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Color superconductivity, Z_N flux tubes and monopole confinement in deformed $\mathcal{N} = 2^*$ super Yang-Mills theories

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Abstract

We study the Z_N flux tubes and monopole confinement in deformed $\mathcal{N} = 2^*$ super Yang-Mills theories. In order to do that we consider an $\mathcal{N} = 4$ super Yang-Mills theory with an arbitrary gauge group G and add some $\mathcal{N} = 2$, $\mathcal{N} = 1$ and $\mathcal{N} = 0$ deformation terms. We analyze some possible vacuum solutions and phases of the theory, depending on the deformation terms which are added. In the Coulomb phase for the $N = 2^*$ theory, G is broken to $U(1)^r$ and the theory has monopole solutions. Then, by adding some deformation terms, the theory passes to the Higgs or color superconducting phase, in which G is broken to its center C_G. In this phase we construct the Z_N flux tubes ansatz and obtain the BPS string tension. We show that the monopole magnetic fluxes are linear integer combinations of the string fluxes and therefore the monopoles can become confined. Then, we obtain a bound for the threshold length of the string-breaking. We also show the possible formation of a confining system with 3 different monopoles for the $SU(3)$ gauge group. Finally we show that the BPS string tensions of the theory satisfy the Casimir scaling law.

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1 Introduction

It is long believed that particle confinement at strong coupling regime should be a phenomenon dual to the monopole confinement in a (color)superconductor in the weak coupling. Therefore, the study of monopole confinement in weak coupling may shed some light on the particle confinement and other phenomena in the dual confined theory. Since dualities are better understood for supersymmetric theories, and in particular for the finite ones, it is interesting to consider the monopole confinement in these theories or deformations of them. Therefore in the present work we study the monopole confinement in an $\mathcal{N} = 2^*$ super Yang-Mills theory with the addition of some $\mathcal{N} = 1$ or $\mathcal{N} = 0$ deformation terms.

Since the papers of Seiberg and Witten [1, 2], quite a lot work [4]-[18] have been done analyzing different aspects of confinement in supersymmetric theories. Usually, one starts with a microscopic $\mathcal{N} = 2$ $SU(N)$ super Yang-Mills (SYM) theory (with some possible matter fields) and then obtains an effective $\mathcal{N} = 2$ $U(1)^{N-1}$ SYM theory with a $\mathcal{N} = 1$ deformation term. In this theory, each $U(1)$ factor is broken to Z, resulting in an $(N-1)$ infinite towers of Nielsen-Olesen flux tubes or strings, which gives rise to confinement of Dirac monopoles. However, as was pointed out in [6], a similar phenomenon is not expected to happen to quark confinement in QCD. It is then believed that only some of these strings might be stable, which could correspond to Z_N strings in the microscopic theory.

On the other hand, in [10, 11] the solitonic monopoles and the Z_k strings were obtained directly as solutions of the same theory with two gauge symmetry breakings. In order to do that, we considered $\mathcal{N} = 2$ super Yang-Mills theories with an arbitrary simple gauge group G, a massive hypermultiplet and an $\mathcal{N} = 0$ deformation mass term. This hypermultiplet was considered to be in the symmetric part of the tensor product of k fundamental representations, with $k \geq 2$. We considered this theory in the weak coupling and showed the existence of vacuum solutions which produce the symmetry breaking

$$G \to G_S \equiv [G_0 \times U(1)]/Z_l \to G_\phi \equiv [G_0 \times Z_{kl}]/Z_l$$

where G_0 is a subgroup of G and Z_l is a common discrete subgroup of G_0 and $U(1)$ as explained in [10, 11]. The first symmetry breaking happens when the $\mathcal{N}_0$ deformation mass parameter m vanishes. Then, the theory has solitonic monopoles which should fill representations of $G_0^\vee$, the dual group of G_0. The second symmetry breaking happens when $m > 0$. Since in this phase $\Pi_1(G/G_\phi) = Z_k$, there exist Z_k strings or flux tubes. Moreover, since the $U(1)$ factor is broken, the monopole magnetic lines in this $U(1)$ direction can no longer spread radially over space. However, since the monopole magnetic flux is an integer multiple of the "fundamental" Z_k string flux, these lines can form Z_k strings and monopoles become confined.

It is interesting to note that, when $k = 2$, the complex scalar ϕ which produces the second symmetry breaking that allows the existence of Z_k strings, is in the same representation as that of a diquark condensate. One then could think of ϕ as being itself this diquark condensate, and therefore we would have a situation quite similar to the one in an ordinary superconductor, described by the Abelian-Higgs theory with the scalar being a Cooper pair. In addition, if the gauge group is $SU(N)$, the scalar in the adjoint representation of the vector supermultiplet could also be thought to be a quark-antiquark condensate. These two kinds of condensates are indeed expected to exist in the color superconducting phase of (dense) QCD at the weak coupling [19, 20]. The effective theory describing these condensates are not well known. It should be a $SU(3)$ Yang-Mills-Higgs (or also called Ginzburg-Landau) theory with some scalars in the color sextet and color octet representations. Therefore, one could think that the theory used in [10, 11] or in the present paper, when the gauge group is $G = SU(3)$, as been a toy model for an effective theory of these condensates. Then, one conclude that the effective theory for these condensates could have monopoles, flux tubes and monopole confinement, depending on the form of the potential.

Although the monopoles in [11] should fill representations of the non-Abelian group G_0^{V}, the monopole confinement happened thought flux tubes in a $U(1)$ direction inside the non-Abelian group G. The motivation of the present paper is to consider monopole confinement through the formation of flux tubes due to breaking of the full non-Abelian group G, although in the present case the (stable) monopoles are not expected to fill representations of non-Abelian groups. In order to do that, we shall consider the bosonic part of $\mathcal{N}=4$ SYM theory in the weak coupling regime and add some $\mathcal{N}=2$, $\mathcal{N}=1$ or $\mathcal{N}=0$ deformation mass terms. These SYM theories are usually denoted by $\mathcal{N}=2^*$, $\mathcal{N}=1^*$ and $\mathcal{N}=0^*$ respectively. In [3, 6] was pointed out that the $\mathcal{N}=1^*$ theory should have a weakly-coupled Higgs phase with magnetic flux tubes and this phase should be dual to a strongly-coupled confining phase in the dual theory. One of the aims of the present paper is to analyze some properties of these magnetic flux tubes. In section 2, we obtain the lower bound for the string tension and corresponding BPS string conditions for a Yang-Mills theory with three complex scalars in the adjoint representation. Then, in section 3, we analyze the possible vacuum solutions and corresponding gauge symmetry breaking which happen depending on the mass deformation terms which are added to the $\mathcal{N}=4$ super Yang-Mills theory. We show that in this theory there are vacuum solutions which produce the spontaneous symmetry breaking,

