i$ are $\pm e_i$. Thus, $Spin(2n+1)$ is of rank n. We can identify the Lie algebra of its maximal torus with $\mathbf{R}^n$ in such a way that the roots are $\pm e_i \pm e_j$ for $i \neq j$ and $\pm e_i$, where e_i is the projection onto the i^{th} coordinate. The dual coroots to these roots are $\pm e_i \pm e_j$ and $\pm 2e_i$, so that the coroot lattice Λ for $Spin(2n+1)$ is the even integral lattice in $\mathbf{R}^n$, whereas the fundamental group of the maximal torus of $SO(2n+1)$ is the full integral lattice $\mathbf{Z}^n$. Of course, the quotient $\Lambda/\mathbf{Z}^n = \mathbf{Z}/2\mathbf{Z}$ is the fundamental group of $SO(2n+1)$. The Weyl group of $Spin(2n+1)$ is the group generated by reflections in the simple roots. It is easy to see that these reflections generate the group of all permutations of the coordinates and also all sign changes of the coordinates. Thus, abstractly the group is $(\{\pm 1\}^n \times \Sigma_n$. Clearly, the standard metric on $\mathbf{R}^n$ is a Weyl invariant metric. Notice that something new happens here – not all the roots have the same length. In our normalization $\pm e_i \pm e_j$ has length squared 2, whereas $\pm e_i$ has length squared 1. In particular, not all the roots are conjugate under the action of the Weyl group. In this case there are two orbits – one orbit for each length. This fact is expressed by saying that the group is *not simply laced.*

Since the center of $Spin(2n+1)$ is the cyclic group of order 2, there are only two groups with this Lie algebra $spin(2n+1)$ and $SO(2n+1)$.

The Dynkin diagram of type B_n is

1 2 2 2 2 2

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1.3 Groups of C_n-type

Let J be the $2n \times 2n$ matrix of block 2×2 diagonal entries

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The symplectic group consists of all matrices A such that $A^{tr} J A = J$. These are the linear transformations that leave invariant the standard skew symmetric pairing on $\mathbf{R}^{2n}$, the one given by J. The Lie algebra of this group consists of all matrices A such that

$$A^{tr} = -JAJ^{-1} = JAJ.$$

For any $n \geq 2$ the symplectic group $\mathbf{Sympl}(2n)$ is a group of type C_n so that the complex symplectic group is a complex semi-simple group. There is a complication in that the real symplectic group is non-compact; it is rather what is called the $\mathbf{R}$-split form of the group. Its maximal algebraic torus is a product of n copies of $\mathbf{R}^*$ and is given by the group of diagonal matrices with diagonal entries $(\lambda_1, \lambda^{-1}, \ldots, \lambda_n, \lambda_n^{-1})$. For example, $Sympl(2)$ is identified with $SL_2(\mathbf{R})$. By general theory there is a compact form for the complex symplectic group $Sympl_{\mathbf{C}}(2n)$. It is given as the group of quaternion linear transformations of $\mathbf{H}^n$, so that as one would expect, the compact form of $Sympl_{\mathbf{C}}(2)$ is $SU(2)$. The maximal torus of this group is again a product of circles so that $\mathfrak{t}$ is again identified with $\mathbf{R}^n$. The roots are $\pm e_i \pm e_j$ for $1 \leq i < j \leq n$ and $\pm 2e_i$. Thus, once again the group is non-simply laced. Its Weyl group is the same as the Weyl group of B_n. The coroots dual to the roots are $\pm e_i \pm e_j$ and $\pm e_i$ so that the coroot lattice Λ is the integral lattice $\mathbf{Z}^n$. The dual to the root lattice consists of all $\{x_1, \ldots, x_n\}$ such that $x_i \in (1/2)\mathbf{Z}$ for all i and $x_i \cong x_j \pmod{\mathbf{Z}}$ for all i, j. This lattice contains the coroot lattice with index two, so that the center of the simply connected form of this group is $\mathbf{Z}/2\mathbf{Z}$ and there is one non-simply connected form of these groups.

The Dynkin diagram of type C_n is

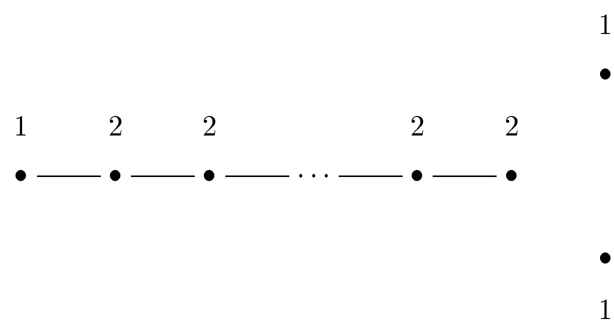
$$\begin{matrix} 1 & & 1 & & & & 1 & & 1 & & 1 \\ \bullet & — & \bullet & — & \cdots & — & \bullet & — & \bullet & =\!<\!= & \bullet \end{matrix}$$

It turns out that $Spin(5)$ is isomorphic to $\mathbf{Sympl}(4)$ which is why we start the B-series at $n = 3$. The group $\mathbf{Symp}(2)$ is isomorphic to $SU(2)$ which is why we begin the C-series at $n = 2$.

1.3.1 Groups of D_n-type

For any $n \geq 4$ the group $Spin(2n)$ is a group of type D_n. The usual maximal torus for $Spin(2n)$ is the subgroup that projects onto $SO(2) \times \cdots \times SO(2) \subset SO(2n)$. Thus, $\mathfrak{t}$ is identified with $\mathbf{R}^{2n}$ with the factors being tangent to the factors in this decomposition. The roots of $Spin(2n)$ are $\pm e_i \pm e_j$ for $1 \leq i < j \leq n$. This group is simply laced and in the given Weyl invariant inner product all roots have length $\sqrt{2}$. Thus, we can identify the roots with their dual coroots in this case. The coroot lattice is then the even integral lattice. The fundamental group of the maximal torus for $SO(2n)$ is the integral lattice which contains the coroot lattice with index two reflecting the fact that the fundamental group of $SO(2n)$ is $\mathbf{z}/2\mathbf{Z}$. The Weyl group consists of all permutations of the coordinates and all even sign changes of the coordinates. This is a simply laced group.

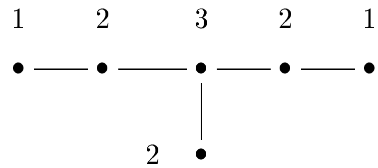
The Dynkin diagram of type D_n is



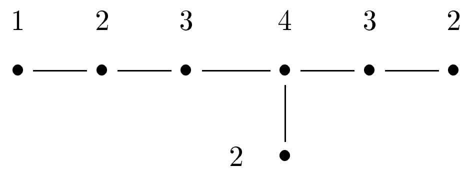
1.3.2 The exceptional groups

A good reference for the exceptional Lie groups is [1]. In addition to the classical groups there are five exceptional simply connected simple groups. Their names are E_6, E_7, E_8, G_2 and F_4. The subscript is the rank of the group. There are natural inclusions $D_5 \subset E_6 \subset E_7 \subset E_8$. The fundamental group of E_r is a cyclic group of order $9 - r$. Both G_2 and F_4 are simply connected.

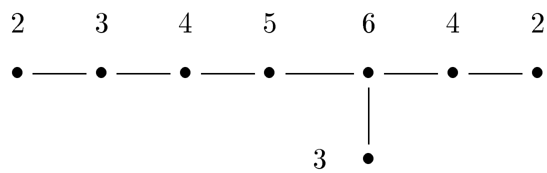
Here are their Dynkin diagrams:



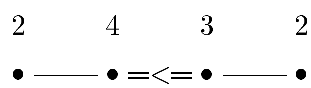
E_6



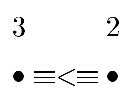
E_7



E_8



F_4



G_2

We shall not say too much about these groups now, but let me give the lattices E_6, E_7, E_8. These are viewed as the fundamental group of the maximal torus of the simply connected form of the group. We give these lattices with an inner product. This is the Weyl invariant inner product. In all cases these groups are simply laced and the coroots are the elements in the lattice of square two. We describe all these lattices at once – for any $r \leq 8$ consider the indefinite integral quadratic form $q(x, a_1, \ldots, a_r) = \sum_{i=1}^{r} a_i^2 - x^2$

on $\mathbf{Z}^{r+1}$. We let $k = (3, 1, 1, \ldots, 1)$. Then $q(k) = r - 9 < 0$. For $6 \leq r \leq 8$, the lattice E_r is the orthogonal subspace in $\mathbf{Z}^{r+1}$ of k. It is of rank r and has the induced quadratic form which is easily seen to be even, positive definite, and of discriminant $9 - r$. For lower values of r it turns out that the lattice defined this way is the lattice of a classical group: $E_5 = D_5$, $E_4 = A_4$, $E_3 = A_2 \times A_1$.

1.4 Principal holomorphic $G_{\mathbf{C}}$-bundles

Let $G_{\mathbf{C}}$ be a complex linear algebraic group and let X be a complex manifold. A holomorphic principal $G_{\mathbf{C}}$-bundle over X is determined by an open covering $\{U_i\}$ of X and transition functions $g_{ij}: U_i \cap U_j \to G_{\mathbf{C}}$. The transition functions are required to be holomorphic and to satisfy the cocycle conditions: $g_{ji} = g_{ij}^{-1}$ and $g_{jk}(z) \cdot g_{ij}(z) = g_{ik}(z)$ for all $z \in U_i \cap U_j \cap U_k$. As usual, we can use the g_{ij} as gluing data to glue $U_i \times G_{\mathbf{C}}$ to $U_j \times G_{\mathbf{C}}$ along $u_{ij} \times G_{\mathbf{C}}$, by the rule $(z, g) \in U_i \times G_{\mathbf{C}}$ maps to $(z, g_{ij}(z) \cdot g)$ in $U_j \times G_{\mathbf{C}}$ provided that $z \in U_i \cap U_j$. The cocycle condition tells us that the triple gluings are compatible so that we have defined an equivalence relation and the result of gluing E is a Hausdorff space. The projection mappings $U_i \times G_{\mathbf{C}} \to U_i$ then fit together to define a continuous map $p: E \to X$. The fact that the g_{ij} are holomorphic implies that the natural complex structures on the $U_i \times G_{\mathbf{C}}$ are compatible and hence define a complex structure on E for which p is a holomorphic submersion with each fiber isomorphic to $G_{\mathbf{C}}$. The complex manifold E is called the total space of the principal bundle and p is called the projection. There is a natural (right) free, holomorphic $G_{\mathbf{C}}$-action on E such that p is the quotient projection of this action. Two open coverings and gluing functions define the isomorphic principal $G_{\mathbf{C}}$-bundles if the resulting total spaces are biholomorphic by a $G_{\mathbf{C}}$-equivariant mapping commuting with the projections to X.

If ξ is a holomorphic principal $G_{\mathbf{C}}$-bundle over X and $\rho: G_{\mathbf{C}} \to \mathrm{Aut}(V)$ is a complex linear representation, then there is an associated holomorphic vector bundle $\xi(\rho)$. If E is the total space of ξ then the total space of $\xi(\rho)$ is $\xi \times_{G_{\mathbf{C}}} V$ where $G_{\mathbf{C}}$ acts on V via the representation ρ.

Example: Let $G_{\mathbf{C}}$ be $\mathbf{C}^*$. (Notice that this is not a simple group.) Then a holomorphic principal $\mathbf{C}^*$ bundle determines a holomorphic line bundle under the natural representation given by complex multiplication $\mathbf{C}^* \times \mathbf{C} \to \mathbf{C}$. The holomorphic line bundles associated to ξ and ξ' are isomorphic as holomorphic line bundles if and only if ξ and ξ' are isomorphic as holomorphic principal $\mathbf{C}^*$-bundles. Notice that the total space of the $\mathbf{C}^*$-bundle can be identified with the complement of the zero section in the corresponding holomorphic line bundle.

Along the same lines, let $G_{\mathbf{C}}$ be $SL_n(\mathbf{C})$. Then using the defining complex n-dimensional representation, a principal $SL_n(\mathbf{C})$-bundle ξ over X determines a holomorphic n-dimensional vector bundle V over X. But this bundle has the property that its determinant line bundle $\wedge^n V$ is trivialized as a holomorphic line bundle. Of course, given a holomorphic n-dimensional vector bundle V over X with a trivialization of its determinant line bundle we can define the associated bundle of special linear frames in V. The fiber over $x \in X$ consists of all bases $\{f_1, \ldots, f_n\}$ for V_x such that $f_1 \wedge \cdots \wedge f_n$ is identified with $1 \in \mathbf{C}$ under the given trivialization of $\wedge^n(V_x)$. The local trivialization of the vector bundle, produces a local trivialization of this bundle of frames. The holomorphic structure on the total space of V determines a holomorphic structure on the bundle of frames. The obvious $SL_n(\mathbf{C})$-action on the bundle of frames then makes it a holomorphic principal $SL_n(\mathbf{C})$-bundle. This sets up a bijection between isomorphism classes of holomorphic principal $SL_n(\mathbf{C})$-bundles over X and holomorphic vector bundles over X with trivialized determinant line bundle.

In the same way we can identify a holomorphic principal $SO(n, \mathbf{C})$-bundle over X with a holomorphic rank n vector bundle V over X with holomorphically trivialized determinant and with a holomorphic symmetric form $V \otimes V \to \mathbf{C}$ which is non-degenerate on each fiber. A holomorphic principal $\mathrm{Symp}(2n)$-bundle over X is identified with a holomorphic rank $2n$ vector bundle V over X with holomorphically trivialized determinant and with a holomorphically varying skew-symmetric bilinear form on the fibers which is non-degenerate on each fiber.

Up to questions of finite covering groups, this exhausts the list of classical simple groups: $SL_n(\mathbf{C})$, $SO(n, \mathbf{C})$, and $Sympl(2n, \mathbf{C})$.

Associated to any complex group $G_{\mathbf{C}}$ there is the adjoint representation of $G_{\mathbf{C}}$ on its Lie algebra $\mathfrak{g}_{\mathbf{C}}$. Thus, associated to any holomorphic principal $G_{\mathbf{C}}$-bundle ξ is a vector bundle denoted $\mathrm{ad}\xi$. Its rank is the dimension of $G_{\mathbf{C}}$ as a group. In the case of E_8 this is the smallest dimensional representation; it is of course of the same dimension as the group 248. All other simple groups have smaller representations: G_2 has a seven dimensional representation even though it has dimension 14; F_4 has a 28-dimensional representation; E_6 has a 27 dimensional representation (which we discuss later), and E_7 has a 54-dimensional representation. Still, it is not clear that the best way to study principal bundles over these groups is to look at the vector bundles associated to these representations. For example, it is not obvious what extra structure a 248-dimensional vector bundle carries if it comes from a principal E_8-bundle, nor what extra information it takes to determine the structure of that bundle.

1.5 Principal G-bundles over S^1

Let us begin with a simple problem. Fix a compact, simply connected, simple group G. Let ξ be a principal G-bundle over S^1 and let A be a flat G-connection on ξ. (The reason for the passage from G_C to G and the introduction of a flat connection will be explained in the next lecture.) The holonomy of A around the base circle is an element of G, determined up to conjugacy, which completely determines the isomorphism class of (ξ, A) and sets up an isomorphism between the space of conjugacy classes of elements in G and the space of isomorphism classes of principal G-bundles with flat connections over S^1. Thus, we have reduced our problem to that of understanding the space of conjugacy classes of elements in a compact group. To some extent this is a classical and well-understood problem, as we show in the next section.

1.5.1 The affine Weyl group and the alcove structure

This leads us to the question of what the space of conjugacy classes of elements in G looks like. To answer this question we introduce the affine Weyl group and the alcove structure on the Lie algebra $\mathfrak{t}$ of the maximal torus T of G. For a root α and $k \in \mathbf{Z}$ we denote by $W_{\alpha,k}$ the codimension-one affine-linear subspace of $\mathfrak{t}$ determined by the equation $\{\alpha = k\}$. By definition, the affine Weyl group, $W_{\rm aff}$, is the group of affine isometries of $\mathfrak{t}$ generated by reflections in all walls of the form $W_{\alpha,k}$. It is easy to see that there is an exact sequence of groups

$$0 \to \Lambda \to W_{\rm aff} \to W \to 0.$$

(Recall that $\Lambda \subset \mathfrak{t}$ is the coroot lattice.) Here, the map $W_{\rm aff} \to W$ is the differential or linearization of the affine map. (Recall that Λ is the coroot lattice, i.e., $\pi_1(T) \subset \mathfrak{t}$.) This sequence is split by including W as the group generated by the reflections in the walls $W_{\alpha,0}$, but the action of W on Λ is non-trivial: it is the obvious action. Thus, $W_{\rm aff}$ is isomorphic to the semi-direct product $\Lambda \times W$ with the natural action of W on Λ.

The set of walls $W_{\alpha,k}$ is a locally finite set and divides $\mathfrak{t}$ into (an infinite number of) regions called *alcoves*. If G is simple (or even semi-simple), then the alcoves are compact. In the case when G is simple, the alcoves are simplices. Clearly, each alcove is contained in some Weyl chamber. An alcove containing the origin in fact contains a neighborhood of the origin in the Weyl chamber that contains it. Its walls are the walls of the Weyl chamber containing it together with one more. This extra wall is given by an equation of the form $\tilde{\alpha} = 1$ for some root α determined by the Weyl chamber (or equivalently by the set of simple roots $\{\alpha_1, \ldots, \alpha_r\}$ determined by the Weyl chamber). It turns out that $\tilde{\alpha}$ is a positive linear combination of the α_i and it has the largest coefficients. In particular, it is the unique nonnegative linear combination of the simple roots with the property that its sum with any simple root is not a root. This root $\tilde{\alpha}$ is called the *highest root* of G. The numbers displayed on the Dynkin diagrams above are the coefficients of the simple roots in their unique linear combination which is the highest root. If Δ is a set of simple roots for G, then the associated set $\Delta \cup \tilde{\alpha}$ is denoted by $\widetilde{\Delta}$ and is called the extended set of simple roots.

As in the case of the Weyl group, it is a nice geometric argument to show that W_{aff} acts simply transitively on the set of alcoves and that the quotient $\mathfrak{t}/W_{\mathrm{aff}}$ is identified with any alcove.

Lemma 1.5.1 *Let G be a compact, simply connected semi-simple group. Then the space of conjugacy classes of elements in G is identified with an alcove A in $\mathfrak{t}$. The identification associates to $t \in A$ the conjugacy class of* $\exp(t) \in T$.

