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# GENERALIZED LOCAL HOMOLOGY AND COHOMOLOGY FOR LINEARLY COMPACT MODULES

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**Abstract**

We study generalized local homology for linearly compact modules. By duality, we get some properties of generalized local cohomology modules and extend well-known properties of local cohomology of A. Grothendieck.

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# 1 Introduction

It is well-known that the local cohomology theory of A. Grothendieck [8] [2] has proved to be an important tool in algebraic geometry and commutative algebra. Recently, its dual theory of local homology is studied by many mathematicians: Greenlees - May [7], Tarrio - Lopez - Lipman [1], Cuong and the author [4] [5] .... The purpose of this paper is to study the generalized local homology and cohomology for linearly compact modules. Let $I$ be an ideal of the ring $R$ and $M, N$ $R-$modules, the $i-$th generalized local homology module $H_i^I(M,N)$ of $M, N$ with respect to $I$ is defined by

$$H_i^I(M,N) = \varprojlim_t \operatorname{Tor}_i^R(M/I^tM, N).$$

This definition is in some sense dual to J. Herzog's definition of generalized local cohomology modules [9] and in fact a generalized one of the usual local homology. Note that the class of linearly compact modules is big, it contains important classes of modules in commutative algebra. Even its subclass of semi-discrete linearly compact modules contains Artinian modules, moreover it also contains finitely generated modules over a complete ring. Fortunately, inverse limits are exact on the category of linearly compact modules.

We know that (generalized) local cohomology was essentially studied on finitely generated modules. Thus by duality we get some properties of generalized local cohomology modules and extend some well-known properties of local cohomology of A. Grothendieck to linearly compact modules. Throughout, $(R, \mathfrak{m})$ will be a local (Noetherian commutative) ring and has a topological structure. The organization of the paper is as follows.

In Section 2 we recall some basic properties of linearly compact modules that we shall use.

In Section 3 we study generalized local homology for linearly compact modules. We show some basic properties of generalized local homology modules such as: the Noetherianness of generalized local homology modules (Theorem 3.8), the characterization of the $\mathrm{Width}_I(N)$ of a semi-discrete linearly compact module by generalized local homology modules (Theorem 3.15).

The last section is devoted to study duality and generalized local cohomology. In this section the topology on $R$ is the $\mathfrak{m}-$adic topology. Theorem 4.6 gives dual formulas of generalized local homology and cohomology modules. By duality we get some properties of generalized local cohomology modules (corollaries 4.8, 4.10) and extend some well-known properties of local cohomology (corollaries 4.9 and 4.11).

In this paper, the terminology "isomorphism" means "algebraic isomorphism" and the terminology "topological isomorphism" means "algebraic isomorphism with the homomorphisms (and its inverse) are continuous".

## 2 Preliminaries

We begin by recalling briefly the definition of linearly compact modules by the terminology of I.G. Macdonald [11] and some of its basic properties that we shall use.

Let $M$ be a topological $R-$module. $M$ is said to be *linearly topologized* if $M$ has a base of neighborhoods of the zero element $\mathcal{M}$ consisting of submodules. $M$ is called *Hausdorff* if the intersection of all the neighborhoods of the zero element is 0. A Hausdorff linearly topologized $R-$module $M$ is said to be *linearly compact* if $\mathcal{F}$ is a family of closed cosets (i.e., cosets of closed submodules) in $M$ which has the finite intersection property, then the cosets in $\mathcal{F}$ have a non-empty intersection.

It is clear that Artinian $R-$modules are linearly compact and discrete. If $(R, \mathfrak{m})$ is a complete ring, then finite $R-$modules are also linearly compact and discrete ([11, 3.10,7.3]).

**Lemma 2.1.** ([11, §3])

*(i) Let $M$ be a Hausdorff linearly topologized $R-$module and $N$ a closed submodule of $M$. Then $M$ is linearly compact if and only if $N$ and $M/N$ are linearly compact.*

*(ii) Let $f : M \longrightarrow N$ be a continuous homomorphism of Hausdorff linearly topologized $R-$modules. If $M$ is linearly compact, then $f(M)$ is linearly compact and $f$ is a closed map.*

*(iii) The inverse limit of a system of linearly compact $R-$modules and continuous homomorphisms is linearly compact.*

Denote by $\varprojlim\limits_t{}^i$ the $i-th$ right derived functor of the inverse limit $\varprojlim\limits_t$. If $\{M_t\}$ is an inverse system of linearly compact modules with continuous homomorphisms, then $\varprojlim\limits_t{}^1 M_t = 0$ by [10, 7.1]. Therefore we have the following immediate result.

**Lemma 2.2.** *Let*

$$0 \longrightarrow \{M_t\} \longrightarrow \{N_t\} \longrightarrow \{P_t\} \longrightarrow 0$$

*be a short exact sequence of inverse systems of $R-$modules. If $\{M_t\}$ is an inverse system of linearly compact modules with continuous homomorphisms, then the sequence of inverse limits*

$$0 \longrightarrow \varprojlim_t M_t \longrightarrow \varprojlim_t N_t \longrightarrow \varprojlim_t P_t \longrightarrow 0$$

*is exact.*

**Lemma 2.3.** ([5, 2.6]) *Let $M$ be a finitely generated $R-$module and $N$ a linearly compact $R-$module. Then for all $i \geq 0$, $\operatorname{Tor}_i^R(M, N)$ is a linearly compact $R-$module. Moreover, if $f : M \longrightarrow M'$ is a homomorphism of finitely generated $R-$modules, then the induced homomorphism $\psi_{i,N} : \operatorname{Tor}_i^R(M, N) \longrightarrow \operatorname{Tor}_i^R(M', N)$ is continuous.*

In general, the functor $\mathrm{Tor}_i^R(M,-)$ cannot commute with inverse limits. However, in the case $M$ is a finitely generated $R-$module $\mathrm{Tor}_i^R(M,-)$ can commute with inverse limits of inverse systems of linearly compact $R-$modules with continuous homomorphisms.