$$G \to U(1)^r \to C_G,$$

where r is the rank of G and C_G its center. The first symmetry breaking happens in the $\mathcal{N}=4$ and $\mathcal{N}=2^*$ theories. Then, the second symmetry breaking happens when one adds to the $\mathcal{N}=2^*$ theory an $\mathcal{N}=1$ or an $\mathcal{N}=0$ deformation term (or both). In section 4, we analyze the Coulomb or free-monopole phase which occurs in the first symmetry breaking. In this phase there are BPS monopole solutions. In section 5, we analyze the Higgs or color superconducting phase which occurs when it happens the second symmetry breaking. In this phase the monopoles chromomagnetic lines can not spread out radially over space. However, since in this phase

$$\Pi_1(G/C_G) = C_G,$$

when C_G is non-trivial, these flux lines can form topologically nontrivial Z_N strings. We then construct the Z_N string ansatz. Some Z_N string solutions have been considered in [21] for different $SU(N)$ gauge theories. We show that the flux of the magnetic monopoles can be expressed as an integer linear combination of the string fluxes. Therefore, in the Higgs phase the monopole magnetic lines can form Z_N strings and the monopole can become confined, as in [11]. We then obtain for the monopole-antimonopole system a bound for the threshold length for the string-breaking. In section 6 we consider $G=SU(N)$ and analyze how the monopoles magnetic flux could be consider to be formed by a set of a string and an antistring in the fundamental representation. For the $SU(3)$ gauge group we show that, besides the monopoles-antimonopole system, the monopoles with strings attached could form a confining system with 3 different monopoles. In section 7, we show that the BPS string tensions satisfy the Casimir scaling law.

2 String BPS conditions

Let us start with a Yang-Mills-Higgs theory with three complex scalars ϕ_s, $s=1,2,3$, in the adjoint representation of an arbitrary gauge group G which we shall consider to be simple, connected and simply-connected. Let $\phi_s = M_s + iN_s$ where M_s and N_s are real scalars and pseudo-scalars respectively. Let us consider the Lagrangian

$$L = -\frac{1}{4} G_{a\mu\nu} G_a^{\mu\nu} + \frac{1}{2} \left(D_\mu \phi_s^*\right)_a \left(D^\mu \phi_s\right)_a - V$$

where V is for the moment an arbitrary positive potential. In [10], it was considered a theory with a complex scalar in the adjoint and another complex scalar initially in an arbitrary representation, and it was obtained that the non-Abelian string BPS conditions for an arbitrary gauge group. Let us repeat this procedure for the case with three scalars in the adjoint. Let $D_\mu = \partial_\mu + ieW_\mu$, $D_\pm = D_1 \pm iD_2$ and $B_i = -\varepsilon_{ijk}G_{jk}/2$ is the non-Abelian magnetic field. Let us consider a static configuration with cylindrical symmetry and not depending on x_3. Then, generalizing the Bogomol'nyi procedure [22], we obtain that the string tension T must satisfy

$$\begin{aligned} T &= \int d^2x \left\{ \frac{1}{2}\left[(E_{ia})^2 + (B_{ia})^2 + \left|(D_\mu \phi_s)_a\right|^2 \right] + V \right\} \\ &\geq \int d^2x \left\{ \frac{1}{2}\left[\left|(D_\mp \phi_s)_a\right|^2 + (B_{3a})^2 \pm e\left(\phi_{sb}^* i f_{abc}\phi_{sc}\right) B_{3a} \right] + V \right\} \\ &\geq \int d^2x \left\{ \frac{1}{2}(B_{3a})^2 \pm d_a B_{3a} \pm X_a B_{3a} + V \right\} \\ &= \int d^2x \left\{ \frac{1}{2}\left[B_{3a} \pm d_a\right]^2 \pm X_a B_{3a} - \frac{1}{2}(d_a)^2 + V \right\} \end{aligned}$$

where

$$d_a \equiv \frac{e}{2}\left(\phi_{sb}^* i f_{abc}\phi_{sc}\right) - X_a, \tag{1}$$

and the quantity X_a is a real scalar which transforms in the adjoint representation. We could consider that

$$X_a = \frac{e}{2}\left[m_{N_s}\mathrm{Im}\left(\phi_{sa}\right) + m_{M_s}\mathrm{Re}\left(\phi_{sa}\right) + c\delta_{a,0}\right]$$

where m_{N_s}, m_{M_s} and c are real mass parameters and the last term could only exist if G contains a $U(1)$ factor generated by T_0 (and therefore G would not be simple). If

$$V \geq \frac{1}{2}(d_a)^2 \ , \tag{2}$$

it follows that

$$T \geq \pm \int d^2x X_a B_{3a} \,. \tag{3}$$

Since $T \geq 0$, we take the upper (lower) sign if the above integral is positive (negative). The equality happens when

$$\begin{aligned} B_{3a} &= \mp d_a & (4)\\ D_\mp \phi_s &= 0 & (5)\\ V - \frac{1}{2}(d_a)^2 &= 0 & (6)\\ E_{ia} &= B_{1a} = B_{2a} = D_0\phi_s = D_3\phi_s = 0 & (7) \end{aligned}$$

and we recover the non-Abelian string BPS conditions in [10] for the particular case in which all scalars are in the adjoint. Like in [10], for simplicity we shall consider that m_{M_3} could be the only non-vanishing mass parameter in X_a ands we shall rename it by m. Note that if we had chosen to set that only $m_{N_3} \neq 0$, then X_a would allow a non-vanishing pseudoscalar vacuum solution which would result that the magnetic charge of the monopole and the flux of the string to be Lorentz scalar and not pseudoscalar as usual. Moreover we shall consider G to be simple since we are interested in string solutions associated to the breaking of non-Abelian group and not due to the breaking of $U(1)$ factors. Therefore we shall consider that X_a do not have the term $c\delta_{a,0}$.

We shall consider the potential

$$V = \frac{1}{2}\left[(d_a)^2 + f_{sa}^\dagger f_{sa}\right] \tag{8}$$

with d_a given by (1) and

$$\begin{aligned} f_1 &\equiv \frac{1}{2}\left(e\left[\phi_3,\phi_1\right]-\mu\phi_1\right) \,, \\ f_2 &\equiv \frac{1}{2}\left(e\left[\phi_3,\phi_2\right]+\mu\phi_2\right) \,, \\ f_3 &\equiv \frac{1}{2}\left(e\left[\phi_1,\phi_2\right]-\mu_3\phi_3\right) \,. \end{aligned} \tag{9}$$

This potential fulfills condition (2). For this potential, the BPS condition (6) is equivalent to the condition

$$f_s = 0 \;\; , \;\; s = 1,\, 2,\, 3\,.$$

This is the potential of the bosonic part of $\mathcal{N}=4$ super Yang-Mills (SYM) theory with some mass term deformations which break completely supersymmetry. If we set $m=0$, $\mathcal{N}=1$ supersymmetry is restored and we obtain the potential considered in [3]. If further $\mu_3=0$ we recover the potential of $\mathcal{N}=2$ with a massive hypermultiplet in the adjoint representation. Finally, if also $\mu=0$, we obtain $\mathcal{N}=4$. As usual, we shall denote by $\mathcal{N}=2^*$, $\mathcal{N}=1^*$ and $\mathcal{N}=0^*$ to the theories which are obtained by adding deformation mass terms to $\mathcal{N}=4$ SYM theory.

