AE-335
UDC 517.94

AE-335

Solution of Large Systems of Linear Equations in the Presence of Errors. A Constructive Criticism of the Least Squares Method

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AKTIEBOLAGET ATOMENERGI

STOCKHOLM, SWEDEN 1968

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by

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ABSTRACT

From the point of view that no mathematical method can ever minimise or alter errors already made in a physical measurement, the classical least squares method has severe limitations which makes it unsuitable for the statistical analysis of many physical measurements. Based on the assumptions that the experimental errors are characteristic for each single experiment and that the errors must be properly estimated rather than minimised, a new method for solving large systems of linear equations is developed. The new method exposes the entire range of possible solutions before the decision is taken which of the possible solutions should be chosen as a representative one. The choice is based on physical considerations which (in two examples, curve fitting and unfolding of a spectrum) are presented in such a form that a computer is able to make the decision. A description of the computation is given. The method described is a tool for removing uncertainties due to conventional mathematical formulations (zero determinant, linear dependence) and which are not inherent in the physical problem as such. The method is therefore especially well fitted for unfolding of spectra.

Printed and distributed in October 1968.

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INTRODUCTION

The classical method of least squares was published by Legendre in 1806 and by Gauss in 1809. As Gauss invented the method as early as 1794-1795 it is normally ascribed to him even if Legendre was the first to publish it.

The method was originally intended to be used for obtaining a unique solution of linearised equations arising from measurements of geodetic nets and astronomical observations. Without exaggeration one can say that during the years the method has served its purpose well; the use of it has been extended to other areas of physics, and many physicists just regard the least squares method as a useful tool when dealing with overdetermined mathematical models. The method is normally considered to be so well known that it is hardly worth discussion, but the fact that it gives non-acceptable solutions in a number of different situations where it is expected to work properly calls for attention. The present paper is a discussion of the limitations of the least squares method and the presentation of a new method for solution of linear equations which overcomes the limitations while still including the least squares method as a special case. The method is based on the thought that linear equations arising from physical experiment due to the experimental errors define a range of possible solutions rather than one single solution. By transforming the original system into a triangular system the range of solutions is clearly exposed and one can, within the range, search for the solutions which fit ones purpose. The present paper at first gives a short description of the least squares method in the matrix language, thereafter the formation of the triangular matrix is shown and a description of the prac-

tical calculation is given in an appendix. The solution of the triangular system or the search for a suitable solution is shown by means of two examples, a curve fitting problem and the unfolding of a spectrum.

I. THE LEAST SQUARES METHOD

As the basis for a discussion of the least squares method we will recapitulate it in the matrix language. The representation and notation follows essentially E. Andersen [1]. Matrices are denoted by underlined capital letters and vectors by underlined small letters. Transponation is denoted by ^{T}.

We wish to determine a series of parameters y_1 through y_n by measuring another series x_1 through x_m. At our disposal we have some condition equations which give the connection between the y's and the x's. We define

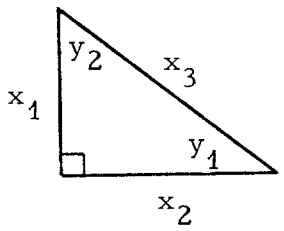
$$\underline{y} = \begin{Bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{Bmatrix} \tag{1}$$

and

$$\underline{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{Bmatrix} \tag{2}$$

The condition equations are divided in two groups, the first group consists of the equations, $f_i(\underline{x}, \underline{y}) = 0$, which contain both x's and y's and the second group consists of the equations, $h_j(\underline{y}) = 0$, which contain y's only.

As an example we might want to determine the angles of a right angled triangle by measuring its sides. The angles are y_1, y_2 and the sides are x_1, x_2, x_3 as shown in the figure.



A set of condition equations is obviously

$$f_1(\underline{x}, \underline{y}) \equiv x_3 \cos y_1 - x_2 = 0$$
$$f_2(\underline{x}, \underline{y}) \equiv x_3 \cos y_2 - x_1 = 0$$
$$h_1(\underline{y}) \equiv y_1 + y_2 - 90^{\circ} = 0$$

We assume there are k functions of the type $f_i(\underline{x}, \underline{y}) = 0$ and ℓ functions of the type $h_j(\underline{y}) = 0$.

