Manifolds of Positive Scalar Curvature

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Contents

1 Survey on the problem of finding a positive scalar curvature metric on a closed manifold

The basic question that we want to address in the first three lectures is the following.

1.1. Question. Which manifolds admit a Riemannian metric of positive scalar curvature?

We will consider this question for smooth, compact manifolds without boundary, and will assume that all manifolds mentioned in these lectures are of this type unless otherwise indicated. The above is just one of the many questions that one might ask relating the geometry and topology of manifolds. Here the 'geometry' of a manifold M is determined by a Riemannian metric on M and its curvature. There are various 'flavors' of curvature (sectional, Ricci, scalar, curvature operator, e.t.c., see [Be]), of which the scalar curvature is the simplest in the sense that the scalar curvature is just real valued function $s\colon M \to \mathbb{R}$ (the other flavors of curvature are described as 'tensors' on M). One might ask finer questions concerning the scalar curvature; for example: Given a manifold M, which smooth functions on M can be realized as the scalar curvature functions of Riemannian metrics on M? However, thanks to results of Kazdan-Warner [KW1], [KW2] (see also [Fu]), it turns out that this question essentially boils down to answering Question 1.1. Unlike most other curvature problems, we know quite a bit about the answer to Question 1.1 as we shall see.

1.1 Scalar curvature

We begin by recalling the definition of the scalar curvature function. Usually, the scalar curvature is defined in terms of the 'curvature tensor', which is the right thing to do if the goal is to do calculations. However, for our purposes the following geometric/conceptual definition seems more adequate. Let M be a manifold of dimension n equipped with a Riemannian metric g. Then the scalar curvature $s(p) \in \mathbb{R}$ at a point $p \in M$ is determined by the volume growth of geodesic balls around p. More precisely, let $B_r(p, M)$ be the geodesic ball of radius $r > 0$ around the point p (consisting of all points $x \in M$ whose distance from p is $\leq r$), and let us write $\operatorname{vol} B_r(p, M)$ for the volume of this ball. Then $s(p)$ is determined by the power series expansion (see [Be, 0.60])

$$\frac{\operatorname{vol} B_r(p, M)}{\operatorname{vol} B_r(0, \mathbb{R}^n)} = 1 - \frac{s(p)}{6(n+2)} r^2 + \dots . \tag{1.2}$$

In particular, $s(p) > 0$ means that geodesic balls around p of sufficiently small radius have *smaller volume* than the balls of the same radius in $\mathbb{R}^n$.

1.3. Examples of manifolds with $s > 0$. The *round sphere* (the geometer's slang for the sphere with its standard metric) has positive scalar curvature (provided of course $n \geq 2$, where n is the dimension of the sphere). Since curvature is local it follows that any manifold that has the round sphere as its universal covering, like the real projective space, or more generally any lens space, has positive scalar curvature.

The complex projective space $\mathbb{CP}^n$ or the quaternionic projective space $\mathbb{HP}^n$ also inherit a Riemannian metric from the round metric on the sphere whose quotients they are (these are referred to as the *Fubini-Study metric* in the geometric literature). It can be shown that their scalar curvature is positive as well.

1.2 Constructions of positive scalar curvature metrics

In order to answer Question 1.1, we have to do two things:

- construct positive scalar curvature metrics on some manifolds, and
- show that other manifolds do not admit such metrics.

We will begin with the 'constructive' part. The following observation allows us to construct a lot of new manifolds with positive scalar curvature metrics from a given manifold with positive scalar curvature metric.

1.4. Observation. Let g be a positive scalar curvature metric on a manifold M. Then the following manifolds admit metrics of positive scalar curvature:

1. The product $M \times N$ with any manifold N.
2. The total space of any fiber bundle $E \to N$ with fiber M, provided the transition functions are isometries of (M, g).

To prove the first claim, pick any metric h (not necessarily with positive scalar curvature) on N. Then g and h combine to determine a Riemannian metric $g \times h$ (called the product metric) on $M \times N$. The scalar curvature of $g \times h$ can be expressed in terms of the scalar curvature of g and h via the formula

$$s((x,y); g \times h) = s(x; g) + s(y; h) \qquad \text{for } (x,y) \in M \times N.$$

Here we write $s(x; g)$ for the scalar curvature at $x \in M$ with respect to the metric g; the meaning of $s(y; h)$ and $s((x,y); g \times h)$ is analogous. Assuming no further information about the metric h, of course this quantity could be positive or negative, and that doesn't seem to bode well for our goal of producing a positive scalar curvature metric on $M \times N$. Here is the trick: shrink M! Phrased more mathematically: replace g by tg where t is a positive real number, and let t approach 0. We see that

$$s((x,y); tg \times h) = s(x; tg) + s(y; h) = \frac{1}{t} s(x; g) + s(y; h).$$

This is *positive* for t sufficiently small thanks to the positivity of $s(x; g)$. Of course, *how* small we need to choose t will depend on (x, y); however, compactness of M and N implies that there is a t that will do the job for all $(x, y) \in M \times N$.

The same trick still works in the case of the 'twisted product' $E \to N$. In addition to a metric on the base N, we need to choose here a 'connection' on this bundle; assuming that the transition functions are isometries, these data determine a 'twisted product metric' on E (which makes $E \to M$ a Riemannian submersion with totally geodesic fibers; see [Be, Ch. 9]); the *O'Neill formulas* express the curvature of this metric in terms of the curvatures of g, h and the curvature of the connection (see [Be, Proposition 9.70]). Replacing g by tg, the dominant term of the scalar curvature of E for small t is again $\frac{1}{t}$ times the scalar curvature of g (see [Be, Formula (9.70d)]). Hence the scalar curvature on E is positive for sufficiently small t.

In differential topology important ways of modifying a given manifold is *surgery* and *attaching a handle.* Of course these modifications are closely related: if W is a manifold with boundary $\partial W = M$, and $\widehat{W}$ is obtained from W by attaching a handle $D^{k+1} \times D^{n-k}$ via an embedding $S^k \times D^{n-k} \subset M^n$, then $\widehat{M} \stackrel{\text{def}}{=} \partial \widehat{W}$ is obtained from M by a surgery (i.e., by removing $S^k \times D^{n-k}$ and replacing it by $D^{k+1} \times S^{n-k-1}$). Independently Gromov-Lawson [GL] and Schoen-Yau [SY] showed that if M admits a positive scalar curvature metric, and $n-k$ (the *codimension of the surgery/handle*) is greater than 2, then $\widehat{M}$ also admits such a metric. Based on their techniques, Gajer [Gaj] later proved the following result:

Theorem 1.5. *Let W be a manifold with boundary and let g be a positive scalar curvature metric on W. Assume that $\widehat{W}$ is obtained from W by attaching a handle of codimension ≥ 3. Then g extends to a positive scalar curvature metric on $\widehat{W}$.*

Here we make the convention that all metrics considered on manifolds with boundary are *product metrics near the boundary.*

Gromov and Lawson made the fundamental observation that this result implies that the answer to our Question 1.1 whether a manifold M admits a positive scalar curvature metric depends only on the *bordism class of* M in a suitable bordism group [GL]. We recall that two closed n-manifolds M, N are called *bordant* if there is a manifold W of dimension $n+1$ whose boundary ∂W is the disjoint union $M \coprod N$.

1.6. Spin structures. (see [LaM, Chap. II, §1]) Let M be an oriented Riemannian manifold of dimension n, and let $SO(M) \to M$ be its *oriented frame bundle*; i.e., the principal $SO(n)$-bundle whose fiber over $x \in M$ consists of all orientation preserving isometries $\mathbb{R}^n \to T_xM$ of $\mathbb{R}^n$ to the tangent space at x (the image of the standard base element in $\mathbb{R}^n$ then gives a 'frame' of T_xM; $SO(n)$ acts on these isometries by precomposition). A *spin structure* on M consists of a double covering of $SO(M)$, whose restriction to each fiber $SO(M)_x$ is a *non-trivial* double covering (the universal covering if $n \geq 3$). A *spin manifold* is a manifold equipped with a spin structure. We note that a spin structure implicitly involves the choice of an orientation; for each spin structure on M there is an 'opposite' spin structure whose underlying orientation is the opposite of the previous one. If M is a spin manifold (resp. oriented manifold), we denote by $-M$ the manifold equipped with the opposite spin structure (resp. orientation).

1.7. Bordism groups. We recall that two n-manifolds M, N are called *bordant* if there is a $(n+1)$-manifold W whose boundary ∂W is the disjoint union $M \coprod N$. If M and N are oriented manifolds (resp. spin manifolds), the requirement is that W is equipped with an orientation (resp. spin structure) such that $\partial W = M \coprod -N$, where the orientation or spin structure on ∂W is induced by that on W. To treat both – orientations and spin structures – on the same footing, it is convenient to refer to them as G-structures, where $G = SO$ if we talk about orientations, and $G = Spin$ for spin structures. More generally, if M, N are n-manifolds with G-structures and $f\colon M \to X$, $g\colon N \to X$ are maps to a topological space X, then the pairs (M, f), (N, g) are *bordant* if there is a $n+1$-manifold with G-structure W with $\partial W = M \coprod -N$, and a map $F\colon W \to X$, which restricts to f on $M \subset W$ and to g on $N \subset W$. We write $[M, f]$ for the bordism class of the pair (M, f), and denote by $\Omega_n^G(X)$ the set consisting of the bordism classes of such pairs. The disjoint union of pairs gives $\Omega_n^G(X)$ the structure of an abelian group; the neutral element is represented by the empty n-dimensional manifold; the inverse of $[M, f]$ is given by $[-M, f]$.

The following result was proved by Gromov-Lawson for simply connected manifolds [GL]; a proof in the general case can be found in [RS1]. In the statement of this result the following subgroup of $\Omega_n^G(X)$ plays a crucial role:

$$\Omega_n^{G,+}(X) \stackrel{\text{def}}{=} \left\{ [N,f] \in \Omega_n^G(X) \;\middle|\; \begin{array}{c} N \text{ admits a metric of} \\ \text{positive scalar curvature} \end{array} \right\}.$$

Theorem 1.8. *Let M be a manifold of dimension $n \geq 5$ with fundamental group π. Let $u\colon M \to B\pi$ be the classifying map of the universal covering $\widetilde{M} \to M$. Assume that*

(a) M admits a Spin-structure, or that

(b) M admits a SO-structure and $\widetilde{M}$ does not admit a Spin-structure,

and let $[M,u] \in \Omega_n^G(B\pi)$ be the element represented by the pair (M,u), where $G = Spin$ in case (a) and $G = SO$ in case (b). Then M admits a positive scalar curvature metric if and only if $[M,u]$ is in $\Omega_n^{G,+}(B\pi)$.

1.9. Remark. There are closed manifolds of dimension $n \geq 5$ which don't satisfy the assumptions of the above theorem; for example non-orientable manifolds, or manifolds without a spin structure whose universal cover admits a spin structure, like the real projective space $\mathbb{RP}^n$ for $n \equiv 1 \bmod 4$. There is a more general version of this theorem that applies to *all* closed manifolds of dimension $n \geq 5$, the proof of which is no harder than the proof of Theorem 1.8; it is just more technical to define the relevant bordism groups, which are spin (resp. oriented) bordism groups of $B\pi$ with 'twisted coefficients' if $\widetilde{M}$ admits a spin structure (if $\widetilde{M}$ does not admit a spin structure) (see [RS1], [St4]).

Outline of the proof of Theorem 1.8. At first glance the statement of the theorem might appear to be tautological. Of course, the existence of a positive scalar curvature metric on M implies that $[M,u]$ is in $\Omega_n^{G,+}(B\pi)$ by definition of that subgroup, but the converse statement is *not* obvious: $[M,u] \in \Omega_n^{G,+}(B\pi)$ means that (M,u) is *bordant* to a pair (N,f), where N admits a positive scalar curvature metric, and the claim is that M itself admits such a metric. To prove this, we would like to argue that a bordism W between M and N is obtained from $N \times [0,1]$ by attaching handles of codimension ≥ 3; then Theorem 1.5 would imply that the positive scalar curvature metric on $N \times [0,1]$ extends to a positive scalar curvature metric on all of W. In particular, its restriction to $M \subset \partial W$ gives a positive scalar curvature metric on M (we recall that all metrics considered on manifolds with boundary are required to be product metrics near the boundary). While not *every* bordism between M and N can be obtained by attaching handles of codimension ≥ 3, it turns out that the conditions of the theorem are carefully chosen in such a way that W can always be modified by surgeries in the interior in order to obtain a bordism for which this *is* the case. □

A striking application of the 'Bordism Theorem' 1.8 is the following result.

Theorem 1.10 (Gromov-Lawson [GL]). *Every simply connected closed non-spin manifold of dimension $n \geq 5$ admits a positive scalar curvature metric.*

Proof. The cartesian product of manifolds gives $\Omega_*^G = \bigoplus_{n=0}^\infty \Omega_n^G$ the structure of a graded ring. C.T.C. Wall constructed explicitly manifolds which are multiplicative generators for Ω_*^{SO}. These manifolds are either projective spaces or total spaces of fiber bundles with projective spaces as fibers whose transition functions are isometries of the Fubini-Study metric (cf. 1.3) on projective space. By part 2 of Observation 1.4 then all of these manifolds admit positive scalar curvature metrics. Since these manifolds multiplicatively generate Ω_*^{SO} it follows from part 1 of Observation 1.4 that the subgroup $\Omega_n^{SO,+}$ is *equal* to Ω_n^{SO} for $n > 0$. Hence Theorem 1.8 implies the corollary. □

1.3 Obstructions to positive scalar curvature metrics

The discussion of the preceding subsection, notably Corollary 1.10 might leave the impression that most manifolds admit metrics of positive scalar curvature. This is not so; in fact, currently, there are three known methods to show that some manifolds do not admits such metrics. These methods are – in chronological order – the following:

Index obstructions. This method, pioneered by Lichnerowicz in the early sixties [Li] and developed since then by many mathematicians is based on the 'Bochner-Lichnerowicz-Weitzenböck formula' (see 1.17) which provides a relationship between positive scalar curvature and the 'Dirac operator' (see 1.15) defined by Atiyah-Singer on any Riemannian manifold equipped with a spin-structure. This method – described below – is the most powerful of the three methods available. Its limitations come from the fact that it requires a spin structure (this can actually be weakened to requiring a spin-structure on the universal cover, see [St4]).

