On dynamics of $5D$ superconformal theories

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Abstract

$5D$ superconformal theories involve vacuum valleys characterized in the simplest case by the vacuum expectation value of the real scalar field σ. If $\langle\sigma\rangle \neq 0$, conformal invariance is spontaneously broken and the theory is not renormalizable. In the conformally invariant sector $\langle\sigma\rangle = 0$, the theory is intrinsically nonperturbative. We study classical and quantum dynamics of this theory in the limit when field dependence of the spatial coordinates is disregarded. The classical trajectories "fall" on the singularity at $\sigma = 0$. The quantum spectrum involves ghost states with unbounded from below negative energies, but such states fail to form complete 16-plets as is dictated by the presence of four complex supercharges and should be rejected by that reason. Physical excited states come in supermultiplets and have all positive energies. We conjecture that the spectrum of the complete field theory hamiltonian is nontrivial and has a similar nontrivial ghost-free structure and also speculate that the ghosts in higher-derivative supersymmetric field theories are exterminated by a similar mechanism.

1 Introduction

Field theories in more than four dimensions attracted recently a considerable attention. Usually they are discussed in the string theory perspective, but to our mind higher dimensional theories are interesting *per se*. In particular, certain arguments may be given [1] that a variant of higher-derivative superconformal theory in higher dimensions (possibly, a theory enjoyng the maximal $\mathcal{N} = 2$ superconformal symmetry in six dimensions) may be the fundamental Theory of Everything.

For a field theory to make sense, the continuum limit of the path integral when the lattice spacing is sent to zero or, what is equivalent, the ultraviolet cutoff Λ is sent to infinity should be well defined. Usually, this property is associated with renormalizability. Indeed, a renormalizable theory is a theory with a finite number of bare couplings defined at ultraviolet scale Λ, which can be sent to infinity in such a way that the effective lagrangian describing the physics at energies $E \ll \Lambda$ also involves a finite number of couplings that do not depend on Λ. Renormalizable theories are abundant in 4 dimensions. They are divided in two types: *(i)* asymptotically free and conformal theories

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with nontrivial continuum limits and *(ii)* the theories like ordinary QED, where the only meaningful continuum limit describes a free theory.

It is difficult, however, to construct a sensible renormalizable theory for $D > 4$. For example, a usual YM theory with the action $\sim \mathrm{Tr}\{F_{\mu\nu}F_{\mu\nu}\}$ involves in higher dimensions a dimensionful coupling and is not renormalizable. If one wishes still to work with higher-dimensional theories, one should choose between the following nonstandard options:

First, one can remark that renormalizability is a statement about the benign perturbative structure of the theory. Though it is *sufficient* for the continuum limit to exist, we do not know whether it is also *necessary*. Even in perturbative framework, one can imagine a theory involving an infinite number of counterterms which absorb all power ultraviolet divergences so that the physical quantities do not depend on Λ [2]. One such theory has actually been studied for a long time, I mean the chiral perturbation theory [3]. A distinguishable property of the latter is that loops involve there not only power UV divergences cancelled out by counterterms, but also so called chiral logarithms which lead to nontrivial contributions $\propto \ln(\mu_{\mathrm{hadr}}/m_\pi)$ in all physicall quantities. For sure, chiral theory is an effective theory describing low-energy dynamics of QCD and not a fundamental theory. But we cannot exclude at present that a fundamental theory with nonrenormalized lagrangian that leads to nontrivial cutoff-independent dynamics at all energy scales can be defined.

Another possibility is to keep renormalizability by adding higher derivatives in the lagrangian. In [4], we constructed a renormalizable $6D$ higher-derivative theory. It is a supersymmetric gauge theory with the terms $\propto (D_\mu F_{\mu\nu})^2$ etc. in the lagrangian so that the coupling constant is dimensionless and only logarithmic ultraviolet divergences (handled by the renormalization procedure) are present. At the classical level, this theory enjoys conformal symmetry, which is broken, however, by quantum anomaly. The effective charge runs there increasing with the energy similar to what happens in ordinary $4D$ QED.

Higher derivative theories have been seldom considered seriously by theorists and the reason is the problem of ghosts. It is very difficult to get rid of the ghosts (physically, they mean instability of the perturbative vacuum and are associated with the absence of the ground state in the hamiltonian) when the lagrangian involves higher derivatives. In [5] we managed, however, to construct a quantum mechanical example where higher derivatives are present and still the ground state is well defined. This and some other observations encouraged us to speculate in [1] that the TOE may represent a $6D$ superconformal higher-derivative theory. The example considered in [5] was not supersymmetric. However, (and this is one of the observations of the present paper), it is much easier to cope with ghosts in a supersymmetric theory than in a non-supersymmetric one. Indeed, supersymmetry usually implies positivity of all energies (only the energy of vacuum can be zero). And this means that the spectrum is bounded from below and ghosts are absent. We return to the discussion of this question in the end of the paper.

Besides two possibilities mentioned above (*(i)* nonrenormalizable theory where a nontrivial continuum limit exists and *(ii)* renormalizable theory with higher derivatives), there is also a third possibility to obtain a viable extra dimensional theory. A theory can be *intrinsically nonperturbative* so that even a question whether it is renormalizable or not cannot be posed. An example of such theory is superconformal theory in five dimensions.