Proof. Every $g \in G$ is conjugate to a point $t \in T$. Two points of T are conjugate in G if and only if they are in the same orbit of the Weyl group action on T. Thus, the space of conjugacy classes of elements in G is identified with T/W. Since G is simply connected, $T = \mathfrak{t}/\Lambda$ and hence $T/W = \mathfrak{t}/W_{\mathrm{aff}}$ which we have just seen is identified with an alcove. □

Example: If $G = SU(n+1)$, then the alcove is the subset $(t_1, \ldots, t_{n+1}) \in \mathbf{R}^{n+1}$ satisfying $\sum_j t_j = 0$ and $t_j \geq 0$ and $t_j \leq t_{j+1}$ for all $1 \leq j \leq n+1$. Every element in $SU(n+1)$ is conjugate to a diagonal matrix with diagonal entries $(\lambda_1, \ldots, \lambda_{n+1})$, the λ_j being elements of S^1. We can do a further conjugation under $\lambda_j = \exp(2\pi i t_j$ for $0 \leq t_j \leq 1$. The determinant condition is $\prod_j \lambda_j = 1$, which translates into $\sum_j t_j = 0$. By a further Weyl conjugation we can arrange that the t_j are in increasing order. This makes explicit the isomorphism between the simplex in $\mathbf{R}^{n+1}$ and the space of conjugacy classes in $SU(n+_1)$.

1.6 Flat G-bundles over T^2

Let G be a compact, simply connected group.

Lemma 1.6.1 *Let $x, y \in G$ be commuting elements. Show that there is a torus $T \subset G$ containing both x and y. If (x, y) and (x', y') are pairs of elements in a torus $T \subset G$ and if there is $g \in G$ such that $g(x, y)g^{-1} = (x', y')$, then there is an element n normalizing T and conjugating (x, y) to (x', y').*

The proof is an exercise.

Corollary 1.6.2 *Let G be a compact simply connected group. Then the space of isomorphism classes of principal G-bundles with flat connections over T^2 is identified with the space $(T \times T)/W$, where T is a maximal torus of G and W is the Weyl group acting on $T \times T$ by simultaneous conjugation.*

Notice that this implies that the space of such bundles is connected. Of course, this is not too surprising since for G simply connected, all G-bundles over T^2 are topologically trivial.

1.7 Exercises

1. Let G be a compact connected Lie group. Show that the exponential mapping from the Lie algebra $\mathfrak{g}$ to G is onto. Use this to show that every element of G is contained in a maximal torus.
2. Show all maximal tori of G are conjugate.
3. Suppose that G is compact. Show that the centralizer of a maximal torus in G is the torus itself.
4. Let G be a compact simply connected group. Show that the center of G is the intersection of the kernels of all the roots. Show that if G is simply connected and semi-simple then the center of G is identified with the quotient $(\Lambda_{\rm root})^*/\Lambda$ where $\Lambda \subset \mathfrak{t}$ is the coroot lattice, where $\Lambda_{\rm root} \subset \mathfrak{t}^*$ is the root lattice, and $(\Lambda_{\rm root})^*$ is the algebraically dual lattice in $\mathfrak{t}$. Use this to show that the center of a semi-simple group is finite.

5. Show that if G is a compact Lie group whose Lie algebra is semi-simple, then any normal subgroup of G is a finite central subgroup.
6. Count the number of roots for groups of A_n, B_n, C_n, and D_n type and determine the dimension of each of these groups.
6. For any $r, 3 \leq r \leq 8$, let $\mathbf{Z}^{r+1}$ be the free abelian group of rank $r+1$ with basis $h, e_1, \ldots, e_r$ and with non-degenerate quadratic form Q with $Q(h) = -1$, $Q(e_i) = 1, 1 \leq i \leq r$, and the basis being mutually orthogonal. Let $k = 3h - \sum_{i=1}^{r} e_i$. Then $k^{\perp}$ is a lattice of rank r with a positive definite pairing of determinant $9-r$. Show that a basis for $k^{\perp}$ is $e_1 - e_2, \ldots e_{r-1} - e_r, H - e_1 - e_2 - e_3$ and that these vectors are all of square 2. Count the number of vectors in this lattice of square two. Show that for each vector of square 2, reflection in the vector is an integral isomorphism of the lattice $k^{\perp}$ and its form. Show that for $r = 3$ the lattice is the coroot lattice of $A_2 \times A_1$, for $r = 4$, the lattice is the coroot lattice of A_4, for $r = 5$ the lattice is the coroot lattice for D_5. In all cases the coroots are exactly the vectors of square 2. It follows of course that the group generated by reflections in the vectors of square two is the Weyl group. It turns out that for $r = 6, 7, 8$ the lattice is the coroot lattice of E_r. The statements about the roots, coroots and the Weyl group remain true for these cases as well. Assuming this, compute the dimensions of the Lie groups E_6, E_7, E_8.
8. Let E be an elliptic curve. Give a 1-cocycle which represents the generator of $H^1(E; \mathcal{O}_E))$. Give a 1-cocycle that represents the principal $\mathbf{C}^*$-bundle $\mathcal{O}(q - p_0)$.
9. Show that if V is a semi-stable vector bundle of degree zero over an elliptic curve E with a non-degenerate skew-form, then $\det(V)$ is trivial. Show that this is not necessarily true it V supports a non-degenerated quadratic form instead.
10. Define the compact form of $\mathrm{Sympl}_{\mathbf{C}}(2n)$ in terms of quaternion linear mappings of a quaternionic vector space.
11. Show that the conjugacy class of the holonomy representation determines an isomorphism between the space of isomorphism classes of principal G-bundles over S^1 with flat connections and the space of conjugacy classes of elements in G.
12. Show that the space of conjugacy classes of elements in G is identified with the alcove of the affine Weyl group action on $\mathfrak{t}$.
13. Show that if x, y are commuting elements in a compact simply connected group G then there is a torus in G containing both x and y. [Hint: Show that the centralizer of x, $Z_G(x)$ is connected.] Show that this fails to be true

if G is compact but not simply connected (e.g. $G = SO(3)$). Show that if X and X' are ordered subsets of T which are conjugate in G, then they are conjugate by an element of the normalizer of T.

14. Show that if G is a compact semi-simple group, then G-bundles over T^2 are classified up to topological equivalence by a single characteristic class $w_2 \in H^2(T^2; \pi_1(G)) = \pi_1(G)$. Show that if a G-bundle admits a flat connection with holonomy $(\overline{x}, \overline{y})$ around the generating circles, then its characteristic class in $\pi_1(G) \subset G$ is computed as follows. One chooses lifts x, y in the universal covering group $\tilde{G}$ of G for $\overline{x}, \overline{y}$. Then $[x, y] \in \tilde{G}$ lies in the kernel of the projection to G, i.e., lies in $\pi_1(G) \subset \tilde{G}$. This is the characteristic class of the bundle.

15. Show that the space of isomorphism classes of $SO(3)$-bundles with flat connection over T^2 has two components. Show that one of these components has dimension 2 and the other is a single point.

2 Semi-Stable $G_{\mathbf{C}}$-Bundles over Elliptic Curves

In this lecture we introduce the classical notions of stability, semi-stability, and S-equivalence for vector bundles. We then consider in some detail semi-stable vector bundles over an elliptic curve and the moduli space of their S-equivalence classes. We extend these results to the one situation where it has an easy and direct analogue – namely complex symplectic bundles. Then we switch and consider flat $SU(n)$-bundles and state the Narasimhan-Seshadri result relating these bundles to semi-stable holomorphic bundles. Then we generalize this result to compare flat G-bundles and semi-stable holomorphic $G_{\mathbf{C}}$-bundles. Lastly, we discuss Looijenga's theorem which, in the case that G is simple and simply connected, describes quite explicitly these moduli spaces in terms of the coroot integers for the group G.

2.1 Stability

Let C be a compact complex curve. The slope of a holomorphic vector bundle $V \to C$ is $\mu(V) = \deg(V)/\mathrm{rank}(V)$ where $\deg(V)$ is the degree of the determinant line bundle of V. A vector bundle $V \to C$ is said to be *stable* if for every proper subbundle $W \subset V$ we have $\mu(W) < \mu(V)$. The bundle V is said to be *semi-stable* if $\mu(W) \leq \mu(V)$ for every proper subbundle $W \subset V$. A subbundle W which violates these inequalities is called destabilizing or de-semistabilizing.

Note: One usually requires $\mu(W) < \mu(V)$, resp., $\leq \mu(V)$ for every vector bundle over C which admits a vector bundle mapping $W \to V$ which is injective on the generic fiber. The image of such a mapping will not necessarily be a subbundle of V, but rather is a subsheaf of its sheaf of local holomorphic sections. Nevertheless, there is a subbundle $W' \subset V$ containing the image of W under the given mapping with the property that W' modulo the image of W is supported at a finite set of points (a sky-scraper sheaf). In this case $\deg(W') = \deg(W) + \ell(W'/W)$ where $\ell(W'/W)$ is the total length of the sky-scraper sheaf. It follows that $\mu(W') \geq \mu(W)$ so that if W destabilizes or de-semistabilizes V then so does W'. Thus, in the case of curves it suffices to work exclusively with subbundles. Let me re-iterate that this is special to the case of curves.

The main reason for introducing stability is that the space of all bundles can be studied in terms on stable bundles (the so-called Harder-Narasimhan filtration) and that the space of isomorphism classes of stable bundles forms a reasonable space.

Over high dimensional manifolds X^n we need a couple of modifications. First of all we need a Kähler class ω so that we can define the degree of a bundle V, $\deg(V)$, to be $\int_X \omega^{n-1} \wedge c_1(V)$. (Of course, the degree, and hence the slope depends on the choice of Kähler class.) Then as indicated above we must also consider all torsion-free subsheaves W of V. Each of these has a first Chern class and hence a degree, again depending on ω and computed by the same formula as above. With these modifications one defines *slope stability* and *slope semi-stability* exactly as before. There are more refined notions, for example Gieseker stability, which are often needed over higher dimensional bases, especially if one hopes to obtain compact moduli spaces.

2.2 Line bundles of degree zero over an elliptic curve

We fix a smooth elliptic curve E. By this we mean that E is a compact complex (smooth) curve of genus 1, and we have fixed a point $p_0 \in E$. This determines an abelian group law on E for which p_0 is the identity element.

Consider a line bundle $L \to E$ of degree zero. According to Riemann-Roch,

$$\operatorname{rank} H^1(E; L) = \operatorname{rank} H^0(E; L)$$

which tells us nothing about whether L has a holomorphic section. However, if we consider $L \otimes \mathcal{O}(p_0)$, then RR tells us that this bundle has at least one holomorphic section. Let $\sigma\colon \mathcal{O} \to L \otimes \mathcal{O}(p_0)$ be such a section. Of course, since the degree of $L\otimes\mathcal{O}(p_0)$ is 1, σ vanishes once, say at a point $q \in E$. This means that σ factors to give a holomorphic mapping $\sigma'\colon \mathcal{O}(q) \to L \otimes \mathcal{O}(p_0)$ which is generically an isomorphism. In general a map between line bundles which is generically one-to-one has torsion cokernel and the total length of the cokernel is the difference of the degrees of the two bundles. In our case, the domain and range both have degree one, so that the cokernel is trivial, i.e., σ' is an isomorphism. This proves that L is isomorphic to $\mathcal{O}(q)\otimes\mathcal{O}(-p_0)$, which is also written as $\mathcal{O}(q - p_0)$. It is easy to see that associating to L the point q sets up an isomorphism between the space of line bundles of degree zero on E, $\mathrm{Pic}^0(E)$, and E itself.

2.3 Semi-stable $SL_n(\mathbf{C})$-bundles over E

Let V be a semi-stable vector bundle of rank n and degree zero. Semi-stability implies that any subbundle of V has non-positive degree. Let us first show that there is a line bundle of degree zero mapping into V. We proceed in the same manner as with line bundles. The bundle $V \otimes \mathcal{O}(p_0)$ is also easily seen to be semi-stable and of slope 1. By RR $V \otimes \mathcal{O}(p_0)$ has n holomorphic sections. Let H^0 be the space of these sections. Then evaluation determines a vector bundle mapping from the trivial bundle $H^0 \otimes \mathcal{O}_E$ to $V \otimes \mathcal{O}_E(p_0)$. If no non-trivial section of $V \otimes \mathcal{O}_E(p_0)$ vanished at any point, then this map would be an isomorphism of vector bundles, which is absurd since the bundles have different degree. We conclude that there is a non-zero section of $V \otimes \mathcal{O}_E(p_0)$ vanishing at some point. Consider the cokernel of this section. Generically it is a vector bundle of rank $n-1$, but it has non-trivial torsion, say T, corresponding to the zeros of the section. Then there is a line bundle L_T fitting into an exact sequence

$$0 \to \mathcal{O} \to L_T \to T \to 0$$

and an extension of σ to a map $\sigma' \colon L_T \to V \otimes \mathcal{O}(p_0)$ whose quotient is a vector bundle W of rank one less than that of V. Of course, the degree of L_T is the total length of T. By the fact that the slope of $V \otimes \mathcal{O}_E(p_0)$ is one and is semi-stability, we know the total length of T is at most one. But on the other hand we know that T is nontrivial, so that it has length exactly one. It follows that it is of the form $\mathcal{O}/\mathcal{O}(q)$ for some point $q \in E$. Thus, we have a map $\mathcal{O}(q) \to V$ with torsion-free cokernel and hence a map $\mathcal{O}(q - p_0) \to V$ whose quotient is a semi-stable bundle of degree $n-1$.

Continuing inductively we see that V is written as a successive extension of n line bundles of degree zero. We can associate to V, the n points that are identified with these line bundles. Let us now think about the extensions. For any line bundle of degree zero, RR tells us that since $H^1(E; L) = 0$ unless L is trivial. It follows immediately that if L and L' are non-isomorphic line bundles of degree zero then any extension

$$0 \to L \to V \to L' \to 0$$

is trivial. An easy inductive argument shows that if W_1 written as a successive extension of line bundles L_i of degree zero and W_2 as a successive extension of line bundles M_j of degree zero, and no L_i is isomorphic to any M_j, then any extension

$$0 \to W_1 \to V \to W_2 \to 0$$

is trivial. Consequently, we can split V into pieces $\oplus_{q \in E} V_q$ where V_q is a successive extension of line bundles isomorphic to $\mathcal{O}(q - p_0)$. Lastly, there is, up to equivalence, exactly one non-trivial extension

$$0 \to \mathcal{O}(q - p_0) \to V \to \mathcal{O}(q - p_0) \to 0.$$

An easy argument shows that if V is such a nontrivial extension, then it has a unique nontrivial extension by $\mathcal{O}(q - p_0)$ and so forth.

Notice that since every vector bundle of degree zero over an elliptic curve E has a subline bundle of degree zero, the only stable degree zero vector bundles over E are line bundles. Thus, the best we can hope for in higher rank and degree zero is that the bundle be semi-stable. As we shall eventually see, there are stable bundles of positive degree.

This allows us to establish the following theorem first proved by Atiyah:

Theorem 2.3.1 *Any semi-stable vector bundle of degree zero over E is isomorphic to a direct sum of bundles of the form $\mathcal{O}(q - p_0) \otimes I_r$ where the I_r are defined inductively as follows; $I_1 = \mathcal{O}$ and I_r is the unique non-trivial extension of I_{r-1} by $\mathcal{O}$.*

Clearly, the determinant of $\oplus_{q \in E} \mathcal{O}(q - p_0) \otimes I_{r(q)}$ is $\otimes_{q \in E} \mathcal{O}(q - p_0)^{\otimes r(q)}$, or under our identification of line bundles of degree zero with points of E, the determinant of V is identified with $\sum_{q \in E} r(q)q$, where the sum is taken in the group law of the elliptic curve. Thus, V has a trivial determinant if and only if $\sum_q r(q)q = 0$ in E.

2.4 S-equivalence

We just classified semi-stable vector bundles of degree zero on an elliptic curve in the sense that we enumerated the isomorphism classes. But we have not produced a moduli space (even a coarse one) of all such bundles. The trouble, as always in these problems, is that the natural space of isomorphism classes is not separated, i.e., not a Hausdorff space. The reason is exemplified by the fact that there is a bundle over $E \times H^1(E; \mathcal{O})$ whose restriction to $E \times \{a\}$ is the bundle which is the extension of $\mathcal{O}$ by $\mathcal{O}$ given by the extension class a. This bundle is isomorphic to I_2 for all $\{a\} \neq 0$ and is isomorphic to $\mathcal{O} \oplus \mathcal{O}$ if $\{a\} = 0$. Thus, we see that any Hausdorff quotient of the space of isomorphism classes of bundles I_2 and $\mathcal{O} \oplus \mathcal{O}$ must be identified.

This phenomenon is an example of S-equivalence. We say that a semi-stable bundle V is S-equivalent to a semi-stable bundle V' if there is a family of $\mathcal{V} \to E \times C$, where C is a connected smooth curve, so that for generic $c \in C$ the bundle $\mathcal{V}|C \times \{c\}$ is isomorphic to V and so that there is $c_0 \in C$

for which $\mathcal{V}|E \times \{c_0\}$ is isomorphic to V'. This relation is not an equivalence relation (it is not symmetric), so we take S-equivalence to be the equivalence relation generated by this relation.

Then we take as the moduli space of semi-stable vector bundles of degree zero the space of S-equivalence classes. Since I_k is S-equivalent to $\oplus_k \mathcal{O}$, it follows immediately from Atiyah's theorem that:

Theorem 2.4.1 *The set of S-equivalence classes of semi-stable bundles of rank n is identified with the set of unordered n-tuples of points $(e_1, \ldots, e_n) \subset E$. The subset of those with trivial determinant is the subset of unordered n-tuples $(e_1, \ldots, e_n)$ for which $\sum_{i=1}^{b} e_i = 0$ in the group law of E.*

2.5 The moduli space of semi-stable bundles

Everything we have done so far is at the level of points – that is to say we are describing all isomorphism classes or S-equivalence classes of bundles. Now we wish to see that the symmetric product of E with itself n-times is actually a coarse moduli space for the S-equivalence classes of semi-stable bundles of rank n and degree zero. Since we have already established a one-to-one correspondence between the points of this symmetric product and the set of S-equivalence classes of bundles, the question is whether this identification varies holomorphically with parameters. By this we mean that any time we have a holomorphic family $\mathcal{V} \to E \times X$ of semi-stable bundles rank n bundles on E (that is to say $\mathcal{V}$ is a rank n vector bundle and its restriction to each slice $E \times \{x\}$ is a semi-stable bundle), there should be a unique holomorphic mapping $X \to (E \times \cdots \times E)/S_n$ which associates to each $x \in X$ the point of the symmetric product which characterizes the S-equivalence class of $\mathcal{V}|E \times \{x\}$. Of course, this does define a function from X to the symmetric product; the only issue is whether it is always holomorphic.

To establish this we need a direct algebraic construction which goes from a vector bundle to an unordered n-tuple of points in E. Let V be a semi-stable vector bundle of degree 0 and rank n. Then $H^0(E; V \otimes \mathcal{O}(p_0))$ is n-dimensional. We have the evaluation map from sections to the bundle which we can view as a map from the trivial bundle $H^0(E; V \otimes \mathcal{O}(p_0)) \otimes \mathcal{O}_E$ to $V \otimes \mathcal{O}(p_0)$. Taking the determinants we get a map of line bundles

$$\wedge^n H^0(E; V \otimes \mathcal{O}) \otimes \mathcal{O}_E \to \det V \otimes \mathcal{O}(np_0).$$

This map is non-trivial. The domain is a trivial line bundle and the range has degree n. Thus, the cokernel of the map is a torsion sheaf of total degree n.