Minimal hypersurface method. Schoen and Yau proved in 1980 [SY] that if M is a Riemannian manifold of dimension n of positive scalar curvature then any stable minimal hypersurface $N \subset M$ (i.e., N is a local minimum of the area functional) admits a positive scalar curvature metric (the induced metric might *not* have positive scalar curvature, but a conformal change produces a positive scalar curvature metric on N). This can lead to interesting restrictions if N represents a non-trivial element in $H_{n-1}(M;\mathbb{Z})$: As Thomas Schick will explain in his second lecture, there is a 5-dimensional manifold for which the index obstruction is zero, but for which the minimal hypersurface method can be used to show that it cannot admit a positive scalar curvature metric. A limitation of this method is that it doesn't give restrictions if $H_{n-1}(M;\mathbb{Z}) \cong H^1(M;\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(M),\mathbb{Z})$ is trivial, for example if the fundamental group $\pi_1(M)$ is finite. Moreover, even if this group is non-trivial, there might only be a stable minimal hypersurface *with singularities* representing a given homology class for $n \geq 8$. It has been claimed [Y] that the technical difficulties associated with the possible singularities can be overcome to prove non-existence of positive scalar curvature metrics for manifolds of arbitrary dimension; however, the author is not aware of a published account of this.

Seiberg-Witten invariants. This is a diffeomorphism invariant of 4-dimensional manifolds, which vanishes if the manifold admits a positive scalar curvature metric. For example, the manifold

$$X^2(d) = \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \;\middle|\; \mathrm{z}_0^{\mathrm{d}} + \cdots + \mathrm{z}_3^{\mathrm{d}} = 0 \right\}$$

is simply connected of real dimension 4; its Seiberg-Witten invariant is non-zero for $d \geq 3$ [Ta2] (the restriction $d \geq 3$ guarantees $b_2^+ > 1$, where b_2^+ is the number of positive eigenvalues of the intersection form, which is a necessary restriction for the definition of the Seiberg-Witten invariant. A 'fancier' version of the invariant is defined for $b_2^+ = 1$, but that doesn't seem to lead to obstructions for positive scalar curvature metrics as the example $X^2(d) = \mathbb{CP}^2$ shows). We note that $X^2(d)$ is a simply connected manifold, which is non-spin for d odd. This shows that Theorem 1.10 does not hold in dimension $n = 4$. We also observe that the non-existence of a positive scalar curvature metric on $X^2(d)$ for d odd cannot be proved by the other methods (the minimal hypersurface method doesn't apply, since $X^2(d)$ is simply connected and there are no index obstructions coming from the Dirac operator since $X^2(d)$ doesn't admit a spin structure). The obvious limitation of this method is that it applies only to 4-dimensional manifolds.

Now we explain the 'Index obstruction' method in more detail. The first result in this direction is the following.

Theorem 1.11 (Lichnerowicz, [Li]). *Let M be a spin manifold of dimension $n = 4k$ which admits a positive scalar curvature metric. Then the $\widehat{A}$-genus $\widehat{A}(M)$ vanishes.*

1.12. Definition of the $\widehat{A}$-genus. (see [LaM, Chap. III, §11]) To define the $\widehat{A}$-genus $\widehat{A}(M)$ of a manifold M, we first recall the definition of the 'characteristic class' $\widehat{A}(E) \in H^*(X;\mathbb{Q})$ associated to any vector bundle $E \to X$. It is characterized by the following properties:

(naturality) for any map $f\colon X' \to X$ we have $\widehat{A}(f^*E) = f^*(\widehat{A}(E))$

(exponential property) $\widehat{A}(E \oplus F) = \widehat{A}(E) \cdot \widehat{A}(F)$.

(normalization) If $L \to X$ is a complex line bundle with Euler class (which equals the first Chern class) $x \in H^2(X;\mathbb{Q})$, then

$$\widehat{A}(L) = \frac{x/2}{\sinh(x/2)} = 1 - \frac{1}{2^3 \cdot 3}\, x^2 + \frac{7}{2^7 \cdot 3^2 \cdot 5}\, x^4 + \cdots \in H^*(X;\mathbb{Q}).$$

If M is a oriented manifold of dimension $n = 4k$, its $\widehat{A}$-genus is defined as

$$\widehat{A}(M) = \langle \widehat{A}(TM), [M] \rangle \in \mathbb{Q},$$

where TM is the tangent bundle of M, and $\langle\ \ , [M]\rangle$ is the evaluation on the fundamental class $[M] \in H_n(M;\mathbb{Z})$.

Example 1.13. $\widehat{A}(X^2(d)) = \frac{d(d-2)(d+2)}{24}$ (see [LaM, Ch. IV, Formula 4.4]), which implies via Lichnerowicz' Theorem 1.11 that $X^2(d)$ does not admit a positive scalar curvature metric for $d \geq 4$ even (d even guarantees that $X^2(d)$ has a spin structure).

1.14. The complex spinor bundle. (see [LaM, Chap. II, §§3–4]). Let M be a Riemannian manifold of dimension $n = 2k$ equipped with a spin structure $Spin(M) \to SO(M)$ (cf. 1.6). We recall that $Spin(M) \to SO(M)$ is a principal $Spin(n)$-bundle, where $Spin(n)$ is the connected Lie group obtained as the non-trivial double covering of $SO(n)$ (the universal covering for $n \geq 3$). The Spinor bundle $S \to M$ is the vector bundle associated to a certain representation Δ of $Spin(n)$ called the *spinor representation.* To construct Δ, the group $Spin(n)$ is identified with a subgroup of units of the *Clifford algebra* $C\ell_n$, and then Δ is a certain $C\ell_n$-module considered as a representation of $Spin(n) \subset C\ell_n^\times$. We recall that the Clifford algebra $C\ell_n = C\ell_n^+ \oplus C\ell_n^-$ is the $\mathbb{Z}/2$-graded $\mathbb{R}$-algebra with unit generated by all vectors $v \in \mathbb{R}^n \subset C\ell_n^-$ subject to the relations $v \cdot v = -|v|^2 \cdot 1$. The subgroup

$Pin(n)$ of $C\ell_n^\times$ generated by all unit vectors $v \in \mathbb{R}^n \subset C\ell_n$ is a double covering group of the orthogonal group $O(n)$ (the double covering map is given by sending v to the reflection at the hyperplane perpendicular to v). The identity component of $Pin(n)$ can then be identified with $Spin(n)$.

It can be shown (see [LaM, Ch. I, §4]) that the complexification $C\ell_{2k} \otimes \mathbb{C}$ is the algebra $\mathbb{C}(2^k)$ of $2^k \times 2^k$-matrices over $\mathbb{C}$. Let Δ be $\mathbb{C}^{2^k}$ with the $\mathbb{C}(2^k)$-module structure given by multiplying a $2^k \times 2^k$-matrix by a 2^k-vector. We consider Δ as a module over $C\ell_{2k} \otimes \mathbb{C}$ and define a $\mathbb{Z}/2$-grading $\Delta \stackrel{\text{def}}{=} \Delta^+ \oplus \Delta^-$ by letting $\Delta^\pm$ be the ± 1-eigenspace of the involution given by multiplication by the *complex volume element* $\omega_{\mathbb{C}} = i^k e_1 \cdots e_{2k} \in C\ell_{2k} \otimes \mathbb{C}$. Then the *complex spinor bundle* $S \to M$ is defined by

$$S = Spin(M) \times_{Spin(n)} \Delta.$$

The crucial feature of the spinor bundle is that there is a *Clifford multiplication*, a vector bundle map

$$TM \otimes S \longrightarrow S.$$

It is induced by the module multiplication map $\mathbb{R}^n \otimes \Delta \subset C\ell_n \otimes \Delta \to \Delta$. The $\mathbb{Z}/2$-grading $\Delta = \Delta^+ \oplus \Delta^-$ induces a corresponding $\mathbb{Z}/2$-grading $S = S^+ \oplus S^-$ on the vector bundle S. In particular, Clifford multiplication by a tangent vector maps S^+ to S^- and vice versa. The Levi-Civita connection on TM induces a principal connection on the frame bundle $SO(M)$, which lifts to a principal connection on $Spin(M)$, which in turn induces a connection on the associated vector bundle $S \to M$ (see [LaM, Ch. II, §4]).

1.15. The Dirac operator. (see [LaM, Ch. II, §5]) The Dirac operator $D\colon C^\infty(S) \to C^\infty(S)$ is the first order elliptic differential operator defined by

$$(D\psi)(x) = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\psi. \tag{1.16}$$

Here $\{e_1, \dots, e_n\}$ is an orthonormal basis of the tangent space T_xM, $\nabla_{e_i}\psi \in S_x$ is the covariant derivative of ψ in the direction of e_i, and $e_i\cdot$ is Clifford multiplication by e_i. We note that D is an *odd* operator in the sense that if ψ is a section of S^+, then $D\psi$ is a section of S^- and vice versa: if ψ is a section of S^+, then $\nabla_{e_i}\psi \in S_x^+$, and hence $e_i \cdot \nabla_{e_i}\psi \in S_x^-$). In particular, restricting D gives operators $D^\pm\colon C^\infty(S^\pm) \to C^\infty(S^\mp)$.

1.17. The Bochner-Lichnerowicz-Weitzenböck Formula (see [LaM, Ch. II, §8]) is the equation

$$D^2 = \nabla^* \nabla + \frac{1}{4} s, \tag{1.18}$$

where ∇ is the connection on the spinor bundle S, considered as a homomorphism $\nabla \colon C^\infty(S) \to C^\infty(T^*M \otimes S)$, and $\nabla^* \colon C^\infty(T^*M \otimes S) \to C^\infty(S)$ is its adjoint with repect to the inner product on these spaces of sections induced by the Riemannian metric on M (T^*M is the cotangent bundle of M). All the terms in the formula above are considered as linear maps $C^\infty(S) \to C^\infty(S)$; the term $\frac{1}{4}s$ is multiplication by $\frac{1}{4}$ times the scalar curvature function.

Proof of Lichnerowicz' Theorem 1.11. Let $\langle \ \ , \ \ \rangle$ be the inner product on each fiber of the spinor bundle $S \to M$ (which is induced by the Riemannian metric on M), and let $(\phi, \psi) \in \mathbb{R}$ be the inner product of sections $\phi, \psi \in C^\infty(S)$ defined by

$$(\phi, \psi) = \int_M \langle \phi(x), \psi(x) \rangle \, dvol(x),$$

where *dvol* is the volume element determined by the Riemannian metric on M. We observe that the Weitzenböck Formula 1.18 has the following consequence: if ψ is in the kernel of the Dirac operator D, then

$$0 = (D^2\psi, \psi) = (\nabla^*\nabla\psi + s\psi, \psi) = \|\nabla\psi\|^2 + (s\,\psi, \psi) \geq (s\,\psi, \psi).$$

Assuming that the scalar curvature function s is everywhere positive, a non-zero ψ would force $(s\,\psi, \psi)$ to be strictly positive in contradiction to the inequality above.

Besides the Weitzenböck formula, the other main input in the proof of Theorem 1.11 is the Atiyah-Singer Index Theorem. Specializing to the Dirac operator on a spin manifold M of dimension $n = 4k$ (see [LaM, Ch. III, Thm. 13.10]), it says

$$\operatorname{index}(D^+) = \widehat{A}(M),$$

where $\operatorname{index}(D_+) = \dim\ker(D^+) - \dim\operatorname{coker}(D^+)$. It can be shown that D^+ is the adjoint operator of D^-, which allows us to identify $\operatorname{coker}(D_+)$ with $\ker(D^-)$. This shows that if the scalar curvature function is strictly positive, then both, $\dim\ker(D^+)$ and $\dim\operatorname{coker}(D^+)$ vanish, and hence so does $\widehat{A}(M)$. □

Lichnerowicz' result 1.11 has been refined by Hitchin [Hit] and later Rosenberg [Ro2]. The idea is to construct a version of the Dirac operator D which commutes with the action of a C^*-algebra A. In the simplest case, the relevant algebra is the Clifford algebra, and the construction is the following.

1.19. The $C\ell_n$-linear Dirac operator (see [LaM, Ch. II, §7]). Let M be a n-dimensional spin manifold. The *$C\ell_n$-linear spinor bundle* is the vector bundle

$$\mathfrak{S} \stackrel{\text{def}}{=} Spin(M) \times_{Spin(n)} C\ell_n.$$

This is a variation of the spinor bundle described in 1.14. We note that the Clifford algebra $C\ell_n$ acts by right multiplication on $\mathfrak{S}$. This action is fiber preserving and hence gives the space of sections $C^\infty(\mathfrak{S})$ the structure of a $\mathbb{Z}/2$-graded right $C\ell_n$-module. As in 1.15 we can define the Dirac operator $\mathfrak{D}\colon C^\infty(\mathfrak{S}) \to C^\infty(\mathfrak{S})$. It commutes with the $C\ell_n$-action, and it is therefore referred to as the *$C\ell_n$-linear Dirac operator*. The kernel of $\mathfrak{D}$ is then a $\mathbb{Z}/2$-graded module over $C\ell_n$ and so represents an element the Grothendieck group $\widehat{\mathfrak{M}}_n$ of $\mathbb{Z}/2$-graded $C\ell_n$-modules. This element might depend on the Riemannian metric on M; however, the class $[\ker \mathfrak{D}] \in \widehat{\mathfrak{M}}_n / i^*\widehat{\mathfrak{M}}_{n+1}$ is independent of the metric, where $i^*\colon \widehat{\mathfrak{M}}_{n+1} \to \widehat{\mathfrak{M}}_n$ is induced by the inclusion $i\colon C\ell_n \to C\ell_{n+1}$. The element

$$\alpha(M) \stackrel{\text{def}}{=} [\ker \mathfrak{D}] \in KO_n(\mathbb{R}) \stackrel{\text{def}}{=} \widehat{\mathfrak{M}}_n / i^*\widehat{\mathfrak{M}}_{n+1} \tag{1.20}$$

is the *Clifford index* of $\mathfrak{D}$ [LaM, Ch. III. Def. 10.4.].