This theory was first discussed qualitatively in [6], and its action was written in the language of harmonic superfields in [7] and in the component form in [8]. The simplest such theory involves an Abelian gauge field A_μ, a scalar field σ, auxiliary fields and fermionic superpartners. The lagrangian is [1]

$$
\begin{aligned}
g^2\mathcal{L} \;=\; & -\frac{\sigma}{4}F_{\mu\nu}F_{\mu\nu} + \frac{\sigma}{2}(\partial_\alpha\sigma)^2 + \frac{i\sigma}{2}\bar{\psi}^j \partial\!\!\!/ \psi_j - \sigma D^{jk}D_{jk} \\
& \frac{i}{8}\bar{\psi}^j\tilde{\sigma}_{\mu\nu}F_{\mu\nu}\psi_j + \frac{1}{24}\epsilon_{\mu\nu\lambda\rho\sigma}A_\mu F_{\nu\lambda}F_{\rho\sigma} + \frac{1}{2}\bar{\psi}^j\psi^k D_{jk}\,,
\end{aligned}
\tag{1}
$$

where ψ_j is the pseudo-Majorana fermion satisfying the constraint

$$
\bar{\psi}^{ia} \equiv (\psi_{ia})^*\tilde{\gamma}^0 = \epsilon^{ij}\tilde{C}^{ab}\psi_{jb}\,. \tag{2}
$$

2 Field theory: perturbative and nonperturbative features.

The lagrangian (1) does not involve higher derivatives. Still, it is scale (actually, conformally) invariant and the coupling constant g^2 is dimensionless. The latter would suggest renormalizability of the theory. However, the lagrangian (1) does not involve a quadratic part and perturbative calculations based on smallness of the interaction term with respect to the free part are impossible ! In this respect, the situation is even worse than in a nonrenormalizable theory. In the latter, perturbative series diverges, but one can at least go into the interaction representation, define asymptotic scattering states, and if not to find what their scattering matrix is, but at least ask this question. Here we seem not to be able to do it .

An option that one has, however, is to capitalize on the presence of the vacuum moduli space in the lagrangian (1). Indeed, any vacuum expectation value $\langle\sigma\rangle = m$ is equally admissible and does not cost energy. If $m \neq 0$, we can pose $\sigma = m + \sigma'$ treating σ' perturbatively. In this case, conformal symmetry of the original lagrangian is broken spontaneously and the scale parameter m is introduced. This allows one to decompose the

[1] Our conventions are here the following. We use the metric $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1,-1)$. The γ-matrices $\tilde{\gamma}_\mu$ (we put tildas to distiguish these γ matrices from the untilded Euclidean γ matrices to be introduced later) satisfy $\tilde{\gamma}_\mu\tilde{\gamma}_\nu + \tilde{\gamma}_\nu\tilde{\gamma}_\mu = 2\eta_{\mu\nu}$, $\tilde{\gamma}_0^\dagger = \tilde{\gamma}_0, \tilde{\gamma}_{1,2,3,4}^\dagger = -\tilde{\gamma}_{1,2,3,4}$. The usual notation $\tilde{\sigma}_{\mu\nu} = \frac{1}{2}(\tilde{\gamma}_\mu\tilde{\gamma}_\nu - \tilde{\gamma}_\nu\tilde{\gamma}_\mu)$ is used. The charge conjugation or symplectic matrix $\tilde{C}$ (the algebra $Spin(4,1)$ is equivalent up to irrelevant complexities to $Sp(4)$) satisfies the properties

$$
\tilde{C}^T = -\tilde{C}, \qquad \tilde{C}^2 = -1 \qquad \tilde{C}\tilde{\gamma}_\mu^T = \tilde{\gamma}_\mu\tilde{C}
$$

so that the form $\tilde{C}^{ab}\psi_a\chi_b$ is invariant with respect to Lorentz transformations for arbitrary spinors ψ_a, χ_b. One of the possible explicit choices for $\tilde{\gamma}_\mu$ and $\tilde{C}$ is

$$
\tilde{\gamma}_0 = \mathbb{1}\otimes\sigma_1, \qquad \tilde{\gamma}_{1,2,3} = i\sigma_{1,2,3}\otimes\sigma_2, \qquad \tilde{\gamma}_4 = -i\mathbb{1}\otimes\sigma_3, \qquad \tilde{C} = i\sigma_2\otimes\mathbb{1}\ .
$$

The convention for the auxiliary fields D_{jk} follows Ref. [4] and is such that $D_{12} = D_{21}$ are real and $D_{11} = -D_{22}^*$. Then $D^{jk}D_{jk} < 0$.

lagrangian into the free and interaction parts and construct the S-matrix in a conventional way. Inverting the quadratic part of the lagrangian, we obtain the propagators

$$\begin{aligned} \langle \sigma\sigma \rangle &= \frac{i}{mp^2}\,, \\ \langle A_\mu A_\nu \rangle &= -\frac{i\eta_{\mu\nu}}{mp^2}\,, \\ \langle \psi_j \bar\psi^k \rangle &= \delta_j^k \frac{i\not p}{mp^2}\,. \end{aligned} \tag{3}$$

The mass parameter in the denominator makes the theory nonrenormalizable. In particular, the vacuum expectation value $\langle\sigma\rangle$ does not want to keep its bare value m, but involves power divergent corrections.

The easiest way to find the one-loop correction to $\langle\sigma\rangle$ is to consider the term $\propto D^{jk}D_{jk}$ in the effective lagrangian. Only one graph depicted in Fig. 1 contributes. The calculation gives

$$\Delta\mathcal{L}_{\rm eff} = -\frac{D^{jk}D_{jk}}{m^2}\int \frac{d^5p_E}{(2\pi)^5 p_E^2} \equiv -\frac{D^{jk}D_{jk}}{m^2}\Lambda^3\ . \tag{4}$$

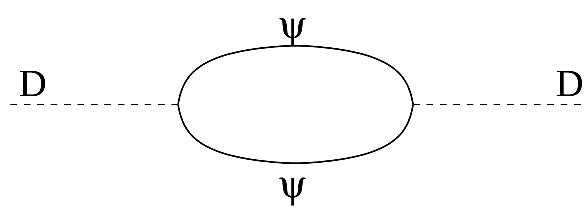

Figure 1:

In other words, the vacuum expectation value acquires a shift involving a cubic ultraviolet divergence

$$\langle\sigma\rangle_{1\ \rm loop} = m + \frac{\Lambda^3}{m^2} \tag{5}$$

The two-loop correction is expected to be of order Λ^5/m^4, which is much larger than $\Lambda^3/m^2 \gg m$, etc. In addition, higher-dimensional operators in the effective lagrangian are generated, which should in turn be added to the tree lagrangian. The situation is the same as in any other nonrenormalizable field theory. Maybe one can handle it, as discussed above, by adding an infinite number of fine-tuned counterterms, but we are not able to say anything more in this respect.