Taking the support of this torsion module, counted with multiplicity gives the unordered n-tuple in E. As one can see directly, for any sum of line bundles of degree zero, this map exactly picks out the unordered n points in E associated with the n-line bundles. More generally, one can easily check that the sections of $I_r \otimes \mathcal{O}(q)$ all vanish to first order at p_0 so that such a factor produces a zero of order r at q in the above determinant map.

Theorem 2.5.1 *The coarse moduli space of S-equivalence classes of semi-stable vector bundles of rank n and degree zero over an elliptic curve E is identified with $(E\times\cdots\times E)/S_n$. The coarse moduli space of those with trivial determinant is identified with the subspace of unordered n-tuples which sum to zero in the group law of E.*

Proof. To each semi-stable vector bundle of degree zero we associate the unordered n-tuple of points in E which corresponds to the set of line bundles of degree zero on E which are the successive quotients of V. By Atiyah's classification we see that this is a well-defined function. Clearly, S-equivalent semi-stable bundles are mapped to the same point and in light of Atiyah's classification, two bundles which map to the same point are S-equivalent.

It remains to see that if $\mathcal{V} \to E \times X$ is an algebraic family of semi-stable rank n bundles of degree zero over E, parametrized by X, then the resulting map $X \to (E\times\cdots\times E)/S_n$ is holomorphic. To see this let $\pi: E\times X \to X$ be the projection and consider the cohomology of along the fibers of $V \otimes \mathcal{O}(p_0)$. Let $p: E\times X \to E$ be the projection to the other component. Since we have already seen that for each $x \in X$, the cohomology $H^0(E; V|(E\times\{x\}\otimes\mathcal{O}(p_0))$ has rank n, it follows that the cohomology along the fibers $R^0\pi_*(V\otimes p^*\mathcal{O}(p_0))$ is a vector bundle of rank n over X. We take its n^{th} exterior power and pull back to a line bundle $\mathcal{L}$ on $E\times X$, trivial on each $E\times\{x\}$. As before, the evaluation mapping induces a map from $\mathrm{ev}: \mathcal{L} \to \wedge^n V \otimes p^*\mathcal{O}(np_0)$, which fiber-by-fiber in X is the map we considered above. In particular, the zero locus of ev is a subvariety of $E\times X$ whose projection to X is an n-sheeted ramified covering. Its intersection (counted with multiplicity) with each $E\times\{x\}$ gives the unordered n-points in E associated with the bundle $\mathcal{V}|E\times\{x\}$. This proves that the map $X \to (E\times\cdots\times E)/S_n$ is algebraic. □

(This argument works in either the classical analytic topology or in the Zariski topology.)

The space we just obtained of S-equivalence classes of semi-stable vector bundles of rank n with trivial determinant has another, extremely useful description.

Theorem 2.5.2 *The coarse moduli space of S-equivalence classes of semi-stable rank n bundles with trivial determinant on an elliptic curve E is identified with the projective space associated with the vector space $H^0(E;\mathcal{O}(np_0))$.*

Proof. Given n points $e_1,\ldots,e_n$ in E whose sum is zero there is a meromorphic function on E vanishing at these n points (with the correct multiplicities) with a pole only at p_0. Of course, the order of this pole is at most n and is exactly equal to the number of $i, 1 \leq i \leq n$, for which e_i is distinct from p_0. (Our evaluation mapping constructed such a function.) But a non-zero meromorphic function on E is determined up multiple by its zeros and poles. $\square$

Corollary 2.5.3 *The coarse moduli space of semi-stable vector bundles on E of rank n and trivial determinant is isomorphic to a projective space $\mathbf{P}^{n-1}$. In particular, as we vary the complex structure on E, the complex structure on this moduli space is unchanged.*

2.6 The spectral cover construction

Let us construct a family of semi-stable bundles on E parametrized by $\mathbf{P}(H^1(E;\mathcal{O}_E(p_0)))$, which to simplify notation we denote by $\mathbf{P}_n$. There is a covering $T \to \mathbf{P}_n$ defined as follows: a point of T consists of a pair $([f] \in |\mathcal{O}_E(np_0)|, e)$ where $e \in E$ is a point of the support of the zero locus of f. In other words, letting $S_{n-1} \subset S_n$ be the stabilizer of the first point, $T = \underbrace{(E\times\cdots\times E)}_{n-\text{times}}/S_{n-1}$. Clearly, $T \to \mathbf{P}_n$ is an n-sheeted ramified covering. There is of course a natural mapping $g\colon T \to E$ which associates to (f,e) to point e. We let $\mathcal{L}$ be pullback of the Poincaré bundles $\mathcal{O}_{E\times E}(\Delta - E\times\{p_0\})$ to $T\times E$ under $g\times \mathrm{Id}$. Then the pushforward $\mathcal{V}_n = (g\times\mathrm{Id})_*(\mathcal{L})$ of $\mathcal{L}$ over $T\times E$ to $\mathbf{P}_n \times E$ is a rank n vector bundle. A generic point of $\mathbf{P}_n$ has n distinct preimages in T and at such points $\mathcal{V}_n$ restricts to be a sum of n distinct line bundles of degree zero – the sum of bundles associated with the n-points in the preimage. If one asks what happens at points where g ramifies, it turns out that the pushforward bundle develops factors of the form $\mathcal{O}(q-p_0)\otimes I_r$

when r points come together. Thus, in fact, this construction produces a family of regular semi-stable bundles on E. Let $q: T \times E \to T$ be the projection to the first factor. Notice that for any line bundle $\mathcal{M}$ on T, we have that $(g \times \mathrm{Id})_*(\mathcal{L} \otimes q^*\mathcal{M})$ is also a rank n vector bundle on $\mathbf{P}_n \times E$ which is isomorphic fiber-by-fiber to the bundle $(g \times \mathrm{Id})_*\mathcal{L}$. Of course, globally these bundles can be quite different, and for example have different characteristic classes over $\mathbf{P}_n \times E$.

This construction is universal in the following sense, which we shall not prove (see Theorem 2.8 in Vector Bundles over elliptic fibrations).

Lemma 2.6.1 *Suppose that S is an analytic space and $\mathcal{U} \to S \times E$ is a rank n vector bundle which is regular and semi-stable with trivial determinant on each fiber $\{s\} \times E$. Let $\varphi_{\mathcal{U}}: S \to \mathbf{P}_n$ be the map that associates to each $s \in S$ the S-equivalence class of $\mathcal{U}|_{\{s\}\times E}$. Let $\tilde{S} = S \times_{\mathbf{P}_n} T$ and let $\tilde{\varphi}: \tilde{S} \to S$ and Φ be the natural maps. Then there is a line bundle $\mathcal{M}$ over $\tilde{S}$ such that $\mathcal{U}$ is isomorphic to*

$$(\varphi \times \mathrm{Id})_*((\Phi \times \mathrm{Id})^*\mathcal{L} \otimes p_1^*\mathcal{M}),$$

where $p_1: \tilde{S} \times E \to \tilde{S}$ is the projection.

2.7 Symplectic bundles

Let us make an analysis of principal $\mathrm{Sympl}_{\mathbf{C}}(2n)$-bundles which follows the same lines as the analysis for SL_n. We can view a holomorphic principal $\mathrm{Sympl}(2n)$-bundle over E as a holomorphic vector bundle V with a nondegenerate skew symmetric (holomorphic) pairing $V \otimes V \to \mathbf{C}$. Equivalently, we can view the pairing as an isomorphism $\varphi: V \to V^*$ which is skew-adjoint in the sense that $\varphi^* = -\varphi$. We say that such a bundle is semi-stable if the underlying vector bundle is. If we consider first a sum of line bundles of degree zero, $\oplus_{i=1}^{2n} L_i$ this bundle will support a skew symmetric pairing if and only if we can number the line bundles so that L_{2i-1} is isomorphic to L_{2i}^*. In this case we take the pairing to be an orthogonal sum of rank two pairings, the individual rank two pairings pairing L_{2i-1} and L_{2i} via the duality isomorphism. This means that if a semi-stable rank $2n$ vector bundle with trivial determinant supports a symplectic form then the associated points $(e_1, \ldots, e_{2n})$ in E are invariant (up to permutation) under the map $e \mapsto -e$. It turns out that we can identify the coarse moduli space of rank $2n$ semi-stable symplectic bundles over E with the subset of n unordered points in $E/\{e \cong -e\}$. Of course, $E/\{e \cong -e\}$ is the projective line P^1.

Then the space of n unordered points in P^1 is the n-fold symmetric product of the projective line, which is well-known to be projective n-space P^n. In terms of linear systems, n unordered points on P^1 is the projective space of $H^0(P^1; \mathcal{O}(n))$.

The double covering map from E to P^1 is given by the Weierstrass p-function, and the set of $2n$ points in E invariant under $e \mapsto -e$ is the zeros of a polynomial of degree at most n in p. This polynomial has a pole only at p_0 and that pole has order at most $2n$.

Once again one can make a spectral covering construction. We define T_{Symp} as the space of pairs $(f \in |\mathcal{O}_E(2np_0) f(-x) = -f(x), \pm e)$ where $\pm e$ is in the support of the zero locus of f. Then over $T \times E$ there is a two plane bundle with a non-degenerate skew symmetric pairing. The pushforward of this bundle to the projective space of even functions on E with pole less than $2np_0$ times E is then a family of symplectic bundles over E.

It turns out that these are the only two families of simply connected groups for which a direct construction like this, relating the moduli space to the projective space of a linear series can be made. For the other groups the coarse moduli space is not a projective space, but rather a space of a type which is a slight generalization called a weighted projective space. For the groups $SO(n)$, which of course are not simply connected, it is possible to make a similar construction producing a projective space, but it is somewhat delicate and I shall not give it here.

2.8 Flat $SU(n)$-connections

Let us switch gears now to state a general result linking principal holomorphic vector bundles to $SU(n)$-bundles equipped with a flat connection. This is a variant of the famous Narasimhan-Seshadri theorem.

Suppose that $W \to E$ is an $SU(n)$-bundle equipped with a connection A. Consider the complex vector bundle associated to the defining n-dimensional representation: $W_{\mathbf{C}} = W \times_{SU(n)} \mathbf{C}^n$. Let $d_A\colon \Omega^0(E; W_{\mathbf{C}}) \to \Omega^1(E; W_{\mathbf{C}})$ be the covariant derivative determined by the connection A. We can take the $(0,1)$-part of the covariant derivative $\overline{\partial}_A\colon \Omega^0(E; W_{\mathbf{C}}) \to \Omega^{0,1}(E; W_{\mathbf{C}})$. Since the base is a curve, for dimension reasons the square of this operator vanishes, and hence it defines a holomorphic structure on $W_{\mathbf{C}}$. It is a standard argument to see that this bundle is semi-stable, and in fact is a sum of stable bundles, i.e. of line bundles of degree zero. The Narasimhan-Seshadri result is a converse to this computation.

Theorem 2.8.1 *Let V be a semi-stable holomorphic vector bundle of rank n with trivial determinant over an elliptic curve E. Then there is an $SU(n)$-bundle $W \to E$ and a flat $SU(n)$-connection on W such that the induced holomorphic structure on the bundle $W_{\mathbf{C}}$ is S-equivalent to V. This bundle and flat connection are unique up to isomorphism. Thus, the space of isomorphism classes of flat $SU(n)$-bundles is identified with the moduli space of S-equivalence classes of semi-stable holomorphic vector bundles of rank n with trivial determinant.*

Remarks: (1) As we have already pointed out the holomorphic bundles produced by this construction are sums of line bundles of degree zero. Generically, of course, these are the unique representatives in their S-equivalence class, but when two or more of the line bundles coincide there are other isomorphism classes in the same S-equivalence class.
(2) This theorem is usually stated for curves of genus at least 2. Over such curves there are stable (as opposed to properly semi-stable) vector bundles. In this context, the notion of S-equivalence is not necessary (or more precisely it coincides with isomorphism). The result is that associating to a flat $SU(n)$-bundle its associated holomorphic vector bundle determines an isomorphism between the space of the space of conjugacy classes of irreducible representations of $\pi_1(C) \to SU(n)$ and the space of stable holomorphic vector bundles of rank n and trivial determinant. In our case, when the base is a curve of genus one, there are now irreducible representations (for $n > 1$) reflecting the fact that there are no stable vector bundles with trivial determinant. In our case we have an equivalence between the S-equivalence classes of properly semi-stable bundles.
(3) A flat connection on an $SU(n)$-bundle over E is given by homomorphisms $\pi_1(E) \to SU(n)$ up to conjugation. Of course, choosing a basis for $H_1(E)$, or equivalently choosing a pair of one-cycles on E which intersect transversely in a single point, with $+1$ interesection at that point, identifies $\pi_1(E)$ with a free abelian group on two generators. A homomorphism of this group into $SU(n)$ is then simply a pair of commuting elements in $SU(n)$. We consider two pairs as equivalent if they are conjugate by a single element of $SU(n)$. As is well-known, two commuting elements in $SU(n)$ are simultaneously conjugate into the maximal torus T (the diagonal matrices) of $SU(n)$. Thus, we can assume that our elements lie in this maximal torus. The only conjugation remaining is simultaneous Weyl conjugation. Thus, the moduli space of homomorphisms of $\pi_1(E) \to SU(n)$ up to conjugation is identified with

$(T \times T)/W$. (Notice that in this description we have ignored the complex structure.) This generalizes the picture we established in the last lecture for flat $SU(n)$-bundles over S^1. Still we have one more step to complete this picture. Before we identified T/W with the alcove for the affine Weyl group action on $\mathfrak{t}$. We have not yet identified $(T \times T)/W$.

Let us give another description of $(T \times T)/W$ in a way that will keep track of the complex structure. We write $E = \mathbf{C}/\pi_1(E)$ and $T = \mathfrak{t}/\Lambda$ where $\mathfrak{t} \subset \mathbf{R}^n$ is the subspace of points whose coordinates sum to zero, and where the coroot lattice Λ is the intersection of this subspace with the integral lattice. A homomorphism $\pi_1(E) \to T$ dualizes under Pontrjagin duality to a homomorphism $\mathrm{Hom}(T, S^1) \to \mathrm{Hom}(\pi_1(E), S^1)$. The first group is identified with the dual Λ^* to the lattice Λ and the second is identified with the dual curve E^* to E. Since we have chosen a point p_0 on E we have identified E and E^* holomorphically. Thus, the Pontrjagin dual of the map $\pi_1(E) \to T$ is a map $\Lambda^* \to E$, or equivalently an element of $\Lambda \otimes E$. We have the exact sequence

$$0 \to \Lambda \to \mathbf{Z}^n \to \mathbf{Z} \to 0$$

where the last map is the sum of the coordinates. Tensoring with E yields

$$0 \to \Lambda \otimes E \to \times_n E \to E$$

where again the last map is the sum of the coordinates. Thus, we see that $\Lambda \otimes E$ is identified with the subset $(e_1, \ldots, e_n) \in \times_n E$ of points which sum to zero. Following through the action of the Weyl group, which is the symmetric group on n letters acting in the obvious way on $\Lambda \subset \mathbf{Z}^n$, we see that the Weyl action on $\mathrm{Hom}(\pi_1(E), T)$ becomes the permutation action on the space on n points summing to zero. Thus, we see a direct isomorphism

$$\mathrm{Hom}(\pi_1(E), T)/W \to (\Lambda \otimes E)/W$$

which realizes the Narasimhan-Seshadri theorem. Notice that the complex structure on E induces a complex structure on $\Lambda \otimes E$ which is clearly invariant under the Weyl action. Thus, this structure descends to a complex structure on $(\Lambda \otimes E)/W$, and hence determines a holomorphic structure on the moduli space. This structure agrees with the usual functorial one of the coarse moduli space. Notice that since, as we have already seen by different methods, the quotient is a projective space, in the end, the complex structure is independent of the complex structure on E.

2.9 Flat G-bundles and holomorphic $G_{\mathbf{C}}$-bundles

The above result for vector bundles generalizes to an arbitrary complex semi-simple group. Let $G_{\mathbf{C}}$ be a complex semi-simple group with compact form G. Suppose that $W \to E$ is a principal G-bundle equipped with a flat G-connection A. We take an open covering of E by contractible open sets $\{U_i\}$. Then for each i, there is a trivialization of $W|U_i$ in which A is the product connection. This induces a trivialization of $W \times_G G_{\mathbf{C}}|U_i$. The overlap functions on $g_{ij}: U_i \cap U_j \to G$ in the given trivializations for W are locally constant and hence holomorphic functions when viewed as maps of $U_i \cap U_j$ into $G \times_G G_{\mathbf{C}}$. Thus, we have produced a holomorphic bundle structure. (A better argument shows that any G-connection produces a holomorphic bundle structure on the associated $G_{\mathbf{C}}$-bundle.)

Definition 2.9.1 A holomorphic $G_{\mathbf{C}}$-bundle over E is *semi-stable* if its adjoint bundle is semi-stable. We say that two semi-stable $G_{\mathbf{C}}$-bundles V_1, V_2 on E are S-equivalent if there is a connected holomorphic family of semi-stable $G_{\mathbf{C}}$-bundles on E containing bundles isomorphic to each of V_1 and V_2.

Here is the general version of the Narasimhan-Seshadri result [8] for holomorphic principal $G_{\mathbf{C}}$-bundles over an elliptic curve.

Theorem 2.9.2 *Let E be an elliptic curve and let $G_{\mathbf{C}}$ be a complex semi-simple group (not necessarily simply connected). Let $V \to E$ be a semi-stable holomorphic $G_{\mathbf{C}}$-bundle. Then there is a flat G-bundle $W \to E$ such that the induced holomorphic $G_{\mathbf{C}}$-bundle structure on $W \times_G G_{\mathbf{C}}$ is S-equivalent to V. This flat G-bundle is uniquely determined up to isomorphism.*

Once again this result is usually stated for smooth curves of genus at least two and establishes an isomorphism between the space of conjugacy classes of irreducible representations of $\pi_1(C)$ into G and the space of isomorphism classes of stable $G_{\mathbf{C}}$-bundles. But over an elliptic curve there are no stable $G_{\mathbf{C}}$-bundles (and no irreducible representations of $\pi_1(E)$ into G) for any semi-simple group G, and we must consider semi-stable bundles. As in the case of vector bundles, we are then forced to work with the weaker equivalence relation of S-equivalence instead of isomorphism.