The same argument as for Lichnerowicz' Theorem 1.11 leads to the following result due to Hitchin (with a somewhat different proof).

Theorem 1.21 (Hitchin [Hit]). *Let M be a spin manifold of dimension n. If M admits a positive scalar curvature metric, then $\alpha(M) \in KO_n(\mathbb{R})$ is zero.*

Remark 1.22. The groups $KO_n(\mathbb{R})$ depend only on n modulo 8, and are given by the following table.

n mod 8	0	1	2	3	4	5	6	7
$KO_n(\mathbb{R})$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0

Moreover, if M is a spin manifold of dimension $n \equiv 0 \bmod 4$, then $\widehat{A}(M) = \alpha(M)$ for $n \equiv 0 \bmod 8$, and $\widehat{A}(M) = 2\alpha(M)$ for $n \equiv 4 \bmod 8$ (if we identify $KO_n(\mathbb{R})$ with $\mathbb{Z}$ by choosing as generator of $KO_n(\mathbb{R})$ the $\mathbb{Z}/2$-graded

$C\ell_n$-module Δ used in the construction of the spinor bundle in those dimensions, see 1.14). This shows that Hitchin's result is a generalization of Lichnerowicz' Theorem 1.11.

In dimensions $n \equiv 1, 2 \mod 8$, $n \geq 9$, there are smooth manifolds Σ homeomorphic, but not diffeomorphic to the n-dimensional sphere with $\alpha(\Sigma) \neq 0 \in KO_n(\mathbb{R}) \cong \mathbb{Z}/2$. This is interesting since it shows that the answer to the question "Does a given manifold M admit a positive scalar curvature metric?" might depend on quite subtle things like the differentiable structure of M.

Hitchin's result can be generalized by 'twisting' the Dirac operator as follows. Suppose $E \to M$ is a real vector bundle with connection. Then we can define the *twisted Dirac operator*

$$\mathfrak{D}_E \colon C^\infty(\mathfrak{S} \otimes E) \longrightarrow C^\infty(\mathfrak{S} \otimes E)$$

by the same Formula 1.16 defining the usual Dirac operator, where now ∇ is the product connection on $\mathfrak{S} \otimes E$ (of the usual connection on $\mathfrak{S}$ induced by the Levi-Civita connection and the given connection on E). The Bochner-Lichnerowicz-Weitzenböck formula 1.18 continues to hold, provided the connection on E is *flat*. In that case, the $C\ell_n$-module $\ker \mathfrak{D}_E$ gives an element

$$[\ker \mathfrak{D}_E] \in KO_n(\mathbb{R}),$$

which must be zero if M admits a positive scalar curvature metric. All flat vector bundles over M are obtained in the following way: Given an orthogonal representation $\rho \colon \pi \to O(V)$ of a discrete group π and a map $f \colon M \to B\pi$, we can form the flat vector bundle $E(\rho) = E\pi \times_\pi V$ over $B\pi = E\pi/\pi$ and pull it back via f to get a flat vector bundle $f^*E(\rho)$ over M.

We can do better: all the obstructions $[\ker \mathfrak{D}_{f^*E(\rho)}] \in KO_n(\mathbb{R})$ corresponding to various orthogonal representations ρ of π can be obtained as the images of a single obstruction

$$[\ker \mathfrak{D}_{f^*\mathcal{V}(\pi)}] \in KO_n(C^*\pi)$$

under homomorphisms $\rho_* \colon KO_n(C^*\pi) \to KO_n(\mathbb{R})$. Here $C^*\pi$ is the *group C^*-algebra* of π, and $\mathcal{V}(\pi) = E\pi \times_\pi C^*\pi$ is the *Miščenko-Fomenko line bundle*. Let us recall the relevant definitions.

1.23. The group C^*-algebra of a discrete group π. We recall that a *C^*-algebra* is an algebra over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ (the more usual case considered is $\mathbb{C}$; the C^*-algebras considered in these lectures are all over $\mathbb{R}$) equipped with an anti-involution $*: A \to A$ which is $*$-isomorphic to a closed subalgebra of the algebra of bounded operators $\mathcal{B}(H)$ on a $\mathbb{F}$-Hilbert space H. Here the involution on $\mathcal{B}(H)$ is given by sending an operator to its adjoint, and the subalgebra is required to be closed with respect to the norm topology on $\mathcal{B}(H)$. It should be mentioned that there are more 'intrinsic' definitions (see [WO, 1.1]), but the above suffices for our purposes.

The main example of a C^*-algebra of interest to us is the (real) C^*-algebra $C^*\pi$ of a discrete group π. It is a norm completion of the real group ring $\mathbb{R}\pi = \{\sum_{g\in\pi} r_g g \mid r_g \in \mathbb{R}\}$ (these sums are *finite* sums and they are multiplied using the product in π) equipped with the anti-involution induced by $g \mapsto g^{-1}$ for $g \in \pi \subset \mathbb{R}\pi$. We note that there is a one-to-one correspondence between orthogonal representations of π on a real Hilbert space H and $*$-homomorphisms from $\mathbb{R}\pi$ to $\mathfrak{B}(H)$ (by extending $\rho\colon \pi \to O(H)$ linearly to an algebra homomorphism $\rho\colon \mathbb{R}\pi \to \mathfrak{B}(H)$). The (maximal, real) *group C^*-algebra* $C^*\pi$ is the completion of $\mathbb{R}\pi$ with respect to the norm on $\mathbb{R}\pi$ defined by

$$\|\sigma\|_{max} = \sup_{\rho} \{\|\rho(\sigma)\|\} \qquad \text{for} \quad \sigma \in \mathbb{R}\pi,$$

where the sup is taken over all $*$-homomorphisms ρ from $\mathbb{R}\pi$ to the bounded operators on some Hilbert space.

A variant of $C^*\pi$ is the *reduced* C^*-algebra $C^*_r\pi$ of a group π; as $C^*\pi$, the C^*-algebra $C^*_r\pi$ is a norm completion of $\mathbb{R}\pi$, but with respect to the norm $\|\sigma\| = \|\rho(\sigma)\|$, where $\rho\colon \mathbb{R}\pi \to \mathfrak{B}(l^2(H))$ is the *regular representation of* $\mathbb{R}\pi$, corresponding to the orthogonal representation of π via translations on the Hilbert space $l^2(\pi) = \{f\colon \pi \to \mathbb{R} | \sum_{g\in\pi} |f(g)|^2 < \infty\}$.

Other C^*-algebras of interest to us are the Clifford algebra $C\ell_n$ with anti-involution given by $v \mapsto -v$ for a generator $v \in \mathbb{R}^n$, and the tensor product $C\ell_n \otimes C^*\pi$.

1.24. A Dirac type operator commuting with an action of $C\ell_n \otimes C^*\pi$. Given a discrete group π, the bundle $\mathcal{V}(\pi) \overset{\text{def}}{=} E\pi \times_\pi C^*\pi$ over $B\pi$ with fiber $C^*\pi$ is called the *Miščenko-Fomenko line bundle.* We note that $\mathcal{V}(\pi)$ is a flat vector bundle, which is infinite dimensional if the group π is infinite. The reason that $\mathcal{V}(\pi)$ is called a *line* bundle is that $C^*\pi$ acts on $\mathcal{V}(\pi)$ by right multiplication, making the fibers 1-dimensional free modules over $C^*\pi$.

As above, we can defined the twisted Dirac operator

$$\mathfrak{D}_{f^*\mathcal{V}(\pi)}\colon C^\infty(\mathfrak{S} \otimes f^*\mathcal{V}(\pi)) \longrightarrow C^\infty(\mathfrak{S} \otimes f^*\mathcal{V}(\pi))$$

for any map $f\colon M \to B\pi$ from a spin manifold M to $B\pi$. We note that the C^*-algebra $C\ell_n \otimes C^*\pi$ acts fiber preserving on $S \otimes f^*\mathcal{V}(\pi)$ (via the $C\ell_n$-action on $\mathfrak{S}$, and the $C^*\pi$-action on $\mathcal{V}(\pi)$), thus making $C^\infty(\mathfrak{S} \otimes f^*\mathcal{V}(\pi))$ a $\mathbb{Z}/2$-graded module over it. Moreover, this action commutes with $\mathfrak{D}_{f^*\mathcal{V}(\pi)}$, giving in particular $\ker \mathfrak{D}_{f^*\mathcal{V}(\pi)}$ the structure of a $\mathbb{Z}/2$-graded module over $C\ell_n \otimes C^*\pi$.

1.25. The K-theory of C^*-algebras. For a real $Z/2$-graded C^*-algebra A its K-theory is defined by

$$KO_0(A) = \left\{ \begin{matrix} \text{equivalence classes of finitely} \\ \text{generated projective } A\text{-modules} \end{matrix} \right\}$$

and $KO_n(A) = KO_0(C\ell_n \otimes A)$. It is tempting to try to define a $KO_n(C^*\pi)$-valued index for the operator $\mathfrak{D}_{f^*\mathcal{V}(\pi)}$ as the class of $KO_n(C^*\pi)$ represented by the kernel of $\mathfrak{D}_{f^*\mathcal{V}(\pi)}$. But this module over $C\ell_n \otimes C^*\pi$ is in general neither finitely generated nor projective. Luckily one can always find a 'compact perturbation' $\mathfrak{D}'_{f^*\mathcal{V}(\pi)}$ of $\mathfrak{D}_{f^*\mathcal{V}(\pi)}$ whose kernel is finitely generated projective (the difference of $\mathfrak{D}$ and $\mathfrak{D}'$ is an operator that is compact in the sense of Hilbert modules over the C^*-algebra $C\ell_n \otimes C^*\pi$; see [WO, Ch. 17]). It turns out that the element in $KO_n(C^*\pi)$ represented by the kernel of $\mathfrak{D}'$ is independent of the choice of the perturbation, and that allows one to define

$$\alpha(M, f) \overset{\text{def}}{=} \left[\ker \mathfrak{D}'_{f^*\mathcal{V}(\pi)}\right] \in KO_n(C^*\pi).$$

If $\rho\colon \pi \to O(\mathbb{R}^d)$ is a finite dimensional orthogonal representation of π, the corresponding $*$-homomorphism $\rho\colon C^*\pi \to \mathbb{R}(d)$ (where $\mathbb{R}(d)$ is the C^*-algebra of $d \times d$-matrices with entries in $\mathbb{R}$) induces a homomorphism $\rho_*\colon KO_n(C^*\pi) \to KO_n(\mathbb{R}(d)) \cong KO_n(\mathbb{R})$, under which $\alpha(M, f)$ maps to $[\ker \mathfrak{D}_{f^*E(\rho)}] \in KO_n(\mathbb{R})$.

Theorem 1.26 (Rosenberg [Ro2]). *Let M be a spin manifold of dimension n and let $f\colon M \to B\pi$ be a map to the classifying space of a discrete group π. If M admits a Riemannian metric with positive scalar curvature then $\alpha(M, f) \in KO_n(C^*\pi)$ vanishes.*

2 The Gromov-Lawson-Rosenberg Conjecture

In the last lecture we defined the invariant $\alpha(M, f) \in KO_n(C^*\pi)$ for any spin manifold M of dimension n equipped with a map $f\colon M \to B\pi$ to the classifying space of a discrete group π. We will refer to $\alpha(M, f)$ as the *index obstruction*, since it is the $KO_n(C^*\pi)$-valued index of a twisted Dirac operator on M, and the vanishing of $\alpha(M, f)$ is a necessary condition for the existence of a positive scalar curvature metric on M by Theorem 1.26. Optimistically, one might hope that the vanishing of the α-invariant is not only necessary, but also *sufficient* for the existence of a positive scalar curvature metric, and this is what the following conjecture asserts.

2.1. The Gromov-Lawson-Rosenberg Conjecture. Let M be a connected spin manifold of dimension $n \geq 5$. Then M admits a metric of positive scalar curvature if and only if all index obstructions $\alpha(M, f) \in KO_n(C^*\pi)$ vanish.

Although Thomas Schick has found a counterexample to this conjecture [Sch], which he described in his lectures at this summer school, we state this conjecture, since

1. it has been a very influential conjecture in the field;
2. it is true for manifolds with certain fundamental groups (cf. Thm. 2.13), as we will discuss below, in particular, for simply connected manifolds (cf. Thm. 2.4);
3. a weaker form of the conjecture is true for large classes of fundamental groups (cf. Thm. 3.3 and Thm. 3.10).