But what if $\langle\sigma\rangle = 0$? In this case, one cannot treat the theory perturbatively whatsoever. This does not mean, however, that the theory does not have a nonperturbative cutoff-independent meaning. It is quite possible that its hamiltonian has well defined ground state and excitations above it. Unfortunately, we do not have the means to tackle a nonlinear quantum field theory problem analytically and the guess above can neither be substantiated not disproved. Let us do few things which we can. To begin with, one can wonder what is the classical dynamics of the lagrangian (1). Let us limit ourselves by the scalar sector. The equations of motion read $\Box\sigma^{3/2} = 0$ with the solution

$$\sigma(x_\mu) \;=\; [\cos(kx+\delta)]^{2/3} \tag{6}$$

with $k^2 = 0$. This is singular at $kx + \delta = \pi/2 + \pi m$.

One can compare this theory to ordinary hydrodynamics. If viscosity is zero, the latter can be cast into lagrangian (or hamiltonian) form, but the quadratic terms in such a hamiltonian (formulated in terms of so called Klebsch variables) are absent. A typical solution to the Euler equations of motion is also singular. Smooth laminar solutions exist, but they are unstable with respect to creation of vortices. The scale of these vortices becomes rapidly very small leading to large values of the velocity gradients etc. In reality, the minimal size of the vortices is controlled by the viscosity playing the role of ultraviolet cutoff. But in the mathematical limit of zero viscosity, classical solutions run into singularity. The latter also happens in simple mechanical systems with strong attractive potential.

In our case, the situation is somewhat worse because in hydrodynamics or in the mechanical system with the potential $V(r) = -\gamma/r^2$, only *some* of the classical trajectories are singular while regular trajectories (laminar motions in hydrodynamics and the trajectories with large enough angular momentum for the problem with $V(r) = -\gamma/r^2$) also exist. In the theory under consideration, *all* classical solutions seem to be singular. In other words, the classical theory with the lagrangian (1) seems to be meaningless.

This is not a final diagnosis yet as classically meaningless (singular) theories *can* acquire meaning when quantized. A classical example is the QM theory with $V(r) = -\gamma/r^2$. If $\gamma < 1/(4m)$, the spectrum of the quantum hamiltonian is well defined in spite of the fact that its classical counterpart involves singular trajectories. Another example was considered in [5] where we showed that a higher-derivative mechanical system involving singular classical trajectories has a well-defined quantum spectrum.

Is it also the case for the theory (1) ?

3 Quantum mechanics.

In this section, we study nonperturbative quantum dynamics of the lagrangian (1) dimensionally reduced in $(0+1)$ dimensions (the only limit when we can do it). The reduced lagrangian involves five dynamical bosonic degrees of freedom (four spatial components of A_μ and σ). After suppressing spatial dependence and excluding auxiliary fields D_{jk} the lagrangian acquires the form

$$L \;=\; \frac{\sigma}{2}\left(\dot{A}_K^2 + \dot{\sigma}^2 + i\psi^{\dagger j}\dot{\psi}_j\right)$$
$$-\frac{i}{4}\psi^{\dagger j}\tilde{\gamma}_K\psi_j\dot{A}_K + \frac{1}{16\sigma}\left(\psi^{\dagger j}\tilde{\gamma}_0\psi^k\right)\left(\psi_j^\dagger\tilde{\gamma}_0\psi_k\right) \tag{7}$$

with $K = 1, 2, 3, 4$. This QM lagrangian involves four complex conserved supercharges in virtue of its 5D origin. It belongs to the class of $\mathcal{N} = 4$ SQM lagrangians suggested in [9] and studied in [10,11]. A generic such lagrangian in component notation is [10] [2]

$$L = \; h\left[\frac{1}{2}\dot{A}_J^2 + \frac{i}{2}\left(\eta^\dagger\dot{\eta} - \dot{\eta}^\dagger\eta\right)\right] - \frac{i}{2}(\partial_K h)\dot{A}_J\eta^\dagger\sigma_{KJ}\eta$$

$$+\frac{1}{24}\left[2\partial_J\partial_K h - \frac{3}{h}(\partial_J h)(\partial_K h)\right]\left[(\eta^\dagger\gamma_J\eta)(\eta^\dagger\gamma_K\eta) - (\eta C\gamma_J\eta)(\eta^\dagger\gamma_K C\eta^\dagger)\right] \;, \qquad (8)$$

where $J = 1, \ldots 5$ with Euclidean hermitian γ_J, and $h(A_J)$ is a 5-dimensional harmonic function, $\partial_J^2 h = 0$. This lagrangian is invariant up to a total derivative with respect to the following SUSY transformations,

$$\begin{aligned}
\delta A_J &= \eta^\dagger\gamma_J\epsilon + \epsilon^\dagger\gamma_J\eta \;, \qquad (9)\\
\delta\eta_\alpha &= -i\dot{A}_J(\gamma_J\epsilon)_\alpha + \frac{\partial_J h}{2h}\eta^\dagger\gamma_J\eta\,\epsilon_\alpha + \frac{\partial_J h}{2h}(\eta C\gamma_J\eta)(\epsilon^\dagger C)_\alpha \;,\\
\delta\eta^{\dagger\beta} &= i\dot{A}_J(\epsilon^\dagger\gamma_J)^\beta + \frac{\partial_J h}{2h}\eta^\dagger\gamma_J\eta\,\epsilon^{\dagger\beta} + \frac{\partial_J h}{2h}(\eta^\dagger\gamma_J C\eta^\dagger)(C\epsilon)^\beta \;.
\end{aligned}$$

