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ŁOJASIEWICZ EXPONENTS AND NEWTON POLYHEDRA

Phạm Tiến Sơn[1]
Department of Mathematics, Dalat University, Dalat, Vietnam
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

In this paper we obtain the exact value of the Łojasiewicz exponent at the origin of analytic map germs on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) under the Newton non-degeneracy condition, using information from their Newton polyhedra. We also give some conclusions on Newton non-degenerate analytic map germs. As a consequence, we obtain a link between Newton non-degenerate ideals and their integral closures, thus leading to a simple proof of a result of Saia. Similar results are also considered to polynomial maps which are Newton non-degenerate at infinity.

MIRAMARE - TRIESTE
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[1]Regular Associate of ICTP. pham_ts@yahoo.co.uk

1. Introduction

1. Let $f := (f_1, f_2, \ldots, f_k) \colon (\mathbb{K}^n, 0) \to (\mathbb{K}^k, 0)$ be an analytic map germ, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We define the *Łojasiewicz exponent* $L_0(f)$ of the germ f = the greatest lower bound of the set of all real numbers $l > 0$ which satisfy the condition: there exists a positive constant c such that

$$\max_{i=1,2,\ldots,k} |f_i(x)| \geq c\|x\|^l \quad \text{for } \|x\| \ll 1.$$

If the set of all the exponents is empty we put $L_0(f) = +\infty$. It is well known that $L_0(f) < +\infty$ if and only if f has an isolated zero at the origin in $\mathbb{K}^n$.

Calculating explicitly the Łojasiewicz exponent $L_0(f)$ is important in the theory of singularities. There are some previous works which give an upper estimate for this number. For instance, when g is a complex analytic function of two variables, some formulae for $L_0(\text{grad } g)$ are obtained as in [20]. In the papers [23], [11], [3], [4], [1], estimates for the Łojasiewicz exponent $L_0(\text{grad } g)$ in terms of the Newton diagram of Newton non-degenerate complex analytic function germs g (in the sense of [18]) are also given. It seems more difficult to obtain effective estimates in the real case (see [19], [10], [15], [16], [17], [2]).

The first aim of this paper is to calculate the Łojasiewicz exponent $L_0(f)$ in terms of the Newton polyhedron of an analytic map germ f, under the Newton non-degeneracy condition. Moreover, motivated by the works of Yoshinaga [28], Saia [27] and Bivià-Ausina [4], we also extract some conclusions on Newton non-degenerate analytic map germs. As a consequence, we obtain a connection between Newton non-degenerate ideals and their integral closures. In particular, we retrieve a result of Saia in [27].

Our method is actually different from the argument of the previous authors: the proof, based on the ideas of Kuo and Łojasiewicz, uses only the Curve Selection Lemma as a tool.

2. We next suppose that $f := (f_1, f_2, \ldots, f_k) \colon \mathbb{K}^n \to \mathbb{K}^k$ is a polynomial mapping. We define the *Łojasiewicz exponent at infinity* $L_\infty(f)$ of the map f = the smallest upper bound of the set of all real numbers $l > 0$ which satisfy the condition: there exists a positive constant c such that

$$\max_{i=1,2,\ldots,k} |f_i(x)| \geq c\|x\|^l \quad \text{for } \|x\| \gg 1.$$

If the set of all the exponents is empty we put $L_\infty(f) = -\infty$.

In the case $n = 2$, Hà [12] (see also [22]) gave an exact formula for the Łojasiewicz exponent at infinity $L_\infty(\text{grad } g)$ of the gradient of a complex polynomial g, and he showed a link between $L_\infty(\text{grad } g)$ and the singularities at infinity of g. In the papers [6], [7] Chadzynski and Krasinski described the Łojasiewicz exponent at infinity of a polynomial mapping $f \colon \mathbb{C}^2 \to \mathbb{C}^2$. In particular, they obtained a characterization of a component of a polynomial automorphism of $\mathbb{C}^2$ from a characterization of $L_\infty(f)$. Kollár [17] (see also [9]) gave an effective estimate for the Łojasiewicz exponent at infinity of real polynomial maps with a compact zero set.

The second aim of this paper is to calculate the Łojasiewicz exponent at infinity $L_\infty(f)$ in terms of the Newton polyhedron of a polynomial map f, under Newton non-degeneracy at infinity assumption.

3. The paper is organized as follows. In Section 2.1 we shall give some formulae for the Łojasiewicz exponents. Their proofs are given in Section 2.2. These results constitute a generalization of ones of [13] on the Newton-Puiseux approximation and Łojasiewicz exponents. In Section 3 we find some characterizations of analytic map germs which are Newton non-degenerate. Finally, a relation between Newton non-degenerate ideals and their integral closures is also obtained.

