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D. Obradović

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BORIS KIDRIĆ INSTITUTE OF NUCLEAR SCIENCES
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A B S T R A C T

In the study of the nuclear reactors space-time behaviour the modal analysis is very often used though some basic mathematical problems connected with application of this methods are still unsolved. In this paper the modal analysis is identified as a set of the methods in the mathematical literature known as the Galerkin methods (or projection methods, or sometimes direct methods). Using the results of the mathematical investigations of these methods the applicability of the Galerkin type methods to the calculations of the eigenvalues and eigenvectors of the stationary and non-stationary diffusion operator, as well as for the solutions of the corresponding functional equations, is established.

1. INTRODUCTION

In recent years investigations in the field of the kinetics and the dynamics of the nuclear reactors have been directed towards overcoming an insufficiently accurate point reactor model. For that purposes the modal analysis is very often employed, though some basic mathematical problems connected with the application of this method are still unsolved. In our paper /1/ we were dealing with solutions of the problems connected with the nature of spectra and with properties of the sets of the eigenvectors of the stationary and non-stationary diffusion operators. In this paper we are going to show that the modal analysis is nothing but a set of methods in the mathematical literature known as the Galerkin type methods (or projection methods, or direct methods). Following this idea, using results of paper /1/ and the results of the mathematical investigations of the Galerkin type methods /2-9/ we will establish the applicability of the Galerkin type methods to the approximate solution of the multigroup diffusion equation.

In Part 2 the summary of the mathematical setting of the problem is given. For more details we refer to Refs./1/ or /10/. Part 3 represents review of the Galerkin method, while in the Part 4 the applicability of the Galerkin method to the approximate solution of the multigroup diffusion equation is established.

2. MATHEMATICAL SETTING OF THE PROBLEM

The details of the mathematical setting of the problem can be found in /1/ or /10/ and we are going to give here only a brief summary in order to be able to follow considerations in the Part 4. The notation used in this part is the same as in the paper /1/ or /10/.

The multigroup neutron diffusion equation with the n energy group and g delayed neutrons groups can be written in the form

$$\frac{\partial}{\partial t} N = QN \tag{2.1}$$

where

$$Q = Q_1 + Q_2 \tag{2.2}$$

and

$$Q_1 = \begin{pmatrix} -T & o & \dots & o \\ o & -\lambda_1 I & \dots & o \\ \vdots & & & \\ o & o & \dots & \lambda_g I \end{pmatrix}; \qquad Q_2 = \begin{pmatrix} K & \lambda_1 VB & \dots & \lambda_g VB \\ \nu\beta_1 M & & & \\ \vdots & & & \\ \nu\beta_g M & o & \dots & o \end{pmatrix}$$

$$\dots\dots \tag{2.2a}$$

with

$$T = V\Sigma - V\nabla \cdot D\nabla; \qquad K = V(C+\nu(1-\beta)BM) \tag{2.2b}$$

$$N = \begin{pmatrix} \phi \\ n_1 \\ n_2 \\ \vdots \\ n_g \end{pmatrix} \tag{2.3}$$

Boundary and initial conditions are as follows

$$\phi(\vec{r},t) = 0 \quad \text{for} \quad \vec{r}\in\Gamma \qquad (\Gamma - \text{outer reactor boundary}) \tag{2.4}$$

$$N(\vec{r},o) = N_o(\vec{r}) \tag{2.5}$$

Let operator Q act in a Hilbert space H which represents n(g+1) orthogonal copies of the space L_2, while the operators T and K act in the Hilbert space h which represents n orthogonal copies of the space L_2. In other words the space H represents (g+1) copies of the space h. The scalar product and the norm in these spaces are defined in the usual way (see/1/).