It is not difficult to see that (8) goes over into (7) if taking $h(A_J) = A_5$ (obviously, it is harmonic), if identifying

$$A_5 \equiv \sigma\,, \quad \eta \equiv \psi_1\,, \quad \gamma_5 \equiv \tilde{\gamma}_0\,, \quad \gamma_{1,2,3,4} \equiv \tilde{\gamma}_0\tilde{\gamma}_{1,2,3,4}, \quad C \equiv \tilde{\gamma}_0\tilde{C} \qquad (10)$$

and using the property (2) with the corollary $C\eta^\dagger = -\eta^\dagger C \equiv \psi_2$. The metric $ds^2 = \sigma(d\sigma^2 + dA_K^2)$ describes an orbifold of conic variety, though it does not represent a flat cone but has a nontrivial curvature. At the "tip of the cone" ($\sigma = 0$) the curvature is singular, $R \sim 1/\sigma^3$.

The canonic hamiltonian can be derived from (7), (8) by a usual procedure. It is convenient to introduce $\xi = h^{1/2}\eta$ such that $\xi^\dagger$ is canonically conjugate to ξ and the Poisson bracket $\{\xi^\dagger, \xi\}_{\rm PB}$ equals to 1. In a generic case, we have

$$H \;=\; \frac{1}{2h}\left[P_J + \frac{i\partial_K h}{2h}\xi^\dagger\sigma_{KJ}\xi\right]^2$$

$$-\frac{1}{24h^2}\left[2\partial_J\partial_K h - \frac{3}{h}(\partial_J h)(\partial_K h)\right]\left[(\xi^\dagger\gamma_J\xi)(\xi^\dagger\gamma_K\xi) - (\xi C\gamma_J\xi)(\xi^\dagger\gamma_K C\xi^\dagger)\right] \;, \qquad (11)$$

where P_J is the canonic momentum

$$P_J \;=\; h\dot{A}_J - \frac{i}{2}(\partial_K h)\eta^\dagger\sigma_{KJ}\eta \;. \qquad (12)$$

The classical supercharges are obtained in a relatively direct though tedious way (see Appendix for some details) by the standard Nöther procedure. The result is

$$\begin{aligned}
Q_\alpha &= fP_I(\gamma_I\xi)_\alpha + \frac{i}{2}Q_{IJ}(\partial_J f)(\gamma_I\xi)_\alpha - \frac{i}{24}\epsilon_{IJKLM}(\partial_I f)Q_{JK}(\sigma_{LM}\xi)_\alpha \;,\\
Q^{\dagger\beta} &= fP_I(\xi^\dagger\gamma_I)^\beta + \frac{i}{2}Q_{IJ}(\partial_J f)(\xi^\dagger\gamma_I)^\beta + \frac{i}{24}\epsilon_{IJKLM}(\partial_I f)Q_{JK}(\xi^\dagger\sigma_{LM})^\beta \;, \qquad (13)
\end{aligned}$$

[2] We corrected the sign error in the second term in Eq.(2.14) of Ref. [10].

where we introduced $f = 1/\sqrt{h}$, $Q_{IJ} = \xi^\dagger \sigma_{IJ} \xi$. By construction, they satisfy the supersymmetry algebra

$$\{Q_\alpha, Q_\beta\}_{\rm P.B.} = \{Q^{\dagger\alpha}, Q^{\dagger\beta}\}_{\rm P.B.} = 0 \ , \qquad \{Q^{\dagger\beta}, Q_\alpha\}_{\rm PB} = 2\delta_\alpha^\beta H \, . \tag{14}$$

To determine quantum supercharges, one has to resolve ordering ambiguities such that the classical SUSY algebra (14) is kept intact at the quantum level. An universal receipt to do it is to use Weyl ordering for supercharges [12], i.e. to write the quantum operators $\hat{Q}_\alpha$ and $\hat{\bar{Q}}^\beta$ in such a way that their Weyl symbols would coincide with the classical expressions (13).

The proof of this statement is simple. Note that the Weyl symbol of the anticommutator of supercharges coincides with the Moyal bracket of their Weyl symbols. The Moyal bracket introduced in [13] was generalized on the case when the phase space involves not only bosonic (P_J, A_J), but also canonically conjugate fermionic variables $(\xi^{\dagger\alpha}, \xi_\alpha)$ in Ref. [12]. The definition is

$$\begin{aligned} i\{A,B\}_{\rm M.B.} \;=\; & 2\sinh\left\{ \frac{1}{2}\sum_\alpha \left(\frac{\partial^2}{\partial\xi_\alpha^{(2)}\partial\xi^{\dagger\alpha(1)}} - \frac{\partial^2}{\partial\xi_\alpha^{(1)}\partial\xi^{\dagger\alpha(2)}} \right) \right. \\ & \left. -\frac{i}{2}\sum_J \left(\frac{\partial^2}{\partial A_J^{(2)}\partial P_J^{(1)}} - \frac{\partial^2}{\partial A_J^{(1)}\partial P_J^{(2)}} \right) \right\} \\ & A\left(P_J^{(1)}, A_J^{(1)}; \xi^{\dagger\alpha\,(1)}, \xi_\alpha^{(1)}\right) B\left(P_J^{(2)}, A_J^{(2)}; \xi^{\dagger\alpha\,(2)}, \xi_\alpha^{(2)}\right)\Big|_{1=2} \, . \end{aligned} \tag{15}$$

In our case, besides the first term of the expansion of hyperbolic sine giving the Poisson bracket, also the second term of the expansion involving the 6-th order derivatives over fermion variables of the product $Q_\alpha^{(1)} Q^{\dagger\beta\,(2)}$ contributes in $\{Q_\alpha, Q^{\dagger\beta}\}_{\rm M.B.}$ (while $\{Q_\alpha, Q_\beta\}_{\rm M.B.} = \{Q_\alpha, Q_\beta\}_{\rm P.B.} = 0$ and the same for $Q^\dagger$). It is not difficult to see that this extra term is proportional to δ_α^β so that the algebra (14) still holds. The fact that this extra contribution does not vanish means, however, that the Weyl symbol of the quantum hamiltonian (in contrast to Weyl symbol of quantum supercharges) does *not* coincide with the classical expression (11), but has corrections. The situation is the same as in conventional σ models [12].