In this lecture we will outline the proofs of the known cases of the Gromov-Lawson-Rosenberg Conjecture. We would like to begin by giving a reformulation of the Conjecture. We observe that any map $f\colon M \to B\pi$ to the classifying space of a discrete group π will factor in the form

$$M \xrightarrow{u} B\pi_1(M) \xrightarrow{B\rho} B\pi,$$

where u is the classifying map of the universal covering of M, and $B\rho$ is a map of classifying spaces induced by a homomorphism ρ from the fundamental group $\pi_1(M)$ to π. Then $\alpha(M, u) \in KO_n(C^*\pi_1(M))$ maps to $\alpha(M, f) \in KO_n(C^*\pi)$ via the map $KO_n(C^*\pi_1(M)) \to KO_n(C^*\pi)$ induced by ρ. In particular, the vanishing of $\alpha(M, u)$ implies the vanishing of $\alpha(M, f)$ for all f.

The motivation for defining $\alpha(M,f)$ for a general map f rather than just $\alpha(M,u)$ is that we get a well-defined homomorphism

$$\alpha\colon \Omega_n^{Spin}(B\pi) \to KO_n(C^*\pi) \qquad \text{by} \quad [M,f] \mapsto \alpha(M,f), \tag{2.2}$$

since it can be shown that the index obstruction $\alpha(M,f)$ depends only on the bordism class of $[M,f] \in \Omega_n^{Spin}(B\pi)$.

In view of the Bordism Theorem 1.8 the Gromov-Lawson-Rosenberg Conjecture is equivalent to the following conjecture:

2.3. Conjecture. $\Omega_n^{Spin,+}(B\pi) = \ker\Big(\alpha\colon \Omega_n^{Spin}(B\pi) \to KO_n(C^*\pi)\Big)$.

Of course, according to Theorem 1.26 we have the inclusion $\Omega_n^{Spin,+}(B\pi) \subset \ker\alpha$, while the converse inclusion is conjectural.

2.1 The simply connected case

Theorem 2.4 (Stolz [St1]). *The Gromov-Lawson-Rosenberg Conjecture holds for simply connected manifolds.*

We want to mention that in dimensions $n \leq 23$ this was proved by Rosenberg [Ro3, Thm. 3.6] in a spirit similar to the proof of Theorem 1.10 by producing explicit spin manifolds whose bordism classes are generators of $\ker\alpha$. For example, the kernel of $\alpha\colon \Omega_n^{Spin} \to KO_n(\mathbb{R})$ is trivial in dimensions $n < 8$. For $n = 8$, it is infinite cyclic and a generator is given by the bordism class of the quaternionic projective plane $\mathbb{HP}^2$. Since this manifold admits a positive scalar curvature metric this proves the Gromov-Lawson-Rosenberg Conjecture in the simply connected case for $n \leq 8$.

The difficulty with this line of argument is that although the bordism groups Ω_n^{Spin} have been computed [ABP], we do *not* know explicit spin manifolds whose bordism classes generate the spin bordism ring (unlike the oriented bordism ring, for which Wall has given explicit generators, which was the key for the proof of Theorem 1.10). So the dilemma when trying to prove Theorem 2.4 is to try to represent bordism classes by manifolds admitting positive scalar curvature metrics *without knowing how to represent these classes by explicit manifolds.* This is not as impossible as it sounds: according to part 2 of Observation 1.4 if M is a Riemannian manifold with positive scalar curvature, then the total space of *any* fiber bundle with fiber M whose transition functions are isometries of M admits a positive scalar curvature metric. This suggests to analyze which bordism classes are represented by such total spaces for some fixed M, e.g. $M = \mathbb{HP}^2$ equipped with the Fubini-Study metric. For simplicity we will call these bundles $\mathbb{HP}^2$-bundles. The answer is this:

Theorem 2.5 (Stolz [St1]). *The subgroup of Ω_n^{Spin} represented by total spaces of $\mathbb{HP}^2$-bundles is equal to the kernel of $\alpha\colon \Omega_n^{Spin} \to KO_n$.*

By our discussion above, this implies Theorem 2.4.

Idea of proof of Theorem 2.5. The isometry group of $\mathbb{HP}^2$ is the projective symplectic group $G = PSp(3) = Sp(3)/\pm 1$. It acts transitively on $\mathbb{HP}^2$ with isotropy group $H = (Sp(2)\times Sp(1))/\pm 1$, allowing us to identify $\mathbb{HP}^2$ with the homogeneous space G/H. The map of classifying spaces $BH \to BG$ induced by the inclusion map $H \to G$ is then a fiber bundle with fiber $\mathbb{HP}^2 = \mathrm{G/H}$. This is the *universal* $\mathbb{HP}^2$-bundle in the sense that *any* $\mathbb{HP}^2$-bundle over a manifold N is the pull-back $\widehat{N} = f^*BH \to N$ of $BH \to BG$ via some map $f\colon N \to BG$. This discussion shows that the subgroup $T_n \subset \Omega_n^{Spin}$ represented by total spaces of $\mathbb{HP}^2$-bundles is the image of the following *transfer map*:

$$\Psi\colon \Omega_{n-8}^{Spin}(BG) \longrightarrow \Omega_n^{Spin} \qquad [N,f] \mapsto [\widehat{N}].$$

Hence the claim of the theorem is equivalent to the exactness of the following sequence at the middle group:

$$\Omega_{n-8}^{Spin}(BG) \xrightarrow{\Psi} \Omega_n^{Spin} \xrightarrow{\alpha} KO_n(\mathbb{R}).$$

It is well-known that the Pontryagin-Thom construction allows us to identify the bordism group Ω_n^{Spin} with the n-th homotopy group of the 'Thom spectrum' $MSpin$ [Sto]. In fact, it turns out that the whole sequence above is isomorphic (for $n \geq 0$) to the following sequence of homotopy groups:

$$\pi_n(MSpin \wedge \Sigma^8 BG_+) \xrightarrow{T_*} \pi_n(MSpin) \xrightarrow{D_*} \pi_n(ko). \tag{2.6}$$

Here $\Sigma^8 BG_+$ is the 8-th suspension of BG furnished with a disjoint base point, and ko is the *connective real K-theory spectrum.* There is a closely related spectrum KO, the *periodic real K-theory spectrum* whose homotopy groups $\pi_n(KO)$ are isomorphic to $KO_n(\mathbb{R})$ for all n. The spectrum ko is the *connective cover* of KO in the sense that $\pi_n(ko)$ is trivial for $n < 0$, and that there is a map per: $ko \to KO$ which induces an isomorphism on π_n for $n \geq 0$.

It can be shown that the composition

$$MSpin \wedge \Sigma^8 BG_+ \xrightarrow{T} MSpin \xrightarrow{D} ko$$

is homotopic to the constant map (by interpreting it as a family index). This implies that T can be factored in the form

$$\begin{array}{ccc} & & \widehat{MSpin} \\ & \overset{\widehat{T}}{\nearrow} & \downarrow \\ MSpin \wedge \Sigma^8 BG_+ & \xrightarrow{T} & MSpin \\ & & \downarrow{\scriptstyle D} \\ & & ko \end{array}$$

where $\widehat{MSpin}$ is the homotopy fiber of D. The exactness of the sequence above is equivalent to the surjectivity of the map induced by $\widehat{T}$ on homotopy groups.

It turns out to be convenient to break the proof that $\widehat{T}_*$ is surjective into two steps: surjectivity of $\widehat{T}_*$ *localized at the prime* 2 (i.e., after tensoring with $\mathbb{Z}_{(2)} = \{\frac{a}{b} \mid b \text{ is prime to } 2\}$) and surjectivity *away from* 2 (i.e., after tensoring with $\mathbb{Z}[\frac{1}{2}]$). Away from 2, the bordism ring $\pi_*(MSpin) \cong \Omega_*^{Spin}$ is a polynomial ring with generators x_n in degrees $n = 4, 8, 12, \dots$, while $\pi_*(ko)$ is the polynomial ring generated by $D_*(x_4)$ (both $MSpin$ and ko are *ring spectra* and D is compatible with this structure, which implies that the homotopy groups of $MSpin$ and ko form a graded ring, and that D_* is a ring homomorphism). In particular, D_* is surjective, and , $\pi_*(\widehat{MSpin})$ can be identified with the ideal generated by $x_8, x_{12}, \dots$. Hence it suffices to show that T_* is onto modulo decomposable elements in degrees $8, 12, \dots$, which is proved by a calculation of characteristic numbers for certain $\mathbb{HP}^2$-bundles (see [KS, §4]).

The proof of surjectivity localized at 2 is technically more involved due to the existence of 2-torsion in $\pi_*(MSpin)$. It is proved using the mod 2 Adams spectral sequence, whose E_2-term depends only on the mod 2 cohomology of the spectrum in question as a module over the Steenrod algebra A; it converges to the homotopy groups of the spectrum localized at 2. A detailed analysis of the action of the Steenrod algebra on $H^*(MSpin \wedge \Sigma^8 BG_+; \mathbb{Z}/2)$ and $H^*(\widehat{MSpin}; \mathbb{Z}/2)$ shows that as A-module the latter can be identified as a direct summand of the former via the map $\widehat{T}^*$. It follows that the map of Adams spectral sequences induced by $\widehat{T}$ is surjective on E_2-terms. The vanishing of all differentials in the Adams spectral sequence of $MSpin \wedge \Sigma^8 BG_+$ then implies that $\widehat{T}$ induces a surjection on E_∞-terms and hence on homotopy groups localized at 2. □

2.2 Positive scalar curvature metrics on non-simply connected spin manifolds

It is tempting to believe that a manifold M with finite fundamental group admits a positive scalar curvature metric if and only if its universal covering $\widetilde{M}$ does using the following line of reasoning. Suppose $\widetilde{g}$ is a positive scalar curvature metric on $\widetilde{M}$. Then the metric $\widetilde{g}$ might not be invariant under the action of $\pi_1(M)$ via deck transformations, but the space of Riemannian metrics on $\widetilde{M}$ is a convex subspace of the vector space of 2-tensors on M, and averaging over the orbit through $\widetilde{g}$, we obtain an *invariant* Riemannian metric $\widetilde{g}'$ on $\widetilde{M}$, which then descents to a Riemannian metric g' on M. It seems reasonable to expect $\widetilde{g}'$ (and hence g') to have positive scalar curvature – after all, it is obtained by averaging positive scalar curvature metrics. However, the following example shows that the average of positive scalar curvature metrics might not have positive scalar curvature.

Example 2.7. Let M be the connected sum of $\mathbb{R}\mathrm{P}^7 \times \mathrm{S}^2$ and a 9-dimensional homotopy sphere Σ^9 with $\alpha(\Sigma^9) \neq 0$. We note that $\mathbb{R}\mathrm{P}^7$ is a spin manifold and hence so is $\mathbb{R}\mathrm{P}^7 \times \mathrm{S}^2$. We note that $\alpha(\mathbb{R}\mathrm{P}^7 \times \mathrm{S}^2)$ is zero since $\mathbb{R}\mathrm{P}^7 \times \mathrm{S}^2$ is zero bordant. Since the connected sum $\left(\mathbb{R}\mathrm{P}^7 \times \mathrm{S}^2\right) \# \Sigma^9$ is spin bordant to the disjoint union of $\mathbb{R}\mathrm{P}^7 \times \mathrm{S}^2$ and Σ^9, we see that

$$\alpha(M) = \alpha(\left(\mathbb{R}\mathrm{P}^7 \times \mathrm{S}^2\right) \# \Sigma^9) = \alpha(\mathbb{R}\mathrm{P}^7 \times \mathrm{S}^2) + \alpha(\Sigma^9) = \alpha(\Sigma^9) \neq 0.$$

Hence by Theorem 1.21 the manifold M does not admit a positive scalar curvature metric. However, the universal covering $\widetilde{M}$ does admit a positive scalar curvature metric, since $\widetilde{M} = (S^7 \times S^2) \# \Sigma^9 \# \Sigma^9$, which is diffeomeorphic to $S^7 \times S^2$, since $\Sigma^9 \# \Sigma^9$ is diffeomorphic to S^9.

While the above example shows that the question whether a spin manifold with finite fundamental group π admits a positive scalar curvature metric cannot be reduced to the *universal* covering, Kwasik and Schultz observed that it can be reduced to the coverings corresponding to the Sylow subgroups of π.

Theorem 2.8 (Kwasik-Schultz [KwS]). *Let M be a spin manifold of dimension $n \geq 5$ with finite fundamental group π. Then M admits a positive scalar curvature metric if and only if all coverings of M corresponding to the p-Sylow groups of π admit such a metric.*

In particular, if the Gromov-Lawson-Rosenberg Conjecture is true for all p-Sylow groups of a finite group π, then it holds for π.

Proof. To prove the non-trivial implication of this theorem, assume that all coverings of M corresponding to p-Sylow subgroups of π admit positive scalar curvature metrics. To show that M also admits such a metric, it suffices by the Bordism Theorem 1.8 to show $[M,u] \in \Omega_n^{Spin,+}(B\pi)$, where $u\colon M \to B\pi$ is the map classifying the universal covering of M.

In order to relate $[M,u]$ to the corresponding bordism class for a covering of M corresponding to a subgroup $H \subset \pi$, we consider the *transfer homomorphism*

$$\Omega_n^{Spin}(B\pi) \xrightarrow{(Bi)^!} \Omega_n^{Spin}(BH) \qquad \text{defined by} \quad [N,f] \mapsto [\widehat{N},\widehat{f}],$$

where $\widehat{N} \overset{\text{def}}{=} f^*BH \to N$ is the covering obtained by pulling back the covering $BH \to B\pi$ via f, and $\widehat{f}$ is the map making the diagram

$$\begin{array}{ccc} \widehat{N} = f^*BH & \xrightarrow{\widehat{f}} & BH \\ \downarrow & & \downarrow \\ N & \xrightarrow{f} & B\pi \end{array}$$

commutative. Now consider the composition

$$\Omega_n^{Spin}(B\pi) \xrightarrow{(Bi)^!} \Omega_n^{Spin}(BH) \xrightarrow{Bi_*} \Omega_n^{Spin}(B\pi)$$

assuming that H is a p-Sylow group of π. Then

- The image of $[M,u]$ under the transfer map is in $\Omega_n^{Spin,+}(BH)$ due to the assumption that p-Sylow coverings of M admit a positive scalar curvature metrics. Hence the image of $[M,u]$ under the composition is in $\Omega_n^{Spin,+}(B\pi)$.