**Lemma 2.4.** ([5, 2.7]) *If $M$ is a finitely generated $R-$module and $\{N_s\}$ is an inverse system of linearly compact $R-$modules with continuous homomorphisms, then for all $i \geq 0$, $\{\mathrm{Tor}_i^R(M,N_s)\}$ forms an inverse system of linearly compact modules with continuous homomorphisms. Moreover, we have*

$$\mathrm{Tor}_i^R(M, \varprojlim_s N_s) \cong \varprojlim_s \mathrm{Tor}_i^R(M,N_s).$$

Let $I$ be an ideal of $R$, the *i-th local homology* module $H_i^I(M)$ of an $R-$module $M$ with respect to $I$ can be defined as in [4] by

$$H_i^I(M) = \varprojlim_t \mathrm{Tor}_i^R(R/I^t, M).$$

This definition of local homology modules coincides with the definition of J. P. C. Greenlees and J. P. May [7] in the case of linearly compact modules. Note that the local homology modules $H_i^I(M)$ of a linearly compact $R-$module $M$ are also linearly compact $R-$modules. Denote by $\Lambda_I(M) = \varprojlim_t M/I^tM$ the $I-$adic completion of $M$, then $H_0^I(M) \cong \Lambda_I(M)$.

**Lemma 2.5.** ([5, 4.9]) *Let $M$ be a linearly compact $R-$module. The following statements are equivalent:*
*(i) $M$ is $I-$separated, i.e., $\bigcap_{t>0} I^tM = 0$.*
*(ii) $\Lambda_I(M) \cong M$.*
*(iii) $H_0^I(M) \cong M$, $H_i^I(M) = 0$ for all $i > 0$.*

# 3 Generalized local homological modules

Let $I$ be an ideal of the ring $R$ and $M, N$ $R-$modules. In [13], the $i-$th generalized local homology module $H_i^I(M,N)$ of $M, N$ with respect to $I$ is defined by

$$H_i^I(M,N) = \varprojlim_t \mathrm{Tor}_i^R(M/I^tM, N).$$

**Remark 3.1.** (i) Since $\mathrm{Tor}_i^R(M/I^tM,N)$ has a natural structure as a module over the ring $R/I^t$ for all $t > 0$, $H_i^I(M,N)$ has a natural structure as a module over the ring $\Lambda_I(R) = \varprojlim_t R/I^t$.

(ii) In general, $H_i^I(M,N) \neq H_i^I(N,M)$.

**Lemma 3.2.** ([13, 2.3]) *Let $M, N$ be $R-$modules. Then the generalized local homology module $H_i^I(M,N)$ is $I-$separated for all $i \geq 0$, i.e.,*

$$\bigcap_{t>0} I^t H_i^I(M,N) = 0.$$

We have the following basic properties of the generalized local homology modules for linearly compact modules.

**Proposition 3.3.** *(i) If $M$ is a finitely generated $R-$module and $N$ is a linearly compact $R-$module, then for all $i \geq 0$, $H_i^I(M,N)$ is a linearly compact $R-$module.*

*(ii) If $M$ is a linearly compact $R-$module and $N$ is a finitely generated $R-$module, then for all $i \geq 0$, $H_i^I(M,N)$ is a linearly compact $R-$module.*

*Proof.* (i) It follows from 2.3 that $\{\mathrm{Tor}_i^R(M/I^tM,N)\}_t$ forms an inverse system of linearly compact modules with continuous homomorphisms. By 2.1 (iii), $H_i^I(M)$ is also a linearly compact $R-$module.

(ii) It is clear that $\{M/I^tM\}_t$ is an inverse system of linearly compact modules with continuous homomorphisms. Then $\{\mathrm{Tor}_i^R(M/I^tM,N)\}_t$ forms an inverse system of linearly compact modules with continuous homomorphisms by 2.4. Therefore $H_i^I(M)$ is also a linearly compact $R-$module. □

The following lemma is used to prove Proposition 3.5.

**Lemma 3.4.** *If $\{f_t\} : \{M_t\}_t \longrightarrow \{M'_t\}_t$ is a continuous homomorphism of inverse systems of linearly topologized modules, then the induced homomorphism on inverse limits $f : \varprojlim_t M_t \longrightarrow \varprojlim_t M'_t$ is also continuous.*

*Proof.* It follows from [11, 2.4] that the inverse limit of an inverse system of linearly topologized modules is a closed submodule of the direct product. As the homomorphisms $f_t : M_t \longrightarrow M'_t$ are continuous for all $t$, the induced homomorphism on the direct products is continuous. Therefore the induced homomorphism on inverse limits is also continuous. □

**Proposition 3.5.** *(i) If $M$ is a finitely generated $R-$module and $f : N \longrightarrow N'$ is a continuous homomorphism of linearly compact modules, then the induced homomorphism*

*$H_i^I(M,f) : H_i^I(M,N) \longrightarrow H_i^I(M,N')$ is also continuous for all $i \geq 0$.*

*(ii) If $g : M \longrightarrow M'$ is a continuous homomorphism of finitely generated $R-$modules and $N$ is a linearly compact $R-$module, then the induced homomorphism $H_i^I(g,N) : H_i^I(M,N) \longrightarrow H_i^I(M',N)$ is also continuous for all $i \geq 0$.*

*Proof.* (i) The continuous homomorphism $f : N \longrightarrow N'$ induces by 2.4 a continuous homomorphism of inverse systems $\{f_{i,t}\}_t : \{\mathrm{Tor}_i^R(M/I^tM,N)\}_t \longrightarrow \{\mathrm{Tor}_i^R(M/I^tM,N')\}_t$ for all $i \geq 0$. Passing to inverse limits, by 3.4, we have the continuous homomorphism $H_i^I(M,f) : H_i^I(M,N) \longrightarrow H_i^I(M,N')$.