An application of the Laplace transform to the Eq. (2.1) allows to writte this equation, for $\lambda \neq -\lambda_j$, in the space h

$$(\lambda I + T - K(\lambda))\phi = G_o \tag{2.6}$$

where

$$K(\lambda) = K + V \sum_{j=1}^{g} \frac{\beta_j \lambda_j}{\lambda+\lambda_j} B_j M \tag{2.7}$$

and

$$G_o = \phi_o + \Sigma \frac{\lambda_j}{\lambda+\lambda_j} V n_{jo} \tag{2.8}$$

Supposing further trat the energy groups can be chosen in such a way that

$$B_j \equiv B \qquad j = 1,2,\ldots,g \tag{2.9}$$

the Eq.(2.7) becomes

$$K(\lambda) = V(C + \nu VBM\ c(\lambda)) \tag{2.7a}$$

where

$$c(\lambda) = 1 - \lambda \sum_{j=1}^{g} \frac{\beta_i}{\lambda+\lambda_j} \qquad (2.7b)$$

In the stationary case the Eq.(2.6) reduces to

$$(T - VC)\phi = \nu VBM\phi \qquad (2.10)$$

3. THE GALERKIN METHOD

In this part we are going to present the mathematical fundamentals of the Galerkin method used for the approximate solution of the functional equations in the separabile Hilbert space. A few mathematical theorems which are of the fundamental interest for our considerations will be also given. More details about the Galerkin method can be found in the references /2-9/.

Let us consider a separabile Hilbert space H, arbitrary linear operators L and T whose domains $D(L)$ and $D(T)$ as well as their ranges $R(L)$ and $R(T)$ are dense in H, and the equation

$$Tx - \lambda Lx = y \,. \tag{3.1}$$

The Galerkin method for the approximate solution of the Eq. (3.1) is as follows. Let $\{H_N\}$ and $\{H_N'\}$ be two sequences of the subspaces which are projectionally complete* in the space H and such that $\dim H_N \equiv \dim H_N'$ ($\dim H$ denotes the dimension of the space H). Let P_N and I_N be projection operators which orthogonaly project H onto H_N and H_N' respectively. The approximate solution to the Eq.(3.1) can be now determined from the next condition

$$I_N (Tx_N - \lambda Lx_N - y) = 0 \qquad x_N \in H_N \,.$$

---

* The sequence of the subspaces H of the Hilbert space H together with the sequence of the projection operators P_n which project H onto these subspaces are called projectionally complete if for arbitrary $f \in H$ the next condition is fulfilled $||f - P_n f|| \to 0, \quad n \to \infty$.

Explicitly this means, if the sequence $\{\rho_k\}_{k=1}^N$ represents a base in the space H_N and the sequence $\{\psi_k\}_{k=1}^N$ represents a base in the space H_N' ($N = \dim H_N = \dim H_N'$) and if the solution of the equation (3.1) should be found in the form

$$x_N = \sum_{k=1}^{N} a_k^N \rho_k \tag{3.2}$$

then the coefficients a_k^N in the expression (3.2) are determined from the system of the algebraic equations

$$(Tx_N - \lambda Lx_N - y, \psi_m) = 0, \quad m=1,2,\ldots,N \tag{3.3}$$

or more explicitly

$$\sum_{k=1}^{N} \{(T\rho_k,\psi_m) - \lambda(L\rho_k,\psi_m)\} a_k^N = (y,\psi_m) \tag{3.3a}$$

If in the Eq.(3.1) $y \equiv 0$, we arrive to the eigenvalue problem

$$Tx - \lambda Lx = 0 \tag{3.4}$$

The approximate solution to the equation (3.4) is obtained in the same manner as the solution to the equation (3.1). The approximate eigenvalues are obtained as the roots of the algebraic equation

$$\det\left((T\rho_k,\psi_m) - \lambda^N(L\rho_k,\psi_m)\right) = 0 \tag{3.5}$$

while the coefficients a_k^N are obtained as the nontrivial solution to the equation

$$\sum_{k=1}^{N} \{(T\rho_k,\psi_m) - \lambda^N(L\rho_k,\psi_m)\} a_k^N = 0 \tag{3.6}$$

$$m=1,2,\ldots,N$$

In order that the solutions of the above equations are possible it is necessary that the sequencies of the subspaces $\{H_N\}$ and $\{H'_N\}$ and the corresponding projection operators P_N and I_N satisfy the condition, which Pol'skii in his papers /4, 5/ calles THE CONDITION A:

> The Condition A: The system of the equations (3.3) and (3.6) can be solved if for the sufficiently large N ($N>N_o$) and for the arbitrary $u \in H_N$ the next condition is fullfiled
>
> $$||u|| \leq C||I_N u|| \; ; \quad C = \text{const} > 0$$
>
> Remark. If $H_N \equiv H'_N$ the CONDITION A is always fulfilled.