- The composition is an isomorphism after tensoring with $\mathbb{Z}_{(p)}$ (replacing spin bordism by homology, the composition above is multiplication by the index of the subgroup H; for a Sylow subgroup, this index is prime to p and consequently the composition on homology is an isomorphism after localizing at p; the Atiyah-Hirzebruch spectral sequence then shows that the same holds for spin bordism).

This implies $[M,u] \in \Omega_n^{Spin,+}(B\pi) \otimes \mathbb{Z}_{(p)}$ for all primes p and hence $[M,u] \in \Omega_n^{Spin,+}(B\pi)$. □

We recall that the Gromov-Lawson-Rosenberg Conjecture for a group π is equivalent to $\Omega_n^{Spin,+}(B\pi) = \ker(\alpha\colon \Omega_n^{Spin}(B\pi) \to KO_n(C^*\pi))$. This shows that when attempting to prove the conjecture, it is useful to understand the kernel of α; this is facilitated by factoring α as follows:

$$\Omega_n^{Spin}(B\pi) \xrightarrow{D} ko_n(B\pi) \xrightarrow{\text{per}} KO_n(B\pi) \xrightarrow{A} KO_n(C^*\pi). \qquad (2.9)$$

Here $KO_n(X) = \pi_n(KO \wedge X_+)$ and $ko_n(X) = \pi_n(ko \wedge X_+)$ are the *generalized homology theories* associated to the periodic (resp. connective) real K-theory spectrum KO (resp. ko), which we mentioned earlier in this lecture (in the proof of Theorem 2.5). The maps D and per are natural transformations between homology theories which are induced by the corresponding maps between spectra $D\colon MSpin \to ko$ and $\text{per}\colon ko \to KO$ (abusing notation we use the same letter for the map between spectra and the natural transformation). The homomorphism A is the *assembly map* which has been mentioned in the talks by Lück and Schick. We will use the following notation.

2.10. Notation. For a spin manifold M, the bordism class $[M, \mathrm{id}_M] \in \Omega_n^{Spin}(M)$ is the *fundamental class* of M in Ω^{Spin}-theory. We define

$$[M]_{ko} \stackrel{\text{def}}{=} D([M, \mathrm{id}_M]) \in ko_n(M) \quad \text{and} \quad [M]_{KO} \stackrel{\text{def}}{=} \text{per}([M]_{ko}) \in KO_n(M).$$

We call $[M]_{ko}$ (resp. $[M]_{KO}$) the *fundamental class of* M in connective (resp. periodic) KO-homology. Given a map $f\colon M \to B\pi$, and the corresponding bordism class $[M, f] \in \Omega_n^{Spin}(B\pi)$, we have

$$D([M,f]) = f_*[M]_{ko} \in ko_n(B\pi) \quad \text{per}(D([M,f])) = f_*[M]_{KO} \in KO_n(B\pi).$$

Finally, we define $ko_n^+(X) \stackrel{\text{def}}{=} D\left(\Omega_n^{Spin,+}(X)\right) \subset ko_n(X)$ and define $KO_n^+(X) \subset KO_n(X)$ to be the subgroup generated by $\text{per}(ko_{n+8k}^+(X)) \subset KO_{n+8k}(X) \cong KO_n(X)$ for all k.

The following result is similar to the Bordism Theorem 1.8.

Theorem 2.11 (Stolz, Jung). *Let M be a spin manifold of dimension $n \geq 5$ with fundamental group π and let $u\colon M \to B\pi$ be the classifying map of the universal covering $\widetilde{M} \to M$. Then M admits a positive scalar curvature metric if and only if $u_*[M]_{ko}$ is in $ko_n^+(B\pi)$.*

In particular, by this result the Gromov-Lawson-Rosenberg Conjecture is equivalent to the following conjecture.

2.12. Conjecture. $ko_n^{Spin,+}(B\pi) = \ker\left(A \circ \text{per}\colon ko_n(B\pi) \to KO_n(C^*\pi)\right)$.

The advantage of this formulation of the Gromov-Lawson-Rosenberg Conjecture over that used in Conjecture 2.3 is that the groups $ko_*B\pi)$ are *a lot* smaller than $\Omega_*(B\pi)$. For example in joint work with Botvinnik and Gilkey [BGS], the author could calculate the groups $ko_n(B\pi)$ and the kernel of $A \circ \mathrm{per}$ in the case of 2-groups which are cyclic or generalized quaternion. It turns out that the kernel of $A \circ \mathrm{per}$ is generated by ko-fundamental classes of lens spaces and lens space bundles over S^2 (in the cyclic case) resp. by lens spaces and quaternionic space forms (for the quaternionic groups). This proves the Gromov-Lawson-Rosenberg Conjecture for these 2-groups.

Previously Kwasik and Schultz [KwS] had proved the Gromov-Lawson-Rosenberg Conjecture 2.1 for cyclic groups of prime order $p \neq 2$, and Rosenberg [Ro2] for cyclic groups odd order. This implies by the Kwasik-Schultz induction result 2.8 that the Conjecture is true for all finite groups whose p-Sylow groups are cyclic or generalized quaternion (for $p = 2$). These are precisely the finite groups with periodic cohomology and so these arguments prove:

Theorem 2.13 (Botvinnik-Gilkey-Stolz [BGS]). *The Gromov-Lawson-Rosenberg Conjecture holds for finite groups with periodic cohomology.*

Next we want to outline the proof of Theorem 2.11. It suffices to show $\ker\left(D\colon \Omega_n^{Spin}(X) \to ko_n(X)\right) \subset \Omega_n^{Spin,+}(X)$. As in the proof of the Gromov-Lawson-Rosenberg Conjecture in the simply connected case, there are two somewhat different arguments, showing that the required inclusion holds after tensoring with $\mathbb{Z}_{(2)}$ and $\mathbb{Z}[\frac{1}{2}]$, respectively (and this of course suffices to prove the inclusion).

To prove the inclusion localized at 2 (i.e., after tensoring with $\mathbb{Z}_{(2)}$), we consider the subgroup $T_n(X) \subset \Omega_n^{Spin}(X)$ consisting of bordism classes of the form $[M \xrightarrow{p} N \xrightarrow{f} X]$, where $p\colon M \to N$ is a $\mathbb{HP}^2$-bundle over a spin manifold N. The following result is proved by strengthening the homotopy theoretic techniques used in the proof of Theorem 2.5.

Theorem 2.14 ([St2]). *The map D induces a 2-local isomorphism*

$$ko_n(X) \cong \Omega_n^{Spin}(X)/T_n(X).$$

This implies in particular that 2-locally the kernel of $D\colon \Omega_n^{Spin}(X) \to ko_n(X)$ is equal to $T_n(X)$. Since total spaces of $\mathbb{HP}^2$-bundles admit positive scalar curvature metrics by Observation 1.4, this implies the desired inclusion 2-locally.

Proof of Theorem 2.14. We recall from the proof of Theorem 2.5 that the map $\widehat{T}\colon MSpin \wedge \Sigma^8 BG_+ \to \widehat{MSpin}$ induces a split injection of A-modules; an fact, more is true and proved in [St2], Proposition 8.3: $\widehat{T}$ is a split surjection of spectra. In particular, for any space X, the map

$$MSpin \wedge \Sigma^8 BG_+ \wedge X_+ \xrightarrow{\widehat{T}\wedge 1} \widehat{MSpin} \wedge X_+$$

induces 2-locally a surjection on homotopy groups. Since $\widehat{MSpin} \wedge X_+$ is the homotopy fiber of $D \wedge 1\colon MSpin \wedge X_+ \to ko \wedge X_+$, this implies that the following sequence is 2-locally exact:

$$\pi_n(MSpin \wedge \Sigma^8 BG_+ \wedge X_+) \xrightarrow{(T\wedge 1)_*} \pi_n(MSpin \wedge X_+) \xrightarrow{(D\wedge 1)_*} \pi_n(ko \wedge X_+).$$

Identifying $\pi_n(MSpin \wedge X_+)$ with $\Omega_n^{Spin}(X)$ via the Pontryagin-Thom construction, the image of $(T \wedge 1)_*$ can be identified as the subgroup $T_n(X) \subset \Omega_n^{Spin}(X)$. It is well-known that the map $(D \wedge 1)_*$ is 2-locally surjective, since $MSpin \to ko$ is a split surjection of spectra (cf. [ABP]). This shows that $ko_n(X)$ is isomorphic to $\Omega_n^{Spin}(X)/\ker D = \Omega_n^{Spin}(X)/T_n(X)$. □

The proof of the inclusion $\ker\Big(D\colon \Omega_n^{Spin}(X) \to ko_n(X)\Big) \subset \Omega_n^{Spin,+}(X)$ after inverting 2 (i.e., tensoring with $\mathbb{Z}[\frac{1}{2}]$), is again a consequence of a *geometric* description of ko-homology, this time as a bordism group of manifolds with singularities a la Baas-Sullivan. This then implies that if $[M, f]$ is in the kernel of D, then M is *zero bordant as a manifold with singularities.* Equivalently, after removing a neighborhood of the singularities from the zero bordism, we obtain a bordism W between M and another manifold N which is constructed inductively from the set Σ of manifolds describing the possible types of the admissible singularities. The essential observation is that in the Baas-Sullivan type description of $ko_n(X)$, the set Σ *can be chosen to consist of manifolds admitting positive scalar curvature metrics.* An inductive argument then shows that N admits a positive scalar curvature metric and then so does M by the Bordism Theorem 1.8.

At the end of this lecture we would like to mention the following result that can be proved quite similarly to Theorem 2.11 above (replacing e.g. $\mathbb{HP}^2$-bundles by $\mathbb{CP}^2$-bundles). To state it, we need the following notation. For an oriented manifold N, let $[N] \in H_n(N;\mathbb{Z})$ be the usual homology fundamental class of N. For any space X, let $H_n^+(X;\mathbb{Z}) \subset H_n(X;\mathbb{Z})$ be the subgroup consisting of all homology classes of the form $f_*[N]$, where N is a manifold with positive scalar curvature metric and $f\colon N \to X$.

Theorem 2.15 (Stolz, Jung). *Let M be an oriented manifold of dimension $n \geq 5$ with fundamental group π and let $u\colon M \to B\pi$ be the classifying map of the universal covering $\widetilde{M} \to M$. Assume that $\widetilde{M}$ does not admit a spin-structure. Then M admits a positive scalar curvature metric if and only if $u_*[M]$ is in $H_n^+(B\pi)$.*

3 The Gromov-Lawson-Rosenberg Conjecture and its relation to the Baum-Connes Conjecture

An important feature of the K-theory of C^*-algebras is the periodicity of these groups; they are 2-periodic for complex C^*-algebras, and 8-periodic for real C^*-algebras. With the definition of K-theory we have adopted this is not a deep fact, but rather reflects the algebraic periodicity of the Clifford algebras. If M is a spin manifold of dimension n with fundamental group π, the periodicity isomorphism

$$KO_n(C^*\pi) \xrightarrow{\cong} KO_{n+8}(C^*\pi)$$

maps the index obstruction $\alpha(M,u)$ to $\alpha(M \times B, u)$, where B is any simply connected spin manifold of dimension 8 with $\widehat{A}(B) = 8$ (we use the letter u for the classifying map of the universal covering of whatever manifold we are talking about) . We pick such a manifold B – the particular choice of it is immaterial for our purposes – and refer to it as 'Bott manifold', since the cartesian product with B corresponds to Bott-periodicity. It should be mentioned that Joyce [J] has constructed manifolds with these properties and metrics thereon with particular interesting geometric properties: they are Ricci flat and their holonomy group reduces to $Spin(7)$.

This shows that the Gromov-Lawson-Rosenberg Conjecture is equivalent to the following two conjectures.

3.1. Cancellation Conjecture. *Let M be a spin manifold of dimension $n \geq 5$. Then M admits a positive scalar curvature metric if and only if $M \times B$ does.*

3.2. Stable Conjecture *Let M be a spin manifold. Then M admits* stably a positive scalar curvature metric *(i.e., the product of M with sufficiently many copies of B has a positive scalar curvature metric) if and only if $\alpha(M,u) = 0$.*

Theorem 3.3 (Rosenberg-Stolz [RS1]). *The stable Gromov-Lawson-Rosenberg Conjecture is true for manifolds with finite fundamental group.*

For the proof of this result we need a K-theoretic reformulation of the Stable Conjecture, which is based on the following geometric description of KO-homology. Let $\left(\Omega_n^{Spin}(X)/T_n(X)\right)[B^{-1}]$ be the direct limit of the homomorphisms

$$\Omega_n^{Spin}(X)/T_n(X) \xrightarrow{\times B} \Omega_{n+8}^{Spin}(X)/T_{n+8}(X) \xrightarrow{\times B} \dots,$$

given by the cartesian product with the Bott manifold B. We note that the groups $\left(\Omega_n^{Spin}(X)/T_n(X)\right)[B^{-1}]$ are 8-periodic; in fact, multiplication by B provides an isomorphism. In other words, we made the non-periodic groups $\Omega_n^{Spin}(X)/T_n(X)$ periodic by *inverting the Bott-element*, which motivates the notation.

Theorem 3.4 (Kreck-Stolz [KS]). *The map* $\mathrm{per}\circ D\colon \Omega_n^{Spin}(X) \to KO_n(X)$ *induces an isomorphism* $\left(\Omega_n^{Spin}(X)/T_n(X)\right)[B^{-1}] \cong KO_n(X)$.