We can examine flat G-bundles analogously to the way we did when G is $SU(n)$. First, it is a classical result [2] that in a simple connected Lie group

G any pair of commuting elements can be conjugated into the maximal torus T, and any two pairs of elements in T are simultaneously conjugate if and only if they are conjugate by a Weyl element. Thus:

Theorem 2.9.3 *Let G be a compact simply connected semi-simple group with maximal torus T, Weyl group W and coroot lattice $\Lambda \subset \mathfrak{t}$. Then, the space of isomorphism classes of G-bundles with flat connections is identified with $(T \times T)/W$. By Pontrjagen duality this space is identified with $(\Lambda \otimes E)/W$ where W acts trivially on E and in the natural fashion on Λ.*

We have already unraveled all this for $SU(n)$. Let us see what it says in the case of Sympl$(2n)$. In this case the coroot lattice Λ is the integral lattice in $\mathbf{R}^n$ and the Weyl group is the group $(\pm 1)^n \rtimes S_n$ with the ± 1's acting as sign changes of the various coordinates and S_n permuting the coordinates. Thus, $\Lambda \otimes E$ is identified with $\times_n E$ and the action of the Weyl group is by ± 1 in each factor and permutations of the factors. The quotient $\times_n E/(\pm 1)^n$ is $\times_n \mathbf{P}^1$ and the symmetric group acts on this to produce a quotient naturally identified with $\mathbf{P}^n$. This recaptures what we saw directly in terms of holomorphic bundles.

Theorem 2.9.3 does not hold for non-simply connected groups. The reason is that we cannot simultaneously conjugate a pair of commuting elements in a non-simply connected group into a maximal torus. For example, if $G = SO(3)$ then rotations by π radians in two perpendicular planes commute but cannot be simultaneously conjugated into a maximal torus (a circle) of $SO(3)$. The reason is that if we lift these elements in any manner to the double covering $SU(2)$, then they generate a quaternion group of order 8, i.e., the commutation of the lifts in $SU(2)$ is the non-trivial central element of $SU(2)$. If we could put both elements in a maximal torus of $SO(3)$, then they would lift to elements in a maximal torus of $SU(2)$, and hence there would be lifts which commuted. A similar phenomenon occurs in any non-simply connected group G/C. It is an interesting problem to determine the dimension of the space of representations of $\pi_1(E) \to G/C$ which produce bundles of a given nontrivial topological type. There is a purely lattice theoretic description of conjugacy classes of commuting elements in a non-simply connected compact group, and hence using the Narasimhan-Seshdari result, a lattice-theoretic description of the coarse moduli space of bundles over a non-simply connected semi-simple complex group. This description is quite interesting but much more complicated.

2.9.1 Looijenga's theorem

We have seen that in the cases of $G = SU(n)$ and $G = \mathrm{Sympl}(2n)$ the space of S-equivalence classes of semi-stable principal $G_{\mathbf{C}}$-bundles or equivalently the space of conjugacy classes of representations of $\pi_1(E) \to G$ is a projective space. In fact, in each case we explicitly identified the moduli space with the projective space of a linear system either on the elliptic curve or on the P^1 quotient of the elliptic curve by ± 1. We now give a theorem which determines the nature of these moduli spaces for an arbitrary simply connected and simple group G.

First, let us recall the notion of a weighted projective space. Suppose that V is a k-dimensional complex vector space with a linear action of $\mathbf{C}^*$. Of course, this action can be diagonalized in an appropriate basis of V and hence the action is completely determined up to isomorphism by k characters on $\mathbf{C}^*$. Any character of $\mathbf{C}^*$ is automatically of the form $\lambda \mapsto \lambda^r$ for some integer r, and in this fashion the characters of $\mathbf{C}^*$ are identified with the integers. The characters arising in the action on V are called the weights of the action. If all the weights are non-zero and have the same sign, let us say positive, then we say that the action is an action with positive weights. In this case, there is a nice compact quotient space: $V - \{0\}/\mathbf{C}^*$ which is called a *weighted projective space*. For any set $(g_0, g_1, \ldots, g_k)$ of positive integers, the symbol $\mathbf{P}(g_0, g_1, \ldots, g_k)$ denotes the quotient of the action of $\mathbf{C}^*$ with weights $(g_0, g_1, \ldots, g_k)$. This quotient space is compact and of dimension k. Indeed, is finitely covered in a ramified fashion by an ordinary projective space $\mathbf{P}^k$. A weighted projective space is not in general a smooth complex variety, since the action of $\mathbf{C}^*$ has finite cyclic isotropy groups along certain subspaces. Rather, the quotient space has cyclic orbifold-type singularities. (The quotient space is locally isomorphic to the quotient of $\mathbf{C}^k$ by a finite cyclic group.)

Theorem 2.9.4 [6,7] *Let G be a compact, simple, simply connected group and let E be an elliptic curve. Then the space of conjugacy classes of homomorphisms $\pi_1(E) \to G$ has a natural complex structure and with this structure is isomorphic to a weighted projective space $\mathbf{P}(1, g_1, \ldots, g_r)$ where $g_1, \ldots, g_r$ are the coefficients that occur when the coroot dual to the highest root of G is expressed as a linear combination of the coroots dual to the simple roots.*

As we have seen, the space of conjugacy classes of homomorphisms $\pi_1(E) \to G$ is identified with $(\Lambda \otimes E)/W$. Since Λ is abstractly a free abelian group of rank r, $\Lambda \otimes E$ is $\times_r E$ and hence has a natural complex structure inherited from that of E. Clearly, the Weyl action is holomorphic, and thus there is a possibly singular complex structure on the quotient space. This is the one referred to in Looijenga's theorem.

2.10 The coarse moduli space for semi-stable holomorphic $G_{\mathbf{C}}$-bundles

As we explained in the case of vector bundles, it is not enough simply to find the set of S-equivalence classes, one would like to identify a coarse moduli space (one has to worry whether or not such a coarse moduli space even exists). As a first step in constructing the coarse moduli space for semi-stable holomorphic $G_{\mathbf{C}}$-bundles, we claim that there is a holomorphic family of $G_{\mathbf{C}}$-bundles over E parametrized by $\Lambda \otimes E$. Actually, this family is a holomorphic family of $T_{\mathbf{C}}$-bundles over E. To construct this family we choose an integral basis for Λ, hence identifying it with $\mathbf{Z}^r$ and hence identifying $\Lambda \otimes E$ with $\times_r E$. This also identifies the complexification $T_{\mathbf{C}}$ of the maximal torus T of G with a product $\times_r \mathbf{C}^*$. Then we let $\mathcal{P} \to E \times E$ be the Poincaré line bundle $\mathcal{O}(\Delta - E \times \{p_0\})$ (where Δ is the divisor given by the diagonal embedding of E). This is a family of line bundles of degree zero over the second factor parametrized by the first factor. Over $(\times_r E) \times E$ we form $\oplus_{i=1}^r p_i^* \mathcal{P}$ where $p_i\colon (\times_r E) \times E \to E \times E$ is given by the product of projection onto the i^{th}-component in the first factor and the identity in the second factor. This sum of line bundles is then equivalent to a family of principal $\times_r \mathbf{C}^*$-bundles, and though our identification, with a family of $T_{\mathbf{C}}$-bundles. It is easy to trace through the identifications and see that the resulting family of $T_{\mathbf{C}}$-bundles is independent of the choice of basis for Λ.

The Weyl group W acts, as we have already used several times, in the obvious way on the parameter space $\Lambda \otimes E$. It also acts as outer automorphism of the $T_{\mathbf{C}}$ and hence changes one $T_{\mathbf{C}}$-principal bundle into a different one. The family is equivariant under these actions: the $T_{\mathbf{C}}$-bundle parametrized by $\lambda \otimes e$ is transformed by $w \in W$ acting on the set of $T_{\mathbf{C}}$-bundles to the $T_{\mathbf{C}}$-bundle parametrized by $w(\lambda) \otimes e$. Thus, when we extend the structure group from $T_{\mathbf{C}}$ to $G_{\mathbf{C}}$ the bundles parametrized by points in the same Weyl orbit are isomorphic. Thus, our family of $G_{\mathbf{C}}$-bundles is equivariant under the Weyl action. Consequently, if there is a coarse moduli space $\mathcal{M}_{G_{\mathbf{C}}}$ for S-equivalence classes of semi-stable holomorphic $G_{\mathbf{C}}$-bundles over E, then we obtain a Weyl invariant holomorphic mapping $\Lambda \otimes E \to \mathcal{M}_{G_{\mathbf{C}}}$, and hence holomorphic mapping $(\Lambda \otimes E)/W \to \mathcal{M}_{G_{\mathbf{C}}}$. Since the set of points of $(\Lambda \otimes E)/W$ are identified with the set of S-equivalence classes of such bundles, this map would then be a bijection. In fact there is such a moduli space (as can be established by rather general algebro-geometric arguments) and it is $(\Lambda \otimes E)/W$. I will not establish this here, but I will assume it in what follows. The precise statement is:

Theorem 2.10.1 *The set of points of $(\Lambda \otimes E)/W$ is naturally identified with the set of S-equivalence classes of semi-stable $G_{\mathbf{C}}$-bundles over E. If $W \to E \times X$ is a holomorphic family of semi-stable $G_{\mathbf{C}}$-bundles over E parametrized by X, then the function $X \to (\Lambda \otimes E)/W$ induced by associating to each $x \in X$ the point of $(\Lambda \otimes E)/W$ identified with the S-equivalence class of $W|E \times \{x\}$ is a holomorphic mapping. This makes $(\Lambda \otimes E)/W$ the coarse moduli space for S-equivalence classes of semi-stable $G_{\mathbf{C}}$-bundles over E.*

This completes the problem of understanding semi-stable holomorphic $G_{\mathbf{C}}$-bundles over E. There is a coarse moduli space which is $(\Lambda \otimes E)/W$ with the identification of its points with bundles as given above. Furthermore, by Looijenga's theorem, this complex space is a weighted projective space with positive weights given by the coefficients of the simple coroots in the linear combination which is the coroot dual to the highest root. The only cases when this weighted projective space is in fact an honest projective space is when all the weights are 1 and this occurs only for the groups $SU(n)$ of A-type and the groups Sympl$(2n)$ of C-type. In these two cases we directly identified with projective space as being associated with an appropriate linear series.

In the next lecture, we will give a construction which will describe the other moduli spaces in terms of a $\mathbf{C}^*$-action on an affine space. In the two special cases given above this affine space will be a linear space and the action will be the usual $\mathbf{C}^*$-action hence producing a quotient projective space. In the other cases, we will show that the affine action can be linearized to produce a quotient which is a weighted projective space. This will provide a proof of Looijenga's theorem, different from his original proof.

2.11 Exercises:

1. Show that if V, W are vector bundles over a smooth curve and that if $W \to V$ is a holomorphic map which is one-to-one on the generic fiber, then there is a subbundle $\hat{W} \subset V$ which contains the image of W and so that

$\hat{W}/W$ is a sky-scraper sheaf. Show that the degree of $\hat{W}$ is the degree of W plus the total length of $\hat{W}/W$.
2. Let E be a smooth projective curve over the complex numbers of genus one. Show that the universal covering of E is analytically isomorphic to $\mathbf{C}$ and that the fundamental group of E is identified with a lattice $\Lambda \subset \mathbf{C}$. Show that fixing a point $p_0 \in E$ there is a holomorphic group law on E which is abelian and for which p_0 is the origin. Show this group law is unique given p_0. Using the Weierstrass p-function associated with this lattice show that E can be embedded as a cubic curve in $\mathbf{P}^2$ with equation of the form

$$y^2 = 4x^3 + g_2x + g_3$$

for appropriate constants g_2, g_3.
3. Using the RR theorem show that if V is a semi-stable vector bundle of positive degree over E, then $H^1(E;V) = 0$ and $H^0(E;V)$ has rank equal to the degree of V. Formulate and prove the corresponding result for semi-stable vector bundles of negative degree.
4. Show that there is a unique non-trivial extension I_2 of $\mathcal{O}_E$ by $\mathcal{O}_E$. Show more generally that there is a unique bundle I_r of rank r which is an iterated extension of $\mathcal{O}_E$ where each extension is non-trivial. Show that $H^1(E;I_r)$ is rank one. The iterated extensions give an increasing filtration $|calF_*$ of I_r by subbundles so that for each $s \leq r$, we have $\mathcal{F}_s/\mathcal{F}_{s-1} = \mathcal{O}_E$. Show that this filtration is stable under any automorphism of I_r and is preserved under any endomorphism of I_r.
5. Show that if V is a vector bundle over any base and if every nonzero section of V vanishes nowhere then the union of the images of the sections of V produce a trivial subbundle of V with torsion-free cokernel. In particular, in this case the number of linearly independent sections is at most the rank of V and if it is equal to the rank of V, then V is a trivial bundle.
6. Show that there is a rank two vector bundle over $E \times H^1(E;\mathcal{O}_E)$ whose restriction to $E \times \{0\}$ is $\mathcal{O}_E \oplus \mathcal{O}_E$ and whose restriction to any other fiber $E \times \{x\}$, $x \neq 0$ is isomorphic to I_2.
7. By a coarse moduli space of equivalence classes of bundles of a certain type we mean the following: we have a reduced analytic space X and a bijection between the points of X and the equivalence classes of the bundles in question. Furthermore, if $V \to E \times Y$ is a holomorphic bundle such that the restriction V_y of V to each slice $E \times \{y\}$ is of the type under consideration, then the map $Y \to X$ defined by sending y to the point of X corresponding to the equivalence class of V_y is a holomorphic mapping. Show that if a coarse

moduli space exists, then it is unique up to unique isomorphism. Show that there cannot be a coarse moduli space for isomorphism classes of semi-stable bundles over E.

8. Show that $H^0(E; \mathcal{O}(q-p_0)\otimes\mathcal{O}(p_0)$ is one-dimensional and that any non-zero section of this bundle vanishes to order one at q. Show that $H^0(E; I_r\otimes\mathcal{O}(q-p_0)\otimes\mathcal{O}(p_0))$ has rank r and that every section of this bundle vanishes to order one at q. Thus, the determinant map for this bundle has a zero of order r at q.

9. Show that points $(e_1,\ldots,e_r)$ in E are the zeros of a meromorphic function $f\colon E\to\mathbf{P}^1$ with a pole only at p_0 if and only if $\sum_i e_i = 0$ in the group law of E.

10. Let E be embedded in $\mathbf{P}^2$ so that it is given by an equation in Weierstrass form:

$$y^2 = 4x^3 + g_2x + g_3.$$

Show that any meromorphic function on E with pole only at infinity in this affine model is a polynomial expression in x and y. Show that a meromorphic function with pole only at infinity which is invariant under $e\mapsto -e$ is a polynomial expression in x.

11. Let $g\colon T\to|np_0|$ be the n-sheeted ramified covering constructed in Section 2.6. Let $\mathcal{L}$ be the line bundle over $T\times E$ obtained by pulling back the Poincaré line bundle over $E\times E$. Show that if $(e_1,\ldots,e_n)$ are distinct points of E then the restriction of $(g\times\mathrm{Id})_*\mathcal{L}$ to $\{e_1,\ldots,e_n\}\times E$ is isomorphic to $\oplus_i\mathcal{O}_E(e_i-p_0)$. Show that if $e_1=e_2$ but otherwise the e_i are distinct then $(g\times\mathrm{Id})_*(\mathcal{L}$ restricted to $\{e_1,\ldots,e_n\}\times E$ is isomorphic to $\mathcal{O}_E(e_1-p_0)\otimes I_2\oplus_{i=3}^n\mathcal{O}(e_i-p_0)$

12. Show that the Pontrjagen dual of a homomorphism $\pi_1(E)\to T$ is a homomorphism $\Lambda\to\mathrm{Hom}(\pi_1(E),S^1)$. Show that the choice of an origin p_0 allows us to identify E with $\mathrm{Hom}(\pi_1(E),S^1)$.

13. Show that the quotient of E by $e\cong -e$ is $\mathbf{P}^1$. Show that the quotient of $(\mathbf{P}^1)^n$ under the action of the symmetric group on n letters is $\mathbf{P}^n$.

14. Show that a weighted projective space with positive weights is a compact complex variety. Show that it is finitely covered by an ordinary projective space. Show that in general these varieties are singular, but that their singularities are modeled by quotients of finite linear group actions on a vector space.

3 The Parabolic Construction

In this section we are going to give a completely different construction of semi-stable bundles over an elliptic curve. We begin first with the case of vector bundles of degree zero.

3.1 The parabolic construction for vector bundles

Lemma 3.1.1 *Let E be an elliptic curve with p_0 as origin. Then for each integer $d \geq 1$ there is, up to isomorphism, a unique vector W_d over E with the following properties:*

1. $\mathrm{rank}(W_d) = d$.
2. $\det(W_d) = \mathcal{O}(p_0)$.
3. *W_d is stable.*

Furthermore, $H^0(E; W_d)$ is of dimension one.

Proof. The proof is by induction on d. If $d = 1$, then it is clear that there is exactly one bundle, up to isomorphism, which satisfies the first and second item, namely $\mathcal{O}(p_0)$. Since this bundle is stable, we have established the existence and uniqueness when $d = 1$. Since $\mathcal{O}(-p_0)$ is of negative degree, it has no holomorphic sections and hence by RR, $H^1(E; \mathcal{O}(-p_0))$ is one-dimensional. By Serre duality, it follows that $H^0(E : \mathcal{O}(p_0))$ is also one-dimensional. This completes the proof of the result for $d = 1$.

Suppose inductively for $d \geq 2$, there is a unique W_{d-1} as required. By the inductive hypothesis and Serre duality we have $H^1(E; W_{d-1}^*)$ is one-dimensional. Thus, there is a unique non-trivial extension

$$0 \to \mathcal{O} \to W_d \to W_{d-1} \to 0.$$

Clearly, using the inductive hypothesis we see that W_d is of rank d and its determinant line bundle is isomorphic to $\mathcal{O}(p_0)$. In particular, the degree of W_d is one. Suppose that W_d has a destabilizing subbundle U. Then $\deg(U) > 0$. The intersection of U with $\mathcal{O}$ is a subsheaf of $\mathcal{O}$ and hence has non-positive degree. Thus, the image $p(U)$ of U in W_{d-1} has positive degree, and hence degree at least one. In particular, it is non-zero. Since the rank of U is at most the rank of W_{d-1}, it follows that $\mu(p(U)) \geq \mu(W_{d-1})$. Since

$p(U)$ is non-trivial, it follows that $p(U) = W_{d-1}$. Thus, the rank of U is either $d-1$ or d. If it is of rank d, the $U = W_d$ and does not destabilize. Thus, it must be of rank $d-1$. This means that $p: U \to W_{d-1}$ is an isomorphism and hence that U splits the exact sequence for W_d. This is a contradiction since the sequence was a nontrivial extension. This contradiction proves that W_d is stable.

A direct cohomology computation using the given exact sequence shows that $H^0(E; W_d)$ has rank one, completing the proof. □

Note (1): We have seen that W_d is given as successive extensions with all quotients except the last one being $\mathcal{O}$. The last one is $\mathcal{O}(p_0)$.

(2): There is a one-parameter family of stable bundles of rank d and degree 1. The determinant of such a bundle is of the form $\mathcal{O}(q)$ for some point $q \in E$ and the isomorphism class of the bundle is completely determined by q.

(3) The bundle W_d^* is then a stable bundle of degree minus one and determinant $\mathcal{O}(-p_0)$.