(ii) The continuous homomorphism $g : M \longrightarrow M'$ induces by 2.3 a continuous homomorphism of inverse systems $\{g_{i,t}\}_t : \{\mathrm{Tor}_i^R(M/I^tM,N)\}_t \longrightarrow \{\mathrm{Tor}_i^R(M'/I^tM',N)\}_t$ for all $i \geq 0$. Now the continuousness of the homomorphism $H_i^I(g,N)$ follows from 3.4. □

In general, the functor $H_i^I(M,-)$ cannot commute with inverse limits. However, in the case $M$ is a finitely generated $R-$module, $H_i^I(M,-)$ can commute with inverse limits of inverse systems of linearly compact $R-$modules with continuous homomorphisms.

**Proposition 3.6.** *Let $M$ be a finitely generated $R-$module. If $\{N_s\}$ is an inverse system of linearly compact $R-$modules with the continuous homomorphisms, then*

$$H_i^I(M, \varprojlim_s N_s) \cong \varprojlim_s H_i^I(M, N_s).$$

*Proof.* From 2.4 we have

$$\begin{aligned} H_i^I(M, \varprojlim_s N_s) &= \varprojlim_t \mathrm{Tor}_i^R(M/I^tM, \varprojlim_s N_s) \\ &\cong \varprojlim_t \varprojlim_s \mathrm{Tor}_i^R(M/I^tM, N_s). \end{aligned}$$

Moreover, any two limits commute by [15, 2.26]. Therefore

$$H_i^I(M, \varprojlim_s N_s) \cong \varprojlim_s \varprojlim_t \mathrm{Tor}_i^R(M/I^tM, N_s) = \varprojlim_s H_i^I(M, N_s).$$

The proof is complete. □

The following proposition shows that if $M$ is a finite $R-$module, then the sequence of functors $\{H_i^I(M,-)\}$ is positive strongly connected on the category of linearly compact $R-$modules.

**Proposition 3.7.** *Let $M$ be a finitely generated $R-$module. If*

$$0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N" \longrightarrow 0$$

*is a short exact sequence of linearly compact modules in which the homomorphisms $f, g$ are continuous, then we have a long exact sequence of generalized local homology modules*

$$\cdots \longrightarrow H_i^I(M, N') \longrightarrow H_i^I(M, N) \longrightarrow H_i^I(M, N") \longrightarrow$$

$$\cdots \longrightarrow H_0^I(M, N') \longrightarrow H_0^I(M, N) \longrightarrow H_0^I(M, N") \longrightarrow 0.$$

*Proof.* The short exact sequence of linearly compact modules $0 \longrightarrow N' \longrightarrow N \longrightarrow N" \longrightarrow 0$ gives rise to a long exact sequence of Tor modules for all $t > 0$

$$\cdots \longrightarrow \mathrm{Tor}_i^R(M/I^tM, N') \xrightarrow{f_{i,t}} \mathrm{Tor}_i^R(M/I^tM, N) \xrightarrow{g_{i,t}} \mathrm{Tor}_i^R(M/I^tM, N") \longrightarrow$$

$$\cdots \longrightarrow M/I^tM \otimes_R N' \xrightarrow{f_{0,t}} M/I^tM \otimes_R N \xrightarrow{g_{0,t}} M/I^tM \otimes_R N" \longrightarrow 0.$$

It induces short exact sequences

$$0 \longrightarrow \mathrm{Ker}\, f_{i,t} \longrightarrow \mathrm{Tor}_i^R(M/I^tM, N') \longrightarrow \mathrm{Ker}\, g_{i,t} \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Ker}\, g_{i,t} \longrightarrow \mathrm{Tor}_i^R(M/I^tM, N) \longrightarrow \mathrm{Im}\, g_{i,t} \longrightarrow 0$$

$$0 \longrightarrow \operatorname{Im} g_{i,t} \longrightarrow \operatorname{Tor}_i^R(M/I^tM, N") \longrightarrow \operatorname{Ker} f_{i-1,t} \longrightarrow 0$$

for all $i \geq 0, t > 0$. It follows from 2.3 that the modules of the long exact sequence are linearly compact and the homomorphisms $f_{i,t}, g_{i,t}$ are continuous. Then $\operatorname{Ker} f_{i,t}, \operatorname{Ker} g_{i,t}, \operatorname{Im} g_{i,t}$ are linearly compact by 2.1 (i) (ii). Pass to inverse limits we have the short exact sequences for all $i \geq 0$ by 2.2,

$$0 \longrightarrow \varprojlim_t \operatorname{Ker} f_{i,t} \longrightarrow H_i^I(M, N') \longrightarrow \varprojlim_t \operatorname{Ker} g_{i,t} \longrightarrow 0$$

$$0 \longrightarrow \varprojlim_t \operatorname{Ker} g_{i,t} \longrightarrow H_i^I(M, N) \longrightarrow \varprojlim_t \operatorname{Im} g_{i,t} \longrightarrow 0$$

$$0 \longrightarrow \varprojlim_t \operatorname{Im} g_{i,t} \longrightarrow H_i^I(M, N") \longrightarrow \varprojlim_t \operatorname{Ker} f_{i-1,t} \longrightarrow 0.$$

Thus we get the long exact sequence of generalized local homology modules as required. □

We know that $H_i^I(M, N)$ has the natural structure as a module over the ring $\Lambda_I(R)$. Especially, $H_i^{\mathfrak{m}}(M, N)$ is an $\widehat{R}-$module. Before stating the Noetherianness of the $\widehat{R}-$module $H_i^{\mathfrak{m}}(M, N)$, we recall the concept of *semi-discrete* modules. A Hausdorff linearly topologized $R-$module $M$ is called *semi-discrete* if every submodule of $M$ is closed. Thus a discrete $R-$module is semi-discrete. The class of semi-discrete linearly compact modules contains all Artinian modules. Moreover, it also contains all finitely generated modules in case $R$ is a complete local ring ([11, 7.3]). The following is the theorem for the Noetherianness of the $\widehat{R}-$module $H_i^{\mathfrak{m}}(M, N)$.