We remark that this is a direct consequence of Theorem 2.14 at the prime 2; an additional argument is needed localized away from 2.

Theorem 3.4 shows in particular that if $[M,u] \in \Omega_n^{Spin}(B\pi_1(M))$, is in the kernel of $\mathrm{per}\circ D$, then the product $M \times B \times \cdots \times B$ with sufficiently many copies of B represents an element in $T_n(B\pi_1(M))$, and hence carries a positive scalar curvature metric. This implies the following result.

Corollary 3.5. *Let M be a spin manifold of dimension $n \geq 5$ with fundamental group π and let $u\colon M \to B\pi$ be the classifying map of the universal covering $\widetilde{M} \to M$. Then M admits stably a positive scalar curvature metric if and only if $u_*[M]_{KO}$ is in $KO_n^+(B\pi)$.*

We recall from (2.10) that $KO_n^+(X) \subset KO_n(X)$ is the subgroup consisting of all elements of the form $f_*[N]_{KO}$ for some manifold N equipped with a positive scalar curvature metric and $\dim N \equiv n \bmod 8$.

Corollary 3.6. *The Stable Conjecture holds for spin manifolds with fundamental group π if and only if $KO_*^+(B\pi)$ is the kernel of the assembly map.*

For finite fundamental groups π the group C^*-algebra $C^*\pi$ is just the real group ring $\mathbb{R}\pi$ which is isomorphic to a product of matrix rings over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. It follows that the $KO_*(C^*\pi)$ is a sum of copies of the real K-theory of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ with as many summands of each kind as the corresponding matrix factors of $\mathbb{R}\pi$.

It is a well-known result of Atiyah that for a finite group π the complex K-theory $K^0(B\pi)$ can be identified with a completion of the representation ring of π. This was later refined by Atiyah and Segal to give a description of $KO^*(B\pi)$ in terms of the representation theory of π (see [AS]). The following result is obtained by dualizing their theorem.

Proposition 3.7 (Rosenberg-Stolz). *If π is a finite p-group, then the assembly map*

$$KO_n(B\pi;\mathbb{Z}/p^\infty) \xrightarrow{A} KO_n(C^*\pi;\mathbb{Z}/p^\infty)$$

with coefficients in $\mathbb{Z}/p^\infty$ is an isomorphism.

Corollary 3.8. *If π is a finite p-group, then there is a long exact sequence*

$$\to KO_n(B\pi) \xrightarrow{A} KO_n(C^*\pi) \to \widetilde{KO}_n(C^*\pi) \otimes \mathbb{Q} \xrightarrow{\partial} KO_{n-1}(B\pi) \to$$

Proof of corollary. The groups $KO_n(B\pi)$ and $KO_n(C^*\pi)$ are the n-th homotopy groups of certain spectra; moreover, the assembly map $A\colon KO_n(B\pi) \to KO_n(C^*\pi)$ is induced by a map between these spectra. This implies that KO-groups of $B\pi$ (resp. $C^*\pi$) with *coefficients* in some abelian group can be defined, and that associated to short exact sequences of coefficients we obtain long exact sequences of KO-groups. Moreover, there is an assembly map for KO-theory with coefficients compatible with these long exact sequences. In particular, the short exact sequence

$$\mathbb{Z}_{(p)} \to \mathbb{Q} \to \mathbb{Z}/p^\infty$$

gives rise to the following commutative diagram, whose rows are long exact sequences.

$$\begin{array}{ccccccccc}
\longrightarrow & KO_{n+1}(B\pi;\mathbb{Z}/p^\infty) & \xrightarrow{\partial} & KO_n(B\pi;\mathbb{Z}_{(p)}) & \longrightarrow & KO_n(B\pi;\mathbb{Q}) & \longrightarrow \\
 & {\scriptstyle A}\downarrow{\scriptstyle \cong} & & {\scriptstyle A}\downarrow & & {\scriptstyle A}\downarrow & \\
\longrightarrow & KO_{n+1}(C^*\pi;\mathbb{Z}/p^\infty) & \xrightarrow{\partial} & KO_n(C^*\pi;\mathbb{Z}_{(p)}) & \longrightarrow & KO_n(C^*\pi;\mathbb{Q}) & \longrightarrow
\end{array}$$

It is well-known and proved by a quick diagram chase in the above diagram, that this leads to a Meyer-Vietoris type long exact sequence

$$\to KO_n(B\pi;\mathbb{Z}_{(p)}) \to KO_n(C^*\pi;\mathbb{Z}_{(p)}) \oplus KO_n(B\pi;\mathbb{Q}) \to KO_n(C^*\pi;\mathbb{Q}) \xrightarrow{\partial},$$

which reduces to the long exact sequence of the corollary. □

Proof of Theorem 3.3. By Theorem 2.8 it is enough to prove the Stable Conjecture for finite p-groups. For *cyclic* groups, the Gromov-Lawson-Rosenberg Conjecture and hence also the weaker Stable Conjecture hold. So the idea of the proof is to compare the assembly map A for a finite p-group π with the assembly maps A_H for its cyclic subgroups $H \subset \pi$ by means of the following diagram.

$$\begin{array}{ccccccccc}
\longrightarrow & \bigoplus_H \widetilde{KO}_{n+1}(C^*H;\mathbb{Q}) & \xrightarrow{\oplus \partial_H} & \bigoplus_H KO_n(BH) & \xrightarrow{\oplus A_H} & \bigoplus_H KO_n(C^*H) & \longrightarrow \\
 & {\scriptstyle \mathrm{Ind}}\downarrow & & {\scriptstyle \mathrm{Ind}}\downarrow & & {\scriptstyle \mathrm{Ind}}\downarrow & \\
\longrightarrow & \widetilde{KO}_{n+1}(C^*\pi)\otimes\mathbb{Q} & \xrightarrow{\partial} & KO_n(B\pi)) & \xrightarrow{A} & KO_n(C^*\pi) & \longrightarrow
\end{array}$$

Here the vertical maps $\mathrm{Ind} = \bigoplus \mathrm{Ind}_H^\pi$ are sums of induction maps for cyclic subgroups $H \subset \pi$; we sum over representatives H of all conjugacy classes of cyclic subgroups of π. The rows are exact by Corollary 3.8.

By Artin induction, the left vertical map is surjective. This implies by a diagram chase that an element in the kernel of A is in the image of $\mathrm{Ind} \circ \bigoplus \partial_H$. Since the Stable Conjecture holds for cyclic groups, the group $\mathrm{image}(\partial_H) = \ker(A_H)$ is equal to $KO_n^+(BH)$. It follows that the image of $\mathrm{Ind} \circ \bigoplus \partial_H$ is contained in $KO_n^+(B\pi)$ which proves the theorem. □

Now we want to discuss groups π which are not necessarily finite. If π is *torsion free*, then according to (a form of) the Novikov-Conjecture, the assembly map $A\colon KO_n(B\pi) \to KO_n(C^*\pi)$ is injective. If this is true for π, then obviously the kernel of A is contained in $KO_n^+(B\pi)$ and the Stable Conjecture holds for π. In general the assembly map is *not* injective, for example for finite groups. However, the Novikov Conjecture can be generalized to any discrete group π in the following way. A s explained in Schick's lectures, the assembly map can be expressed in terms of the equivariant KO-homology and then factored as follows:

$$\begin{array}{ccccc} KO_n(B\pi) & & \xrightarrow{\quad A \quad} & & KO_n(C^*\pi) \\ \downarrow\cong & & & & \downarrow\cong \\ KO_n^\pi(E\pi) & \longrightarrow & KO_n^\pi(E(\pi,\mathcal{F})) & \xrightarrow{\mu} & KO_n^\pi(pt) \end{array}$$

Here $E\pi$ (resp. $E(\pi,\mathcal{F})$) is the universal π-space with trivial (resp. finite) isotropy groups and the map μ is induced by the projection of $E(\pi,\mathcal{F})$ to the point. We note that if π is torsionfree, then $E(\pi,\mathcal{F}) = E\pi$, and hence μ can be identified with the assembly map. The map μ is called the *Baum-Connes map.*

Conjecture 3.9 (Baum-Connes [BCH]). *The map μ is an isomorphism for any discrete group π.*

Theorem 3.10 (Stolz [St5]). *If the Baum-Connes map μ is injective for a group π, then the Stable Conjecture holds for π.*

We recall that the Stable Conjecture for a group π is equivalent to the statement $\mathfrak{K}O_n(B\pi) = KO_n^+(B\pi)$ (cf. 3.6), where from now on we write $\mathfrak{K}O_n(B\pi)$ for the kernel of the assembly map $A\colon KO_n(B\pi) \to KO_n(C^*\pi)$. To prove this equality, it suffices to prove that it holds localized at p for all primes p (localizing an abelian group at p means tensoring it with $\mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} \mid b \text{ is prime to } p\}$). So throughout this section we will fix a prime p and localize all abelian groups and spectra at that prime.

Our strategy to prove $\mathfrak{K}O_n(B\pi) = KO^+_n(B\pi)$ is to use the injectivity of the Baum-Connes map and induction techniques to show that that every element in $\mathfrak{K}O_n(B\pi)$ comes from some finite cyclic p-subgroup of π in a sense made precise by the following theorem.

By Theorem 3.4 any element of $KO_*(X)$ is represented by some spin manifold N and a map $f\colon N \to X$. We will write $[N,f] \in KO_*(X)$ for this element (or $[N,f]_{KO} \in KO_*(X)$ if there might be danger of confusion with the bordism class $[N,f] \in \Omega^{Spin}_*(X)$). Let H be a subgroup of π, and let $C(H) = \{g \in \pi \mid gh = hg \text{ for all } h \in H\}$ be its centralizer. Then there is a pairing

$$KO_*(BH) \otimes KO_*(BC(H)) \longrightarrow KO_*(B\pi)$$

given by sending $[M,f] \otimes [N,g]$ to the class represented by

$$M \times N \xrightarrow{f\times g} BH \times BC(H) = B(H \times C(H)) \xrightarrow{Bj} B\pi,$$

where $j\colon H \times C(H) \to \pi$ maps (h,c) to the product hc (note that this is a homomorphism since the elements of $C(H)$ commute with the elements of H). The multiplicative properties of the assembly map imply that the above pairing restricts to a pairing $\mathfrak{K}O_*(BH) \otimes KO_*(BC(H)) \to \mathfrak{K}O_*(B\pi)$.

Theorem 3.11. *Assume that the Baum-Connes map for the group π is injective. Then the homomorphism*

$$\bigoplus_H \mathfrak{K}O_*(BH) \otimes KO_*(BC(H)) \longrightarrow \mathfrak{K}O_*(B\pi),$$

is p-locally surjective, where H runs through representatives of all conjugacy classes of cyclic p-subgroups of π.

Proof of Theorem 3.10 assuming Theorem 3.11. We have

$$\mathfrak{K}O_n(BH) = KO^+_n(BH)$$

for every cyclic p-subgroup $H \subseteq \pi$, since the Stable Conjecture holds for cyclic groups. We observe that the cartesian product of manifolds $M \times N$ admits a positive scalar curvature metric if M does, which implies that the image of the above pairing restricted to $KO^+_*(BH) \otimes KO_*(BC(H))$ is contained in $KO^+_*(B\pi)$. This proves Theorem 3.10. □

Outline of the proof of Theorem 3.11. For the proof it is necessary to express the groups $KO_n(B\pi) = KO^\pi_n(E\pi)$, $KO^\pi_n(E(\pi,\mathcal{F}))$ and $KO_n(C^*\pi) =$

$KO_n^\pi(pt)$ as the n-th homotopy group of homotopy limits as explained in the lectures by Thomas Schick. More precisely, let $\mathrm{Or}(\pi)$ be the *orbit category* of π, whose objects are orbits of π (i.e., transitive π-sets); morphisms from an orbit V to an orbit U are the π-equivariant maps $U \to V$. There is a functor

$$KO\colon \mathrm{Or}(\pi) \to SPECTRA$$

to the category of spectra with the property that for a π-orbit π/H we have $\pi_n(KO(\pi/H)) \cong KO_n(C^*H)$ [DL]. Let

$$\mathrm{Or}(\pi,\mathcal{T}) \subset \mathrm{Or}(\pi,\mathcal{F}(p)) \subset \mathrm{Or}(\pi,\mathcal{F}) \subset \mathrm{Or}(\pi)$$

be the the full subcategories of the orbit category consisting of those orbits whose isotropy groups are trivial (resp. finite p-groups, resp. finite groups). Restricting the functor KO to these subcategories, and abusing notation by writing KO again for the restricted functor, the inclusions above induce the following maps of homotopy colimits:

$$\operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{T})} KO \xrightarrow{f} \operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{F}(p))} KO \longrightarrow \operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{F})} KO \longrightarrow \operatorname*{hocolim}_{\mathrm{Or}(\pi)} KO.$$

On the n-th homotopy group, this sequence of maps induces the following sequence of homomorphisms whose composition is the assembly map

$$KO_n(B\pi) \to KO_n^\pi(E(\pi,\mathcal{F}(p))) \to KO_n^\pi(E(\pi,\mathcal{F})) \to KO_n(C^*\pi).$$

By assumption the right map is injective, and it can be shown that the middle map is p-locally injective by constructing a p-local splitting for the corresponding map of homotopy colimits. Hence the elements in the kernel of A are in the image of the map

$$\pi_n(\mathrm{fiber}(f)) \to \pi_n(\operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{T})} KO) = KO_n(B\pi).$$

To identify the homotopy fiber $\mathrm{fiber}(f)$, it is useful to rewrite

$$\operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{T})} KO \qquad \text{in the form} \qquad \operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{F}(p))} i_*KO,$$

where the functor $i_*KO\colon \mathrm{Or}(\pi,\mathcal{F}(p)) \to SPECTRA$ is the *Kan extension* of the functor $KO\colon \mathrm{Or}(\pi,\mathcal{T}) \to SPECTRA$ via the inclusion $i\colon \mathrm{Or}(\pi,\mathcal{T}) \to \mathrm{Or}(\pi,\mathcal{F}(p))$. More explicitly, $i_*KO(\pi/H)$ can be identified with $KO \wedge BH_+$ (the domain of the assembly map for the finite p-subgroup $H \subset \pi$), and the natural transformation $i_*KO(\pi/H) \to KO(\pi/H) = KO(C^*H)$ is just the (spectrum level) assembly map for H. Corollary 3.8 implies that the homotopy fiber of this map is $\Sigma^{-1}\widetilde{KO}_{\mathbb{Q}}$, where the subscript $\mathbb{Q}$ indicates the rationalization of the spectrum in question. Thus we obtain the following homotopy fibration

$$\operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{F}(p))} \Sigma^{-1}\widetilde{KO}_{\mathbb{Q}} \xrightarrow{g} \operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{F}(p))} i_* KO \xrightarrow{f} \operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{F}(p))} KO.$$

Let $\mathrm{Or}(\pi, \mathcal{C}(p)) \subset \mathrm{Or}(\pi, \mathcal{F}(p))$ be the subcategory consisting of all orbits whose isotropy subgroups are cyclic p-groups. Artin induction implies that the map

$$\operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{C}(p))} \Sigma^{-1}\widetilde{KO}_{\mathbb{Q}} \xrightarrow{h} \operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{F}(p))} \Sigma^{-1}\widetilde{KO}_{\mathbb{Q}}$$

induced by the inclusion of categories is a homotopy equivalence.