The Weyl ordered quantum supercharges are

$$\begin{aligned} \hat{Q}_\alpha \;&=\; \hat{P}_I f(\gamma_I\xi)_\alpha + \frac{i}{2}(\partial_J f)(\gamma_I\xi)_\alpha Q_{IJ} - \frac{i}{24}\epsilon_{IJKLM}(\partial_I f)(\sigma_{LM}\xi)_\alpha Q_{JK} \ , \\ \hat{Q}^{\dagger\beta} \;&=\; \hat{P}_I f(\hat{\xi}^\dagger\gamma_I)^\beta + \frac{i}{2}(\partial_J f)(\xi^\dagger\gamma_I)^\beta Q_{IJ} + \frac{i}{24}\epsilon_{IJKLM}(\partial_I f)(\hat{\xi}^\dagger\sigma_{LM})^\beta Q_{JK} \ , \end{aligned} \tag{16}$$

where $\hat{P}_J$ and $\hat{\xi}^{\dagger\alpha}$ are differential operators, $\hat{P}_J = -i\partial_J$, $\hat{\xi}^{\dagger\alpha} = \partial/\partial\xi_\alpha$. The quantum hamiltonian is obtained from the anticommutator $\{\hat{Q}_\alpha, \hat{Q}^{\dagger\beta}\}$. It can be written in the form

$$\begin{aligned} \hat{H} \;=\; & \frac{1}{2} f \hat{P}_I^2 f - i f(\partial_K f)\hat{P}_J(\hat{\xi}^\dagger\sigma_{KJ}\xi) \\ & -\frac{1}{12}\left[6(\partial_J f)(\partial_K f) + f(\partial_J\partial_K f)\right](\hat{\xi}^\dagger\sigma_{JP}\xi)(\hat{\xi}^\dagger\sigma_{KP}\xi) \end{aligned} \tag{17}$$

We are interested in the case $h = A_5 \equiv \sigma$, $f = 1/\sqrt{\sigma}$. The wave functions Ψ may depend on σ, on $A_{1,2,3,4}$, and on holomorphic fermion variables ξ_α.

Let us first find the vacuum states. They should satisfy the conditions

$$\hat{Q}_\alpha \Psi_{\rm vac} \;=\; \hat{Q}^{\dagger\beta}\Psi_{\rm vac} \;=\; 0 \tag{18}$$

The only solutions to this equation system are

$$\Psi_1 = \sqrt{\sigma}, \qquad \Psi_2 = \sqrt{\sigma}\,\xi^4 = \frac{\sqrt{\sigma}}{24}\epsilon^{\alpha\beta\gamma\delta}\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta, \qquad \Psi_3 = \sqrt{\sigma}\,\xi C\xi\,. \tag{19}$$

All these states are bosonic. Neither of them is normalized, however. This could be expected in advance as the allowed range $(0,\infty)$ for the variable σ implies infinite motion and continuous spectrum with not normalizable wave functions. Note that the normalization integral diverges at $\sigma = \infty$, but not at $\sigma = 0$. In other words, the singularity of the moduli space at $\sigma = 0$ does not lead to trouble in this case.

Let us study now excited states. Consider first the sector of zero fermionic charge. In that case, only the first term in the quantum hamiltonian (17) contributes and the eigenvalue equation is

$$-\frac{1}{2\sqrt{\sigma}}\left[\frac{\partial^2}{\partial\sigma^2} + \frac{\partial^2}{\partial A_M^2}\right]\frac{1}{\sqrt{\sigma}}\Psi(\sigma, A_M) \;=\; \lambda\Psi(\sigma, A_M)\;, \tag{20}$$

where $M = 1,2,3,4$. The solutions are $\Psi(\sigma, A_M) = g(\sigma)e^{ik_M A_M}$, with $g(\sigma)$ satisfying

$$-\frac{1}{\sqrt{\sigma}}\left[\frac{\partial^2}{\partial\sigma^2} - k_M^2\right]\frac{g(\sigma)}{\sqrt{\sigma}} = 2\lambda g(\sigma)\;. \tag{21}$$

Mathematically, this is the textbook problem: the Schrödinger equation for the function $g(\sigma)/\sqrt{\sigma}$ in a homogeneous field. The physics is different, however. First, the physical wave function is still $g(\sigma)$ rather than $g(\sigma)/\sqrt{\sigma}$. Second, the spectral parameter λ corresponds not to the energy of the conventional problem, but to the "field strength" $\partial V/\partial\sigma$. Let first $k_M = 0$ and assume the energy λ to be positive. Then a general solution to Eq.(21) is

$$g(\sigma) \;=\; \sigma\left[AJ_{1/3}\left(\frac{2\sqrt{2\lambda}}{3}\sigma^{3/2}\right) + BJ_{-1/3}\left(\frac{2\sqrt{2\lambda}}{3}\sigma^{3/2}\right)\right] \tag{22}$$

It seems to be a benign continuous spectrum wave function. It vanishes at $\sigma = 0$. For any A, B the normalization integral $\int d\sigma |g(\sigma)|^2$ converges at small σ (and diverges for large σ, as it should).