Corollary 3.1.2 *The automorphism group of W_d is $\mathbf{C}^*$.*

Proof. Suppose that $\varphi: W_d \to W_d$ is an endomorphism of W_d. Then we see that if φ is not an isomorphism, then $\deg(\mathrm{Ker}(\varphi)) = \deg(\mathrm{Coker}(\varphi))$. But by stability, $\deg(\mathrm{Ker}(\varphi)) \leq 0$ or $\varphi = 0$. Similarly, stability implies that $\deg(\mathrm{Coker}(\varphi)) \geq 1$ or φ is trivial. We conclude that either φ is an isomorphism or $\varphi = 0$. If φ is an endomorphism and λ is an eigenvalue of φ, then applying the previous to $\varphi - \lambda \cdot \mathrm{Id}$ we conclude that $\varphi = \lambda \cdot \mathrm{Id}$. This shows that all endomorphisms of W_d are multiplication by scalars. The result follows. □

Now we are ready to construct semi-stable vector bundles of rank n and degree zero.

Proposition 3.1.3 *Let E be an elliptic curve and $p_0 \in E$ an origin for the group law on E. Fix integers $d, n; 1 \leq d \leq n-1$. Let W_d and W_{n-d} be the bundles of the last lemma. Then any vector bundle V over E which fits in a non-trivial extension*

$$0 \to W_d^* \to V \to W_{n-d} \to 0$$

is semi-stable.

Proof. Clearly, any bundle V as above has rank n and degree zero. Suppose that $U \subset V$ is a destabilizing subbundle. Then $\deg(U) \geq 1$. Since W_d^* is stable, the intersection $U \cap W_d^*$ has negative degree (or is trivial). In either case, it is not all of U. This means that the image of U in W_{n-d} is nontrivial and has degree at least one. Since W_{n-d} is stable, this means the image of U in W_{n-d} is all of W_{n-d}. Hence, the degree of U is one more than the degree of $U \cap W_d^*$. Since $U \cap W_d^*$ has negative degree or is trivial, the only way that U can have positive degree is for $U \cap W_d^* = 0$, which would mean that U splits the sequence. This proves that all nontrivial extensions of the form given are semi-stable bundles. □

Lemma 3.1.4 *Suppose that V is a non-trivial extension of W_{n-d} by W_d^*. Then for any line bundle λ over E of degree zero, we have* $\mathrm{Hom}(V, \lambda)$ *has rank either zero or one.*

Proof. The bundle $Hom(W_{n-d}, \lambda)$ is of degree -1 and is semi-stable. Thus, it has no sections. Similarly, $Hom(W_d^*, \lambda)$ has a one-dimensional space of sections. From the long exact sequence

$$0 \to \mathrm{Hom}(W_{n-d}, \lambda) \to \mathrm{Hom}(W, \lambda) \to \mathrm{Hom}(W_d^*, \lambda) \to \cdots$$

we see that $\mathrm{Hom}(V, \lambda)$ has rank at most one. □

Corollary 3.1.5 *If V is a non-trivial extension of W_{n-d} by W_d^*, then V is isomorphic to a direct sum of bundles of the form $\mathcal{O}(q_i - p_0) \otimes I_{r_i}$ for distinct points $q_i \in E$.*

Proof. If V has two irreducible factors of the form $\mathcal{O}(q-p_0) \otimes I_{r_1}$ and $\mathcal{O}(q-p_0) \otimes I_{r_2}$ then $\mathrm{Hom}(V, \mathcal{O}(q-p_0)$ would be rank at least two, contradicting the previous result. □

3.2 Automorphism group of a vector bundle over an elliptic curve

The following is an easy direct exercise:

Lemma 3.2.1 *Let E be an elliptic curve and let L and M be non-isomorphic line bundles of degree zero over E. Then* $\mathrm{Hom}(L,M)=0$. *Also,* $\mathrm{Hom}(L,L)=\mathbf{C}$.

Let V be a semi-stable bundle of degree zero over E. The *support* of V is the subset of points $e\in E$ at which some non-zero section of V vanishes. As we have seen in Atiyah's theorem, every semi-stable vector bundle of degree zero over E decomposes as a direct sum of bundles with support a single point.

Corollary 3.2.2 *Let q,q' be distinct points of E and let V_q and $V'_{q'}$ be vector bundles of degree zero over E supported at q and q' respectively. Then* $\mathrm{Hom}(V_q,V'_{q'})=0$.

Corollary 3.2.3 *Let V be a semi-stable bundle of degree zero over and elliptic curve E and let $V=\oplus_{q\in E}V_q$ be its decomposition into bundles with support at single points. Then* $\mathrm{Hom}(V,V)=\oplus_{q\in E}\mathrm{Hom}(V_q,v_q)$.

Now let us analyze the individual terms in this decomposition.

Lemma 3.2.4 *With I_r as in Lecture 1,* $\mathrm{Hom}(I_r,I_r)$ *is an abelian algebra* $\mathbf{C}[t]/(t^{r+1})$ *of dimension r.*

Proof. Recall that I_r comes equipped with a filtration $\mathcal{F}_0\subset\mathcal{F}_1\subset\cdots\subset\mathcal{F}_r=I_r$ with associated quotients $\mathcal{O}$. The quotient of $I_r/\mathcal{F}_s$ is identified with I_{r-s}. The first thing to prove is that this filtration is preserved under any endomorphism. It suffices by an straightforward inductive argument to show that $\mathcal{F}_1$ is preserved by an endomorphisms. But $\mathcal{F}_1$ is the image of the unique (up to scalar multiples) non-zero section of I_r.

Let $t\colon I_r\to I_r$ be the map of the form $I_r\to I_r/\mathcal{F}_1=I_{r-1}=\mathcal{F}_{r-1}\subset I_r$. Clearly, the image of t^k is contained in $\mathcal{F}_{r-k}$ so that $t^{r+1}=0$. We claim that every endomorphism of I_r is a linear combination of $\{1,t,t^2,\ldots,t^r\}$. Suppose that $f\colon I_r\to\mathcal{F}_{r-s}\subset I_r$. Then there is an induced mapping $I_r/I_{r-1}\to I_{r-s}/I_{r-s-1}$. Since both of these quotients are isomorphic to $\mathcal{O}$, this map and some multiple of the map between these quotients induced by t^s are equal. Subtracting this multiple of t^s from f we produce a map $I_r\to\mathcal{F}_{r-s-1}$. Continuing inductively proves the result. □

Notice that an endomorphism is an automorphism if and only if its image is not contained in I_{r-1} if and only if it is not an element of the ideal generated by t.

From this description the following is easy to establish.

Corollary 3.2.5 *Let V be a semi-stable bundle of degree zero and rank n over an elliptic curve E. Then the dimension of the automorphism group is at least n. It is exactly n if and only if for each $q \in E$, the subbundle $V_q \subset V$ supported by q is of the form $\mathcal{O}(q - p_0) \otimes I_{r(q)}$ for some $r(q) \geq 0$.*

Proof. We decompose V into a direct sum of bundles V_i which themselves are indecomposable under direct sum. Thus, each V_i is of the form $\mathcal{O}(q - p_0) \otimes I_r$ for some $q \in E$ and some $r \geq 1$. Thus, by the previous result the automorphism group of V_i has dimension equal to the rank of V_i. Clearly, then the automorphism group of V preserving this decomposition has dimension equal to the rank of V. This will be the entire automorphism group if and only if $\mathrm{Hom}(V_i, V_j) = 0$ for all $i \neq j$. This will be the case if and only if the V_i have disjoint support. □

Note that if, in the above notation, two or more of the V_i have the same support then the automorphism group has dimension at least two more than the rank of V.

Definition 3.2.6 A semi-stable vector bundle over an elliptic curve whose automorphism group has dimension equal to the rank of the bundle is called a *regular* bundle.

We are now in a position to prove the main result along these lines.

Theorem 3.2.7 *Fix n, d with $1 \leq d < n$. A vector bundle V of rank n can be written as a non-trivial extension*

$$0 \to W_d^* \to V \to W_{n-d} \to 0$$

if and only if

1. *The determinant of V is trivial.*

2. *V is semi-stable.*

3. *V is regular.*

Proof. The first condition is obviously necessary and in Proposition 3.1.3 and Corollary 3.1.5 we established that the second and third are also necessary.

Suppose that V satisfies all the conditions. Condition (iii) says that $V = \oplus_{q\in E} V_q$ where each V_q is of the form $\mathcal{O}(q-p_0)\otimes I_{r(q)}$. We claim that there is a map $W_d \to \mathcal{O}(q-p_0)\otimes I_{r(q)}$ whose image is not contained in $\mathcal{O}(q-p_0)\otimes I_{r(q)-1}$. The reason for this is that $W_d\otimes\mathcal{O}(q-p_0)\otimes I_r$ is stable and has degree r. Thus, its space of sections has dimension exactly r. Applying this with $r = r(q)$ and $r = r(q)-1$ we see that there is a homomorphism $W_d \to \mathcal{O}(q-p_0)\otimes I_{r(q)}$ which does not factor through $\mathcal{O}(q-p_0)\otimes I_{r(q)-1}$.

Claim 3.2.8 *If $\mathcal{F}$ is a subsheaf of V of degree zero and if the projection of $\mathcal{F}$ onto each V_q is not contained in a proper subsheaf, then $\mathcal{F} = V$.*

Proof. Let W be the smallest subbundle of V containing $\mathcal{F}$. It has degree at least zero and is equal to $\mathcal{F}$ if and only if its degree is zero. By stability it has degree zero and hence is equal to $\mathcal{F}$. This shows that $\mathcal{F}$ is in fact a subbundle. Since there are no non-trivial maps between subbundles of V_q and $V_{q'}$ for $q \neq q'$, it follows that any subbundle of $\oplus V_q$ is in fact a direct sum of its intersections with the various V_q. If the image of the subbundle under projection to V_q is all of V_q then its intersection with V_q is all of V_q. The claim now follows. □

Let $W_d^* \to V$ be a map whose projection onto each V_q is not contained in any proper subbundle of V_q. Let us consider the image of this map. It is a subsheaf of V which is proper since $d < n$. This means it has degree at most zero. By the above it cannot be of degree zero. Thus, it is of degree at most -1. Hence the kernel of the map is either trivial or has degree at least 0. This latter possibility contradicts the stability of W_d^*. This shows that the map is an isomorphism onto its image; that is to say it is an embedding of $W_d^* \subset V$.

Next, let us consider the cokernel X. We have already seen that the cokernel is a bundle. Clearly, its determinant is $\mathcal{O}(p_0)$ and its rank is $n-d$. To show that it is W_{n-d} we need only see that it is stable. Suppose that $U \subset X$ is destabilizing. Then the degree of U is positive. Let $\tilde{U} \subset V$ be the preimage of U. It has degree one less than U and hence has degree at least zero. But since it contains the image of W_d, the previous claim implies that it is all of V, which implies that U is all of W_{n-d}, contradicting the assumption that U was destabilizing. □

Now we shall show that, given V, there is only one such extension with V in the middle, up to automorphisms of V.

Proposition 3.2.9 *Let V be a semi-stable rank n-vector bundle with trivial determinant. Suppose that the group of automorphisms of V has dimension n. Then V can be written as an extension*

$$0 \to W_d^* \to V \to W_{n-d} \to 0.$$

This extension is unique up to the action of the automorphism group of V.

Proof. Suppose we have an extension as above for V. First notice that if the image of W_d^* were contained in a proper subsheaf of degree zero of V, then this subsheaf would project into W_{n-d} destabilizing it. Thus, the only subsheaf of degree zero of V that contains the image of W_d^* is all of V. That is to say the image of W_d^* in each V_q is not contained in a lower filtration level, i.e. a proper subsheaf of degree zero. Now we need to show that all maps $W_d^* \to V_q$ which are not contained in a proper subsheaf of degree zero are equivalent under the action of the automorphisms of V_q. This is easily established from the structure of the automorphism sheaf of V_q given above. □

Corollary 3.2.10 *For any $1 \leq d < n$, the projective space of $H^1(E; Hom(W_{n-d}, W_d^*)) = H^1(E; W_{n-d}^* \otimes W_d^*)$ is identified with the space of isomorphism classes of regular semi-stable vector bundles of rank n and trivial determinant.*

Notice that it is not apparent, *a priori*, that for different $d < n$ that the above projective spaces can be identified in some natural manner.

The association to each regular semi-stable vector bundle of rank n and trivial determinant of its S-equivalence class then induces a holomorphic map from $\mathbf{P}(H^1(E; W_{n-d}^* \otimes W_d^*))$ to the coarse moduli space $\mathbf{P}(\mathcal{O}(np_0))$ of S-equivalence classes of such bundles. It follows immediately from Atiyah's theorem that each S-equivalence class contains a unique regular representative up to isomorphism, so that this map is bijective. Since it is a map between projective spaces it is in fact a holomorphic isomorphism. Thus, for any d, $1 \leq d < n$, we can view the projective space of $H^1(E; W_{n-d}^* \otimes W_d^*)$ as yet another description of the coarse moduli space of S-equivalence classes of semi-stable rank n vector bundles of trivial determinant. Notice that the actual bundles produced by this construction are the same as those produced by the spectral covering construction since both are families of regular semi-stable bundles. On the other hand, using the Narasimhan-Seshadri result gives a different set of bundles – namely the direct sum of line bundles. These families agree generically, but differ along the codimension-one subvariety of the parameter space where two or more points come together. We have seen two constructions of holomorphic families of semi-stable vector bundles – the spectral covering construction and the parabolic construction and both create regular semi-stable bundles. It is not clear that the flat connection point of view can be carried out holomorphically in families (indeed it cannot). A hint to this fact is that it is producing different bundles and these do not in general fit together to make holomorphic families.

The reason for the name 'parabolic' will become clear after we extend to the general semi-simple group. Before we can give this generalization we need to discuss parabolic subgroups of a semi-simple group.

3.3 Parabolics in $G_{\mathbf{C}}$

Let $G_{\mathbf{C}}$ be a complex semi-simple group. A *Borel subgroup* of $G_{\mathbf{C}}$ is a connected complex subgroup whose Lie algebra contains a Cartan subalgebra (the Lie algebra of a maximal complex torus) together with the root spaces of all positive roots with respect to some basis of simple roots. All Borel subgroups in $G_{\mathbf{C}}$ are conjugate. By definition *a parabolic subgroup* is a connected complex proper subgroup of $G_{\mathbf{C}}$ that contains a Borel subgroup. Up to conjugation parabolic subgroups of $G_{\mathbf{C}}$ are classified by proper (and possibly empty) subdiagrams of the Dynkin diagram of G. Fix a maximal torus of $G_{\mathbf{C}}$ and a set of simple roots $\{\alpha_1, \ldots, \alpha_n\}$. A subdiagram is given simply by a subset $\{\alpha_1, \ldots, \alpha_r\}$ of the set of simple roots. The Lie algebra of the parabolic is the Cartan subalgebra tangent to the maximal torus, together with all the positive root spaces and all the root spaces associated with negative linear combinations of the $\{\alpha_1, \ldots, \alpha_r\}$. Thus, a Borel subgroup corresponds to the empty subdiagram. The full diagram gives $G_{\mathbf{C}}$ and hence is not a parabolic subgroup. Up to conjugation a parabolic P is contained in a parabolic P' if and only if the diagram corresponding to P is a subdiagram of that corresponding to P'. It follows that the maximal parabolic subgroups of $G_{\mathbf{C}}$ up to conjugation are in one-to-one correspondence with the subdiagrams of the Dynkin diagram of G obtained by deleting a single vertex. This sets up a bijective correspondence between conjugacy classes of maximal parabolic subgroups of $G_{\mathbf{C}}$ and vertices of the Dynkin diagram, or equivalently with the set of simple roots for $G_{\mathbf{C}}$.

A parabolic subgroup P has a maximal unipotent subgroup U whose Lie algebra is the sum of the roots spaces of positive roots whose negatives are not roots of P. This subgroup is normal and its quotient is a reductive group called the Levi factor L of P. There is always a splitting so that P can be written as a semi-direct product $U \cdot L$. The derived subgroup of L is a semi-simple group whose Dynkin diagram is the subdiagram that determined P in the first place. A maximal torus of P is the original maximal torus of G. If P is a maximal parabolic then the character group of P is isomorphic to the integers, and the component of the identity of the center of P is $\mathbf{C}^*$, and any nontrivial character of P is non-trivial on the center. On the level of the Lie algebra the generating character of P is given by the weight dual to the coroot associated with the simple root α_i that is omitted from the Dynkin diagram in order to create the subdiagram that determines P. The value of this weight on any root α is simply the coefficient of α_i in the linear combination of the simple roots which is α. The root spaces of the Lie algebra of P are those ones which the character is non-negative, and the Lie algebra of the unipotent radical is the sum of the root spaces of roots on which this character is positive.

Example: The maximal parabolics of $SL_n(\mathbf{C})$ correspond to nodes of its diagram. Counting from one end we index these by integers $1 \leq d < n$. The parabolic subgroup corresponding to the integer d is the subgroup of block diagonal matrices with the lower left $d \times (n-d)$ block being zero. The Levi factor is the block diagonal matrices or equivalently pairs $(A, B) \in GL_d(\mathbf{C}) \times GL_{n-d}(\mathbf{C})$ with $\det(A) = \det(B)$. A vector bundle with structure reduced to this parabolic is simply a bundle with a rank d subbundle, or equivalently a bundle written as an extension of a rank d bundle by a rank $(n-d)$ bundle. In this case, the unipotent subgroup is a vector group $Hom(\mathbf{C}^{n-d}, \mathbf{C}^d)$.

3.4 The distinguished maximal parabolic

For all simple groups except those of A_n type we shall work with a distinguished maximal parabolic. It is described as follows: If the group is simply laced, then the node of the Dynkin diagram that is omitted is the trivalent one. If the group is non-simply laced, then either vertex which is omitted is the long one connected to the multiple bond. It is easy to see that in all cases the Levi factor of this parabolic is written as the subgroup of a product of GL_{k_i} of matrices with a common determinant.

Examples: (i) For a group of type C_n, there is a unique long root. The Levi factor of the corresponding subgroup is $GL_n(\mathbf{C})$ and the unipotent radical is the self-adjoint maps $\mathbf{C}^n$ to its dual. In terms of complex symplectic $2n$-dimensional bundles, a reduction of structure group of V^{2n} to this parabolic means the choice of a self-annihilating n-dimensional subbundle W^n. This bundle has structure group the Levi factor $GL_n(\mathbf{C})$ of the parabolic. The quotient of the bundle by this subbundle is simply the dual bundle W_n^*. The extension class that determines the bundle and its symplectic form is an element in $H^1(E; \mathrm{SymHom}(W, W^*))$, where SymHom means the self-adjoint homomorphisms.

(ii) For a group of type B_n the distinguished maximal parabolic has Levi factor the subgroup of $GL_{n-1}(\mathbf{C}) \times GL_2(\mathbf{C})$ consisting of matrices of the same determinant. Let us consider the orthogonal group instead of the spin group. Then a reduction in the structure group of an orthogonal bundle V^{2n+1} to this parabolic is a self-annihilating subspace $W_1 \subset V^{2n+1}$ of dimension n. This produces a three term filtration $W_1 \subset W_2 \subset W_3$ where $W_2 = W_1^{\perp}$. Under the orthogonal pairing W_1 and W_3/W_2 are dually paired and W_2/W_1, which is three-dimensional, has a self-dual pairing and is identified with the adjoint of the bundle over the $GL_2(\mathbf{C})$-factor. The subbundle W_1 is the bundle over the $GL_n(\mathbf{C})$-factor of the Levi. There are two levels of extension data one giving the extension comparing $W_1 \subset W_2$ which is an element of $H^1(E; (W_2/W_1)^* \otimes W_1))$ and the other an extension class in $H^1(E; \mathrm{SkewHom}(W_3/W_2, W_1))$, where SkewHom refers to the anti-self adjoint mappings under the given pairing.