If the Pol'skii CONDITION A is fulfilled there are a few theorems which ensure the convergence of the Galerkin method. We are going to give here only those theorems which are of the interest for our further considerations.

Theorem A. If the equation (3.1) has an unique solution in the space H, if the operator T^{-1} exists and if the operator $T^{-1}L$ is compact in the space H then an application of the Galerkin method to the approximate solution of the equation (3.1) leads to the convergent process.

Theorem B. The approximate eigenvalues λ_k^N obtained from the equation (3.5), if the operator $T^{-1}L$ is compact in the space H, will converge only to the eigenvalues of the eigenvalue problem (3.4). Inversly, each eigenvalue of the eigenvalue problem (3.4) represents a limit value of the approximate eigenvalues obtained from the algebraic eigenvalue problem (3.6), $\lambda_k^N \to \lambda_k$.

Theorem C. From the arbitrary sequence $\{u_N\}$ of the approximate eigenvectors which correspond to the approximate eigenvalue $\lambda^N \to \lambda$ it can be chosen, at least one strongly convergent subsequence and any such a subsequence converges to the eigenvector of the eigenvalue problem (3.4) which corresponds to the eigenvalue λ.

Remark. As it is shown by Pol'skii (see e.g./2/)from the theorem C. does not follow an inverse statement that

each eigenvector of the eigenvalue problem (3.4) can be obtained applying the Galerkin Method. In order that each eigenvector of the eigenvalue problem (3.4) can be obtained applying the Galerkin method it is necessary that the invariant subspace of the operator $T^{-1}L$ consists of the eigenvectors only, or in other words it is necessary that the resolvent of the operator $T^{-1}L$ has simple poles (see e.g./2/).

Let us mention that if the sequences of the subspaces $\{H_N\}$ and $\{H'_N\}$ are identical then this method is called the Bubnov and Galerkin method, otherwise it is called the Galerkin and Petrov method.

4. APPLICATION OF THE GALERKIN METHOD TO THE APPROXIMATE SOLUTION OF THE MULTIGROUP DIFFUSION EQUATION

The basic idea of the modal analysis is to represent an unknown space and time dependent function as a linear combination of the known space functions (harmonics, modes) with unknown time dependent coefficients

$$\phi(\vec{r},t) = \sum_{n=1}^{N} A_n(t)\rho_n(\vec{r}) \qquad (4.1)$$

and problem reduces now to the determination of the coefficients $A_n(t)$, $n=1,2,\ldots,N$. A comparison of the Eqs.(3.2) and (4.1) leads to the conclusion that the modal analysis is nothing but the Galerkin method.

The sets of the expanding functions $\{\rho_n(r)\}$ usually used in the study of the space-time reactor behaviour are as follows:

- eigenvectors of the non-stationary diffusion operator,
- eigenvectors of the stationary diffusion operator and
- eigenvectors of the Laplace operator $(-\nabla^2)$.

It is a well known fact that the eigenvectors of the operator $-\nabla^2$ form a complete set in the space L_2, while the completness of the eigenvectors of the stationary and non-stationary diffusion operators have been studied in /1/. So, we will suppose further that the set $\{\rho_k\}$ is complete in the corresponding functional space.

Some results of the paper /1/ are summarized in the next lemmae:

Lemma 1. The Eq.(2.1) is a properly posed abstract

Cauchy problem.

Lemma II. The operators T^{-1}, $(T-VC)^{-1}$, $(T-VC)^{-1}VBM$ are completely continuous (compact). These operators and their adjoints have the same spectra.

Let us prove noew the next lemma:

Lemma III. The resolvent of the operators $(T-VC)^{-1}VBM$ and Q have a simple poles.