However, in a supersymmetric theory, the states should come in multiplets. As we have four different complex supercharges, the dimension of such multiplets should be $2^4 = 16$ in our case. The states of the multiplet with fermion charge $F = 1,2,3,4$ are obtained by the action of the supercharges on the state with $F = 0$. In particular, the four states of unit fermion charge associated with the state (22) are

$$\Psi_\alpha^{F=1} \;=\; \hat{Q}_\alpha\Psi^{F=0} \;\sim\; (\gamma_5\xi)_\alpha\sigma\left[AJ_{-2/3}\left(\frac{2\sqrt{2\lambda}}{3}\sigma^{3/2}\right) - BJ_{2/3}\left(\frac{2\sqrt{2\lambda}}{3}\sigma^{3/2}\right)\right]\;. \tag{23}$$

The first term in Eq.(23) behaves as a constant at $\sigma = 0$. We will shortly see that non-vanishing of the wave function at the boundary may lead to trouble, but in this particular case it does not. The states are normalizable and admissible, as the states (22) are. Consider now the states $\hat{Q}_\alpha \hat{Q}_\beta \Psi^{F=0}$ in the sector $F = 2$. The state $\propto A$ in Eq. (22) leads to the following six states,

$$\Psi^{F=2} \sim (\xi C \xi)\, \sigma J_{1/3}\left(\frac{2\sqrt{2\lambda}}{3}\sigma^{3/2}\right); \quad \sim (\xi C \gamma_M \xi)\, \sigma J_{1/3}\left(\frac{2\sqrt{2\lambda}}{3}\sigma^{3/2}\right); \\ (\xi C \gamma_5 \xi)\, \sigma J_{-5/3}\left(\frac{2\sqrt{2\lambda}}{3}\sigma^{3/2}\right) . \tag{24}$$

and the state $\propto B$ leads to the states of the same structure, but with the sign of the Bessel indices reversed. We observe that five of six states in (24) are benign, but the sixth $\propto \xi C \gamma^5 \xi$ behaves as $\sigma^{-3/2}$ at small σ, is not renormalizable there, and not admissible by that reason. But if we want to keep supersymmetry, we cannot reject only one state in the supersymmetric 16-plet. We should also throw fifteen others away ! [3]

On the other hand, all the states stemming from the state $\sim \sigma J_{-1/3}(z)$ in (22) — the states

$$\begin{aligned} F=0: &\quad \sim \sigma J_{-1/3}(z); \\ F=1: &\quad \sim (\gamma_5\xi)_\alpha\, \sigma J_{2/3}(z); \\ F=2: &\quad \sim (\xi C \xi)\, \sigma J_{-1/3}(z), \quad \sim (\xi C \gamma_M \xi)\, \sigma J_{-1/3}(z), \quad \sim (\xi C \gamma_5 \xi)\, \sigma J_{5/3}(z); \\ F=3: &\quad \sim (\ \gamma_5)^\alpha_\beta \epsilon^{\beta\gamma\delta\epsilon} \xi_\gamma \xi_\delta \xi_\epsilon\, \sigma J_{2/3}(z); \\ F=4: &\quad \sim \xi^4\, \sigma J_{-1/3}(z) . \end{aligned} \tag{25}$$

($z = 2\sqrt{2\lambda}\,\sigma^{3/2}/3$), obtained from each other by the action of supercharges (16) are admissible and are present in the supersymmetric spectrum of our system.

What happens if λ is negative ? If limiting ourselves with the sector $F = 0$, one obtains the solutions

$$g(\sigma) \sim \sigma K_{1/3}\left(\frac{2\sqrt{-2\lambda}}{3}\sigma^{3/2}\right) . \tag{26}$$

Being normalizable not only at $\sigma = 0$, but also at infinity, they seem to be quite respectable. But the presense of the states with negative energy is in obvious contradiction with supersymmetry. Indeed, there are at least two reasons to reject them.

1. First, recall the standard proof that the energies of all the states in a supersymmetric system are positive or zero. We have

$$\langle \Psi | \hat{H} | \Psi \rangle = \langle \hat{Q}\Psi | \hat{Q}\Psi \rangle + \langle \hat{Q}^\dagger \Psi | \hat{Q}^\dagger \Psi \rangle \geq 0 . \tag{27}$$

[3] The observation that, when defining Hilbert space in SUSY theories, one has to filter out not only the "bad" (nonrenormalizable, not gauge-invariant etc.) states, but also the superpartners of such states even though these superpartners look benign by themselves was made in [14]. We refer the reader to that paper for more examples and discussion.

But this is based on the assumption that $\hat{Q}^\dagger$ is adjoint to $\hat{Q}$. One can note, however, that this property does not hold if including in the spectrum the state (26). Indeed, only the first term in the expression (16) for $\hat{Q}_\alpha$ acts nontrivially on the states in the sector $F = 0$. Disregarding the irrelevant spinor structure, consider the operators

$$\hat{S} = \frac{1}{\sqrt{2}}\partial_J f, \qquad \hat{S}^\dagger = -\frac{1}{\sqrt{2}} f \partial_J; \qquad \hat{H}' = \hat{S}^\dagger \hat{S} \; . \tag{28}$$

Then the identity

$$\langle g|\hat{H}'|g\rangle = -\frac{1}{2}\int_0^\infty d\sigma \, g(\sigma) \frac{1}{\sqrt{\sigma}} \frac{d^2}{d\sigma^2} \frac{g(\sigma)}{\sqrt{\sigma}} = \int_0^\infty d\sigma \, [\hat{S} g(\sigma))]^2 > 0 \tag{29}$$

would be correct if the boundary term

$$\frac{1}{2} \frac{g(\sigma)}{\sqrt{\sigma}} \frac{d}{d\sigma} \left(\frac{g(\sigma)}{\sqrt{\sigma}} \right) \Bigg|_{\sigma=0} \tag{30}$$

vanished. But it does not ! One can be convinced that the contribution (30) to $\langle g|\hat{H}'|g\rangle$ is negative, which overcomes the positive contribution of the R.H.S. of Eq.(29) and allows for the eigenvalue of $\hat{H}'$ to be negative.