(iii) There is a similar description for D_{2n}. Here the Levi factor of the distinguished parabolic is the subgroup of matrices in $GL_{n-2}(\mathbf{C}) \times GL_2(\mathbf{C}) \times GL_2(\mathbf{C})$ consisting of matrices with a common determinant. This time a reduction of the structure group to P corresponds to a self-annihilating subspace W_1 of dimension $n-2$, it is the bundle over the $GL_{n-2}(\mathbf{C})$-factor of the Levi. The quotient W_2/W_1 is four-dimensional and self-dually paired. It is identified with the tensor product of the bundle over one of the $GL_2(\mathbf{C})$-factor with the inverse of the bundle over the other. Once again the cohomology describing the extension data is two step – one giving the extension which is $W_1 \subset W_2$ and the other a self-dual extension class for W_3/W_2 by W_1.

(iv) In the case of E_r, $r = 6, 7, 8$ the Levi factor is the subgroup of matrices in $GL_2(\mathbf{C}) \times GL_3(\mathbf{C}) \times GL_{r-3}(\mathbf{C})$ with the same determinants. It is more difficult to describe what a reduction of the structure group to this parabolic means since we have no standard linear representation to use. For E_6 there is the 27-dimensional representation, which would then have a three-step filtration with various properties.

In the case of $SL_n(\mathbf{C})$ we began with a particular, minimally unstable vector bundle $W_d \times W_{n-d}$ whose structure group has been reduced to the Levi factor of the parabolic subgroup. We then considered extensions

$$0 \to W_d^* \to V \to W_{n-d} \to 0.$$

These extensions have structure group the entire parabolic. They also have the property that modulo the unipotent subgroup they become the unstable bundle $W_d^* \oplus W_{n-d}$ with structure group reduced to the Levi factor L of this maximal parabolic.

3.5 The unipotent subgroup

Let us consider the unipotent subgroup of a maximal parabolic group. Fix a maximal torus T of $G_{\mathbf{C}}$ and a set of simple roots $\{\alpha_1, \ldots, \alpha_n\}$. Suppose that this parabolic is the one determined by deleting the simple root α_i. We begin with its Lie algebra. Consider the direct sum of all the root spaces $\mathfrak{g}^u$ associated with positive roots whose negatives are not roots of P. These are exactly the positive roots which, when expressed as a linear combination of the simple roots have a positive coefficient times α_i. Clearly, these roots form a subset which is closed under addition, in the sense that if the sum of two roots of this type is a root, then that root is also of this type. This means that the sum $\tilde{U}$ of the root spaces for these roots makes a Lie subalgebra of $\mathfrak{g}_{\mathbf{C}}$. Furthermore, there is an integer $k > 0$ such that any sum of at least k roots of this type is not a root. (The integer k can be taken to be the largest coefficient of α_i in any root of $\mathfrak{g}_{\mathbf{C}}$.) This means that the Lie algebra $\tilde{U}$ is in fact nilpotent of index of nilpotency at most k. It follows that the restriction of the exponential map to $\tilde{U}$ is a holomorphic isomorphism from $\tilde{U}$ to a unipotent subgroup $U \subset G_{\mathbf{C}}$. The dimension of this group is equal to the number of roots with positive coefficient on α_i. Furthermore, U is filtered by a chain of normal subgroups $\{1\} \subset U_k \subset U_{k-1} \subset \cdot \subset U_1$ where U_i is the unipotent subgroup whose Lie algebra is the root spaces of roots whose α_i-coefficient is at least i. Clearly, U_k is contained in the center of the group and U_i/U_{i-1} is contained in the center of U/U_i. The entire structure of the unipotent group can be directly read off from the set of roots with positive coefficient on α_i together with the information about which sums of roots are roots.

Examples: (i) For $SL_n(\mathbf{C})$ and any maximal parabolic the filtration is trivial $U_1 = U; U_2 = \{1\}$. The reason is of course that all positive roots are linear combinations of the simple roots with coefficients $0, 1$ only. Thus, U is a vector group. It is $\mathrm{Hom}(\mathbf{C}^d, \mathbf{C}^{n-d})$.

(ii) For groups of type C_n and maximal parabolics obtained by deleting the vertex corresponding to the unique long root, again $U = U_1; U_2 = \{1\}$ (all roots have coefficient ± 1 or 0 on this simple root). The unipotent radical U is the vector group $\wedge^2 \mathbf{C}^n$. For all other maximal parabolics of groups of type C_n, the unipotent radical has a two-step filtration and is not abelian.

(iii) For groups of type B_n and for maximal parabolics obtained by deleting the simple root α_i which is long and which corresponds to a vertex of the double bond in the Dynkin diagram, the filtration is $\{1\} \subset U_2 \subset U_1 = U$. The dimension of U_2 is $(n-1)(n-2)/2$ and the dimension of U_1/U_2 is $2(n-1)$. The Lie bracket mapping $U_1/U_2 \otimes U_1/U_2 \to U_2$ is onto, so that the unipotent group is not a vector group, i.e., it is unipotent but not abelian.

(iv) For groups of type D_n and maximal parabolics obtained by deleting the simple root corresponding to the trivalent vertex, once again the filtration is of length 2: we have $\{1\} \subset U_2 \subset U_1 = U$. The dimension of U_2 is $(n-2)(n-3)/2$ and the dimension of U_1/U_2 is $4(n-2)$. Once again the bracket mapping $U_1/U_2 \otimes U_1/U_2 \to U_2$ is onto, and hence the group is non-abelian.

(iv) Once we leave the classical groups, the filtrations become more complicated. For E_6 the filtration of the unipotent subgroup of the distinguished maximal parabolic is $\{1\} \subset U_3 \subset U_2 \subset U_1$ where the dimension of U_3 is two, the dimension of U_2/U_3 is 9 and the dimension of U_1/U_2 is 18. For E_7 and the distinguished maximal parabolic, the filtration of the unipotent radical begins at U_4 which is three-dimensional and descends to U_1 with U_1/U_2 being 24 dimensional. For E_8 and the distinguished maximal parabolic, the filtration begins at U_6 which is five-dimensional and descends all the way to U_1 with U_1/U_2 being of dimension 30.

3.6 Unipotent cohomology

Fix a simple group $G_{\mathbf{C}}$ and a maximal parabolic subgroup P, and fix a splitting $P = U \cdot L$. Also, fix a holomorphic principal bundle $\eta_L \to E$ with structure the Levi factor L of P. We wish to study holomorphic bundles $\xi \to E$ with structure group P with a given isomorphism $\xi/U \to \eta_L$. Let us choose a covering of the elliptic curve by small analytic open subsets $\{U_i\}$. The bundle η_L is described by a cocycle $n_{ij}: U_i \cap U_j \to L$. A bundle ξ and isomorphism $\xi/U \to \eta_L$ is given by maps $u_{ij}: U_i \cap U_j \to U$ satisfying the cocycle condition:

$$u_{ij}n_{ij}u_{jk}n_{jk} = u_{ik}n_{ik}.$$

Since the $\{n_{ij}\}$ are already a cocycle, we can rewrite this condition as

$$u_{ij}u_{jk}^{n_{ij}} = u_{ik},$$

where $u^n = nun^{-1}$ for $u \in U$ and $n \in L$. This is the twisted cocycle condition associated with the bundle η_L and the action of L (by conjugation) on the unipotent subgroup U. A zero cochain is simply a collection of holomorphic maps $v_i: U_i \to U$. Varying a twisted cocycle $\{u_{ij}\}$ by replacing the coboundary of this zero cochain means replacing it by $v_i u_{ij}(v_j^{-1})^{n_{ij}}$.

The set of cocycles modulo the equivalence relation of coboundary makes a set, denoted $H^1(E; U(\eta_L))$. In fact, it is a pointed set since we have the trivial cocycle: $u_{ij} = 1$ for all i, j. In the case when U is abelian, associated to η_L and the action of L on U (which is linear), there is a vector bundle $U(\eta_L)$. The twisted cocycles modulo coboundaries are exactly the usual Cech cohomology of this vector bundle, $H^1(E; U(\eta_L))$, and hence this cohomology space is in fact a vector group. The general situation is not quite this nice. But since U is filtered by normal subgroups with the associated gradeds being vector groups, we can filter the twisted cohomology and the associated gradeds are naturally the usual cohomology of the vector bundles $H^1(E; (U_i/U_{i-1}(\eta_L))$. In this situation, the entire cohomology $H^1(E; U(\eta_L))$ can be given the structure on an affine space which has an origin, and which is filtered with associated gradeds being vector bundles.

The center of P is $\mathbf{C}^*$ (more precisely, the component of the identity of the center of P) and hence acts on U and on $H^1(E; U(\eta_L))$. This action preserves the origin, and the filtration and on each associated graded is a linear action of homogeneous weight. That weight is given by the index of that filtration level (weight i on U_i/U_{i-1}). It is a general theorem that since all these weights of the $\mathbf{C}^*$-action are positive, there is in fact an isomorphism of this affine space with a vector space in such a way that the $\mathbf{C}^*$-action becomes linearized. In particular, the quotient of $\left(H^1(E;U(\eta_L)) - \{0\}\right)/\mathbf{C}^*$ is isomorphic as a projective variety to a weighted projective space. The dimension of the subprojective space of weight i is equal to one less that the dimension of $H^1(E;U_i/U_{i-1}))$.

This is a fairly formal construction and it is not clear that it has anything to do with stable $G_{\mathbf{C}}$-bundles. Here is a theorem that tells us that using the distinguished maximal parabolic identified above and a special unstable bundle with structure group the Levi subgroup of this parabolic in fact leads to semi-stable $G_{\mathbf{C}}$-bundles. It is a generalization of what we have established directly for vector bundles.

Theorem 3.6.1 *Let $G_{\mathbf{C}}$ be a simply connected simple group and let $P \subset G_{\mathbf{C}}$ be the distinguished parabolic subgroup as above. Then the Levi factor of P is isomorphic to the $L \subset \prod_i GL_{n-i}$ consisting of all $\{A_i \in GL_{n_i}\}_i$ such that $\det(A_i) = \det(A_j)$ for all i,j. Let W_{n_i} be the unique stable bundle of rank n_i and determinant $\mathcal{O}(p_0)$. Then $\eta_L = \times_i W^*_{n_i}$ is naturally a holomorphic principal L-bundle over E. Every principal P-bundle which is obtained from a non-trivial cohomology class in $H^1(E;U(\eta_L))$ becomes semi-stable when extended to a $G_{\mathbf{C}}$-bundle. Cohomology classes in the same $\mathbf{C}^*$-orbit determine isomorphic $G_{\mathbf{C}}$-bundles. This sets up an isomorphism between $\left(H^1(E;U(\eta_L)) - \{0\}\right)/\mathbf{C}^*$ and the coarse moduli space of S-equivalence classes of semi-stable $G_{\mathbf{C}}$-bundles over E. Every $G_{\mathbf{C}}$-bundle constructed this way is regular in the sense that its $G_{\mathbf{C}}$-automorphism group has dimension equal to the rank of G, and any regular semi-stable $G_{\mathbf{C}}$-bundle arises from this construction. Any non-regular semi-stable $G_{\mathbf{C}}$-bundle has automorphism group of dimension at least two more than the rank of G.*

This result gives a different proof of Looijenga's theorem. It identifies the coarse moduli space as weighted projective space associated with a non-abelian cohomology space. It is easy to check given the information about the roots and their coefficients over the distinguished simple root that the weights of this weighted projective space are as given in Looijenga's theorem.

3.7 Exercises:

1. Suppose that we have a non-trivial extension

$$0 \to \mathcal{O} \to X \to W_{d-1} \to 0,$$

where W_{d-1} is as in the first lemma of this lecture. Show that $H^0(E;X)$ is one-dimensional and hence that $H^1(E;X^*)$ is also of dimension one.
2. Let V be a holomorphic vector bundle. Show that $\mathrm{Aut}(V)$ is a complex Lie group and that its Lie algebra is identified with $\mathrm{End}(V) = H^0(V \otimes V^*)$.
3. Show that if V_1 and V_2 are semi-stable bundles over E, then so is $V_1 \otimes V_2$. Compute the degree of $V_1 \otimes V_2$ in terms of the degrees and ranks of V_1 and V_2.
4. Prove Lemma 3.2.1 and Corollary 3.2.2.
5. Show that if V is a semi-stable vector bundle of rank n over E which is not regular, then the dimension of the automorphism group of V is at least $r+2$.
6. Let V_{q_i} be semi-stable vector bundles over E of degree zero and disjoint support. Show that any subbundle of degree zero in $\oplus V_{q_i}$ is in fact a direct sum of subbundles of the V_{q_i}.
7. Show that if any two homomorphisms $W_d^* \to I_r$ which have image not contained in I_{r-1} differ by an automorphism of I_r.
8. Show that a Borel subgroup of $G_{\mathbf{C}}$ is determined by a choice of a maximal torus for $G_{\mathbf{C}}$ and a choice of simple roots for that torus. Show all Borel subgroups of $G_{\mathbf{C}}$ are conjugate.
9. Up to conjugation, describe explicitly all parabolic subgroups of $SL_n(\mathbf{C})$.
10. Let $G_{\mathbf{C}}$ be a semi-simple group. Show that the character group of a maximal parabolic subgroup of $G_{\mathbf{C}}$ is isomorphic to $\mathbf{Z}$. Show that the center of a maximal parabolic subgroup of $G_{\mathbf{C}}$ is one-dimensional.
11. For E_6, E_7, E_8, G_2, F_4 work out the dimensions of the various filtration levels in the unipotent subgroups associated with the distinguished maximal parabolic subgroups.
12. Check that in the formula given for the action by a coboundary on a twisted cocylce that the resulting one-cochain is still a twisted cocycle.
13. For groups of type B_n and D_n and the distinguished parabolic and the given bundle η_L over the Levi factor, compute the cohomology vector spaces $H^1(E;U_2(\eta_L))$ and $H^1(E;U_1/U_2(\eta_L))$.

4 Bundles over Families of Elliptic Curves

In this lecture we will generalize the constructions for the case of vector bundles over an elliptic curve to vector bundles over families of elliptic curves.

4.1 Families of elliptic curves

The first thing that we need to do is to decide what we shall mean by a family of elliptic curves. The best choice for our context is a family of Weierstrass cubic curves. Recall that a single Weierstrass cubic is an equation of the form

$$y^2 = 4x^3 + g_2x + g_3,$$

or written in homogeneous coordinates is given by:

$$zy^2 = 4x^2 + g_2xz^2 + g_3z^3.$$

This equation defines a cubic curve in the projective plane with homogeneous coordinates (x, y, z). The point at infinity, i.e., the point with homogeneous coordinates $(0, 1, 0)$ is always a smooth point of the curve. In the case when the curve is itself smooth, this point is taken to be the identity element of the group law on the curve. More generally, there are only two types of singular curves which can occur as Weierstrass cubics – a rational curve with a single node – which occurs when $\Delta(g_2, g_3) = 0$ where $\Delta(g_2, g_3) = g_2^3 + 27g_3^2$ is the discriminant, and the cubic cusp when $g_2 = g_3 = 0$. In each of these cases the subvariety of smooth points of the curve forms a group ($\mathbf{C}^*$ in the nodal case and $\mathbf{C}$ in the cuspidal case), and again we use the point at infinity as the origin of the group law on the subvariety of smooth points.

Now suppose that we wish to study a family of such cubic curves parametrized by a base B which we take to be a smooth variety. Then we fix a line bundle L over B. We interpret the variables x, y, z as follows: let $\mathcal{E}$ be the three-plane bundle $\mathcal{O}_B \oplus L^2 \oplus L^3$ over B; $z: \mathcal{E} \to \mathcal{O}_B$, $x: \mathcal{E} \to L^2$, and $y: \mathcal{E} \to L^3$ are the natural projections. Furthermore, g_2 is a global section of L^4 and g_3 is a global section of L^6. With these definitions

$$zy^2 - (4x^3 + g_2xz^2 + g_3z^3)$$

is a section of $\mathrm{Sym}^3(\mathcal{E}^*) \otimes L^6$. Its vanishing locus projectivizes to give a subvariety $Z \subset \mathbf{P}(\mathcal{E})$ over B, which fiber-by-fiber is the elliptic curve (possible singular) given by trivializing the bundle L over the point $b \in B$ in question and viewing $g_2(b)$ and $g_3(b)$ as complex numbers so that the above cubic equation with values in L^6 becomes an ordinary cubic equation depending on b. While the actual equation associated to b will of course depend on the trivialization of $L|_{\{b\}}$, the homogeneous cubic curve it defines will be independent of this choice.

Thus, as long as the sections g_2 and g_3 are generic enough so as not to always lie in the discriminant locus, $Z \to B$ is an elliptic fibration (which by definition is a flat family of curves over B whose generic member is an elliptic curve). This family of elliptic curves comes equipped with a choice of base point, i.e., there is a given section of $Z \to B$. It is the section given by $\{z = x = 0\}$ or equivalently, by the section $[L^3] \in \mathbf{P}(\mathcal{O}_B \oplus L^2 \oplus L^3)$. (This is the globalization of the point $(0,1,0)$ in a single Weierstrass curve.) This does indeed define a section σ of $Z \to B$. The image of this section is always a smooth point of the fiber. If we use local fiberwise coordinates $(u = x/y, v = z/y)$ near this section, then the local equation is $v = 4u^3 + g_2uv^2 + g_3v^3$, and its gradient at the point $(0,0)$ points in the direction of the v-axis. This means that along σ the surface Z is tangent to the u-axis. Since what we are calling the u-axis actually has coordinate x/y, these lines fit together to form the line bundle $L^2 \otimes (L^3)^{-1} = L^{-1}$, which then is the normal bundle of σ in Z. This bundle is also of course the relative tangent bundle of the fibers along the section σ. Since the tangent bundle of each fiber is trivialized, it follows that the pushforward, $\pi_* T_{\text{fibers}}$, is isomorphic to L^{-1}. Also important for us will be the relative dualizing sheaf. It is $\pi_* T^*_{\text{fibers}}$. (Of course, as I have presented it, we are working only at smooth fibers. But because the singular curves have sufficiently mild singularities the relative dualizing sheaf is still a line bundle, and in fact is the bundle L.)