2. Łojasiewicz exponents

2.1. **Notations and statement of the results.** We first recall the definitions of Newton polyhedron and Newton non-degeneracy. These are essentially standard, and were established in [18]. Let $\mathbb{N} \subset \mathbb{R}_+ \subset \mathbb{R}$ be the sets of all nonnegative integers, all nonnegative real numbers, and all real numbers respectively. Let $J \subseteq \{1, 2, \ldots, n\}$. We write $\mathbb{K}^J := \{\alpha \in \mathbb{K}^n \mid \alpha_j = 0 \text{ if } j \notin J\}$. For any map germ $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^k, 0)$ we denote by $f|_{\mathbb{K}^J}$ the map germ f where the indeterminate x_j is zero whenever $j \notin J$.

Let $f := (f_1, f_2, \ldots, f_k)\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^k, 0)$ be an analytic map germ. If the Taylor expansions of f_i are $\sum_{\alpha\in\mathbb{N}^n} a_\alpha(i)x^\alpha, i = 1, 2, \ldots, k$, (where x^α denotes the monomial $x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}$), the *support* $\mathrm{supp}(f)$ is defined to be $\cup_{i=1}^k\{\alpha \in \mathbb{N}^n \mid a_\alpha(i) \neq 0\}$. We define $\Gamma_+(f)$ to be the convex hull of the set $\cup_{\alpha\in\mathrm{supp}(f)}(\alpha+\mathbb{R}^n_+)$. For any $m \in \mathbb{R}^n_+, m \neq 0$, we consider a *supporting hyperplane* $\{\alpha \in \mathbb{R}^n \mid \langle m, \alpha\rangle = \nu\}$ of $\Gamma_+(f)$ such that

$$\langle m, \alpha\rangle \geq \nu \quad \text{for all } \alpha \in \Gamma_+(f).$$

These conditions determine ν uniquely, while $\Gamma_+(f)$ is given by the system of inequalities[2] $\langle m, \alpha\rangle \geq \nu, m \in \mathbb{R}^n_+$. A *face* of the boundary of the Newton polyhedron $\Gamma_+(f)$ is an intersection of $\Gamma_+(f)$ with some supporting hyperplane. The union of the compact faces of $\Gamma_+(f)$ is called the *Newton diagram* $\Gamma(f)$ of f.

For a face $\gamma \in \Gamma(f)$ we put $f_{i,\gamma}(x) := \sum_{\alpha\in\gamma} a_\alpha(i)x^\alpha$, for $i = 1, 2, \ldots, k$. We say that f is *Newton non-degenerate* if for any face $\gamma \in \Gamma(f)$, the functions $f_{i,\gamma}$ have no common zero in $(\mathbb{K} - \{0\})^n$. It is easily seen from Sard's lemma that the Newton non-degenerate condition is generic in the sense of Kouchnirenko (see [18]).

The germ f is said to be *convenient* if the Newton diagram $\Gamma(f)$ meets all coordinate axes. It is worth noting that if f has isolated zero at the origin then f is convenient. In this case let $l_j, j = 1, 2, \ldots, n$, be the length from the origin to the intersection point of $\Gamma(f)$ and the α_j-axis.

[2]The system of inequalities is infinite; however, there exists a finite number of inequalities of which the remaining inequalities are a consequence.

Define a positive integer number $l_0(f)$ by the following formula

$$l_0(f) := \max_{j=1,2,\ldots,n} l_j.$$

A version of the following result for the case $n = 2$ can be found in [13].

Theorem 2.1. *Let $f := (f_1, f_2, \ldots, f_k) \colon (\mathbb{K}^n, 0) \to (\mathbb{K}^k, 0)$ be a convenient analytic map germ. If f is Newton non-degenerate, then $l_0(f)$ is equal to the Łojasiewicz exponent $L_0(f)$ of f.*

Here we assume that $f := (f_1, f_2, \ldots, f_k) \colon \mathbb{K}^n \to \mathbb{K}^k$ is a polynomial map. We express f as follows: $f_i(x) := \sum_{|\alpha| \le d_i} a_\alpha(i) x^\alpha, i = 1, 2, \ldots, k$, (where $d_i := \deg f_i$ is the degree of f_i). The *support* $\mathrm{supp}(f)$ is defined to be $\cup_{i=1}^k \{|\alpha| \le d_i \mid a_\alpha(i) \ne 0\}$. We define $\Gamma_-(f)$ to be the convex hull of the set $\{0\} \cup \mathrm{supp}(f)$. The *Newton diagram at infinity* of f, denoted by $\Gamma_\infty(f)$, is the polyhedron formed by the closed faces of $\Gamma_-(f)$ which do not contain the origin. As before, for each closed face γ of the polyhedron $\Gamma_\infty(f)$ we denote by $f_{i,\gamma}$ the polynomial $\sum_{\alpha \in \gamma} a_\alpha(i) x^\alpha$. f is called *Newton non-degenerate at infinity* if for each face $\gamma \in \Gamma_\infty(f)$, the polynomial functions $f_{i,\gamma}$ have no common zero in $(\mathbb{K} - \{0\})^n$.