**Theorem 3.8.** *Let $M$ be a finitely generated $R-$module and $N$ a semi-discrete linearly compact $R-$module, then $H_i^{\mathfrak{m}}(M, N)$ is a Noetherian $\widehat{R}-$module for all $i \geq 0$.*

To prove Theorem 3.8, we need the following lemmas.

**Lemma 3.9.** *Let $N$ be a semi-discrete linearly compact $R-$module. Then $IN = N$ if and only if there is an $x \in I$ such that $xN = N$.*

*Proof.* It should be noted that $N$ can be written as a finite sum of sum-irreducible $R-$modules by [18, 1(L3)]. Then the result follows from [3, 2.9]. □

**Lemma 3.10.** *Let $N$ be a linearly compact $R-$module and $K$ a closed submodule of $N$. Then $N$ is complete in $I-$adic topology (i.e., $\Lambda_I(N) \cong N$) if and only if $K$ and $N/K$ are complete in $I-$adic topology.*

*Proof.* ($\Rightarrow$) Assume that $N$ is complete in $I-$adic topology. By 2.5 we only need to show that $K$ and $N/K$ are $I-$separated. It is clear that $\bigcap_t I^tK \subset \bigcap_t I^tN = 0$. On the other hand, $\bigcap_t I^t(N/K) = \bigcap_t (I^tN + K)/K = ((\bigcap_t I^tN) + K)/K = 0$ by [11, 3.13]. Hence $K$ and $N/K$ are complete in $I-$adic topology.

($\Leftarrow$) Suppose that $K$ and $N/K$ are complete in $I-$adic topology. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & K & \longrightarrow & N & \longrightarrow & N/K & \longrightarrow 0 \\ & \downarrow \varphi_K & & \downarrow \varphi_N & & \downarrow \varphi_{N/K} & \\ 0 \longrightarrow & \Lambda_I(K) & \longrightarrow & \Lambda_I(N) & \longrightarrow & \Lambda_I(N/K) & \longrightarrow 0. \end{array}$$

The first row is clear exact. The exactness of the second row follows from the long exact sequence of local homology modules and 2.5(iii). Since the homomorphisms $\varphi_K$ and $\varphi_{N/K}$ are isomorphisms, the homomorphism $\varphi_N$ is also an isomorphism. $\square$

**Lemma 3.11.** *Let $M, N$ be $R-$modules. There is a surjective homomorphism*

$$M \otimes_R \Lambda_I(N) \longrightarrow H_0^I(M, N).$$

*Proof.* We have $\Lambda_I(N) = \varprojlim_t N/I^tN$ and $H_0^I(M,N) = \varprojlim_t (M/I^tM \otimes_R N) \cong \varprojlim_t (M \otimes_R N/I^tN)$. By [12, 6.2], there is a surjective homomorphism $M \otimes_R \varprojlim_t N/I^tN \longrightarrow \varprojlim_t (M \otimes_R N/I^tN)$. This finishes the proof. $\square$

In [13, 2.6] we showed that if $M$ is a finitely generated $R-$module and $N$ an Artinian $R-$module such that $N$ is complete in $I-$adic topology (i.e., $\Lambda_I(N) \cong N$), then $\mathrm{Tor}_i^R(M,N) \cong H_i^I(M,N)$ for all $i \geq 0$. We have the following general result for linearly compactmodules.

**Lemma 3.12.** *Let $M$ be a finitely generated module and $N$ a linearly compact $R-$module. If $N$ is complete in $I-$adic topology (i.e., $\Lambda_I(N) \cong N$), then there is an isomorphism for all $i \geq 0$,*

$$\mathrm{Tor}_i^R(M,N) \cong H_i^I(M,N).$$

*Proof.* Let $\mathcal{C}$ be a nuclear base of $N$. It follows from [11, 4.7] that $N = \varprojlim_{U \in \mathcal{C}} (N/U)$ where $N/U (U \in \mathcal{C})$ are Artinian $R-$modules. By 3.6, $H_i^I(M,N) \cong \varprojlim_{U \in \mathcal{C}} H_i^I(M, N/U)$. In virtue of 3.10, $N/U$ is complete in $I-$adic topology. Then $H_i^I(M, N/U) \cong \mathrm{Tor}_i^R(M, N/U)$ by [13, 2.6]. From 2.4, we have

$$H_i^I(M,N) \cong \varprojlim_{U \in \mathcal{C}} \mathrm{Tor}_i^R(M, N/U) \cong \mathrm{Tor}_i^R(M, \varprojlim_{U \in \mathcal{C}} N/U) \cong \mathrm{Tor}_i^R(M,N). \qquad \square$$

The following is a criterion for a module to be Noetherian.

**Lemma 3.13.** ([5, 6.3]) *Let $J$ be a finitely generated ideal of a commutative ring $A$ such that $A$ is complete in the $J-$adic topology and $M$ an $A-$module. If $M/JM$ is a Noetherian $A-$module and $M$ is $J-$separated (i.e., $\bigcap_{t>0} J^tM = 0$), then $M$ is a Noetherian $A-$module.*

We can now prove Theorem 3.8.