We have proved:

Lemma 4.1.1 *Let ν_σ be the normal bundle of σ in Z. Let $\pi\colon Z \to B$ be the natural projection. Then $\nu_\sigma = \mathcal{O}_Z(\sigma)|_\sigma$ and $\pi_*(\mathcal{O}_Z(\sigma)|_\sigma) = \pi_*(\nu_\sigma) = L^{-1}$. The bundle L is the relative dualizing line bundle.*

N.B. The subvariety of B consisting of $b \in B$ for which the Weierstrass curve parametrized by b is singular, resp., a cuspidal curve, is a subvariety. For generic g_2 and g_3 the codimension of these subvarieties are one and two, respectively.

4.2 Globalization of the spectral covering construction

Having said how we shall replace our single elliptic curve by a family of elliptic curves with a section, we now turn to globalizing the vector bundle constructions. Our first attempt at globalizing the previous constructions would be to try to find the analogue for $\underbrace{(E \times \cdots \times E)}_{n-\text{times}}/S_n$. The obvious candidate is $\underbrace{(Z \times_B \cdots \times_B Z)}_{n-\text{times}}/S_n$. This works fine as long as Z is smooth over B but does not give a good result at the singular fibers. There is in fact a way to globalize this construction, at least across the nodes. It involves considering $Z_{\text{reg}} \times_B \cdots \times_B Z_{\text{reg}}$, where Z_{reg} is the open subvariety of points regular in their fibers, and then given an appropriate toroidal compactification at the nodal fibers. I shall not discuss this construction here.

There is however another way to view n points on E which sum to zero, up to permutation. Namely, as we have already seen, these points are naturally the points of the projective space $H^0(E; \mathcal{O}_E(np_0))$. Thus, a better way to globalize is to replace $\mathcal{O}(p_0)$ by $\mathcal{O}_Z(\sigma)$ and thus consider $R^0\pi_*(\mathcal{O}_Z(n\sigma))$. This is a vector bundle of rank n on B. Its associated projective space bundle is then a locally trivial $\mathbf{P}^{n-1}$ bundle over B. The fiber of this projective bundle over a point $b \in B$ is canonically identified with the projective bundle of the linear system $|np_0|$ on E.

As the next result shows, this pushed-forward bundle splits naturally as a sum of line bundles.

Claim 4.2.1 *The bundle $R^0\pi_*(\mathcal{O}_Z(n\sigma))$ is naturally split as a sum of line bundles: $\mathcal{O}_B \oplus L^{-2} \oplus L^{-3} \oplus \cdots \oplus L^{-n}$.*

Proof. By definition we are considering the bundle whose sections over an open subset $U \subset B$ are the analytic functions on $Z|_U$ with poles only along $\sigma \cap (Z|_U)$ and those being of order at most n. We have already at our disposal functions with this property: $1, x, x^2, \ldots, x^{[n/2]}, y, xy, \ldots, x^{[(n-3)/2]}y$. Given any function with this property over U, we can subtract (uniquely) a multiple of one of these basic functions, x^a or $x^a y$, so that the order of the pole is reduced by at least one. The multiple will have a coefficient which is a section of the line bundle L^{-2a} in the first case and L^{-2a+3} in the second. In this way we identify the sections of our vector bundle over U with expressions of the form

$$a_0 + a_1 x + \cdots + a_{[n/2]} x^{[n/2]} + b_0 y + \cdots + b_{[(n-3)/2]} x^{[(n-3)/2]} y.$$

The coefficient of x^a lies in L^{-a} and the coefficient of $x^a y$ lies in $L^{-(2a+3)}$. This identifies the space of sections with the sum $\mathcal{O}_B \oplus L^{-2} \oplus L^{-3} \oplus \cdots \oplus L^{-n}$. $\square$

Notice that a section of this n-plane bundle is then a family of S-equivalence classes of semi-stable bundles on the fibers of $Z//B$, but that it is not yet a vector bundle on Z. Nevertheless, the spectral covering construction generalizes to produce a vector bundle. Let $\mathbf{P}_n$ be the bundle of projective spaces associated to the vector bundle $R^0\pi_*(\mathcal{O}_Z(n\sigma))$. This is the bundle whose fiber over $b \in B$ is the projective space of the linear system $\mathcal{O}_{E_b}(np_0)$. Consider the natural map $\pi^*\pi_*\mathcal{O}_Z(n\sigma) \to \mathcal{O}_Z(n\sigma)$. It is surjective and we denote by $\mathcal{E}$ its kernel which is a vector bundle of rank $n-1$. Define $\mathcal{T} = \mathbf{P}(\mathcal{E})$. A point of $\pi^*\pi_*\mathcal{O}_Z(n\sigma)$ consists of an element $f \in |\mathcal{O}_{E_b}(n\sigma(b))|$ together with a point $z \in E_b$. The fiber $\mathcal{E}$ consists of all pairs for which $f(z) = 0$. The bundle $\mathcal{T}$ is a $\mathbf{P}^{n-2}$-bundle over Z whose fiber over any $z \in E_b$ is the projective space of the linear system $\mathcal{O}_{E_b}(n\sigma(b) - z)$ on E_b. The composition of the inclusion $\mathcal{T} \to \mathbf{P}_n \times_B Z$ followed by the projection onto $\mathbf{P}_n$ is a ramified n-sheeted covering denoted g, which fiber-by-fiber is the map we constructed before for a single elliptic curve.

Using this map we can construct a family of vector bundles over Z semi-stable on each fiber. Namely, we consider the pullback Δ to $\mathcal{T} \times_B Z$ of the diagonal $\Delta_0 \subset Z \times_B Z$. Then we have a line bundle

$$\mathcal{L} = \mathcal{O}_{\mathcal{T} \times_B Z}(\Delta - \mathcal{T} \times_B \sigma).$$

The pushforward $(g \times_B \mathrm{Id})_*(\mathcal{L})$ is a rank n vector bundle on Z which is regular semi-stable and of trivial determinant on each fiber. Analogous to our result for a single curve we have the following universal property for this construction.

Theorem 4.2.2 *Let $\mathcal{U} \to Z$ be a vector bundle which is regular, semi-stable with trivial determinant on each fiber of $Z//B$. Then associating to each $b \in B$ the class of $\mathcal{U}|E_b$ determines a section $s_A\colon B \to \mathbf{P}_n$. Let $\mathcal{T}_A$ be the pullback of $\mathcal{T} \to \mathbf{P}_n$ via this section. Then the natural projection $T_A \to B$ is an n-sheeted ramified covering. Let $\mathcal{L}_A$ be the pullback to $\mathcal{T}_A \times_B Z$ of the line bundle $\mathcal{L}$ over $\mathcal{T} \times_B Z$ by $s_A \times_B \mathrm{Id}$. Then there is a line bundle M over $\mathcal{T}_A$ such that $\mathcal{U}$ is isomorphic to $(g \times \mathrm{Id})_*(\mathcal{L}_A \otimes p_1^*M)$, where $p_1\colon \mathcal{T}_A \times_B Z \to \mathcal{T}_A$.*

Notice that there are in essence two ingredients in this construction: the first is a section A of $\mathbf{P}_n \to B$ and the second is a line bundle over the induced ramified covering T_A of B. The section A is equivalent to the information of the isomorphism class of the bundle on each fiber of $Z//B$. The line bundle over T_A gives us the allowable twists of the bundle on Z which do not change the isomorphism class on each fiber.

This completes the spectral covering construction. It has the advantage that it produces all vector bundles over Z which are regular and semi-stable with trivial determinant on each fiber. Its main drawback is that it does not easily generalize to other simple groups. The construction that does generalize easily is the parabolic construction to which we turn now.

4.3 Globalization of the parabolic construction

It turns out that (except in the case of E_8-bundles and cuspidal fibers) that the parabolic construction of vector bundles globalizes in a natural way. The first step in establishing this is to globalize the bundles W_d which are an essential part of the construction, both for vector bundles and for more general principal G-bundles.

4.3.1 Globalization of the bundles W_d

We define inductively the global versions of the bundles W_d. The globalization of $W_1 = \mathcal{O}_E(p_0)$ is of course $\mathcal{W}_1 = \mathcal{O}_Z(\sigma)$, so that the way we have chosen to globalize curves has already given us a natural globalization of W_1. Clearly, the restriction of this line bundle to any fiber E of $Z//B$ is the bundle $\mathcal{O}_E(p_0)$. (Notice that even if the fiber is singular, p_0 is a smooth point of it, so that $\mathcal{O}_E(p_0)$ still makes sense as a line bundle.)

Claim 4.3.1 *There is, up to non-zero scalar multiples, a unique non-trivial extension*

$$0 \to \pi^*L \to \mathcal{X} \to \mathcal{W}_1 \to 0.$$

The restriction of $\mathcal{X}$ to any fiber is isomorphic to W_2 of that fiber.

Proof. Let us compute the global extension group $\mathrm{Ext}^1(\mathcal{O}_Z(\sigma), \pi^*L)$. Since both the terms are vector bundles, the extension group is identified with the cohomology group $H^1(Z; \mathcal{O}_Z(\sigma)^* \otimes \pi^*L)$. The local-to-global spectral sequence produces an exact sequence

$$0 \to H^1(B; \pi_*(\mathcal{O}_Z(\sigma)^* \otimes \pi^*L) \to H^1(Z; \mathcal{O}_Z(\sigma)^* \otimes \pi^*L)$$
$$\to H^0(B; R^1\pi_*\mathcal{O}_Z(\sigma)^* \otimes \pi^*L) \to H^2(B; \pi_*\mathcal{O}_Z(\sigma)^* \otimes \pi^*L) \to .$$

Since the restriction of $\mathcal{O}_Z(\sigma)^*$ to each fiber is semi-stable of negative degree, it follows that the first term and the fourth term are both zero, and hence we have an isomorphism

$$H^1(\mathcal{O}_Z(\sigma)^*\otimes\pi^*L) \to H^0(B;R^1(\pi_*\mathcal{O}_Z(\sigma)\otimes\pi^*L)) = H^0(B;R^1\pi_*(\mathcal{O}_Z(\sigma))\otimes L).$$

But we have already seen that $R^1\pi_*(\mathcal{O}_Z(-\sigma)) = L^{-1}$, so that we are considering $H^0(B;(L^{-1}\otimes L) = H^0(B;\mathcal{O}_B) = \mathbf{C}$. Since any non-trivial section of this bundle is nonzero at each point, any non-trivial extension class has non-trivial restriction to each fiber and hence any non-trivial extension of the form $0 \to L \to \mathcal{W}_1 \to 0$ restricts to each fiber E_b to give a nontrivial restriction of W_1 by $\mathcal{O}_{E_b}$ and hence restricts to each fiber to give a bundle isomorphic to W_2 on that fiber. □

Now let us continue this construction. The following is easily established by induction.

Proposition 4.3.2 *For each integer $n \geq 1$ there is a bundle $\mathcal{W}_n$ over Z with the following properties:*

1. $\mathcal{W}_1 = \mathcal{O}_Z(\sigma)$
2. *For any $n \geq 2$ we have a non-split exact sequence*
$$0 \to L^{n-1} \to \mathcal{W}_n \to \mathcal{W}_{n-1} \to 0.$$
3. $R^1\pi_*\mathcal{W}_n^* = L^{-n}$.
4. $R^0\pi_*\mathcal{W}_n^* = 0$.

For these bundles the restriction of $\mathcal{W}_n$ to any fiber of $Z//B$ is isomorphic to the bundle W_n of that Weierstrass cubic curve.

Proof. The proof is by induction on d, with the case $d = 1$ being the last claim. Suppose inductively we have constructed W_{d-1} as required. Since W_{d-1}^* is semi-stable of negative degree on each fiber, and since $R^1\pi_*W_{d-1} = L^{1-d}$, it follows by exactly the same local-to-global spectral sequence argument as in the claim that $H^1(W_{d-1}^*\otimes\pi^*L^{d-1}) = H^0(B;L^{1-d}\otimes L^{d-1}) = H^0(B;\mathcal{O}_B) = \mathbf{C}$. Thus, there is a unique (up to scalar multiples) nontrivial extension of the form

$$0 \to L^{d-1} \to \mathcal{X} \to W_{d-1} \to 0$$

and the restriction of this extension to each fiber of $Z//B$ is nontrivial. We let $\mathcal{W}_d$ be the bundle which is such a nontrivial extension. The computations of $R^i\pi_*\mathcal{W}_d^*$ are straightforward from the extension sequence. □

Notice that $\mathcal{W}_n$ is not the only bundle that restricts to each fiber to give W_n. Any bundle of the form $\mathcal{W}_n \otimes \pi^*M$ for any line bundle M on B will also have that property. Since the endomorphism group of W_n is $\mathbf{C}^*$, one shows easily that these are the only bundles with that property.

N.B If we assume that B is simply connected then there are no torsion line bundles on B. In this case requiring that the determinant of $\mathcal{W}_d$ be $\pi^*L^{d(d-1)/2} \otimes \mathcal{O})Z(\sigma)$ will determine $\mathcal{W}_d$ up to isomorphism.

4.3.2 Globalizing the construction of vector bundles

Lemma 4.3.3 $\mathrm{Ext}^1(\mathcal{W}_{n-d}, \mathcal{W}_d^*)$ *is identified with the space of global sections of the sheaf*

$$R^1\pi_*(\mathcal{W}_{n-d}^*, \mathcal{W}_d^*)$$

on B.

Proof. First of all since $\mathcal{W}_{n-d}$ and $\mathcal{W}_d^*$ are vector bundles, we can identify $\mathrm{Ext}^1(\mathcal{W}_{n-d}, \mathcal{W}_d^*)$ with $H^1(Z; \mathcal{W}_{n-d}^* \otimes \mathcal{W}_d^*)$. The local-to-global spectral sequence produces an exact sequence

$$0 \to H^1(B; R^0\pi_*(\mathcal{W}_{n-d}^* \otimes \mathcal{W}_d^*)) \to H^1(Z; \mathcal{W}_{n-d}^* \otimes \mathcal{W}_d^*)$$
$$\to H^0(B; R^1\pi_*(\mathcal{W}_{n-d} \otimes \mathcal{W}_d^*)) \to H^2(B; R^0\pi_*\mathcal{W}_{n-d}^* \otimes \mathcal{W}_d^*).$$

Since $\mathcal{W}_d^*$ and $\mathcal{W}_{n-d}^*$ are both semi-stable and of negative degree on each fiber, the restriction of their tensor product to each fiber has no sections. It follows that $R^0\pi_*(\mathcal{W}_{n-d}^* \otimes \mathcal{W}_d^*)$ is trivial. Thus, we have an isomorphism

$$H^1(Z; \mathcal{W}_{n-d}^* \otimes \mathcal{W}_d^*) \to H^0(B; R^1\pi_*(\mathcal{W}_{n-d} \otimes \mathcal{W}_d^*)),$$

as claimed in the statement. □

Next we need to compute the sheaf $R^1\pi_*(B; \mathcal{W}_d^* \otimes \mathcal{W}_{n-d}^*)$ on B.

Proposition 4.3.4 $R^1\pi_*(B; \mathcal{W}_d^* \otimes \mathcal{W}_{n-d}^*)$ *is a vector bundle and is isomorphic to the direct sum of line bundles* $L \oplus L^{-1} \oplus L^{-2} \oplus \cdots \oplus L^{1-n}$.

First we consider a special case:

Lemma 4.3.5 *$R^1\pi_*(B;\mathcal{O}_Z(-\sigma)\otimes\mathcal{W}^*_{n-1})$ is isomorphic to $L\oplus L^{-1}\oplus L^{-2}\cdots\oplus L^{1-n}$.*

Proof. Let $R^0_{n-1}=R^0\pi_*(\mathcal{O}_Z(\sigma)\otimes\mathcal{W}_{n-1})$. Since the restriction of $\mathcal{O}_Z(\sigma)\otimes\mathcal{W}_{n-1}$ to each fiber is a semi-stable bundle of degree $-n$, R^0_{n-1} is a vector bundle of rank n over B.

The relative dualizing sheaf for $Z//B$ is π^*L and $R^1\pi_*L=\mathcal{O}_B$. Thus, relative Serre duality is a map

$$S\colon R^1\pi_*(\mathcal{O}_Z(-\sigma)\otimes\mathcal{W}^*_{n-1})\to R^0\pi_*(\mathcal{O}_\mathcal{Z}(\sigma)\otimes\mathcal{W}_{\backslash-\infty}\otimes\pi^*\mathcal{L}^{-\infty})$$
$$\otimes R^1\pi_*L=R^0_{n-1}\otimes L^{-1}.$$

Consider the composition of S with the map

$$(R^0_{n-1})^*\otimes L^{-1}\xrightarrow{A}\bigwedge\nolimits^{n-1}R^0_{n-1}\otimes\det(R^0_{n-1})^{-1}\otimes L^{-1}$$
$$\xrightarrow{\mathrm{ev}\otimes\mathrm{Id}\otimes\mathrm{Id}} R^0\pi_*(\det(\mathcal{O}_Z(\sigma)\otimes\mathcal{W}_{n-1}))\otimes\det(R^0_{n-1})^{-1}\otimes L^{-1}$$
$$=R^0\pi_*(\mathcal{O}_Z(n\sigma))\otimes L^{(n-1)(n-2)/2}\otimes\det(R^0_{n-1})^{-1}\otimes L^{-1},$$

where the map A is induced by taking adjoints from the natural pairing

$$R^0_{n-1}\otimes\bigwedge^{n-1}R^0_{n-1}\to\det(R^0_{n-1}),$$

and ev is the map

$$\mathrm{ev}\colon\bigwedge^{n-1}R^0\pi_*(\mathcal{O}_Z(\sigma\otimes\mathcal{W}_{n-1})\to R^0\pi_*(\bigwedge^{n-1}\mathcal{O}_Z(\sigma)\otimes\mathcal{W}_{n-1})$$

obtained by evaluating sections. Clearly, both S and A are isomorphisms. It is not so clear, but it is still true that ev is also an isomorphism. I shall not prove this result – it is somewhat involved but fairly straightforward. A reference is Proposition 3.13 in Vector Bundles over Elliptic Fibrations.

Assuming this result, we see that the vector bundle we are interested in computing differs from $R^0\pi_*(\mathcal{O}_Z(n\sigma)$ by twisting by the line bundle $L^{-1}\otimes\det R^0_{n-1}$.

According to Claim 4.2.1 $R^0(\mathcal{O}_Z(n\sigma))$ splits as a sum of line bundles $\mathcal{O}\oplus L^{-2}\oplus L^{-3}\oplus\cdots\oplus L^{1-n}$. Now to complete the evaluation of $R^1\pi_*(\mathcal{W}_d\otimes\mathcal{W}^*_{n-d})$ we need only to compute the line bundle $\det R^0\pi_*(\mathcal{O}_Z(n\sigma)\otimes\mathcal{W}_{n-1})$.

Claim 4.3.6 $\det R^0\pi_*(\mathcal{O}_Z(n\sigma)\otimes\mathcal{W}_{n-1})$ *is equal to* $L^{(n-2)(n-1)/2-2}$.