Note that the term X_a is necessary if one want to have a BPS string which is not tensionless. Therefore this term generalize the rôle of the Fayet-Iliopolous terms in theories with $U(1)$ factors, by given tension to the BPS string. However X_a in general breaks supersymmetry.

3 Phases of the theory

The vacua of the theory are solutions of

$$G_{\mu\nu} = D_\mu \phi_s = V = 0\,. \tag{10}$$

The condition $V(\phi_s)=0$ is equivalent to

$$d_a = 0 = f_{sa}\,. \tag{11}$$

We shall only consider the theory in the weak coupling regime, and therefore we shall not consider the quantum corrections to the potential. We are looking for vacuum solutions which produce the symmetry breaking

$$G \;\rightarrow\; U(1)^r \;\rightarrow\; C_G$$

where r is the rank of G and C_G its center. For the first phase transition it will appear (solitonic) magnetic monopoles. Then, in the second phase transition it will appear magnetic flux tubes or strings (if C_G is non-trivial) and the monopoles will become confined. In order to produce this symmetry breaking we shall look for vacuum solutions of the form

$$\begin{aligned} \phi_1^{\rm vac} &= a_1 T_+\,, \\ \phi_2^{\rm vac} &= a_2 T_-\,, \\ \phi_3^{\rm vac} &= a_3 T_3\,, \\ W_\mu^{\rm vac} &= 0\,, \end{aligned} \tag{12}$$

where a_1 and a_2 are complex constants, a_3 is a real constant and

$$\begin{aligned} T_3 &= \delta\cdot H \;\;,\;\; \delta \equiv \sum_{i=1}^{r}\frac{2\lambda_i}{\alpha_i^2} = \frac{1}{2}\sum_{\alpha>0}\frac{2\alpha}{\alpha^2}, \\ T_\pm &= \sum_{i=1}^{r}\sqrt{c_i}E_{\pm\alpha_i}, \end{aligned}$$

with α_i and λ_i being simple roots and fundamental weights respectively and

$$c_i \equiv \sum_{j=1}^{r} \left(K^{-1}\right)_{ij}$$

with $K_{ij} = 2\alpha_i \cdot \alpha_j/\alpha_j^2$ being the Cartan matrix. The generators $T_\pm$, T_3 form the so called principal $SU(2)$ subalgebra of G. The vacuum configuration $\phi_3^{\rm vac}$ breaks G into $U(1)^{\rm r}$ and then $\phi_1^{\rm vac}$ or $\phi_2^{\rm vac}$ breaks it further to C_G. Note that this is not the only possible vacuum configuration which produce the above symmetry breaking. However in this paper we shall restrict to analyze this configuration. We shall adopt the conventions

$$\begin{aligned} [H_i, E_\alpha] &= \alpha^i E_\alpha, \\ [E_\alpha, E_{-\alpha}] &= \frac{2\alpha}{\alpha^2} \cdot H, \end{aligned}$$

where α^i means the i component of the root α. Let

$$\alpha_i^{\rm v} \equiv \frac{2\alpha_i}{\alpha_i^2} \ , \quad \lambda_i^{\rm v} \equiv \frac{2\lambda_i}{\alpha_i^2},$$

be the simple coroots [2] and fundamental coweights respectively. Then using the relations

$$\begin{aligned} \lambda_j^{\rm v} &= \alpha_i^{\rm v} \left(K^{-1}\right)_{ij}, \\ \lambda_i^{\rm v} \cdot \alpha_j &= \delta_{ij}, \end{aligned}$$

we obtain from the vacuum equations $d_a = 0 = f_s$, that

$$\begin{aligned} \left(a_3 - \frac{\mu}{e}\right) a_i &= 0 \ , \quad \text{for } i = 1, 2\,, \\ a_1 a_2 &= \frac{\mu_3 a_3}{e}\,, \\ m a_3 &= |a_2|^2 - |a_1|^2 \ . \end{aligned}$$

Independently of the values of the mass parameters, this system always has the trivial solution $a_1 = a_2 = a_3 = 0$, which correspond to the vacuum in which the G is unbroken. Let us analyze other possible vacuum solutions in which G is broken.

From this system we can conclude that if $\mu = 0$, there exist non-vanishing solution only if $\mu_3 = 0 = m$, which means that we recover $\mathcal{N} = 4$ SYM theory. In this case, $a_1 = 0 = a_2$ and a_3 can be arbitrary which implies that G is broken to $U(1)^r$ if a_3 is non-vanishing. But then if we add a $\mathcal{N} = 1$ or $\mathcal{N} = 0$ deformation to the $\mathcal{N} = 4$ potential, by considering either μ_3 or m non-vanishing, then the only solution is the trivial $a_1 = a_2 = a_3 = 0$ and G returns to be unbroken. Therefore it doesn't happen monopole confinement when $\mu = 0$, at least for vacuum configurations like (12).

For the $\mathcal{N} = 2^*$ theory, in which $\mu \neq 0$ and $\mu_3 = 0 = m$, the situation is like in the $\mathcal{N} = 4$ case, with the solution $a_1 = 0 = a_2$ and a_3 arbitrary, which results in G broken to $U(1)^r$ for $a_3 \neq 0$. Let us analyze the vacuum solutions when we add deformation terms to $\mathcal{N} = 2^*$:

i) Adding the $\mathcal{N} = 1$ deformation term ($\mu \neq 0$, $\mu_3 \neq 0$ and $m = 0$).

In this case, there are non trivial solutions satisfying

$$\begin{aligned} a_3 &= \frac{\mu}{e}\,, \\ a_1 a_2 &= \frac{\mu_3 \mu}{e}\,, \\ |a_1|^2 &= |a_2|^2\,. \end{aligned}$$

[2] In this paper the definition of $\alpha_i^{\rm v}$ differs to the one adopted in [10] and [11] by a factor of two.

which results in a vacuum which breaks $G \to C_G$.

ii) Adding the $\mathcal{N}=0$ deformation term ($\mu \neq 0$, $\mu_3 = 0$ and $m \neq 0$).

If $\mu_3 = 0$, then either $a_1 = 0$ or $a_2 = 0$. We shall take $a_1 = 0$. Then, there are two possible situations:

- $m\mu < 0 \Rightarrow$ If we consider $a_2 \neq 0$, then $a_3 = \mu/e$ which would imply $|a_2|^2 < 0$. Therefore in this case, we must have $a_2 = 0 = a_3$ and G remains unbroken.
- $m\mu > 0 \Rightarrow$ In this case there is the non-trivial solution

$$a_3 = \frac{\mu}{e}\,, \tag{13}$$

$$|a_2|^2 = \frac{m\mu}{e}\,, \tag{14}$$

  which also results in a vacuum which breaks $G \to C_G$.

iii) Adding the $\mathcal{N}=1$ and $\mathcal{N}=0$ deformation terms ($\mu \neq 0$, $\mu_3 \neq 0 \neq m$).