The polynomial map f is said to be *convenient* if the Newton diagram $\Gamma_\infty(f)$ meets all coordinate axes. In this case let $l_{j,\infty}, j = 1, 2, \ldots, n$, be the length from the origin to the intersection point of $\Gamma_\infty(f)$ and the α_j-axis. We define $l_\infty(f)$ by

$$l_\infty(f) := \min_{j=1,2,\ldots,n} l_{j,\infty}.$$

The next theorem was proved in [13] for $n = 2$, that is, for polynomial maps in two variables.

Theorem 2.2. *Let $f := (f_1, f_2, \ldots, f_k) \colon \mathbb{K}^n \to \mathbb{K}^k$ be a convenient polynomial map. Suppose that f is Newton non-degenerate at infinity. Then $l_\infty(f)$ is equal to the Łojasiewicz exponent at infinity $L_\infty(f)$ of f.*

2.2. **Proofs.** We prove only Theorem 2.1. The proof of Theorem 2.2 follows by entirely analogous arguments but instead of working in a small sphere we work in the complement of a large sphere.

The proof of the following lemma is clear from the definitions.

Lemma 2.3. *Let $f := (f_1, f_2, \ldots, f_k) \colon (\mathbb{K}^n, 0) \to (\mathbb{K}^k, 0)$ be an analytic map germ. Suppose that f is convenient. For every $\emptyset \ne J \subset \{1, 2, \ldots, n\}$ then*

(i) $f|_{\mathbb{K}^J}$ is convenient. Moreover, if the germ f is Newton non-degenerate, then so is $f|_{\mathbb{K}^J}$.

(ii) $\Gamma_+(f|_{\mathbb{K}^J}) = \Gamma_+(f) \cap \mathbb{R}^J$.

Let $\{\alpha \in \mathbb{R}^n \mid \langle m, \alpha \rangle = \nu\}$ be the supporting hyperplane of a given face $\gamma \in \Gamma(f)$. The next lemma indicates a convenient way to determine $f_{i,\gamma}$ from f_i.

Lemma 2.4. *Let $x \in \mathbb{K}^n, x \ne 0$. We have*

$$f_i(t^m \bullet x) \;=\; t^\nu f_{i,\gamma}(x) + o(t^\nu) \quad \textit{as} \quad t \to 0,$$

where $t^m \bullet x := (t^{m_1} x_1, t^{m_2} x_2, \ldots, t^{m_n} x_n)$.

Proof. By definition, $\langle m, \alpha\rangle \geq \nu$ for all $\alpha \in \Gamma_+(f)$ with equality if and only if $\alpha \in \gamma$. Moreover, it is obvious that

$$f_{i,\gamma}(t^m \bullet x) = t^\nu f_{i,\gamma}(x).$$

This implies the lemma. □

Lemma 2.5. *Let $f := (f_1, f_2, \ldots, f_k)\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^k, 0)$ be a convenient analytic map germ. If f is Newton non-degenerate, then the origin in $\mathbb{K}^n$ is an isolated zero of f.*

Proof. This proof is due to Wall [30] (see also [29], [14]).

Suppose that the claim does not hold. Then, by the Curve Selection Lemma [25], there exists an analytic curve

$$\varphi\colon [0, \epsilon) \to \mathbb{K}^n, \quad t \mapsto (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)),$$

such that $\varphi(t) = 0$ if and only if $t = 0$, and

$$f_1[\varphi(t)] = f_2[\varphi(t)] = \cdots = f_k[\varphi(t)] = 0 \quad \text{for } t \in [0, \epsilon).$$

Let J be the set of all the indices $j \in \{1, 2, \ldots, n\}$ such that φ_j does not vanish identically. For $j \in J$, expand the coordinate φ_j in terms of the parameter: say

$$\varphi_j(t) = x_j^0 t^{m_j} + \text{ higher order terms in } t,$$

where x_j^0 is non-zero number and $m_j > 0$.

We consider the set Γ', obtained by intersecting the Newton diagram $\Gamma(f)$ and the subspace $\mathbb{R}^J$. By Lemma 2.3, $\Gamma' \neq \emptyset$ and Γ' is the Newton diagram of $f|_{\mathbb{K}^J}$. Let ν be the least value attained by the linear function $\alpha \mapsto \sum_{j\in J} m_j\alpha_j$ on $\Gamma_+(f|_{\mathbb{K}^J})$. Let γ denote the face of $\Gamma_+(f|_{\mathbb{K}^J})$ along which this value is attained. Then, by Lemma 2.4, the functions $f_i, i = 1, 2, \ldots, k$, restricted on φ have the form

$$f_i[\varphi(t)] = t^\nu f_{i,\gamma}(x_1^0, x_2^0, \ldots, x_n^0) + \text{ higher order terms in } t,$$

where $x_j^0 := 1$ whenever $j \notin J$. (Note that the functions $f_{i,\gamma}$ are independent on the variables x_j for all $j \notin J$.)