There is a simpler category $\mathrm{Or}(\pi, \mathcal{C}(p))'$, whose objects are cyclic p-subgroups $H \subset \pi$, one for each conjugacy class of such subgroups. The endomorphisms of the object H is the centralizer $C(H)$ of H in π, and there are no other morphisms in $\mathrm{Or}(\pi, \mathcal{C}(p))'$. Let $F\colon \mathrm{Or}(\pi, \mathcal{C}(p))' \to \mathrm{Or}(\pi, \mathcal{C}(p))$ be the functor which sends a subgroup H to the orbit π/H and an element $c \in C(H)$ to the π-map $\pi/H \to \pi/H$ given by $gH \mapsto cgH$. It can be shown that the functor F induces a surjection on homotopy groups of the corresponding homotopy colimits

$$\operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{C}(p))'} \Sigma^{-1}\widetilde{KO}_{\mathbb{Q}} \xrightarrow{k} \operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{C}(p))} \Sigma^{-1}\widetilde{KO}_{\mathbb{Q}}.$$

(here the fact that $KO_{\mathbb{Q}}$ is *rational* is of central importance).

It is easy to obtain the isomorphism

$$\pi_n\left(\operatorname*{hocolim}_{\mathrm{Or}(\pi,\mathcal{C}(p))'} \Sigma^{-1}KO_{\mathbb{Q}}\right) \cong \bigoplus_H \widetilde{KO}_{n+1}(C^*H;\mathbb{Q}),$$

where we sum over representatives of the conjugacy classes of cyclic p-subgroups of π. Moreover, the image of the pairing of Theorem 3.11 can be identified with the image of the composition $g\,h\,k$ on homotopy groups. However, the induced map $(g\,h\,k)_*$ surjects onto the kernel of f_*, which agrees with the kernel of the assembly map if we assume the injectivity of the Baum-Connes map. This finishes the outline of the proof of Theorem 3.11. □

4 Survey on the problem of finding a positive Ricci curvature metric on a closed manifold

4.1 Ricci curvature

In this lecture we will talk about Ricci curvature, which is a finer curvature invariant of a Riemannian manifold than the scalar curvature $s(p)$: For each point $p \in M$ the Ricci curvature is a quadratic form

$$Ric\colon T_pM \longrightarrow \mathbb{R}$$

on the tangent space at this point. Averaging $Ric(v)$ over all unit tangent vector $v \in T_pM$ gives the scalar curvature $s(p)$ at the point p (up to a factor). More precisely,

$$s(p) = \sum_{i=1}^{n} Ric(e_i),$$

where $\{e_1, \dots, e_n\}$ is an orthonormal basis for T_pM.

It is interesting that the geometric interpretation of scalar curvature in terms of volume measurement can be refined to a description of the Ricci curvature as follows. We recall that the geometric description of $s(p)$ is based on comparing the volume of the ball $B_r(0, \mathbb{R}^n)$ of radius r in euclidean space with the volume of the ball $B_r(p, M)$ of radius r around p. We note that we might identify $B_r(0, \mathbb{R}^n)$ with the ball of radius r around the origin in the tangent space T_pM, and $B_r(p, M)$ with its image under the exponential map $\exp_p\colon T_pM \to M$. This suggests to refine $s(p)$ by comparing the standard volume form on T_pM with $\exp_p^*(dvol)$, where $dvol$ is the volume form on M determined by the Riemannian metric. To do this, it is convenient to work with polar coordinates on T_pM; in other words, we identify $T_pM \setminus 0$ with $U_p \times \mathbb{R}_+$, where $U_p \subset T_pM$ denotes the unit tangent bundle, and define a function $f(v, t)$ on $U_p \times \mathbb{R}_+$ by the equation

$$\exp_p^*(dvol) = \theta(v, t)\, \mu \wedge dt,$$

where μ is the canonical volume form on U_p. Then the Ricci curvature $Ric(v)$ shows up when expanding $\theta(v, t)$ in t (see [Be, 0.61]):

$$\theta(v, t) = t^{n-1}(1 - \frac{1}{3} Ric(v) t^2 + \dots).$$

By integrating over U_p we obtain from this the geometric description of scalar curvature 1.2.

In this lecture which is based on [St6], we want to adress the following question:

4.1. Question. Which manifolds admit metrics of positive Ricci curvature (i.e., $Ric(v) > 0$ for all non-zero tangent vectors v)?

There is a classical result that restricts the topology of manifolds with $Ric > 0$ (see [Be, Cor. 6.52]):

Theorem 4.2 (Myers). *If M is a Riemannian manifold with $Ric > 0$, then the fundamental group $\pi_1(M)$ is finite.*

One would expect that there are many manifolds with positive scalar curvature metric, which do not admit a Riemannian metric with $Ric > 0$, since the scalar curvature of a manifold M is the trace over the Ricci curvature, and since the requirement that a quadratic form is positive definite is so much stronger than the requirement that its trace is positive. So it is surprising that in some sense Myer's Theorem is the *only* known restriction for simply connected manifolds with positive scalar curvature of dimension $n \geq 5$ to admit a metric with $Ric > 0$: There are no known examples of simply connected manifolds with positive scalar curvature metrics, which don't admit metrics with $Ric > 0$. The following conjecture would imply in particular the existence of such examples (see part 2 of Remark 4.5).

4.2 A Conjecture concerning Ricci curvature

4.3. Conjecture [St6]. Let M be a spin manifold of dimension $n = 4k$ and assume that $\frac{p_1}{2}(M) \in H^4(M;\mathbb{Z})$ vanishes. If M carries a Riemannian metric with $Ric > 0$, then the Witten genus $\phi_W(M)$ vanishes.

Explanation of $\frac{p_1}{2}(M)$: for vector bundles with spin structure, the first Pontryagin class p_1 is divisible by 2 (in fact canonically: do it for the universal spin bundle over $BSpin(n)$). Short of a better name for it, this class is denoted $\frac{p_1}{2}$. We should stress that due to the possible torsion in $H^4(M;\mathbb{Z})$, the condition $\frac{p_1}{2}(M) = 0$ might be *stronger* than the requirement $p_1(M) = 0$.

4.4. The Witten genus (see [Wi]) If M is an oriented manifold of dimension $n = 4k$, its *Witten genus* is the power series $\phi_W(M) \in \mathbb{Q}[[q]]$ defined by

$$\phi_W(M) = \left(\prod_{l=1}^{\infty}(1-q^l)\right)^n \sum_{l=1}^{\infty} a_l q^l \qquad a_l = \langle \widehat{A}(TM)ch(R_l), [M]\rangle,$$

where the complex vector bundles $R_l \to M$ are constructed from the complexified tangent bundle $E = TM_{\mathbb{C}}$ and its symmetric powers S^kE in the following way: Combine all symmetric powers of E to the *total symmetric power*

$$S_t E \overset{\text{def}}{=} 1 + Et + (S^2E)t^2 + (S^3E)t^3 + \dots,$$

where t is a formal variable. Then expand the following expression as a powerseries of q whose coefficients are vector bundles over M:

$$\bigotimes_{l=1}^{\infty} S_{q^l} E = R_0 + R_1 \cdot q + R_2 \cdot q^2 + \dots$$

To illustrate the procedure, we calculate R_l for $0 \le l \le 3$ explicitly by expanding $\bigotimes_{l=1}^{\infty} S_{q^l} E$ while ignoring all terms involving q^l for $l > 3$:

$$\begin{aligned}(1 + Eq + S^2Eq^2 + S^3Eq^3 + \dots)(1 + Eq^2 + \dots)(1 + Eq^3 + \dots)\\ = 1 + Eq + (S^2E + E)q^2 + (S^3E + E \otimes E + E)q^3 + \dots\end{aligned}$$

This shows

$$R_0 = 1 \qquad R_1 = E \qquad R_2 = S^2E \oplus E \qquad R_3 = S^3E \oplus (E \otimes E) \oplus E$$

where 1 is the trivial complex line bundle.

4.5. Remarks.

1. The conjecture is true for formal reasons for manifolds of dimension $n < 24$ by the following argument. It can be shown that the Witten genus of any n-manifold M with $p_1(M) = 0$ is the *q-expansion of a modular form of weight $n/2$* (for a definition of these terms, we refer to [HBJ, §6.3]). If we denote by $\mathfrak{M}_k$ the $\mathbb{C}$-vector space of modular forms of weight k, the direct sum $\mathfrak{M}_* = \bigoplus_{k=0}^{\infty} \mathfrak{M}_k$ is a graded $\mathbb{C}$-algebra, which turns out to be isomorphic to the polynomial algebra generated by the modular forms

$$C_4 = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \quad \text{and} \quad C_6 = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

of weight 4 and 6, respectively. In particular, $\mathfrak{M}_k$ has dimension ≤ 1 for $k < 12$, which implies that the Witten genus of a manifold M of dimension $n < 24$ (assuming $p_1(M) = 0$) is determined by its constant term, which equals $\widehat{A}(M)$. This of course vanishes by Lichnerowiz' result Theorem 1.11 if we assume $Ric > 0$, which proves the Conjecture for $n < 24$.

2. There are simply connected spin manifolds M of dimension $= 24$ and $n = 4k \geq 30$ with $\frac{p_1}{2}(M) = 0$, $\widehat{A}(M) = 0$ and $\phi_W(M) \neq 0$. By Theorem 2.5 any such manifold admits a positive scalar curvature metric, but – if the Conjecture holds – M does not admit a metric with $Ric > 0$.

3. The standard Fubini-Study metric on the quaternionic projective plane $\mathbb{HP}^2$ has $Ric > 0$ (even the sectional curvature is positive), however $\phi_W(\mathbb{HP}^2) \neq 0$. This is not a counterexample to Conjecture 2.3, but it shows that the condition $\frac{p_1}{2}(M) = 0$ cannot be dropped. It is interesting to compare this to what happens in the case of the complex projective plane $\mathbb{CP}^2$: the standard metric has positive scalar curvature (even positive sectional curvature), yet $\widehat{A}(\mathbb{CP}^2) \neq 0$. Of course this is not a contradiction to Lichnerowicz' Theorem, since $\mathbb{CP}^2$ is not a spin manifold.

4.3 Evidence for the Conjecture

Supporting evidence for the Conjecture 2.3 is provided by the fact that it has been checked for some classes of manifolds, including the following:

4.6. Examples of manifolds with $Ric > 0$:

1. Homogeneous spaces G/H with G compact and semi-simple. To obtain a Ricci positive Riemannian metric on G/H, pick a bi-invariant metric on G; this induces a metric on G/H, referred to as the *normal homogeneous metric* which is characterized by the property that $G \to G/H$ is a Riemannian submersion [Be, Def. 9.8].

2. Complete intersections $X^n \subset \mathbb{CP}^{n+r}$ with positive first Chern class (i.e., transverse intersections of r hyperplanes in $\mathbb{CP}^{n+r}$). The standard Fubini-Study metric on $\mathbb{CP}^{n+r}$ induces a Kähler metric on X, which might not have positive Ricci curvature. However, thanks to the positivity of the first Chern class and the Calabi-Yau Theorem [Be, Thm. 11.15, 11.16(ii)], X admits a Kähler metric with positive Ricci curvature.

In the case of homogeneous spaces, the vanishing of the Witten genus is a consequence of the following result which was proved independently by Dessai [De] and Höhn (unpublished), based on work of Liu [Liu]. This result is analogous to the classical result of Atiyah and Hirzebruch saying that the $\widehat{A}$-genus of a spin manifold with non-trivial S^1-action vanishes.

Theorem 4.7 (Dessai, Höhn). *Let M be a spin manifold of dimension $n = 4k$ with vanishing first Pontryagin class $p_1(M)$. If M admits a non-trivial S^3-action, then its Witten genus $\phi_W(M)$ is zero.*

The Witten genus of complete intersections was calculated by Landweber and Stong (see [HBJ], the last Example in section 6.3). Their result is:

Theorem 4.8 (Landweber-Stong). *If $X^n \subset \mathbb{C}\mathrm{P}^{\mathrm{n+r}}$ is a complete intersection with vanishing first Pontryagin class, then its Witten genus is zero.*

4.4 Towards a proof of the Conjecture

At first sight, one hope for proving the vanishing of the Witten genus for a manifold M or – equivalently – the vanishing of all the numbers a_l is to interpret the latter as indices of 'twisted' Dirac operators.