Proof. In computing the determinants we can assume that all sequences split. This allows us to replace $\mathcal{W}_{n-1}$ by $\mathcal{O}_Z(\sigma)\oplus L\oplus L^2\cdots\oplus L^{n-2}$. Since $\mathcal{O}_Z(2\sigma)$ sits in an exact sequence

$$0\to\mathcal{O}_Z(\sigma)\to\mathcal{O}_Z(\sigma)\to\mathcal{O}_Z(\sigma)|_\sigma\to 0$$

and since $R^0\pi_*\mathcal{O}_Z(\sigma))=L$ and $R^0\pi_*\mathcal{O}_Z(\sigma)|_\sigma=L^{-1}$, and $R^0\pi_*(\mathcal{O}_Z(\sigma)\otimes\pi^*L^a)=L^{a-1}$, the result follows easily. □

Putting all this together we see that

$$R^1\pi_*(B;\mathcal{O}_Z(-\sigma)\otimes\mathcal{W}_{n-1})=R^0\pi_*(B;\mathcal{O}_Z(n\sigma))\otimes L.$$

This completes the proof of Lemma 4.3.5 □

Now we are ready to complete the proof of Proposition 4.3.4. This is done by induction on d. The case $d=1$ is exactly the case covered by Lemma 4.3.5. Suppose inductively that we have established the result for $\mathcal{W}_d^*\otimes W_{n-d}^*$ for some $d\geq 1$. We consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{W}_d^*\otimes\mathcal{W}_{n-d-1}^* & \longrightarrow & \mathcal{W}_{d+1}^*\otimes\mathcal{W}_{n-d-1}^* & \longrightarrow & L^{-d}\otimes\mathcal{W}_{n-d-1}^* & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{W}_d^*\otimes\mathcal{W}_{n-d}^* & \longrightarrow & \mathcal{W}_{d+1}^*\otimes\mathcal{W}_{n-d}^* & \longrightarrow & L^{-d}\otimes\mathcal{W}_{n-d}^* & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{W}_d^*\otimes L^{1+d-n} & \longrightarrow & \mathcal{W}_{d+1}^*\otimes L^{1+d-n} & \longrightarrow & L^{-d}\otimes L^{1+d-n} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The natural maps $R^1\pi_*(\mathcal{W}_{d+1}^*\otimes L^{1+d-n})\to R^1\pi_*(L^{1-d}\otimes L^{1+d-n})$ and $R^1\pi_*(L^{-d}\otimes\mathcal{W}_{n-d}^*)\to R^1\pi_*(L^{-d}\otimes L^{1+d-n})$ are both isomorphisms. It follows that the images of $R^1\pi_*(\mathcal{W}_{d+1}\otimes\mathcal{W}_{n-d-1}^*)$ and of $R^1\pi_*(\mathcal{W}_d^*\otimes\mathcal{W}_{n-d}^*)$ in $R^1\pi_*(\mathcal{W}_{d+1}^*\otimes\mathcal{W}_{n-d})$ are equal to the kernel of the natural map $R^1\pi_*(\mathcal{W}_{d+1}^*\otimes$

$\mathcal{W}_{n-d}^* \to R^1\pi_*(L^{-d} \otimes L^{1+d-n})$. Since all the bundles in question are semi-stable and of negative degree on each fiber, they all have trivial $R^0\pi_*$. Thus the maps $R^1\pi_*(\mathcal{W}_{d+1} \otimes \mathcal{W}_{n-d-1}^*)$ and $R^1\pi_*(\mathcal{W}_d^* \otimes \mathcal{W}_{n-d}^*)$ to $R^1\pi_*(\mathcal{W}_{d+1}^* \otimes \mathcal{W}_{n-d})$ are injections. It follows that $R^1\pi_*(\mathcal{W}_{d+1} \otimes \mathcal{W}_{n-d-1}^*)$ and $R^1\pi_*(\mathcal{W}_d^* \otimes \mathcal{W}_{n-d}^*)$ are identified This completes the inductive step and hence the proof of the theorem.

4.4 The parabolic construction of vector bundles regular and semi-stable with trivial determinant on each fiber

Let $Z \to B$ be a family of Weierstrass cubic curves with σ the section at infinity. Fix a line bundle M on B and sections t_i of $L^{-i} \otimes M$ for $i = 0, 2, 3, 4, \ldots, n$. Supposing that there is no point of B where all these sections vanish we can construct a vector bundle as follows.

The identification of $\mathrm{Ext}^1(\mathcal{W}_{n-d}, \mathcal{W}_d^*)$ with $L \oplus L^{-1} \oplus L^{-2} \cdots \oplus L^{1-n}$ can be twisted by tensoring with M so as to produce an identification of $\mathrm{Ext}^1(\mathcal{W}_{n-d}, W_d^* \otimes \pi^*M)$ with $M \otimes L \oplus M \otimes L^{-1} \oplus \cdots \oplus M \otimes L^{1-n}$. Thus, the sections t_i determine an element of $\mathrm{Ext}^1(\mathcal{W}_{n-d}, \mathcal{W}_d^* \otimes \pi^*M)$ and hence determine an extension

$$0 \to \mathcal{W}_d^* \otimes \pi^*M \to \mathcal{V} \to \mathcal{W}_{n-d} \to 0.$$

Since we are assuming that not all the sections t_i vanish at the same point of B, the restriction of $\mathcal{V}$ to each fiber is a non-trivial extension of W_{n-d} by W_d^*. Thus, the restriction of $\mathcal{V}$ to each fiber is in fact semi-stable, regular and with trivial determinant.

This parabolic construction thus produces one particular vector bundle associated with each line bundle M on B and each non-zero section of $R^0\pi_*(\mathcal{O}_{\mathcal{Z}}(n\sigma)) \otimes M$. This bundle is automatically regular and semi-stable on each fiber and has trivial determinant on each fiber. Conversely, given the bundle regular and semi-stable and with trivial determinant of each fiber, it determines a section of the projective bundle $\mathbf{P}_n \to B$, to which we can apply the parabolic construction. The result of the parabolic construction may not agree with the original bundle – but they will have isomorphic restrictions to each fiber. Thus, they will differ by twisting by a line bundle on the spectral covering corresponding to the section. That is to say to construct all bundles corresponding to a given section we begin with the one produced by the parabolic construction. The section also gives us a spectral covering $T \to B$. We are then free to twist the bundle constructed by the parabolic construction by any line bundle on T, just as in the spectral covering construction. Thus, the moduli space of bundles that we are constructing fibers over the projective space of $H^1(Z; \mathcal{W}_d^* \otimes \mathcal{W}_{n-d}^*)$ with fibers being Jacobians of the spectral coverings $T \to B$ produced by the section. This twisting corresponds to finding all bundles which agree with the given one fiber-by-fiber. By general theory all such bundles are obtained by twisting with the sheaf of groups $H^1(B; \pi_*(Aut(\mathcal{V}, \mathcal{V}))$.

4.5 Exercises:

1. Show that a Weierstrass cubic has at most one singularity, and that is either a node or a cusp. Show that the cusp appears only if $g_2 = g_3 = 0$. Show that the node appears when $\Delta(g_2, g_3)$, as defined in the lecture, vanishes. Show that the point at infinity is always a smooth point.
2. Show that for any Weierstrass cubic the usual geometric law defines a group structure on the subset of smooth points with the point at infinity being the origin for the group law. Show that this algebraic group is isomorphic to $\mathbf{C}^*$ if the curve is nodal and isomorphic to $\mathbf{C}$ if the curve is cuspidal.
3. Show that any family of Weierstrass cubics is a flat family of curves over the base.
4. Prove Lemma 4.1.1.
5. Describe the singularities of $Z \times_B \times \cdots \times_B Z$ at the nodes and cusps of $Z//B$.
6. Show that if $V \to Z$ is a vector bundle and for each fiber E_b of $Z//B$ we have $H^i(E_b; V|E_b)$ is of dimension k, show that $R^i\pi_*(V)$ is a vector bundle of rank k on B.
7. Let M be a line bundle over B and let $\mathcal{V}$ fit in an exact sequence

$$0 \to \mathcal{W}_d^* \otimes \pi^* M \to \mathcal{V} \to \mathcal{W}_{n-d} \to 0.$$

Compute the Chern classes of $\mathcal{V}$.
8. Show that if V and U are vector bundles over a smooth variety, then $\mathrm{Ext}^1(U, V) = H^1(U^* \otimes V)$.
9. Show that if V is a rank n vector bundle then $\bigwedge^{n-1} V$ is isomorphic to $V^* \otimes \det(V)$.
10. State relative Serre duality and show that it is correctly applied to produce the map S given in the proof of Lemma 4.3.5.
11. Suppose that V is a vector bundle. Show that to first order the deformations of V are given by $H^1(Hom(V, V))$.

5 The Global Parabolic Construction for Holomorphic Principal Bundles

In this section we wish to generalize the parabolic construction to families of Weierstrass cubics. In the last lecture we did this for vector bundles, here we consider principal bundles over an arbitrary semi-simple group $G_{\mathbf{C}}$. This construction will produce holomorphic principal bundles on the total space Z of the family of Weierstrass cubics which have the property that they are regular semi-stable $G_{\mathbf{C}}$-bundles on each fiber of $Z//B$. Of course, this construction can also be viewed as a generalization of the construction given in the third lecture for a single elliptic curve. It is important to note that we do not give an analogue of the spectral covering construction for $G_{\mathbf{C}}$-bundles. We do not know whether such a construction exists for groups other than $SL_n(\mathbf{C})$ and $\mathrm{Sympl}(2n)$.

5.1 The parabolic construction in families

We let $Z \to B$ be a family of Weierstrass cubics with section $\sigma\colon B \to Z$ as before. Let $G_{\mathbf{C}}$ be a simply connected simple group. Fix a maximal torus and a set of simple roots for G, and let $P \subset G$ be the distinguished maximal parabolic subgroup with respect to these choices. Then the Levi factor L of P is isomorphic to the subgroup of a product of general linear groups $\prod_{i=1}^{s} GL_{n_i}$ consisting of matrices with a common determinant. The character group of P and of L is $\mathbf{Z}$ and the generator is the character that takes the common determinant. We consider the bundle $\mathcal{W}_{n_1}^* \times \cdots \times \mathcal{W}_{n_s}^*$. This naturally determines a holomorphic principal L-bundle η_L over Z. Viewed as a bundle over $G_{\mathbf{C}}$ it is unstable since the $G_{\mathbf{C}}$-adjoint bundle associated with this L bundle splits into three pieces: the adjoint $\mathrm{ad}(\eta_L)$ of the L-bundle, the vector bundle associated with the tangent space to the unipotent radical $U_+(\eta_L)$ and the vector bundle associated to the root spaces negative to those in U_+, $U_-(\eta_L)$. The first bundle has degree zero, the second has negative degree and the third has positive degree. The degree of the entire bundle is zero. This makes it clear that $\mathrm{ad}(\eta_L \times_L G_{\mathbf{C}})$ is unstable, and hence according to our definition that $\eta_{\mathbf{C}} \times_L G_{\mathbf{C}}$ is an unstable principal $G_{\mathbf{C}}$-bundle. (Notice that the L-bundle is stable as an L-bundle.)

Once again we are interested in deformations of η_L to P-bundles ξ with identifications $\xi/U = \eta_L$. Just as in the case of a single elliptic curve, these deformations are classified by equivalence classes of twisted cocycles, which we denote by $H^1(Z; U(\eta_L))$. Recall that U is filtered $0 \subset U_n \subset U_{n-1} \subset \cdots \subset U_1 = U$ where the center $\mathbf{C}^*$ acts on U and has homogeneous weight i on U_i/U_{i-1}. Furthermore, U_i/U_{i-1} is abelian and hence is a vector space which lies in the center of U/U_{i-1}. Thus, once again we can filter the cohomology by $H^1(Z; U_i(\eta_L))$ with the associated gradeds being ordinary cohomology of vector bundles $H^1(Z; U_i/U_{i-1}(\eta_L))$. Since $\det(\eta_L)$ as measured with respect to the generating dominant character of P is negative, it follows that $R^0\pi_*(U_i/U_{i-1}(\eta_L))$ is trivial for all i. A simply inductive argument then shows that $R^0\pi_*(U(\eta_L))$ is a bundle of zero dimensional affine spaces over B, and hence $H^0(Z; U(\eta_L))$ has only the trivial element.

A similar inductive discussion shows that $R^1\pi_*(U(\eta_L))$ is filtered with the associated gradeds being the vector bundles $R^1\pi_*(U_i/U_{i-1}(\eta_L))$. This implies that $R^1\pi_*(U(\eta_L))$ is in fact a bundle of affine spaces over B, with a distinguished element – the trivial cohomology class on each fiber. The local-to-global spectral sequence, the vanishing of the $R^0\pi_*(U(\eta_L))$ and an inductive argument shows that in fact the cohomology set $H^1(Z; U(\eta_L))$ is identified with the global sections of $R^1\pi_*(U(\eta_L))$ over B.

5.2 Evaluation of the cohomology group

In all cases except $G = E_8$ and over the cuspidal fibers we can in fact split the bundle $R^1\pi_*(U(\eta_L))$ of affine spaces so that it becomes a direct sum of vector bundles. Under this splitting the $\mathbf{C}^*$ action becomes linear.

Theorem 5.2.1 *Let G be a compact simply connected, simple group and let $Z \to B$ be a family of Weierstrass cubic curves. Assume either that G is not isomorphic to E_8 or that no fiber of $Z//B$ is a cuspidal curve. Then there is an isomorphism $R^1\pi_*U(\eta_L)$ with a direct sum of line bundles $\oplus_i L^{1-d_i}$ where $d_1 = 0$ and $d_2, \ldots, d_r$ are the Casimir weights associated to the group G. Furthermore, the $\mathbf{C}^*$ action that produces the weighted projective space is diagonal with respect to this decomposition and is a linear action on each line bundle.*

Corollary 5.2.2 *The cohomology $H^1(Z; U(\eta_L))$ is identified with the space of sections of a sum of line bundles over B, and hence the space of extensions is identified with a bundle of weighted projective spaces over B. The fibers are weighted projective spaces of type $\mathbf{P}(g_0, g_1, \ldots, g_r)$ where $g_0 = 1$ and for $i = 1, \ldots, r$ the g_i are the coroot integers.*

Here is a table of the Casimir weights grouped by $\mathbf{C}^*$-weights

Group	1	2	3	4
A_n	$0, 2, 3, \ldots, n$			
B_n	$0, 2, 4$	$6, 8, \ldots, 2n$		
C_n	$0, 2, 4, \ldots, 2n$			
D_n	$0, 2, 4, n$	$6, 8, \ldots, 2n-2$		
E_6	$0, 2, 5$	$6, 8, 9$	12	
E_7	$0, 2$	$6, 8, 10$	$12, 14$	18
G_2	$0, 2$	6		
F_4	$0, 2$	$6, 8$	12	

Thus, with this choice of splitting for the unipotent cohomology, a choice of a line bundle M over B and sections t_i of $M \otimes L^{1-d_i}$ will determine a section of $R^1\pi_*(U(\eta_L))$ and a P-bundle over Z deforming the original L-bundle η_L. By construction there will be a given isomorphism from the quotient of the deformed bundle modulo the unipotent subgroup back to η_L. Furthermore, if the sections t_i never all vanish at the same point of B, then the resulting P-bundle will extend to a $G_{\mathbf{C}}$-bundle which is regular and semi-stable on each fiber of $Z//B$. The resulting section of the weighted projective space bundle is equivalent to the data of the S-equivalence class of the restriction of the principal $G_{\mathbf{C}}$-bundle to each fiber of $Z//B$. Of course, since these bundles are regular, it is equivalent to the isomorphism class of the restriction of the $G_{\mathbf{C}}$-bundle to each fiber.

5.3 Concluding remarks

Thus, for each collection of sections we are able to construct a G_{bfC}-bundle which is regular semi-stable on each fiber. The study of all bundles which agree with one of this type fiber-by-fiber is more delicate. From the parabolic point of view, it requires a study of the sheaf $R^1\pi_*(Aut(\xi))$ which can be quite complicated, and is only partially understood at best.

Even assuming this, we are far from knowing the entire story – one would like to have control over the automorphism sheaf so as to find all bundles which are the same fiber-by-fiber. Then one would like to complete the space of bundles by adding those which become unstable on some fibers (but remain semi-stable on the generic fiber). Finally, to complete the space it is surely necessary to add in torsion-free sheaves of some sort. All these issues are ripe for investigation – little if anything is currently known.

The study of these bundles is an interesting problem in its own right. After varieties themselves bundles are probably the next most studied objects in algebraic geometry. Constructions, invariants, classification, moduli spaces are the main sources of interest. The study we have been describing here fits perfectly in that pattern. Nevertheless, from my point of view, there is another completely different motivation for this study. That motivation is the connection with other differential geometric, algebro-geometric, and theoretical physical questions.

The study of stable G-bundles over surfaces is closely related to the study of anti-self-dual connections on G-bundles (this is a variant of the Narasimhan-Seshadri theorem in for surfaces rather than curves and was first established by Donaldson [4]) and whence to the Donaldson polynomial invariants of these algebraic surfaces. Thus, the study described here can be used to compute the Donaldson invariants of elliptic surfaces. These were the first such computations of those invariants, see [5].

More recently, there has been a connection proposed, see [9], between algebraic n-manifolds elliptically fibered over a base B with $E_8 \times E_8$-bundle and algebraic $(n+1)$-dimensional manifolds fibered over the same base with fiber an elliptically fibered $K3$ with a section. The physics of this later set-up is called F-theory. The precise mathematical statements underlying this physically suggested correspondence are not well understood yet, and this work is an attempt to clarify the relationship between these two seemingly disparate mathematical objects. All the evidence to date is extremely positive – the two theories $E_8 \times E_8$-bundles over families elliptic curves line up perfectly as far as we can tell with families of elliptically fibered $K3$-surfaces with sections over the same base. Yet, there is still much that is not understood in this correspondence. Sorting it out will lead to much interesting mathematics around these natural algebro-geometric objects.

References

[1] Adams, J., *Exceptional Lie Groups,*

[2] Borel, A. *Sous-groupes commutatifs et torsion de groupes de Lie compacts connexes,* Tohoku Journal, Ser. 2 **13** (1961), 216-240.

[3] Bourbaki, N., *Eléments de Mathématique, groupes et algèbres de Lie,* Chapt. 4,5,6. Hermann, Paris, 1960 -1975.

[4] Donaldson, S., *Antiselfdual Yang-Mills connections on complex algebraic surfaces and stable bundles,* Proc. London Math. Soc. **3** (1985), 1.

[5] Friedman, R. and Morgan, J., *Smooth Four-Manifolds and Complex Surfaces,* Ergebnisse der Mathematik und ihrer Grenzgebiete 3 Folge, vol. 27, Springer-Verlag, Berlin, Heidelberg, New York, 1994. Friedman

[6] Looijenga, E., *Root systems and elliptic curves,* Invent. Math. **38** (1997), 17.

[7] ______, *Invariant theory for generalized root systems,* Invent. Math. **61** (1980), 1.

[8] Narasimhan, M. and Seshadri, C., *Deformations of the moduli space of vector bundles over an algebraic curve,* Ann. Math (2) **82** (1965), 540.

[9] Vafa, C. *Evidence for F-theory,* hep-th 9602065, Nucl. Phys. B 469 (1996), 403.