In this case there are non-trivial solutions with

$$a_3 = \frac{\mu}{e}$$

and a_1 and a_2 satisfying

$$\begin{aligned} a_1 a_2 &= \frac{\mu_3 \mu}{e}\,, \\ \frac{m\mu}{e} &= |a_2|^2 - |a_1|^2\,. \end{aligned}$$

Once more G is broken to C_G.

In summary, in the $\mathcal{N}=4$ and $\mathcal{N}=2^*$ theory, the gauge group G can be broken to $U(1)^r$ which corresponds to the Coulomb phase. If we add to $\mathcal{N}=2^*$ a $\mathcal{N}=1$ or $\mathcal{N}=0$ deformation (or both), the gauge group can be further broken to C_G, which gives rise to the Higgs or color superconducting phase. Let us analyze each of these phases in the next sections.

4 The Coulomb phase

In this phase G is broken to $U(1)^r$ and there exist solitonic monopole solutions. As we have seen,that phase can only occur for the $\mathcal{N}=4$ and $\mathcal{N}=2^*$ cases. That could happen for example for energy scales in which one can consider $\mu_3 = 0 = m$. In this phase $a_1 = 0 = a_2$ and $a_3 \neq 0$. In principle a_3 is an arbitrary non-vanishing constant. However, we shall fix

$$a_3 = \frac{\mu}{e}$$

in order to have the same value as in the Higgs phase. The vacuum solution $\phi_3^{\rm vac}$ singles out a particular $U(1)$ direction which we call $U(1)_\delta$. Since for any root α, $\delta \cdot \alpha \neq 0$, we can construct a monopole solution for each root α. The asymptotic field configuration for the monopole associated to the root α can be written as [23]

$$\begin{aligned} \phi_3(\theta,\varphi) &= g_\alpha(\theta,\varphi)\phi_3^{\rm vac} g_\alpha(\theta,\varphi)^{-1} = a_3 g_\alpha(\theta,\varphi)\delta\cdot H g_\alpha(\theta,\varphi)^{-1}\,, \\ B_i(\theta,\varphi) &= \frac{r_i}{er^2} g_\alpha(\theta,\varphi) T_3^\alpha g_\alpha(\theta,\varphi)^{-1}\,, \\ \phi_1(\theta,\varphi) &= \phi_2(\theta,\varphi) = 0\,, \end{aligned} \tag{15}$$

where

$$g_\alpha(\theta,\varphi) = \exp(i\varphi T_3^\alpha)\ \exp(i\theta T_2^\alpha)\ \exp(-i\varphi T_3^\alpha)\,,$$

with the $SU(2)$ generators

$$T_1^\alpha = \frac{E_\alpha + E_\alpha}{2}\ ,\quad T_2^\alpha = \frac{E_\alpha - E_{-\alpha}}{2i}\ ,\quad T_3^\alpha = \frac{1}{2}\alpha^{\rm V}\cdot H\,.$$

The associated monopole magnetic charge is

$$g \equiv \frac{1}{|\phi_3^{\rm vac}|}\oint dS_l {\rm Tr}\left[{\rm Re}\left(\phi_3\right) B_l\right] = \frac{2\pi}{e}\frac{\delta\cdot\alpha^{\rm V}}{|\delta|}\,. \qquad (16)$$

with

$$\delta^2 = \frac{h^{\rm V}\left(h^{\rm V}+1\right)r}{24}\,,$$

where $h^{\rm V}$ is the dual Coxeter number of G. Clearly g is equal to the monopole magnetic flux in the $U(1)_\delta$ direction, $\Phi_{\rm mon}$. It is also convenient to define magnetic fluxes associated to each element of the little group $U(1)^r$. As explained in [24], in the sphere with $r \to \infty$, the little group of $\phi_3(\theta,\varphi)$ varies within G by conjugation with the gauge transformations $g_\alpha(\theta,\varphi)$. Therefore the magnetic flux associated to the Cartan generator $\lambda_i^{\rm V}\cdot H$, $i = 1$, 2, ..., r, can be defined as

$$\Phi_{\rm mon}^{(i)} \equiv \oint dS_l\, {\rm Tr}\left[g_\alpha(\theta,\varphi)\lambda_i^{\rm V}\cdot H g_\alpha(\theta,\varphi)^{-1}B_l\right] = \frac{2\pi}{e}\lambda_i^{\rm V}\cdot\alpha^{\rm V}\,, \qquad (17)$$

which are topologically conserved charges [25].

These are BPS monopoles with masses given by the central charge of the $\mathcal{N} = 2$ algebra [26, 2]. For the monopoles with vanishing fermion number, their masses are $M_{\rm mon} = |g||\phi_3^{\rm vac}|$. Not all of these monopoles are stable. The stable or fundamental are the ones with lightest masses [25]. For the present symmetry breaking, the fundamental monopoles are associated to the simple roots for the simple-laced algebras or to the long simple roots for the non simply-laced ones. Their masses are

$$M_{\rm mon}^{\rm L} = \frac{2\pi}{e|\delta|}|\phi_3^{\rm vac}|. \qquad (18)$$

Note that, since G is completely broken to $U(1)^{\rm r}$, differently from [11], here the stable monopoles do not fill representations of a non-Abelian unbroken group.

5 The Higgs or color superconducting phase

In the Higgs or color superconducting phase, G is broken to its center C_G. That can happen when $\mathcal{N} = 2^*$ is broken by a $\mathcal{N} = 1$ or $\mathcal{N} = 0$ deformation term (or both). In this phase, the monopole chromomagnetic flux lines can not spread out radially over space. A phenomena like that is expected to happen in the interior of very dense neutron stars [19]. However, since for simply connected G

$$\Pi_1\left(G/C_G\right) = C_G\,, \qquad (19)$$

when C_G is non-trivial, these flux lines can form topologically nontrivial Z_N strings, with N being the order of C_G.Then, the monopoles of $\mathcal{N} = 2^*$ become confined in this phase, as we shall show bellow.

In order to have finite string tension, these string solution must satisfy the vacuum equations asymptotically where the radial coordinate $\rho \to \infty$, which implies that

$$\begin{aligned}\phi_s(\varphi,\rho\to\infty) &= g(\varphi)\phi_s^{\rm vac}g(\varphi)^{-1},\\ W_I(\varphi,\rho\to\infty) &= g(\varphi)W_I^{\rm vac}g(\varphi)^{-1} - \frac{1}{ie}\left(\partial_I g(\varphi)\right)g(\varphi)^{-1},\end{aligned}$$

where capital Latin letters I, J denote the coordinates 1 and 2; ϕ_s^{vac} and W_I^{vac} are given by Eq. (12) and $g(\varphi) \in G$. In order for the field configuration to be single valued, $g(\varphi+2\pi)g(\varphi)^{-1} \in C_G$. Considering

$$g(\varphi) = \exp i\varphi M \ ,$$

then $\exp 2\pi i M \in C_G$. That implies that M must be diagonalizable and we shall consider that

$$M = \omega \cdot H.$$

Then, in order to $\exp 2\pi i\omega \cdot H \in C_G$,

$$\omega = \sum_{i=1}^{r} l_i \lambda_i^{\text{v}} ,$$

where l_i are integer numbers, that is, ω must be a vector in the coweight lattice of G, which has the fundamental coweights λ_i^{v} as basis vectors, and is equivalent to the weight lattice $\Lambda_{\text{W}}\left(\widetilde{G}^{\text{v}}\right)$ of the covering group of the dual group $\widetilde{G}^{\text{v}}$ [27]. In principle, we could have other possibilities for M like some combinations of step operators, which however we shall not discuss here.