*Proof of Theorem 3.8.* We use induction on $i$. When $i = 0$, there is a surjective homomorphism by 3.11

$$M \otimes_R \Lambda_{\mathfrak{m}}(N) \longrightarrow H_0^{\mathfrak{m}}(M,N).$$

Thus it is sufficient to show that $M \otimes_R \Lambda_{\mathfrak{m}}(N)$ is a Noetherian $\widehat{R}-$module. Indeed, as $N$ is a semi-discrete linearly compact $R-$module, $N/\mathfrak{m}N$ is also a semi-discrete linearly compact $k-$module ($k = R/\mathfrak{m}$ the residue field). From [11, 5.2] $N/\mathfrak{m}N$ is a finite dimensional vector $k-$space. Then $\Lambda_{\mathfrak{m}}(N)$ is a Noetherian $\widehat{R}-$module by [6, 7.2.9]. As $M$ is a finitely generated $R-$module, there is a surjective homomorphism $R^n \longrightarrow M$ where $n$ is a positive integer. It induces a surjective homomorphism $\Lambda_I(N)^n \longrightarrow M \otimes_R \Lambda_I(N)$. Hence $M \otimes_R \Lambda_{\mathfrak{m}}(N)$ is also a Noetherian $\widehat{R}-$module.

Let $i > 0$. Set $K = \bigcap_{t>0} \mathfrak{m}^t N$, the short exact sequence of linearly compact $R-$modules

$$0 \longrightarrow K \longrightarrow N \longrightarrow N/K \longrightarrow 0$$

induces by 3.7 an exact sequence of generalized local homology modules

$$\ldots \longrightarrow H_i^{\mathfrak{m}}(M,K) \longrightarrow H_i^{\mathfrak{m}}(M,N) \longrightarrow H_i^{\mathfrak{m}}(M,N/K) \longrightarrow \ldots.$$

It is clear that $N/K$ is $\mathfrak{m}-$separated, thus $N/K$ is complete in $\mathfrak{m}-$adic topology by 2.5. In virtue of 3.12, there is an isomorphism $\mathrm{Tor}_i^R(M,N/K) \cong H_i^I(M,N/K)$ for all $i \geq 0$. By our claim above, $N/K$ is a Noetherian $\widehat{R}-$module, hence $\mathrm{Tor}_i^R(M,N/K)$ is a Noetherian $\widehat{R}-$module. It follows $H_i^I(M,N/K)$ is also a Noetherian $\widehat{R}-$module. Therefore the proof will be complete if we show that $H_i^{\mathfrak{m}}(M,K)$ is a Noetherian $\widehat{R}-$module. From [11, 3.15] we have $\mathfrak{m}K = K$. Thus there is an element $x \in \mathfrak{m}$ such that $xK = K$ by 3.9 and we have the short exact sequence

$$0 \longrightarrow 0 :_K x \longrightarrow K \xrightarrow{.x} K \longrightarrow 0.$$

It gives rise to a long exact sequence of generalized local homology modules

$$\ldots \longrightarrow H_i^{\mathfrak{m}}(M,K) \xrightarrow{.x} H_i^{\mathfrak{m}}(M,K) \xrightarrow{\alpha} H_{i-1}^{\mathfrak{m}}(M, 0 :_K x) \longrightarrow \ldots.$$

If $0 :_K x = 0$, then $H_i^{\mathfrak{m}}(M,K) = xH_i^{\mathfrak{m}}(M,K) = \bigcap_{t>0} x^t H_i^{\mathfrak{m}}(M,K) = 0$ by 3.2(i). In the case $0 :_K x \neq 0$, it follows from the inductive hypothesis that $H_{i-1}^{\mathfrak{m}}(M, 0 :_K x)$ is a Noetherian $\widehat{R}-$module. Set $H = H_i^{\mathfrak{m}}(M,K)$, we have $H/xH \cong \operatorname{Im}\alpha \subseteq H_{i-1}^{\mathfrak{m}}(M, 0 :_K x)$. Thus $H/xH$ is a Noetherian $\widehat{R}-$module, hence $H/\hat{\mathfrak{m}}H$ is also a Noetherian $\widehat{R}-$module ($\hat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$). Moreover, from 3.2(i) we have $\bigcap_{t>0} \hat{\mathfrak{m}}^t H = \bigcap_{t>0} \mathfrak{m}^t H = 0$. Therefore $H$ is a Noetherian $\widehat{R}-$module by 3.13. The proof is complete. □

A sequence of elements $x_1, \ldots, x_r$ in $R$ is said to be an $N-$*coregular* sequence ([14, 3.1]) if $0 :_N (x_1, \ldots, x_r) \neq 0$ and $0 :_N (x_1, \ldots, x_{i-1}) \xrightarrow{x_i} 0 :_N (x_1, \ldots, x_{i-1})$ is surjective for $i = 1, \ldots, r$. We denote by $\mathrm{Width}_I(N)$ the length of the longest $N-$coregular sequence in $I$. In case $N$ is a semi-discrete linearly compact $R-$module, we know that $\mathrm{Width}_I(N) < \infty$ ([5, §5]). To state the characterization of the $\mathrm{Width}_I(N)$ by generalized local homology, the following lemma is necessary.

**Lemma 3.14.** *Let $M$ be a finitely generated $R-$module and $N$ a semi-discrete linearly compact $R-$module. Then $H_0^I(M,N)=0$ if and only if $xN=N$ for some $x\in I$.*

*Proof.* Assume that there is an $x\in I$ such that $xN=N$. Then $IN=N$, thus $\Lambda_I(N)=0$. On the other hand, by 3.11, we have a surjective homomorphism $M\otimes_R\Lambda_I(N)\longrightarrow H_0^I(M,N)$. It follows that $H_0^I(M,N)=0$.