Thus, if we want to keep $\hat{Q}_\alpha$ and $\hat{Q}^{\dagger\beta}$ conjugate to each other and $\hat{H}$ hermitian, the state (26) should be rejected.

2. The states in the sector $F = 2$ derived from the state (26) by the action of supercharges include the state $\sim (\xi C \gamma_5 \xi) \, \sigma K_{5/3}(2\sqrt{-2\lambda}\,\sigma^{3/2}/3)$, which is not normalizable at the origin and not admissible. Hence, the whole lame would-be multiplet should be rejected, as discussed above.

Up to now we have only discussed the states with $k_M = 0$. Let us consider now the case of nonzero k_M. Let for definiteness $k_1 \equiv k \neq 0, k_{2,3,4} = 0$. The solutions with negative λ should surely be rejected by the argument 1 above. If λ is positive, the solutions to the equation (21) change their nature at the point $\sigma_0 = k^2/2\lambda$, they represent oscillatory Bessel functions for $\sigma \geq \sigma_0$ and modified Bessel functions (related to Airy functions) at $0 \leq \sigma \leq \sigma_0$. To find whether a state is admissible or not, we have to study the behaviour of the solution at the vicinity of $\sigma = 0$ where a generic solution to Eq. (21) has the form

$$\begin{aligned}\Psi^{F=0} \equiv g(\sigma) \;&=\; \sqrt{\sigma}\left[A\sqrt{\sigma_0 - \sigma}\, I_{1/3}\left(\frac{2\sqrt{2\lambda}}{3} (\sigma_0 - \sigma)^{3/2} \right) \right.\\ &\left. + B\sqrt{\sigma_0 - \sigma} I_{-1/3}\left(\frac{2\sqrt{2\lambda}}{3} (\sigma_0 - \sigma)^{3/2} \right) \right] \; .\end{aligned} \tag{31}$$

The superpartners of this state in the sector $F = 1$ are

$$\Psi_\alpha^{F=1} \;=\; \hat{Q}_\alpha \Psi^{F=0} \;=\; \frac{k g(\sigma)}{\sqrt{\sigma}} (\gamma_1 \xi)_\alpha - i \frac{d}{d\sigma}\left(\frac{g(\sigma)}{\sqrt{\sigma}} \right) (\gamma_5 \xi)_\alpha \tag{32}$$

and

$$\Psi^{F=2}_{\alpha\beta} = \hat{Q}_\alpha \hat{Q}_\beta \Psi^{F=0} = \frac{k^2 g(\sigma)}{\sigma} (\gamma_1 \xi)_\alpha (\gamma_1 \xi)_\beta - \frac{d}{d\sigma} \left(\frac{1}{\sqrt{\sigma}} \frac{d}{d\sigma} \frac{g(\sigma)}{\sqrt{\sigma}} \right) (\gamma_5 \xi)_\alpha (\gamma_5 \xi)_\beta$$
$$- \frac{1}{4\sigma^{3/2}} \frac{d}{d\sigma} \left(\frac{g(\sigma)}{\sqrt{\sigma}} \right) (\gamma_M \xi)_\alpha (\gamma_M \xi)_\beta + \frac{1}{24\sigma^{3/2}} \frac{d}{d\sigma} \left(\frac{g(\sigma)}{\sqrt{\sigma}} \right) (\sigma_{MN} \xi)_\alpha (\sigma_{MN} \xi)_\beta \,. \quad (33)$$

(the summation runs over $M, N = 1, 2, 3, 4$). For generic A, B, the projection of the wave function (33) on the structure $\sim \xi C \gamma_5 \xi$ behaves as $\sigma^{-3/2}$ at small σ and is not normalizable. But for one particular choice of the ratio A/B, this leading singularity vanishes and the wave function behaves at the origin as $\sim \sigma^{-1/2}$, which is "almost normalizable". The members of this supermultiplet with fermion charge $F = 3$ and $F = 4$ can be obtained by duality transformation from the states in the sectors $F = 1$ and $F = 0$, respectively and have the same σ dependence. This is best seen by noting that the states (33) can be alternatively obtained by acting with $\hat{Q}^{\dagger\alpha} \hat{Q}^{\dagger\beta}$ on the state $g(\sigma)\xi^4$.

To include in the supersymmetric spectrum the multiplets involving the states, where the normalization integral diverges logarithmically at the origin, or to reject them is a matter of taste and convention. We believe that it is more natural to include them. In this case, the full spectrum involves the 16-plets labelled by five quantum numbers $(\lambda \geq 0, k_M)$. This corresponds to the presence of five bosonic dynamic variables in our system.

4 Discussion

We can return now to the question posed at the end of Sect. 2. As far as the dimensionally reduced system (7) is concerned, the answer is definitely positive — in spite of the presense of singularity at $\sigma = 0$, the *supersymmetric* spectrum of the theory is nontrivial and bonded by zero from below, as it should. In other words, the situation is similar, indeed, to the QM problem with potential $\sim -\gamma/r^2$ for small γ. Though classical trajectories are singular, the quantum problem is well defined. We would *not* be able, however, to define the quantum problem in this case without invoking supersymmetry. Ghost states having unbounded negative energies do exist as solutions of the Schrödinger equation, and only the fact that they do not have normalizable superpartners, allows one to reject them.