But by hypothesis, all functions f_i vanish along φ. So in particular,

$$f_{i,\gamma}(x_1^0, x_2^0, \ldots, x_n^0) = 0 \quad \text{for } i = 1, 2, \ldots, k,$$

which contradicts Newton non-degeneracy of f because $(x_1^0, x_2^0, \ldots, x_n^0) \in (\mathbb{K} - \{0\})^n$. This completes the proof. □

Proof of Theorem 2.1. By Lemma 2.5, the Łojasiewicz exponent $L_0(f)$ is finite.

Without loss of generality we may assume $l_0(f) = l_1$-the length from the origin to the intersection point of $\Gamma(f)$ and the α_1-axis. Let H denote the α_1-axis. Then it easy to check that

$$\min_{i=1,2,\ldots,k} O(f_i|_H) = l_1,$$

where $O(f_i|_H)$ is the multiplicity of the restriction of f_i to H. Hence

$$L_0(f) \geq l_1 = l_0(f).$$

By the above inequality, one has only to prove that $L_0(f) \leq l_0(f)$. By the definition of $L_0(f)$, it suffices to show that

$$\max_{i=1,2,\ldots,k} |f_i(x)| \geq c\|x\|^{l_0(f)}$$

for $\|x\|$ sufficiently small and for $c > 0$. Suppose that this is not the case. By standard argument, based again on the Curve Selection Lemma [25], there exists an analytic curve

$$\varphi\colon [0, \epsilon) \to \mathbb{K}^n, \quad t \mapsto (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)),$$

passing through the origin, such that

$$\max_{i=1,2,\ldots,k} |f_i[\varphi(t)]| \ll \|\varphi(t)\|^{l_0(f)} \quad \text{as } t \to 0. \tag{1}$$

Let J be the set of all the indices $j \in \{1, 2, \ldots, n\}$ such that φ_j does not vanish identically. For $j \in J$, expand the coordinate φ_j in terms of the parameter: say

$$\varphi_j(t) = x_j^0 t^{m_j} + \text{ higher order terms in } t,$$

where x_j^0 is the non-zero number and $m_j > 0$. Let $m_* := \min_{j \in J} m_j > 0$. Then, we have, asymptotically as $t \to 0$,

$$\|\varphi(t)\| \simeq |t|^{m_*}.$$

We consider the set Γ', obtained by intersecting the Newton diagram $\Gamma(f)$ and the subspace $\mathbb{R}^J$. By Lemma 2.3, $\Gamma' \neq \emptyset$ and Γ' is the Newton diagram of $f|_{\mathbb{K}^J}$. Let ν be the least value attained by the linear function $\alpha \mapsto \sum_{j \in J} m_j \alpha_j$ on $\Gamma_+(f|_{\mathbb{K}^J})$. Let $\gamma \in \Gamma'$ denote the face of $\Gamma_+(f|_{\mathbb{K}^J})$ along which this value is attained. Then it is obvious that ν/m_* is equal to the maximum of the lengths from the origin to the intersection points of Γ' and the α_j-axis, $j \in J$. From this observation, together with the definition of $l_0(f)$, we obtain the following inequality

$$\nu \leq m_* l_0(f). \tag{2}$$

On the other hand, from Lemma 2.4 we get

$$f_i[\varphi(t)] \quad = \quad t^{\nu} f_{i,\gamma}(x_1^0, x_2^0, \ldots, x_n^0) + o(t^{\nu}),$$

where $x_j^0 := 1$ if $j \notin J$. Since f is Newton non-degenerate, not all the values $f_{i,\gamma}(x_1^0, x_2^0, \ldots, x_n^0)$ are zero. Consequently,

$$\max_{i=1,2,\ldots,k} |f_i[\varphi(t)]| \simeq |t|^{\nu} \quad \text{as } t \to 0.$$

It follows from (1), asymptotically as $t \to 0$, that

$$|t|^{\nu} \ll \|\varphi(t)\|^{l_0(f)} \simeq |t|^{m_* l_0(f)}.$$

Therefore

$$\nu > m_* l_0(f),$$

which contradicts (2). This ends the proof. □

Example 2.6. (See [27]). Let $f := (f_1, f_2): (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$, where $f_1 = x^8 + xy^5$ and $f_2 = y^8 + yx^5$. Then f is convenient and $l_0(f) = 8$. The 1-dimensional compact faces of $\Gamma_+(f)$ are γ_1, γ_2 and γ_3 with vertices $\{(0, 8), (1, 5)\}$, $\{(1, 5), (5, 1)\}$ and $\{(5, 1), (8, 0)\}$ respectively. The ideal I is Newton non-degenerate since there is no common solution in $(\mathbb{K} - \{0\})^2$ for the equations $f_{1,\gamma_i} = f_{2,\gamma_i} = 0, i = 1, 2, 3$. By Theorem 2.1, $L_0(f) = l_0(f) = 8$.