We now suppose that there is no element $x\in I$ such that $xN=N$. It follows from 3.9 that $IN\neq N$, so $\Lambda_I(N)\neq 0$. As $M$ is finitely generated, there is a surjective homomorphism $R^n\longrightarrow M$ where $n$ is a positive integer. It induces a surjective homomorphism $\Lambda_I(N)^n\longrightarrow M\otimes_R\Lambda_I(N)$. Thus $M\otimes_R\Lambda_I(N)\neq 0$. Again from the surjective homomorphism $M\otimes_R\Lambda_I(N)\longrightarrow H_0^I(M,N)$, we have $H_0^I(M,N)\neq 0$. □

The following is the characterization of the $\mathrm{Width}_I(N)$ by generalized local homology.

**Theorem 3.15.** *Let $M$ be a finitely generated $R-$module and $N$ a semi-discrete linearly compact $R-$module such that $0:_N I\neq 0$. Then all maximal $N-$coregular sequences in $I$ have the same length. Moreover*

$$\mathrm{Width}_I(N)=\inf\{i/H_i^I(M,N)\neq 0\}.$$

*Proof.* It is sufficient to show that if $(x_1,x_2,\ldots,x_n)\subseteq I$ is a maximal $N-$coregular sequence, then

(i) $H_i^I(M,N)=0$ for all $i<n$, and

(ii) $H_n^I(M,N)\neq 0$.

Let us prove both (i) and (ii) by induction on $n$.

When $n=0$, there does not exist any $x$ in $I$ such that $xN=N$. Thus $H_0^I(M,N)\neq 0$ by 3.14.

Let $n>0$. The short exact sequence of linearly compact $R-$modules

$$0\longrightarrow 0:_N x_1\longrightarrow N\xrightarrow{x_1} N\longrightarrow 0$$

gives rise to a long exact sequence of generalized local homology modules

$$\ldots\longrightarrow H_i^I(M,0:_N x_1)\longrightarrow H_i^I(M,N)\xrightarrow{x_1} H_i^I(M,N)\longrightarrow H_{i-1}^I(M,0:_N x_1)\longrightarrow\ldots.$$

By the inductive hypothesis, we have $H_i^I(M,0:_N x_1)=0$ for all $i<n-1$ and $H_{n-1}^I(M,0:_N x_1)\neq 0$. It follows that $H_i^I(M,N)=x_1H_i^I(M,N)$ for all $i<n$. Hence $H_i^I(M)=\bigcap_{t>0}x_1^tH_i^I(M)=0$ for all $i<n$ by 3.2(ii). We now have the exact sequence

$$\ldots\longrightarrow H_n^I(M,N)\xrightarrow{x_1} H_n^I(M,N)\longrightarrow H_{n-1}^I(M,0:_N x_1)\longrightarrow 0.$$

As $H_{n-1}^I(M,0:_N x_1)\neq 0$, we get $H_n^I(M,N)\neq 0$. The proof is complete. □

## 4 Duality and generalized local cohomology

We now study the Matlis and Macdonald dualities of generalized local homology and cohomology for linearly compact modules. Suppose now that the topology on $R$ is the $\mathfrak{m}-$adic topology.

Let $M$ be an $R-$module and $E(R/\mathfrak{m})$ the injective envelope of $R/\mathfrak{m}$. It is well-known that the *Matlis dual* of $M$ is the module $D(M) = \mathrm{Hom}(M, E(R/\mathfrak{m}))$. If $M$ is a Hausdorff linearly topology $R-$module, then the Macdonald dual of $M$ is defined by $M^* = Hom(M, E(R/\mathfrak{m}))$ the set of continuous homomorphisms of $R-$modules ([11, §9]). It is clear that $M^* \subseteq D(M)$, the following lemma shows that the equality is true when $M$ is semi-discrete.

**Lemma 4.1.** ([11, 5.8]) *A Hausdorff linearly topologized $R-$module $M$ is semi-discrete if and only if $D(M) = M^*$.*

A Hausdorff linearly topologized $R-$module is $\mathfrak{m}-$*primary* if each element of $M$ is annihilated by a power of $\mathfrak{m}$. A Hausdorff linearly topologized $R-$module $M$ is *linearly discrete* if every $\mathfrak{m}-$primary quotient of $M$ is discrete. It should be noted that if $M$ is linearly discrete, then $M$ is semi-discrete. The direct limit of a direct system of linearly discrete $R-$modules is linearly discrete ([11, 6.2, 6.7]).

**Lemma 4.2.** ([11, 9.12, 9.13]) *Let $(R, \mathfrak{m})$ be a complete ring.*

*(i) If $M$ is linearly compact, then $M^*$ is linearly discrete (hence semi-discrete). If $M$ is semi-discrete, then $M^*$ is linearly compact.*

*(ii) If $M$ is linearly compact or linearly discrete, then we have a topological isomorphism $M^{**} \cong M$.*

To state the dual theorem, we need the following lemmas.

**Lemma 4.3.** ([5, 7.7]) *Let $M$ be a finitely generated $R-$module and $N$ a linearly compact $R-$module. Then*

$$(\mathrm{Tor}_i^R(M, N))^* \cong \mathrm{Ext}_R^i(M, N^*),$$

$$\mathrm{Tor}_i^R(M, N^*) \cong (\mathrm{Ext}_R^i(M, N))^*$$

*for all $i \geq 0$.*

It is well known that the generalized local cohomology modules $H_I^i(M, N)$ of $M, N$ was introduced by J. Herzog by defining $H_I^i(M, N) = \varinjlim_t \mathrm{Ext}_R^i(M/I^tM, N)$ ([9], [17]). The following lemma shows that generalized local cohomology modules $H_I^i(M, N)$ are linearly discrete when $M$ is a semi-discrete linearly compact $R-$module.