With the above choice for $g(\varphi)$, the asymptotic string configuration can be written as,

$$\phi_s(\varphi, \rho \to \infty) = e^{i\varphi\omega\cdot H} \phi_s^{\text{vac}} e^{-i\varphi\omega\cdot H}, \tag{20}$$

$$W_I(\varphi, \rho \to \infty) = \frac{\varepsilon_{IJ} x^J}{e\rho^2} \omega \cdot H \ , \quad I = 1, 2. \tag{21}$$

Note that not all of these strings are necessarily stable.

We can consider the string ansatz

$$\begin{aligned}
\phi_1(\varphi, \rho) &= e^{i\varphi\omega\cdot H} \sum_{i=1}^{r} \left(f_1^i(\rho) E_{\alpha_i} \right) e^{-i\varphi\omega\cdot H} = \sum_{i=1}^{r} e^{i\varphi\omega\cdot\alpha_i} f_1^i(\rho) E_{\alpha_i} , \\
\phi_2(\varphi, \rho) &= e^{i\varphi\omega\cdot H} \sum_{i=1}^{r} \left(f_2^i(\rho) E_{-\alpha_i} \right) e^{-i\varphi\omega\cdot H} = \sum_{i=1}^{r} e^{-i\varphi\omega\cdot\alpha_i} f_2^i(\rho) E_{-\alpha_i} , \\
\phi_3(\varphi, \rho) &= e^{i\varphi\omega\cdot H} \sum_{i=1}^{r} \left(f_3^i(\rho) \lambda_i^{\text{v}} \cdot H \right) e^{-i\varphi\omega\cdot H} = \sum_{i=1}^{r} f_3^i(\rho) \lambda_i^{\text{v}} \cdot H, \\
W_I(\varphi, \rho) &= \frac{\varepsilon_{IJ} x^I}{e} \sum_{i=1}^{r} g_i(\rho) \lambda_i^{\text{v}} \cdot H \ , \quad W_0(\varphi, \rho) = 0 = W_3(\varphi . \rho) \ ,
\end{aligned}$$

which results that

$$B_3(\varphi, \rho) = \frac{1}{e\rho^2} \sum_{i=1}^{r} \lambda_i^{\text{v}} \cdot H \frac{\partial g_i(\rho)}{\partial \rho}.$$

These functions must satisfy the boundary condition

$$\begin{aligned}
f_n^i(\rho \to \infty) &= a_n \sqrt{c_i} \ , \ \text{ for } n = 1, 2 \ , \\
f_3^i(\rho \to \infty) &= a_3 \ , \\
g_i(\rho \to \infty) &= l_i \ ,
\end{aligned}$$

for $i = 1, ..., r$, in order to recover the asymptotic configuration (20), (21) and

$$\begin{aligned}
f_n^i(\rho = 0) &= 0 \ , \quad \text{for } n = 1, 2 \text{ and } i \text{ such that } \omega \cdot \alpha_i = l_i \neq 0 \ , \\
g_i(\rho = 0) &= 0 \ , \quad \text{for } i = 1, ..., r \ ,
\end{aligned}$$

in order to eliminate the singularities at $\rho = 0$. One can put the ansatz in the BPS conditions (4)-(7) and obtain first order differential equations similar to the ones in [10]. Otherwise one can put in the equations of motion for the non-BPS cases. Like in [10], the BPS conditions are consistent with the equations of motion only when m vanishes. However in the case in which $\mathcal{N} = 2^*$ is broken by a $\mathcal{N} = 0$ deformation, we must take the limit $m \to 0$ in order to maintain the symmetry breaking $G \to C_G$, similarly to the Prassard-Sommerfeld limit [28] for the BPS monopole.

Like in [21] we shall consider that ϕ_3 is constant and equal to its asymptotic value, i e,

$$\phi_3(\varphi, \rho) = \frac{\mu}{e} T_3. \tag{22}$$

For the BPS string, using the above ansatz and boundary conditions, one obtains Eq. (22) directly from the BPS condition $D_\mp \phi_3 = 0$, which implies that $f_3^i(\rho) = \text{const.} = a_3 = \mu/e$. Hence the lower bound for the string tension given by Eq. (3), for $X = em\,\mathrm{Re}(\phi_3)/2$, can be written as

$$T \geq \frac{me}{2} |\phi_3^{\text{vac}}| \, |\Phi_{\text{st}}| = \frac{m\mu}{2} |\delta| \, |\Phi_{\text{st}}| \tag{23}$$

where

$$\Phi_{\text{st}} \equiv \frac{1}{|\phi_3^{\text{vac}}|} \int d^2x \mathrm{Tr}\left(\mathrm{Re}\left(\phi_3\right) B_3\right) = \frac{-e}{\mu\,|\delta|} \oint dl_I \mathrm{Tr}\left(\mathrm{Re}\left(\phi_3\right) W_I\right) = \frac{2\pi}{e} \frac{\delta \cdot \omega}{|\delta|} \tag{24}$$

is the string flux in the $U(1)_\delta$ direction. The bound in Eq.(23) holds for the BPS strings. For the case of $\mathcal{N} = 2^*$ broken by an $\mathcal{N} = 0$ deformation, the limit $m \to 0$ would imply $T \to 0$. Then, if one wants to have a BPS string with finite T, one should also take $\mu \to \infty$, similarly to the case of the BPS Z_k strings in [10]. A similar limit was considered in [5, 9]. That is exactly like the London limit in the Abelian-Higgs theory describing superconductors where one takes to infinite the mass of the scalar. On the other hand, when $\mathcal{N} = 2^*$ is broken by an $\mathcal{N} = 1$ deformation (i.e. $m = 0$, $\mu \neq 0$ and $\mu_3 \neq 0$), from (23) we see that the BPS string will be tensionless. The same happens in general for the BPS strings associated to a coweights ω such that $\delta \cdot \omega = 0$, and therefore $\Phi_{\text{st}} = 0$.