Remark 2.7. (i) After the preparation of this paper we have learnt that Theorem 2.1 was also proved in the case $\mathbb{K} = \mathbb{C}$ by Bivià-Ausina [4] using a different argument.

(ii) In general, as we see in the next example, the conditions $f^{-1}(0) = \{0\}$ and $L_0(f) = l_0(f)$ do not imply that f is Newton non-degenerate.

Example 2.8. Let $f := (f_1, f_2): (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$, where $f_1 = xy - y^2$ and $f_2 = x^3$. The 1-dimensional compact faces of $\Gamma_+(f)$ are γ_1 and γ_2 with vertices $\{(0, 2), (1, 1)\}$ and $\{(1, 1), (3, 0)\}$ respectively. The germ f is non Newton non-degenerate since any point (t, t) with $t \neq 0$ is a solution of the equations $f_{1,\gamma_1} = f_{2,\gamma_1} = 0$. On the other hand, it is not difficult to verify that $f^{-1}(0, 0) = \{(0, 0)\}$ and $L_0(f) = l_0(f) = 3$. This works both over $\mathbb{R}$ and $\mathbb{C}$.

3. The Newton Non-degenerate condition

We are motivated by the works of Yoshinaga [28], Saia [27] and Bivià-Ausina [4] on the characterization of Newton non-degenerate complex analytic map germs. In this section, following this procedure, we will give some conclusions of the class of (complex or real) analytic map germs which are Newton non-degenerate. However, our arguments are based on other ideas, more precisely, we use only the Curve Selection Lemma. Firstly, we have

Theorem 3.1. *Let $f := (f_1, f_2, \ldots, f_k): (\mathbb{K}^n, 0) \to (\mathbb{K}^k, 0)$ be a convenient analytic map germ. Then the following conditions are equivalent:*

(i) f is Newton non-degenerate.

(ii) Take any monomial x^α with $\alpha \in \Gamma_+(f)$. There exists a positive constant c such that

$$\max_{i=1,2,\ldots,k} |f_i(x)| \geq c\|x^\alpha\| \quad \text{for } \|x\| \ll 1.$$

Proof. Suppose, by contradiction, that there exists a monomial x^{α^0} with $\alpha^0 \in \Gamma_+(f)$ such that the claim (ii) does not hold. Then, by the Curve Selection Lemma [25], there exists an analytic curve

$$\varphi: [0, \epsilon) \to \mathbb{K}^n, \quad t \mapsto (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)),$$

such that $\varphi(t) = 0$ if and only if $t = 0$, and

$$\max_{i=1,2,\ldots,k} |f_i[\varphi(t)]| \ll \|[\varphi(t)]^{\alpha^0}\| \quad \text{as } t \to 0. \tag{3}$$

Let J be the set of all the indices $j \in \{1, 2, \ldots, n\}$ such that φ_j does not vanish identically. It is clear that $J \neq \emptyset$ and for $j \notin J$ the j-th component, α^0_j, of α^0 is zero.

For $j \in J$, expand the coordinate φ_j in terms of the parameter: say

$$\varphi_j(t) = x_j^0 t^{m_j} + \text{ higher order terms in } t,$$

where x_j^0 is non-zero number and $m_j > 0$.

Let ν be the least value attained by the linear function $\alpha \mapsto \sum_{j \in J} m_j \alpha_j$ on $\Gamma_+(f|_{\mathbb{K}^J})$. Let γ denote the face of $\Gamma_+(f|_{\mathbb{K}^J})$ along which this value is attained. Then, by Lemma 2.4, the functions $f_i, i = 1, 2, \ldots, k$, restricted on φ have the form

$$f_i[\varphi(t)] = t^\nu f_{i,\gamma}(x_1^0, x_2^0, \ldots, x_n^0) + \text{ higher order terms in } t.$$

Since f is Newton non-degenerate, not all the values $f_{i,\gamma}(x_1^0, x_2^0, \ldots, x_n^0)$ are zero. Thus,

$$\max_{i=1,2,\ldots,k} |f_i[\varphi(t)]| \simeq |t|^\nu \quad \text{as } t \to 0.$$

Then, it follows from (3), asymptotically as $t \to 0$, that

$$|t|^\nu \ll \|[\varphi(t)]^{\alpha^0}\| \simeq |t|^{\langle m, \alpha^0 \rangle}.$$

This gives $\nu > \langle m, \alpha^0 \rangle$, which contradicts the fact that $\alpha^0 \in \Gamma_+(f)$.