**Lemma 4.4.** *Let $(R, \mathfrak{m})$ be a complete ring and $M$ a finitely generated $R-$module. If $N$ is a semi-discrete linearly compact $R-$module, then the generalized local cohomology module $H_I^i(M, N)$ is a linearly discrete $R-$module for all $i \geq 0$.*

*Proof.* We first note that if $N$ is a semi-discrete linearly compact $R-$module, then $N$ is linearly discrete. Indeed, as $N$ is a linearly compact $R-$module, we have a topological isomorphism $N \cong N^{**}$ by 4.2 (ii). Also by 4.2 (i), $N^{**}$ is linearly discrete.

Now as $M/I^tM$ is finitely generated, there is a free resolution $\mathbf{F}_{\bullet,t}$ of $M/I^tM$ with the finitely generated free modules for all $t > 0$,

$$\mathbf{F}_{\bullet,t} : \ldots \longrightarrow F_{i,t} \longrightarrow F_{i-1,t} \longrightarrow \ldots.$$

For each $i \geq 0$, we have $\mathrm{Ext}^i_R(M/I^tM, N) = H^i(\mathrm{Hom}_R(\mathbf{F}_{\bullet,t}, M))$, a quotient of closed submodules of $\mathrm{Hom}(F_{i,t}, M)$. Since $\mathrm{Hom}(F_{i,t}, M)$ is a finite product of copies of $M$, it is a semi-discrete linearly compact $R-$module. Hence $\{\mathrm{Ext}^i_R(M/I^tM, N)\}_t$ is a direct system of semi-discrete linearly compact $R-$modules. Moreover, the homomorphisms of this direct system are continuous. By the notation above, $\{\mathrm{Ext}^i_R(M/I^tM, N)\}_t$ is the direct system of linearly discrete modules. Therefore $H^i_I(M) = \varinjlim_t \mathrm{Ext}^i_R(R/I^t, M)$ is linearly discrete for all $i \geq 0$ by [11, 6.7]. □

**Corollary 4.5.** *Let $(R, \mathfrak{m})$ be a complete ring and $M$ a finitely generated $R-$module. If $N$ is a semi-discrete linearly compact $R-$module, then $(H^i_I(M, N))^* = D(H^i_I(M, N))$.*

*Proof.* By [11, 6.2], a linearly discrete $R-$module is semi-discrete. Thus the equality follows from 4.4 and 4.1. □

The following is the dual theorem.

**Theorem 4.6.** *Let $M$ be a finitely generated $R-$module.*

*(i) If $N$ is a linearly compact $R-$module, then for all $i \geq 0$,*

$$H^I_i(M, N^*) \cong (H^i_I(M, N))^*.$$

*Moreover, if $(R, \mathfrak{m})$ is a complete ring, then*

$$H^i_I(M, N^*) \cong (H^I_i(M, N))^*.$$

*(ii) If $(R, \mathfrak{m})$ is a complete ring and $N$ a semi-discrete linearly compact $R-$module, then we have topological isomorphisms of $R-$modules for all $i \geq 0$,*

$$H^i_I(M, N^*) \cong (H^I_i(M, N))^*,$$

$$H^I_i(M, N^*) \cong (H^i_I(M, N))^*.$$

*Proof.* (i) We begin by proving the first isomorphism. An analysis similar to that in the proof of 4.4 shows that $\{\mathrm{Ext}^i_R(M/I^tM, N)\}$ is a direct system of linearly compact $R-$modules with continuous homomorphisms. Then $\varprojlim_t (\mathrm{Ext}^i_R(M/I^tM,N))^* \cong (\varinjlim_t \mathrm{Ext}^i_R(M/I^tM,N))^*$ by [11, 2.6]. Thus we have by 4.3,

$$\begin{aligned} H^I_i(M^*) &= \varprojlim_t \mathrm{Tor}^R_i(M/I^tM,N^*) \\ &\cong \varprojlim_t (\mathrm{Ext}^i_R(M/I^tM,N))^* \\ &\cong (\varinjlim_t \mathrm{Ext}^i_R(M/I^tM,N))^* = H^i_I(M)^*. \end{aligned}$$

Now we prove the second isomorphism. From 2.3 $\{\mathrm{Tor}^R_i(M/I^tM,N)\}$ is an inverse system of linearly compact $R-$modules with continuous homomorphisms. Then $\varinjlim_t (\mathrm{Tor}^R_i(M/I^tM,N))^* \cong (\varprojlim_t \mathrm{Tor}^R_i(M/I^tM,N))^*$ by [11, 9.14]. Thus we get

$$\begin{aligned} H^i_I(M,N^*) &= \varinjlim_t \mathrm{Ext}^i_R(M/I^tM,N^*) \\ &\cong \varinjlim_t (\mathrm{Tor}^R_i(M/I^tM,N))^* \\ &\cong (\varprojlim_t \mathrm{Tor}^R_i(M/I^tM,N))^* = (H^I_i(M,N))^*. \end{aligned}$$

(ii) Let us prove the first topological isomorphism. We proved in part (i) that it is the algebraic isomorphism. By [11, 6.8], we only need to show that both $H^i_I(M,N^*)$ and $(H^I_i(M,N))^*$ are linearly discrete. Indeed, combining 3.3(i) with 4.2 (i) yields that $H^I_i(M,N)^*$ is linearly discrete. Also by 4.2 (i), $N^*$ is semi-discrete linearly compact. Then the local cohomology module $H^i_I(M,N^*)$ is linearly discrete, because of 4.4. Thus the first topological isomorphism is proved completely. The second topological isomorphism follows from the first isomorphism and 4.2(ii). ☐

**Corollary 4.7.** *Let $(R,\mathfrak{m})$ be a complete local ring.*
*(i) If $M$ is a linearly compact $R-$module, then for all $i \geq 0$,*

$$H^i_I(M,N) \cong (H^I_i(M,N^*))^*,$$

$$H^I_i(M,N) \cong (H^i_I(M,N^*))^*.$$

*(i) If $M$ is a semi-discrete linearly compact $R-$module, then we have topological isomorphisms of $R-$modules for all $i \geq 0$,*

$$H^i_I(M,N) \cong (H^I_i(M,N^*))^*,$$

$$H^I_i(M,N) \cong (H^i_I(M,N^*))^*.$$

*Proof.* (i) Follows from 4.6(i) and 4.2(ii).
(ii) Follows from 4.6(ii) and 4.2(ii). ☐

From the properties of the generalized local homology module $H^{\mathfrak{m}}_i(M,N)$, by duality, we get some properties of the generalized local cohomology module $H^i_{\mathfrak{m}}(M,N)$.