Similarly to the monopole, we can define string fluxes associated to each Cartan element $\lambda_i^{\text{v}} \cdot H$,

$$\Phi_{\text{st}}^{(i)} \equiv \int d^2x \mathrm{Tr}\left[\lambda_i^{\text{v}} \cdot H\, B_3\right] = \frac{2\pi}{e} \lambda_i^{\text{v}} \cdot \omega\,. \tag{25}$$

Therefore, from (24) or (25), we can conclude that the string fluxes take values in the coweight lattice of G. Let us now check if the magnetic fluxes of the monopoles are compatible with the ones of the strings. Since an arbitrary coroot α^{v} can always be expanded in the coweight basis as $\alpha^{\text{v}} = \sum_{i=1}^r n_i \lambda_i^{\text{v}}$ where n_i are integer numbers, one can conclude that the magnetic fluxes (16) or (17) of the monopoles can in principle be expressed as an integer linear combinations of the string fluxes (24) or (25). Therefore in the Higgs phase, the monopole magnetic flux lines can no longer spread radially over the space, since G is broken to the discrete group C_G. However they can form one or more flux tubes or strings, and the monopoles can become confined. We shall call this set of strings attached to a monopole as confining strings. This set of confining strings must have total flux given by Eq. (24) or (25) with $\omega = \alpha^{\text{v}}$. That means that this set of confining magnetic strings belongs to the trivial topological sector of $\Pi_1(G/C_G)$ since $\exp 2\pi i \alpha^{\text{v}} \cdot H = 1$ in G. The fact that the set of confining strings must belong to the trivial sector is consistent with the fact that the set is not topologically stable and therefore can terminate at some point. However since it has a non-vanishing flux it must terminate in a magnetic source, i. e., a monopole. It is important to stress the fact that a string configuration belonging to topological trivial sector doesn't imply that its flux must vanishes as we can see from (24). All these results are generalizations of some well known results for the Z_2-string of $SU(2)$ Yang-Mills-Higgs theory, as explained in [29, 30]. In this theory there are at least two complex scalars in the adjoint representation which produce the symmetry breakings $SU(2) \rightarrow U(1) \rightarrow Z_2$, similarly to our case. In the Higgs phase, it can in principle exist string configurations with flux $2\pi n/e$ for any integer n, although only the ones with $n = \pm 1$ are topologically stable. The ones with odd n belong to the topologically non-trivial sector while the ones with even n belong to the trivial sector. Therefore string configurations belonging to the same topological sector does not have necessarily the same flux and therefore are not related by (non-singular) gauge transformations [29][31]. The string configuration with $n = 2$, belonging to the trivial sector and which can be formed by two strings with $n = 1$, is the one which can terminate in the 't Hooft-Polyakov monopole with magnetic charge $g = 4\pi/e$, and can produce the monopole-antimonopole confinement [32]. In more algebraic terms one can say that this set of integer numbers n form the coweight lattice $\Lambda_{\rm w}$ of $SU(2)$, the subset of even numbers $2n$ form the $SU(2)$ coroot lattice $\Lambda_{\rm r}$ and the quotient $\Lambda_{\rm w}/\Lambda_{\rm r} \simeq Z_2$ correspond to the center of $SU(2)$. Therefore this quotient has two elements which are represented by the cosets $\Lambda_{\rm r}$ and $1 + \Lambda_{\rm r}$. Each coset corresponds to a string topological sector, with $\Lambda_{\rm r}$ been the trivial one. These results also holds for an arbitrary group G, where [27]

$$C_G \simeq \frac{\Lambda_{\rm w}\left(\widetilde{G}^{\rm v}\right)}{\Lambda_{\rm r}\left(G^{\rm v}\right)}, \tag{26}$$

with $\Lambda_{\rm r}(G^{\rm v})$ been the root lattice of the dual group $G^{\rm v}$ or, equivalently, the coroot lattice of G, which has the simple coroots $\alpha_i^{\rm v}$ as basis vectors. If N is the order of C_G, then this quotient can be represented by the N cosets

$$\Lambda_{\rm r}(G^{\rm v}) \text{ and } \lambda_{i,\min}^{\rm v} + \Lambda_{\rm r}(G^{\rm v}), \tag{27}$$

where $\lambda_{i,\min}^{\rm v}$ are the minimal fundamental coweights of G. A fundamental coweight is minimal if

$$\lambda_{i,\min}^{\rm v} \cdot \psi = 1$$

where ψ is the highest root and there exist exactly $(N-1)$ of them. The minimal coweights $\lambda_{i,\min}^{\rm v}$ are associated to a special outer automorphism of the extended Dynkin diagrams [33]. For $SU(N)$, all fundamental weights λ_i are minimal.

From Eqs. (19) and (26) it implies that

$$\Pi_1(G/C_G) = \frac{\Lambda_{\rm w}\left(\widetilde{G}^{\rm v}\right)}{\Lambda_{\rm r}\left(G^{\rm v}\right)}, \tag{28}$$

and we can conclude that each string topological sectors is associated to a coset in (27), with $\Lambda_{\rm r}(G^{\rm v})$ been the trivial topological sector. It is important to note that for $G = SU(N)$, this result is equivalent to consider the string topological sectors to be associated with the N-ality of the representations. However the above result holds for arbitrary G.

Since the confining string configuration linking a monopole to an antimonopole belongs to the trivial topological sector, it can break when it has enough energy to create a new monopole-antimonopole pair. Like was done in [11], one can obtain a bound for the threshold length $d^{\rm th}$for the string breaking, using the relation

$$2M_{\rm mon}^{\rm L} = E^{\rm th} = Td^{\rm th} \geq \frac{me}{2}\left|\phi_3^{\rm vac}\right|\left|\Phi_{\rm st}\right| d^{\rm th}\,, \tag{29}$$

where $E^{\rm th}$ is the string threshold energy and $M_{\rm mon}^{\rm L}$ is the mass of the lightest monopoles, given by Eq. (18). In the above relation we used the string bound given by Eq. (23) and did not consider a possible energy term proportional to the inverse of the monopoles distance, known as the Lücher term. The modulus of the string flux, $|\Phi_{\rm st}|$, must be equal to the modulus of the magnetic charges $|g|$ of each confined monopoles. Let us consider that $|g| = 2\pi|\delta\cdot\beta^{\rm v}|/|\delta|$ with $\beta^{\rm v}$ being an arbitrary coroot. Therefore one can conclude from Eq. (29), using Eq. (18), that

$$d^{\rm th} \leq \frac{4}{me|\delta\cdot\beta^{\rm v}|}.$$

6 Monopole confinement for $SU(N)$ broken to Z_N

Let us consider $G = SU(N)$. Since $SU(N)$ is simply laced, we don't need to distinguish between weights and coweights, roots and coroots. We have seen that the magnetic lines of a given monopole with magnetic charge $g = 2\pi\delta\cdot\alpha^{\rm v}/|\delta|$, can form a set of flux tubes or strings. However there are countless different string configurations with this magnetic flux. It is not clear at the moment which could be the preferable one. The most "economical" sets would be the ones formed by a strings and an antistring as we shall see bellow.