Suppose now that (i) fails. Then there exists $\gamma \in \Gamma(f)$ and $x^0 \in (\mathbb{K} - \{0\})^n$ such that

$$f_{1,\gamma}(x^0) = f_{2,\gamma}(x^0) = \cdots = f_{k,\gamma}(x^0) = 0. \tag{4}$$

Let $J \subset \{1, 2, \ldots, n\}$ be the smallest set of indices such that the subspace $\mathbb{R}^J$ contains γ. Let $\{\alpha \in \mathbb{R}^J \mid \langle m, \alpha \rangle = \nu\}$ be a supporting hyperplane of $\gamma \subset \Gamma_+(f|_{\mathbb{K}^n})$. That means $\langle m, \alpha \rangle \geq \nu$ for all $\alpha \in \Gamma_+(f|_{\mathbb{K}^n})$ with equality if and only if $\alpha \in \gamma$. Define the monomial curve $\varphi \colon [0, \epsilon) \to \mathbb{K}^n, t \mapsto \varphi(t)$, by

$$\varphi_j(t) = \begin{cases} x_j^0 t^{m_j} & \text{if } j \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Take any $\alpha \in \gamma \cap \operatorname{supp}(f) \subset \Gamma_+(f) \cap \mathbb{N}^n$. By definition, it is clear that

$$\|[\varphi(t)]^\alpha\| \simeq |t|^{\langle m, \alpha \rangle} = |t|^\nu.$$

On the other hand, it follows from Lemma 2.4 and the relation (4) that

$$f_i[\varphi(t)] = t^\nu f_{i,\gamma}(x^0) + o(t^\nu) = o(t^\nu) \quad \text{for } i = 1, 2, \ldots, k.$$

These imply that

$$\max_{i=1,2,\ldots,k} |f_i[\varphi(t)]| \ll \|[\varphi(t)]^\alpha\| \quad \text{as } t \to 0,$$

which contradicts (ii). This completes the proof. □

We denote by $\Lambda(f)$ the convex hull in $\mathbb{R}_+^n$ of the set

$$\{\alpha \in \mathbb{N}^n \mid \exists\, c > 0 \text{ such that } \max_{i=1,2,\ldots,k} |f_i(x)| \geq c\|x^\alpha\| \quad \text{for } \|x\| \ll 1\}.$$

A version of the following lemma for the case $\mathbb{K} = \mathbb{C}$ can be found in [27].

Lemma 3.2. *Let $f := (f_1, f_2, \ldots, f_k) \colon (\mathbb{K}^n, 0) \to (\mathbb{K}^k, 0)$ be a convenient analytic map germ. Then $\Lambda(f) \subset \Gamma(f)$.*

Proof. Suppose that $\alpha^0 := (\alpha^0_1, \alpha^0_2, \ldots, \alpha^0_n) \notin \Gamma_+(f) \cap \mathbb{N}^n$. Let $J := \{j \mid \alpha^0_j > 0\}$. It is obvious that $\alpha^0 \notin \Lambda(f)$ when $J = \emptyset$. Hence one has only to consider the case $J \neq \emptyset$. Then there exists a supporting hyperplane $\{\alpha \in \mathbb{R}^J \mid \langle m, \alpha \rangle = \nu\}$ of $\Gamma_+(f|_{\mathbb{K}^J})$ such that

$$\langle m, \alpha^0 \rangle < \nu \leq \langle m, \alpha \rangle \quad \text{for all } \alpha \in \Gamma_+(f|_{\mathbb{R}^J}).$$

Define the monomial curve $\varphi\colon [0, \epsilon) \to \mathbb{K}^n, t \mapsto \varphi(t)$, by

$$\varphi_j(t) = \begin{cases} t^{m_j} & \text{if } j \in J, \\ 0 & \text{otherwise.} \end{cases}$$

By a direct calculation, then

$$\begin{aligned} \|[\varphi(t)]^{\alpha^0}\| &= |t|^{\langle m, \alpha^0 \rangle} \gg |t|^\nu, \\ f_i[\varphi(t)] &= t^\nu f_{i,\gamma}(1, 1, \ldots, 1) + o(t^\nu) \quad \text{for } i = 1, 2, \ldots, k. \end{aligned}$$

These give

$$\max_{i=1,2,\ldots,k} |f_i[\varphi(t)]| \ll \|[\varphi(t)]^{\alpha^0}\|.$$

As a consequence, $\alpha^0 \notin \Lambda(f)$. The lemma is proved. □

From Theorem 3.1 and Lemma 3.2 we obtain immediately:

Corollary 3.3. *Let $f := (f_1, f_2, \ldots, f_k)\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^k, 0)$ be a convenient analytic map germ. Then the following conditions are equivalent:*

(i) f is Newton non-degenerate.

(ii) $\Gamma_+(f) = \Lambda(f)$.