**Corollary 4.8.** *Let $M$ be a finitely generated $R-$module and $N$ a semi-discrete linearly compact $R-$module. Then the generalized local cohomology module $H^i_{\mathfrak{m}}(M,N)$ is an Artinian $R-$module for all $i \geq 0$.*

*Proof.* Let us first prove in the case $(R,\mathfrak{m})$ is a complete ring. It follows from 4.2(i) that $M^*$ is a semi-discrete linearly compact $R-$module. By 3.8, $H_i^{\mathfrak{m}}(M,N^*)$ is a Noetherian $R-$module for all $i \geq 0$. In virtue of 4.6(i), $(H^i_{\mathfrak{m}}(M,N))^*$ is also a Noetherian $R-$module. Thus $D(H^i_{\mathfrak{m}}(M,N))$ is a Noetherian $R-$module by 4.5. From [16, 3.4.12] $H^i_{\mathfrak{m}}(M,N)$ is an Artinian $R-$module.

Let $(R,\mathfrak{m})$ be a local ring. From [17, 1.3,1.5,1.6] we have an isomorphim $D_{\hat{R}}(H^i_{\mathfrak{m}}(M,N)) \cong D_{\hat{R}}(H^i_{\hat{\mathfrak{m}}}(\widehat{M},\widehat{N}))$. By our claim above $H^i_{\hat{\mathfrak{m}}}(\widehat{M},\widehat{N})$ is a Noetherian $\widehat{R}-$module, so $D_{\hat{R}}(H^i_{\mathfrak{m}}(M,N))$ is an Artinian $\widehat{R}-$module. Then $H^i_{\mathfrak{m}}(M,N)$ is an Artinian $\widehat{R}-$module by [16, 3.4.12]. It should be noted that the generalized local cohomology module $H^i_{\mathfrak{m}}(M,N)$ is $\mathfrak{m}-$primary (i.e., $H^i_{\mathfrak{m}}(M,N) = \bigcup_{t>0}(0 :_{H^i_{\mathfrak{m}}(M,N)} \mathfrak{m}^t)$). Thus a subset of $H^i_{\mathfrak{m}}(M,N)$ is an $R-$submodule if and only if it is an $\widehat{R}-$submodule. Therefore $H^i_{\mathfrak{m}}(M,N)$ is also an Artinian $R-$module. □

In the special case $M = R$, it is clear that $H_i^I(M,N) = H_i^I(N)$. Thus from Corollary 4.8, replacing the module $M$ with the ring $R$, we have the immediate consequence which is an extension of the well-known property of local cohomology of A. Grothendieck.

**Corollary 4.9.** *Let $N$ be a semi-discrete linearly compact $R-$module. Then the local cohomology module $H^i_{\mathfrak{m}}(N)$ is an Artinian $R-$module for all $i \geq 0$.*

A sequence of elements $x_1,\ldots,x_r$ in $R$ is said to be an $N-$*regular* sequence if $N/(x_1,\ldots,x_r)N \neq 0$ and $N/(x_1,\ldots,x_{i-1})N \xrightarrow{x_i} N/(x_1,\ldots,x_{i-1})N$ is injective for $i = 1,\ldots,r$. Denote by $\mathrm{depth}_I(N)$ the length of the longest $N-$regular sequence in $I$. Note that $x_1,\ldots,x_r$ is an $N-$regular sequence if and only if it is a $D(N)-$coregular sequence ([14, §3]).

From the characterization of the $\mathrm{Width}_I(N)$ by generalized local homology we have the following characterization of the $\mathrm{depth}_I(N)$ by generalized local cohomology.

**Corollary 4.10.** *Let $M$ be a finitely generated $R-$module and $N$ a semi-discrete linearly compact $R-$module such that $N/IN \neq 0$. Then*

$$\mathrm{depth}_I(N) = \inf\{i / H^i_I(M,N) \neq 0\}.$$

*Proof.* It should be noted that $x_1,\ldots,x_r \in I$ is an $N-$regular sequence if and only if it is a $D(N)-$coregular sequence. It follows that $\mathrm{depth}_I(N) = \mathrm{Width}_I(D(N)) = \mathrm{Width}_I(N^*)$, as $D(N) = N^*$. By 4.2(i), $N^*$ is also a semi-discrete linearly compact $R-$module such that $0 :_{N^*} I \neq 0$. Now combining 3.15 with 4.6(i) yields $\mathrm{Width}_I(N^*) = \inf\{i/H_i^I(M,N^*) \neq 0\} = \inf\{i/H^i_I(M,N)^* \neq 0\}$. In virtue of [11, 5.6], $H^i_I(M,N)^* \neq 0$ if and only if $H^i_I(M,N) \neq 0$. Therefore $\mathrm{depth}_I(N) = \inf\{i/H^i_I(M,N) \neq 0\}$. □

Replacing the module $M$ in Corollary 3.15 with the ring $R$, we also get an extension of the well-known property of local cohomology.

**Corollary 4.11.** *Let $N$ be a semi-discrete linearly compact $R-$module such that $N/IN \neq 0$. Then*

$$\mathrm{depth}_I(N) = \inf\{i / H_I^i(N) \neq 0\}.$$

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