For $SU(3)$, the quotient (26), which is equivalent to $C_{SU(3)} = Z_3$, possesses three elements which can be represented by the cosets $\Lambda_{\rm r}(SU(3))$, $\lambda_1+\Lambda_{\rm r}(SU(3))$ and $\lambda_2+\Lambda_{\rm r}(SU(3))$. One can, for example, construct string solutions associated to each of the three weights λ_1, $\lambda_1-\alpha_1$, $\lambda_1-\alpha_1-\alpha_2$ of the fundamental representation. Since all of them belong to the coset $\lambda_1+\Lambda_{\rm r}$, these string solutions belong to the same topological sector. However one can observe from Eq. (24) that they don't have same flux $\Phi_{\rm st}$, similarly to the Z_2 strings of $SU(2)$ theory. Therefore these string solutions are *not* related by gauge transformations since $\Phi_{\rm st}$ is gauge invariant. One can construct the corresponding antistring solutions associated the the negative of these weights, which form the complex conjugated representation $\overline{3}$ and which belong to the coset $\lambda_2+\Lambda_{\rm r}$. The magnetic fluxes of the monopoles associated to the 6 non-vanishing roots of $SU(3)$ can easily be written using these strings in the following way: for the monopole α_1 we can attach the strings λ_1 and $-\lambda_1+\alpha_1$. For the monopole α_2 we can attach strings $\lambda_1-\alpha_1$ and $-\lambda_1+\alpha_1+\alpha_2$. For the monopole $\alpha_1+\alpha_2$ we can attach the strings λ_1 and $-\lambda_1+\alpha_1+\alpha_2$. And similarly for the other 3 monopoles associated to the negative roots, just changing the sings. The remaining 3 combinations of strings and antistring has vanishing fluxes $\Phi_{\rm st}^{(i)}$.

One could draw the above set of strings attached to monopoles as shown in Fig.1, where the circles represent the monopoles and the arrows are the string flux $\Phi_{\rm st}^{(i)}$. We represented the strings associated to weights in the fundamental representation by an arrow going out of the monopole and for the antistrings we reversed the sense of the arrow and simultaneously changed the sign of the weight. Then, in addition to the monopole-antimonopole pairs one could also conjecture the formation of a system with the monopoles α_1, α_2 and $-\alpha_1-\alpha_2$ as shown in Fig. 2. Note that since these monopoles are not expected to fill the 3 dimensional fundamental representation of $SU(3)$, that system is not exactly like a baryon. With this configuration of monopole with strings attached, one could also think to put one string in the north pole and the on the other in the south pole, forming a configuration similar to the bead described in [31].

In principle one could also think to attach to the monopole α_1 the strings $2\lambda_1$ and $-2\lambda_1+\alpha_1$ which belong to the 6 dimensional symmetric tensor representation and its complex conjugated. However one can conclude directly from the expression for the fluxes and string tension that, in the BPS case, the string $2\lambda_1$ can decay in 2 strings λ_1 and the string $-2\lambda_1+\alpha_1$ can decay in the strings $-\lambda_1$ and $-\lambda_1+\alpha_1$. A similar result have been conjectured in some different theories [6, 16, 13].

One can easily extend the above construction of strings attached to monopoles to the $SU(N)$ case. In this case the quotient (26) has N elements which can be represented by the cosets

$$\Lambda_{\rm r}(SU(N)) \text{ and } \lambda_i+\Lambda_{\rm r}(SU(N)),\ i=1,2,\ \dots,\ N-1.$$

The representation corresponding to the fundamental weight λ_k can be obtained by the antisymmetric tensor product of k fundamental representations associated to λ_1. Once more one can consider the strings associated to the weights of the fundamental N dimensional representation,

$$\lambda_1 \quad \text{and} \quad \lambda_1 - \sum_{j=1}^{l} \alpha_j \quad , \quad l = 1, 2, ..., N-1,$$

which belong to the $\lambda_1 + \Lambda_{\rm r}(SU(N))$ coset and to the negative of these weights which form the complex conjugated representation $\overline{N}$, which belongs to the $\lambda_{N-1} + \Lambda_{\rm r}(SU(N))$ coset. Since $N \times \overline{N} = \text{adj.} + 1$, one can write the fluxes of monopoles (not all of them stable) associated to the $N(N-1)$ non-vanishing roots, in terms of a string and an antistring. For example for the monopole associated to the $SU(N)$ root $\alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+q}$ one could attach strings associated to the weights $\lambda_1 - \alpha_1 - \alpha_2 - \cdots - \alpha_{p-1}$ and $-\lambda_1 + \alpha_1 + \cdots + \alpha_{p+q}$. Like for the $SU(3)$ case, one could in principle form a confining configuration formed by the monopoles associated to the $N-1$ simple roots α_i and the negative of the highest root $-\alpha_1 - \alpha_2 - \cdots - \alpha_{N-1}$.

Since for $SU(N)$, $\lambda_{N-k} = -\lambda_k + \beta$, where $\beta \in \Lambda_{\rm r}(SU(N))$, each weight in the coset $\lambda_{N-k} + \Lambda_{\rm r}(SU(N))$ is the negative of a weight in $\lambda_k + \Lambda_{\rm r}(SU(N))$, and therefore the bound of a string tension, given by Eq. (23), associated to a weight in $\lambda_k + \Lambda_{\rm r}(SU(N))$ should be equal to the one associated to the negative weight in $\lambda_{N-k} + \Lambda_{\rm r}(SU(N))$.

7 String tension and Casimir scaling law

The string tension is one of the main quantities to be determined in quark confinement in QCD. In these last 20 years quite a lot of work have been done trying to determine this quantity. There are mainly two conjectures for the string tension: the "Casimir scaling law" [34] and the "sine law" [4]. In these two conjectures it is considered the gauge group $G = SU(N)$ and a string in the representation associated to the fundamental weight λ_k which can be obtained by the antisymmetric tensor product of k fundamental representations associated to λ_1. For the Casimir scaling conjecture, the string tension should satisfy

$$T_k = T_1 \frac{k(N-k)}{N-1}, \quad k = 1, 2, ..., N-1, \tag{30}$$

where T_1 would be the string tension in the λ_1 fundamental representation. On the other hand, for sine law conjecture,

$$T_k = T_1 \frac{\sin\left(\frac{\pi k}{N}\right)}{\sin\left(\frac{\pi}{N}\right)}, \quad k = 1, 2, ..., N-1.$$

There are some papers [5, 6, 14, 35] using different approaches like MQCD, AdS/CFT, etc, given some support to this last conjecture. On the other hand several lattice studies [36] have appeared in the literature in the last years given support to both conjectures.

All these conjectures are concerned to the chromoelectric strings. However, as we mentioned in the introduction, one expects that chromomagnetic strings could be related to chromoelectric string by a duality transformation. Therefore one could ask if the tension of our chromomagnetic string satisfy one of the two conjectures.