Remark 3.4. It is worth noting that the proofs of the above results are based on the Curve Selection Lemma. Therefore, by entirely analogous arguments but instead of working in a small sphere we work in the complement of a large sphere and then using the Curve Selection Lemma at infinity [26], we may obtain similar results for polynomial maps $f\colon \mathbb{K}^n \to \mathbb{K}^k$. We will leave to the reader to verify these facts.

In the rest of this note, we establish a relation between Newton non-degenerate ideals and their integral closures. In order to do this, we need some definitions.

Let $\mathcal{A}(\mathbb{K}^n)$ be the ring of analytic function germs from $(\mathbb{K}^n, 0)$ onto $(\mathbb{K}, 0)$. If $S \subset \mathcal{A}(\mathbb{K}^n)$, we denote by $V(S)$ the zero set germ at the origin of S in $\mathbb{K}^n$.

Let I be an ideal of $\mathcal{A}(\mathbb{K}^n)$ and $g \in \mathcal{A}(\mathbb{K}^n)$ such that $V(I) \subseteq V(g)$. Let $f_1, f_2, \ldots, f_k \in \mathcal{A}(\mathbb{K}^n)$ be a system of generators of I. By [24] (see also [5]), we can consider the greatest lower bound of those $l > 0$ such that

$$\max_{i=1,2,\ldots,k} |f_i(x)| \geq c\|g(x)\|^l \quad \text{for } \|x\| \ll 1,$$

with some positive constant c. We denote this number by $L_0(g, I)$. We observe that this definition does not depend on the chosen system of generators of I.

By the works of Lejeune-Teissier [21] and Bochnak-Risler [5], $L_0(g, I)$ is a positive rational number.

We will say that g is *integral over* I when $L_0(g, I) \leq 1$ (see [8], [3]). The set of integral elements over I forms an ideal of $\mathcal{A}(\mathbb{K}^n)$ called the *integral closure* of I. This ideal is denoted by $\bar{I}$. Clearly, we have the inclusion $I \subseteq \bar{I}$.

The ideal $I := \langle f_1, f_2, \ldots, f_k \rangle$ is said to be *Newton non-degenerate* if the germ

$$(f_1, f_2, \ldots, f_k) \colon (\mathbb{K}^n, 0) \to (\mathbb{K}^k, 0)$$

is Newton non-degenerate. It is easy to check that the above definition does not depend on the chosen system of generators of I.

We say that I has *finite codimension* in $\mathcal{A}(\mathbb{K}^n)$ if $\dim_{\mathbb{K}} \mathcal{A}(\mathbb{K}^n)/I < \infty$. This is equivalent to saying that $V(I) = \{0\}$.

The following is an characterization of the integral closure of an ideal of $\mathcal{A}(\mathbb{K}^n)$.

Corollary 3.5. *Let I be an ideal of finite codimension in $\mathcal{A}(\mathbb{K}^n)$. Then the following conditions are equivalent:*

(i) I is Newton non-degenerate.

(ii) The integral closure $\bar{I}$ is equal to the ideal generated by the monomials x^α such that $\alpha \in \Gamma_+(f)$.

Proof. Let $f_1, f_2, \ldots, f_k \in \mathcal{A}(\mathbb{K}^n)$ be a system of generators of I. Let $f \colon (\mathbb{K}^n, 0) \to (\mathbb{K}^k, 0)$ be the germ such that its components are $f_1, f_2, \ldots, f_k$. Then f is convenient because $V(I) = \{0\}$. Therefore the claim comes from Corollary 3.3. □

Remark 3.6. To end this paper, we consider the case $\mathbb{K} = \mathbb{C}$. Let I be an ideal of $\mathcal{A}(\mathbb{C}^n)$ and $g \in \mathcal{A}(\mathbb{C}^n)$. It is proved by Lejeune and Teissier [21] that the following conditions are equivalent:

(i) $g \in \bar{I}$.

(ii) g satisfies an equation of the form

$$g^d + a_1 g^{d-1} + \cdots + a_{d-1} g + a_d = 0,$$

where $a_i \in I^i, i = 1, 2, \ldots, d$, for some $d \geq 1$.

Therefore, in the case $\mathbb{K} = \mathbb{C}$, Corollary 3.5 is just the result of Saia [27, Theorem 3.4].

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References

[1] O. M. Abderrahmane. *On the Łojasiewicz exponent and Newton polyhedron,* Kodai Math. Journal, Vol. **28**, No. 1 (2005), 106-110.

[2] D. D'Acunto, K. Kurdyka, *Explicit bounds for the Łojasiewicz exponent in the gradient inequality for polynomials,* Ann. Pol. Math. LXXXVII. 1 (2005).

[3] C. Bivià-Ausina, *Łojasiewicz exponents, the integral closure of ideals and Newton polyhedra,* J. Math. Soc. Japan, Vol. **55**, No. 3 (2003), 655-668.

[4] C. Bivià-Ausina, *Jacobian ideals and the Newton non-degeneracy condition,* Proc. Edinb. Math. Soc. (2) Vol. **48**, No. 1 (2005), 21-36.