4.9. Twisted Dirac operators. Let M be a spin manifold with spinor bundle S and let E be a complex vector bundle over M equipped with a connection. Then we can define an elliptic first order differential operator

$$D_E \colon C^\infty(S \otimes E) \longrightarrow C^\infty(S \otimes E)$$

by the *same* formula 1.16 as the Dirac operator; the only difference is that now ∇ is the product connection on $S \otimes E$ induced by the usual connection on S and the given connection on E. The operator D_E is called the *Dirac operator twisted by E*. There is a Bochner-Lichnerowicz-Weitzenböck formula for D_E (cf. [LaM, Ch. II, Thm. 8.17]) of the form

$$D^2 = \nabla^*\nabla + \frac{s}{4} + \mathfrak{R}^E,$$

where $\mathfrak{R}^E \colon S \otimes E \to S \otimes E$ is a vector bundle homomorphism determined by the curvature tensor of the connection on E.

As in the untwisted case, we denote by $D_E^+ \colon C^\infty(S^+ \otimes E) \longrightarrow C^\infty(S^- \otimes E)$ the restriction of D_E. According to the Atiyah-Singer Index Theorem (cf. [LaM, Ch. III, Thm. 13.10])

$$\mathrm{index}(D_E^+) = \langle \widehat{A}(TM)ch(E), [M] \rangle.$$

Coming back to the case of interest to us, we note that for a spin manifold M we have $a_l = \mathrm{index}(D_{R_l})$. Moreover, since R_l is build from symmetric powers of $TM_{\mathbb{C}}$, the Levi-Civita connection on TM induces a connection on R_l. The curvature term $\mathfrak{R}^{R_l}$ can then (at least in principle) be expressed in terms of the curvature tensor of M. One might hope to prove the conjectured vanishing of a_l by arguing that $Ric > 0$ implies that $F = \frac{s}{4} + \mathfrak{R}^{R_l}$ is a *positive* endomorphism of each fiber of $S \otimes R_l$ (i.e., $\langle Fv, v\rangle > 0$ for each non-zero $v \in S \otimes R_l$), which by the same argument as in the proof of Lichnerowicz' Theorem would imply the vanishing of D_{R_l}).

Alas, this strategy can't work, since nowhere in this line of argument did we use the assumption $\frac{p_1}{2}(M) = 0$, without which the conjecture is false as we've seen in part 3 of Remark 4.5.

4.5 Relation with the loop space

What we have said so far in this lecture is bound to appear quite mysterious, and the reader might have wondered about some of the following questions:

(i) Where does the Witten genus come from?

(ii) Why the assumption $\frac{p_1}{2}(M) = 0$?

(iii) Even if the conjecture happens to hold for homogeneous spaces and complete intersections, is there some heuristic argument that should let us expect it to be true in general?

Thinking of the conjecture as analogous to Lichnerowicz' Theorem, let us imagine we go back in time to the late fifties after Hirzebruch defined the $\widehat{A}$-genus, but before Atiyah and Singer constructed the 'Dirac' operator on a general spin manifold and before Lichnerowicz proved his formula (1.17). Imagine being lectured to about the definition of the $\widehat{A}$-genus (as in our first lecture, cf. 1.12), and being presented with the conjecture that the $\widehat{A}$-genus vanishes for manifolds with $w_2(M) = 0$ which admit a positive scalar curvature metric. As supporting evidence for this 'conjecture' it is observed that it is true for homogeneous spaces and complete intersections. Then the following questions might come to mind:

(i') Where does the $\widehat{A}$-genus come from?

(ii') Why the assumption $w_2(M) = 0$?

(iii') Even if the conjecture happens to hold for homogeneous spaces and complete intersections, is there some heuristic argument that should let us expect it to be true in general?

Questions (i') and (ii') are basically answered by the construction of the Dirac operator by Atiyah and Singer: the condition $w_2(M) = 0$ is needed to construct the spinor bundle M and hence the Dirac operator which acts on the sections of this bundle. The index of the Dirac operator is the $\widehat{A}$-genus.

Similarly, the questions (i) and (ii) are essentially answered by Witten's construction of the 'Dirac operator' on the free loop space LM consisting of all 'loops' $\gamma\colon S^1 \to M$: the condition $\frac{p_1}{2}(M) = 0$ is needed to construct the 'spinor bundle' over LM, on whose sections the 'Dirac operator' D_{LM} acts. The S^1-equivariant index of D_{LM} can be identified with the Witten genus of M. Unfortunately, Witten's considerations in [Wi] with regard to D_{LM} are very much on a formal/heuristic level and to date there is no mathematically rigorous construction of D_{LM} (except for homogeneous spaces [La], as we discuss below). It should be mentioned that Taubes [Ta1] has constructed an operator with the correct equivariant index, which can be interpreted as the 'Dirac operator on small loops'; unfortunatly, one cannot hope to use this operator to prove the conjecture, since its construction does not need the condition $\frac{p_1}{2} = 0$. In his paper, Witten does not discuss the question of how to construct D_{LM}, but rather *assumes* it has been constructed, that its S^1-equivariant index can be defined, and that the fixed point formula which expresses the equivariant index of an elliptic differential operator on a finite dimensional compact manifold continues to hold for the infinite dimensional manifold LM.

4.10. Digression on the fixed point formula. Let M be a spin manifold of dimension $n = 4k$, on which S^1 acts by isometries. We further assume that the S^1-action is compatible with the spin structure in the sense that the induced S^1-action on the oriented frame bundle $SO(M)$ lifts to an S^1-action on the double covering $Spin(M) \to SO(M)$ given by the spin structure. The induced action on $C^\infty(S)$ commutes with the Dirac operator D; in particular, $\ker D^+$ and $\operatorname{coker} D^+$ are representation of S^1,and we define the *S^1-equivariant index*

$$\operatorname{index}^{S^1}(D^+) \stackrel{\text{def}}{=} \sum_{l\in\mathbb{Z}} \left(\dim\,[\ker D^+]_l - \dim\,[\operatorname{coker} D^+]_l\right)\, q^l,$$

where for any representation of S^1, we denote by $V_l \subset V$ is the subspace where $z \in S^1$ acts by multiplication by z^l.

According to the *Fixed Point Formula*, also referred to as the *equivariant Atiyah-Singer Index Theorem*, the equivariant index can be computed in terms of the fixed point set of the S^1-action and the equivariant normal bundle of the fixed point set. To describe the explicit formula, let F be a component of the fixed point set. Then S^1 acts on the normal bundle $N \to F$, and the real vector bundle N can be written uniquely in the form $N = \bigoplus_{l>0} N_l$, where $N_l \to F$ is a *complex* vector bundle over F, on which

$z \in S^1$ acts by multiplication by z^l. Then

$$\operatorname{index}^{S^1}(D) = \sum_F \pm \langle \widehat{A}(TF)\, ch \left(\bigotimes_{l=1}^{\infty} (\Lambda^{n_l} q^l N_l)^{1/2} S_{q^l} N_l \right), [F] \rangle,$$

where $n_l = \dim_{\mathbb{C}} N_l$, and we sum over the connected components of the fixed point set. The determination of the sign for each component is quite involved and we refer to Atiyah-Bott for details [AB1], [AB2].

4.11. Witten's 'Index Theorem' for the Dirac operator on the loop space [Wi], [HBJ, §6.1]. Now we will 'apply' the Fixed Point Formula to the Dirac operator on the free loop space LM of a manifold M, equipped with the S^1-action given by rotating the parametrization of the loops. It should be stressed that this is a 'formal' calculation, since neither has a Dirac operator on LM be constructed, nor is the Fixed Point Formula a priory valid when applied to an infinite dimensional manifold like the free loop space.

We note that the fixed point set of the S^1-action on LM consists of the constant loops. Moreover, if γ is a constant loop, say $\gamma(t) = x_0 \in M$ for all $t \in [0,1]$, then the tangent space $T_\gamma LM$ consists of all loops $\{s\colon [0,1] \to T_{x_0}M \mid \gamma(0) = \gamma(1)\}$. Such a loop s has a Fourier decomposition

$$s(t) = \frac{a_0}{2} + \sum_{l>0} (a_l \cos 2\pi l t + b_l \sin 2\pi l t) \qquad \text{with} \quad a_l, b_l \in T_{x_0}M.$$

This implies that we have an isomorphism

$$T_\gamma LM \cong T_{x_0}M \oplus \bigoplus_{l>0} T_{x_0}M_{\mathbb{C}}$$

given by sending a loop s to its Fourier components (thinking of $a_l + ib_l$ as an element of the l-th copy of $T_{x_0}M_{\mathbb{C}} = T_{x_0}M \otimes_{\mathbb{R}} \mathbb{C}$). It is easy to check (cf. [HBJ, §6.1]) that with this identification $z \in S^1$ acts trivially on $T_{x_0}M$ and by multiplication by z^l on the l-th copy of $T_{x_0}M_{\mathbb{C}}$. This shows that the normal bundle N of the fixed point set $M \subset LM$ decomposes equivariantly as $N = \bigoplus_l N_l$, where $N_l = TM_{\mathbb{C}}$. It follows that $(\Lambda^{n_l} q^l N_l)^{1/2} = q^{\frac{ln}{2}}$, since $\Lambda^n TM_{\mathbb{C}}$ is the complexification of $\Lambda^n TM$, which is trivial since M is assumed orientable. After pulling out all these factors we obtain

$$\operatorname{index}^{S^1}(D_{LM}) = \left(\prod_{l=1}^{\infty} q^l \right)^{n/2} \langle \widehat{A}(TM)\, ch \left(\bigotimes_{l=1}^{\infty} S_{q^l}(TM_{\mathbb{C}}) \right), [M] \rangle. \tag{4.12}$$

To the earthbound eyes of a mathematician, the factor

$$\left(\prod_{l=1}^{\infty} q^l\right) = q^{\sum_{l=1}^{\infty} l}$$

appears to make no sense. However a physicist, used to dealing with ugly infinities showing up when trying to sum certain series, would proceed to 'regularize' the sum $\sum_{l=1}^{\infty} l$ by considering the Riemann ζ-function $\zeta(s) = \sum_{l=1}^{\infty} l^{-s}$ which converges to a holomorphic function if the real part of the complex number s is sufficiently large. Then this function can be extended to a meromorphic function $\zeta(s)$ for $s \in \mathbb{C}$. It turns out that $\zeta(s)$ has no pole at $s = -1$, and $\zeta(-1) = -\frac{1}{12}$. Note that if we formally substitute -1 for s in the sum defining $\zeta(s)$ for s with large real part, we obtain $\sum_{l=1}^{\infty} l$; this is the motivation behind considering $-\frac{1}{12}$ as the 'regularized' value of this sum.

We recall from Definition 4.4 that

$$\phi_W(M) = \left(\prod_{l=1}^{\infty}(1-q^l)\right)^n \langle \widehat{A}(TM)\, ch\left(\bigotimes_{l=1}^{\infty} S_{q^l}(TM_{\mathbb{C}})\right), [M]\rangle.$$

Comparison with formula 4.12 shows that

$$\operatorname{index}^{S^1}(D_{LM}) = \frac{\phi_W(M)}{\eta(q)^n}, \tag{4.13}$$

where $\eta(q) = q^{1/24}\prod_{l=1}^{\infty}(1-q^l)$ is Dedekind's η-function.

This Index 'Theorem' of Witten shows that heuristically the Witten genus $\phi_W(M)$ should be thought of as the equivariant index of the Dirac operator on the free loop space LM. As an optimist, one might believe that it should be possible to imitate the argument used in the proof of Lichnerowicz' Theorem 1.11. In other words, the hope is to prove a 'Bochner-Lichnerowicz-Weitzenböck Formula' for D_{LM} which implies that if M has positive Ricci curvature, then D_{LM} is positive which in conjunction with Witten's 'Index Theorem' (Formula 4.13) would imply the vanishing of the Witten genus for manifolds with $\frac{p_1}{2}(M) = 0$ and $Ric > 0$.

In analogy with the finite dimensional case, one might suspect that D_{LM} is invertible if the scalar curvature of LM is positive, and might hope that the scalar curvature of LM at a loop γ is given by integrating the Ricci curvature $Ric(\dot{\gamma})$ applied to the tangent vector $\dot{\gamma}$ to the loop γ over S^1. However, this is too naive for various reasons: first, D_{LM} is more analogous to Dirac operator associated to a $Spin^c$-structure than the Dirac operator of a spin manifold, which produces an extra term in the Weitzenböck formula. Secondly, there are many possible Riemannian metrics on LM induced by a fixed Riemannian metric on M; for the simplest one (the L^2-metric), the sectional curvature of the loop space is easy to calculate, but the sum describing the scalar curvature is divergent. For the other 'Sobolev-metrics', it seems that the (very complicated) expression for the scalar curvature of LM depends not only on the curvature tensor along γ, but also its covariant derivative, thus making it seem unlikely to be able to prove positivity when given only control over the Ricci tensor (but not its derivatives).

An interesting test case for this line of argument was provided by the recent construction of a 'Dirac operator' D_{LM} for the loop space of a homogeneous space $M = G/H$ of a compact semi-simple Lie group G by G. Landweber [La]. Of course, we know that Conjecture 4.3 is true for homogeneous spaces by Theorem 4.7; however, with the proposed line of argument, not just the *index of* D^+_{LM}, but the *kernel of* D_{LM} should be trivial. This is indeed the case for Landweber's operator.

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