Proof. In order to prove this lemma let us mention first that it can be easily proved /10/ that the eigenvectors N_k of the operator Q and of its adjoint N_k^* form biorthogonal sets in the space H, i.e.

$$(N_k, N_n^*) = \delta_{nk} \tag{4.2}$$

and that the eigenvectors ϕ_k of the operator $(T-VC)^{-1}VBM$ and of its adjoint ϕ_k^* form biorthogonal sets in the space h_1 in which scalar product is defined as

$$|u,v| = (VBMu,v)$$

i.e. the next condition is fulfilled

$$|\phi_k, \phi_n| = (VBM\phi_k, \phi_n) = \delta_{nk} \tag{4.3}$$

Now, to prove Lemma III. we will use the next results of Taylor /11/. In the vicinity of the isolated point $\lambda_k \in \sigma L$ the Laurent expansion is valued

$$(\lambda I - L)^{-1} = \sum_{n=1}^{\infty} B_{kn}(\lambda-\lambda_k)^{-n} + \sum_{n=o}^{\infty} A_{kn}(\lambda-\lambda_k)^{n} \tag{4.4}$$

where A_{kn} and B_{kn} are certain bounded linear operators in whose properties we are only partially interested in here. Obviously, if the resolvent of the operator L has a simple pole at $\lambda = \lambda_k$ the operators B_{kn}, $n > 1$ have to be zero

operators. In other words, if the resolvent of the operator L has simple poles it means that the invariant subspace of the operator L consists only of eigenvectors, or there is no such a vector $f \in D(L)$ $(f \not\equiv 0)$ that

$$(\lambda_k I - L)((\lambda_k I - L)f) = 0$$
$$(\lambda_k I - L)f \not\equiv 0 \qquad (4.5)$$

Let us see whether the condition (4.5) is fulfilled for the operators $(T-VC)^{-1}VBM$ and Q.

If the condition (4.5) is fulfilled for the stationary diffusion operator then for the certain $f \in D(T)$ one has

$$(\nu_k((T-VC)^{-1}VBM - I)f = \phi_k; \quad (\nu_k = \frac{1}{\lambda_k}) \qquad (4.6)$$

where ϕ_k is an eigenvector corresponding to the eigenvalue λ_k. From (4.6) one has

$$(T-VC)f - \nu_k VBMf = (T-VC)\phi_k \qquad (4.6a)$$

Taking into account the stationary diffusion equation (Eq.(2.10)) Eq.(4.6a) becomes

$$(T-VC)f - \nu_k VBMf = \nu_k VBM\phi_k \qquad (4.7)$$

A scalar multiplication of the Eq.(4.7) by ϕ_k^* leads to

$$((T-VC)f, \phi_k^*) - \nu_k(VBMf, \phi_k^*) = \nu_k(VBMf, \phi_k^*)$$

or

$$(f, (T-VC)^*\phi_k^*) - \nu_k(VBMf, \phi_k^*) = \nu_k$$

and taking into account the stationary diffusion equation (2.10) one finally has

$$(f,(VBM)^*\phi_k^*) - (VBMf,\phi_k^*) = 1 \qquad (4.8)$$

The expression (4.8) is obviously impossible therefore there is no such a vector $f \in D(T)$ satisfying the condition (4.5). Accordingly the resolvent of the stationary diffusion operator has simple poles.

The simplicity of the poles of the resolvent of the operator Q can be easily proved in the same manner. That completes the proof of the Lemma III.

Let us prove now a few theorems summarizing the applicability of the Galerkin method to the approximate solution of the diffusion equations.

Theorem I. An application of the Galerkin method to the approximate solution of the stationary and non-stationary diffusion equation leads to the convergent process.

Proof. For the stationary diffusion operator this theorem follows from the Theorem A and Lemma II. For the non-stationary diffusion operator (see Eq.(2.6)) this theorem follows from the Theorem A, Theorem 1 of the paper /3/, Lemmae I and II and boundness of the operator $\lambda I+K(\lambda)$ for $\lambda \neq -\lambda_j$, $j=1,2,\ldots,g$ and $|\lambda| < \infty$.

Theorem II. All eigenvalues of the stationary and non-stationary diffusion operator, and only they, can be obtained as limits of the all possible sequences of the approximate eigenvalues.

Proof. This theorem follows from Theorem B, Theorem II of the paper /3/, Lemma II and boundness of the operator $\lambda I+K(\lambda)$ for $\lambda \neq -\lambda_j$, $j=1,2,\ldots,g$ and $|\lambda| < \infty$.

Theorem III. All eigenvectors of the stationary and non-stationary diffusion operator can be obtained as limits of the sequences of the approximate eigenvectors, i.e. at least one subsequence from the sequence of the approximate eigenvectors can be chosen in such a way that it converges strongly to the

exact eigenvector.

Proof. This theorem follows from the Theorem C, Theorem II and Lemma III.

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