Maybe this is the most important observation of the paper. We suggest that the same is true in the TOE (representing, according to our hypothesis [1], a field theory in higher dimensions with higher derivatives). One can speculate that, though ghost states associated with higher derivatives are formally present in the spectrum, the *physical* Hilbert space $\mathcal{H}$ of such a theory involves only supermultiplets of the states with positive energies. Now, $\mathcal{H}$ should be closed such that when acting by physically admissible operators on any state $\Psi \in \mathcal{H}$, we stay within $\mathcal{H}$ (in more physical language, ghosts are not created in collisions of usual particles). We tend to prefer the mechanism of getting rid of the ghosts discovered in this paper to the mechanism unravelled in [5] — a QM higher-derivative model considered there was not supersymmetric and the bottom in the spectrum appeared by the reasons not related to supersymmetry. The arguments based on supersymmetry have more aesthetic appeal and have the advantage of being *universal*. On the other hand, the ghost-free QM system considered in [5] has one important common feature with the system (7). In both theories, the spectrum is essentially nonperturbative. Obviously, further studies of this question are necessary.

We can conjecture that the hamiltonian has a well defined nonperturbative spectrum not only for the reduced system, but also for the field theory (1) in the sector with zero scalar expectation value $\langle \sigma \rangle = 0$. As this spectrum is essentially not perturbative and we do not have analytic tools to study field theory nonperturbatively, we cannot say much about the nature of this spectrum and about the quantum dynamics of this theory. The only thing that we still can suggest (based on the absence of the scale parameter in the lagrangian) is that this dynamics *is* nontrivial and stays nontrivial in continuum limit. On can recall in this respect $2D$ conformal theories. Like the theory (1), they do not have a scale parameter. They do not have well defined asymptotic states and the scattering matrix cannot be defined (on the other hand, one can define S-matrix for *deformed* conformal theories, like the Sine-Gordon model or Ising model at critical temperature and nonzero magnetic field [15]). $2D$ conformal theories are known to have nontrivial interesting dynamics. We do not see reasons why the $5D$ conformal theory considered in this paper should not have one.

I am indebted to E. Ivanov for fruitful discussions.

Appendix

For references purposes, we will give here some useful formulae referring to the component formulation of the generic DE model. The Euclidean γ matrices satisfy

$$\gamma_I^\dagger = \gamma_I, \qquad \gamma_I \gamma_J + \gamma_J \gamma_I = 2\delta_{IJ}\,, \qquad C\gamma_I^T = \gamma_I C\,.$$

The matrices C and $\Gamma_I = C\gamma_I$ are antisymmetric while $\Gamma_{IJ} = C\sigma_{IJ}$ are symmetric in the spinor indices. One of the possible explicit representations is

$$\gamma_{1,2,3} = \sigma_{1,2,3} \otimes \sigma_3\,, \quad \gamma_4 = 1\!\!1 \otimes \sigma_1\,, \quad \gamma_5 = 1\!\!1 \otimes \sigma_1\,, \quad C = i\sigma_2 \otimes \sigma_1\,.$$

Let us write the lagrangian in the form analogous to (1) so that the $SU(2)$ R-symmetry of the model would be explicitly seen. We have

$$\begin{aligned} L \;=\; & \frac{h}{2}\left(\dot{A}_I^2 - i\psi^k C\dot{\psi}_k\right) + \frac{i}{4}(\partial_J h)\dot{A}_I \psi^k \Gamma_{JI}\psi_k - \frac{h}{4}D^{jk}D_{jk} + \\ & \frac{1}{4}D^{jk}(\partial_I h)\psi_j \Gamma_I \psi_k - \frac{1}{24}(\partial_I\partial_J h)(\psi^j \Gamma_I \psi^k)(\psi_j \Gamma_J \psi_k)\,. \end{aligned} \tag{34}$$

This lagrangian is unvariant up to a total derivative with respect to the SUSY transformations

$$\begin{aligned} \delta A_I &= E^k \Gamma_I \psi_k\,, \\ \delta\psi_j &= -i\dot{A}_I \gamma_I E_j + D_{jk}E^k\,, \\ \delta D_{jk} &= i\dot{\psi}_j C E_k + i\dot{\psi}_k C E_j\,. \end{aligned} \tag{35}$$

Expressing out the auxiliary fields and using the substitutions (10), we reproduce the result (8) above.

Finding *the* total derivative in δL represents a technically most difficult part in deriving the expression for the supercharges Q^j. It can still be done by using various Fierc identities, for example

$$(\psi^j \Gamma_{(I} \psi^k)\, (\psi_j \Gamma_{J)} \psi_k) \;=\; \frac{1}{2} (\psi^j \Gamma_{IK} \psi_j)\, \psi^k \Gamma_{JK} \psi_k) - \frac{1}{8} \delta_{IJ} (\psi^j \Gamma_{KL} \psi_j)^2 \;, \tag{36}$$

the harmonicity of h, and the useful identity

$$\epsilon_{IJKLN} S_{MN} + \epsilon_{JKLMN} S_{IN} + \epsilon_{KLMIN} S_{JN} + \epsilon_{LMIJN} S_{KN} + \epsilon_{MIJKN} S_{LN} = \epsilon_{IJKLM} S_{NN} \tag{37}$$

valid for any tensor S_{IJ}. We obtain finally (note that the contributions involving auxiliary fields cancel)

$$Q^j_\alpha \;=\; P_I (\gamma_I \psi^j)_\alpha - \frac{i\partial_J h}{8} (\psi^k \Gamma_{IJ} \psi_k)\, (\gamma_I \psi^j)_\alpha + \epsilon_{IJKLM} \frac{i\partial_I h}{96} (\psi^k \Gamma_{JK} \psi_k)\, (\sigma_{LM} \psi^j)_\alpha \;, \tag{38}$$

which gives (13).

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