[5] Bochnak J., Risler J. J., *Sur les exposants de Łojasiewicz,* Comment. Math. Helv. Vol. **50** (1975), 493-507.

[6] J. Chadzynski, T. Krasinski, *Exponent of growth of polynomial mappings of* $\mathbb{C}^2$ *into* $\mathbb{C}^2$, in: Singularities, Banach Center Publications, Vol. **20** (1988), PWN, Warszawa 147-160.

[7] J. Chadzynski,T. Krasinski, *On the Łojasiewicz exponent at infinity for polynomial mappings of* $\mathbb{C}^2$ *into* $\mathbb{C}^2$ *and components of polynomial automorphisms of* $\mathbb{C}^2$*,* Annales Polonici Mathematici, Vol. **57**, No. 3 (1992), 291-302.

[8] T. Gaffney, *Integral closure of modules and Whitney equisingularity,* Invent. Math. Vol. **107**, No. 2 (1992), 301-322.

[9] J. Gwoździewicz, *Growth at infinity of a polynomial with a compact zero set,* Singularities Symposium Łojasiewicz 70, Banach Center Publ. Warsawa, 1998.

[10] J. Gwozdźiewicz, *The Łojasiewicz exponent of an analytic function at an isolated zero,* Comment. Math. Helv. Vol. **74**, No. 3 (1999), 364-375.

[11] T. Fukui, *Łojasiewicz type inequalities and Newton diagrams,* Proc. Am. Math. Soc., Vol. **112**, No. 4 (1991), 1169-1183.

[12] H. V. Hà, *Nombres de Łojasiewicz et singularités à l'infini des polynômes de deux variables complexes,* C.R. Acad. Sci. Paris, T. **311**, Serie I (1990), 429-432.

[13] H. V. Hà and T. S. Phạm, *Newton-Puiseux approximation and Łojasiewicz exponents,* Kodai Math. Journal, Vol. **26**, No. 1 (2003), 1-15.

[14] H. V. Hà and T. S. Phạm, *On local Pareto optima of real analytic mappings,* Acta Math. Vietnam., Vol. **30**, No. 2 (2005), 191-202.

[15] J. Kollár, *Sharp effective Nullstellensatz,* J. Amer. Math. Soc. Vol. **1**, No. 4 (1988), 963-975.

[16] S. Ji, J. Kollár and B. Shiffman, *A global Łojasiewicz inequality for algebraic varieties,* Trans. Amer. Math. Soc., Vol. **329**, No. 2 (1992), 813-818.

[17] J. Kollár, *An effective Łojasiewicz inequality for real polynomials,* Periodica Mathematica Hungarica, Vol. **38**, No. 3 (1999), 213-221.

[18] A. G. Kushnirenko, *Polyèdres de Newton et nombres de Milnor,* Inventiones Mathematicae, Vol. **32** (1976), 1-31.

[19] T. C. Kuo, *Computation of Łojasiewicz exponent of* $f(x, y)$*,* Comment. Math. Helvet. Vol. **49** (1974), 201-213.

[20] T. C. Kuo, Y. C. Lu, *On analytic functions germs of two complex variables,* Topology, Vol. **16** (1977), 299-310.

[21] Lejeune-Jalabert M., Teissier B., *Clôture intégrale des idéaux et equisingularité,* Centre de Mathématiques, Université Scientifique et Médicale de Grenoble, 1974.

[22] A. Lenarcik, *On the Łojasiewicz exponent of the gradient of a polynomial function,* Ann. Polon. Math., Vol. **71**, No. 3 (1999), 211-239.

[23] B. Lichtin, *Estimation of Łojasiewicz exponents and Newton polygons,* Inventiones Mathematicae, Vol. **64** (1981), 417-429.

[24] S. Łojasiewicz, *Ensembles semi-analytiques,* preprint IHES, 1965.

[25] J. Milnor, *Singular points of complex hypersurfaces,* Annals of Mathematics Studies **61**, Princeton University Press, 1968.

[26] A. Némethi and A. Zaharia, *Milnor fibration at infinity,* Indagationes Mathematicae, Vol. **3** (1992), 323-335.

[27] M. J. Saia, *The integral closure of ideals and the Newton filtration,* J. Alg. Geom. Vol. **5** (1996), 1-11.

[28] E. Yoshinaga, *Topologically principal part of analytic functions,* Trans. Amer. Math. Soc., Vol. **314**, No. 2 (1989), 803-814.

[29] V. A. Vassiliev, *Asymptotic exponential integrals, Newton's diagram, and the classification of minimal points,* Functional Analysis and Its Applications, Vol. **11**, No. 3 (1977), 163-172.

[30] C. T. C. Wall, *Newton polytopes and non-degeneracy,* J. Reine Angew. Math., Vol. **509** (1999), 1-19.