Let us start with a general gauge group G. From Eqs. (23) and (24), we obtain that the string tension satisfies the bound

$$T_\omega \geq \frac{m\mu\pi}{e} |\delta \cdot \omega|$$

where ω must belong to one of the cosets (27). Let us

$$\omega = \lambda_k^{\rm v} - \beta_\omega$$

where λ_k^{V} is a minimal fundamental coweight and $\beta_\omega \in \Lambda_{\mathrm{r}}(G^{\mathrm{V}})$. Remembering that λ_k^{V} is a fundamental weight of $\widetilde{G}^{\mathrm{V}}$, and the quadratic Casimir associated to this fundamental representation is

$$C(\lambda_k^{\mathrm{V}}) = \lambda_k^{\mathrm{V}} \cdot (\lambda_k^{\mathrm{V}} + 2\delta) \,,$$

it follows that,

$$T_\omega \geq \frac{m\mu\pi}{e} \left| \frac{1}{2} \left[C(\lambda_k^{\mathrm{V}}) - \lambda_k^{\mathrm{V}} \cdot \lambda_k^{\mathrm{V}} \right] - \delta \cdot \beta_\omega \right| . \tag{31}$$

Clearly for a string configuration in the trivial topological sector, i.e. $\omega = -\beta_\omega \in \Lambda_{\mathrm{r}}(G^{\mathrm{V}})$, the above string tension bound does not have the first term.

Let us now consider $G = SU(N)$. The quadratic Casimir associated to the representation with fundamental weight λ_k is

$$C(\lambda_k) = \frac{N^2 - 1}{2N} \left[\frac{k\,(N-k)}{N-1} \right] .$$

Moreover

$$\lambda_k = e_1 + e_2 + \cdots + e_k - \frac{k}{N} \sum_{j=1}^{N} e_j$$

where $e_i \cdot e_j = \delta_{i,j}$. Therefore

$$\lambda_k \cdot \lambda_k = \frac{k\,(N-k)}{N}.$$

Hence, for $SU(N)$

$$T_{\lambda_k - \beta_\omega} \geq \frac{m\mu\pi}{e} \left| \frac{1}{2} \left(\frac{(N-1)^2}{2N} \frac{k\,(N-k)}{N-1} \right) - \delta \cdot \beta_\omega \right| \tag{32}$$

Therefore the first term in the RHS of this inequality or equivalently the BPS string tension associated to $\omega = \lambda_k$, can be written as

$$T_{\lambda_k}^{\mathrm{BPS}} = T_{\lambda_1}^{\mathrm{BPS}} \frac{k\,(N-k)}{N-1} \quad , \; k = 1, 2, ..., N-1 \tag{33}$$

where

$$T_{\lambda_1}^{\mathrm{BPS}} = \frac{m\mu\pi}{2e} \frac{(N-1)^2}{2N}$$

is the BPS string tension associated to $\omega = \lambda_1$. Hence we explicitly showed that the BPS string tensions associated to an arbitrary $SU(N)$ fundamental weight λ_k satisfy the Casimir scaling conjecture, given by Eq. (30). However, in the Casimir scaling law conjecture (and also in the sine law conjecture), it is believed that the string tension should be the same for all weights in a given topological sector. But from Eq. (31) or (32), we can see that the first term depends only on the coset or topological sector but the second term is proportional to $\delta \cdot \beta_\omega$ and therefore depends explicitly on which weight is being considered. As we have seen, that result is exactly like the $SU(2)$ case, where the strings in a given topological sector (i.e. n even or odd) do not have same magnetic flux and consequently string tension. On the other hand only the strings with $n = \pm 1$ are stable and satisfy (33). As we have mentioned before, not all of the strings associated to weights in a given coset are expected to be stable. Therefore, it would be interesting to determine the stability conditions for these string solutions, similarly to what was done for the BPS monopoles [25] and BPS $U(1)$ string solutions [37].

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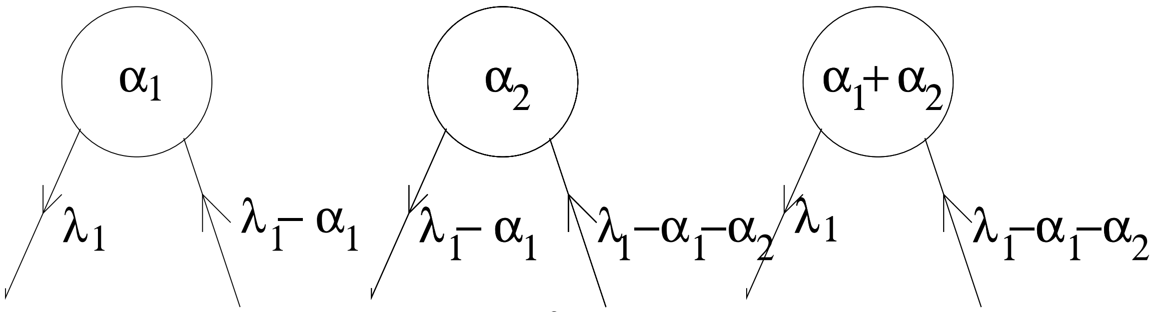
α1
λ1
λ1− α1
α2
λ1− α1
λ1−α1−α2
α1+α2
λ1
λ1−α1−α2

fig. 1

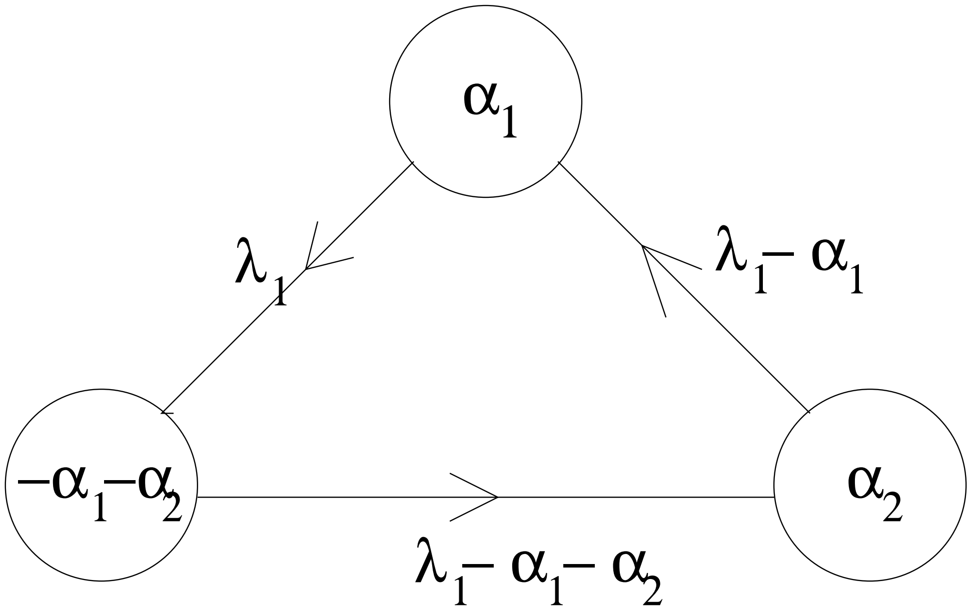
α1
λ1
λ1− α1
−α1−α2
α2
λ1− α1− α2

fig. 2