A measurement of any x_i will be in error to some degree and we will consider x_i as a sum of the measured value x_{oi} and an unknown error v_i

$$x_i = x_{oi} + v_i \tag{4}$$

or in vector notation

$$\underline{x} = \underline{x}_o + \underline{v} \tag{5}$$

We assume that we know an approximative value $\underline{y}_o$ of the wanted $\underline{y}$ and that $\underline{y}$ can be obtained as the sum of $\underline{y}_o$ and a correction $\underline{z}$ which is to be found. One has

$$y_i = y_{oi} + z_i \tag{6}$$

or

$$\underline{y} = \underline{y}_o + \underline{z} \tag{7}$$

Now the functions $f_i(\underline{x}, \underline{y})$ and $h_j(y)$ are linearised by means of a Taylor expansion around $\underline{x}_o$, $\underline{y}_o$. The elements of $\underline{v}$ and $\underline{z}$ are assumed to be so small that the zero order and the first order terms of the expansion describe the system with sufficient accuracy.

To simplify the notations we introduce

$$\underset{(k,m)}{A} = \{a_{ij}\} \quad , \quad a_{ij} = \left(\frac{\delta f_i}{\delta x_j} \right)_o \text{ at } \underline{x} = \underline{x}_o \text{ and } \underline{y} = \underline{y}_o \tag{8}$$

$$\underset{(k,n)}{B} = \{b_{ij}\} \quad , \quad b_{ij} = \left(\frac{\delta f_i}{\delta y_j} \right)_o \quad \text{- " -} \tag{9}$$

$$\underset{(k,1)}{d} = \{d_i\} \quad , \quad d_i = \left(f_i \right)_o \quad \text{- " -} \tag{10}$$

$$\underset{(\ell,n)}{C} = \{c_{ij}\} \quad , \quad c_{ij} = \left(\frac{\delta h_j}{\delta y_j} \right)_o \quad \text{at } \underline{y} = \underline{y}_o \tag{11}$$

$$\underset{(\ell,1)}{e} = \{e_j\} \quad , \quad e_j = \left(h_j \right)_o \quad \text{- " -} \tag{12}$$

and the Taylor expansion of the f_i and h_j functions can be written

$$\{f_i(\underline{x}, \underline{y})\} = \underline{d} + \underline{A}\underline{v} + \underline{B}\underline{z} = \underline{0} \tag{13}$$

$$\{h_j(\underline{y})\} = \underline{e} + \underline{C}\underline{z} = \underline{0} \tag{14}$$

The task is now to find $\underline{v}$ and $\underline{z}$ and hence $\underline{x}$ and $\underline{y}$. Provided the equations (14) do not contain any intrinsic contradiction we assume, despite the uncertainty introduced by the errors, the existence of a representative solution. The least squares method chooses as a representative value of $\underline{v}$ and $\underline{z}$ the one which makes $\Sigma p_i v_i^2 = \min$. Here p_i is a weight factor > 0 inversely proportional to the expected (mean) square value of v_i.

The pair of $\underline{v}$ and $\underline{z}$ which fit into Eqs. (13) and (14) and grant the minimum condition can be found from a linear system of equations with $k + \ell + m + n$ equations and just as many unknowns. This system is found as follows. First we introduce

$$\underline{P} = \begin{Bmatrix} p_1 & & & \\ & p_2 & & \underline{0} \\ & & \ddots & \\ & \underline{0} & & p_m \end{Bmatrix} \tag{15}$$

and write the minimum condition as

$$\underline{v}^T \underline{P}\, \underline{v} = \min.$$

or

$$\underline{v}^T \underline{P}\, d\underline{v} = 0 \tag{16}$$

From (13) we get

$$\underline{A}\, d\underline{v} + \underline{B}\, d\underline{z} = 0 \tag{17}$$

and from (14)

$$\underline{C}\ \underline{dz} = 0 \tag{18}$$

By introducing the Lagrangeian multipliers

$$\underline{q} = \left\{ \begin{array}{l} q_1 \\ q_2 \\ . \\ . \\ . \\ q_k \end{array} \right\} \tag{19}$$

and

$$\underline{r} = \left\{ \begin{array}{l} r_1 \\ r_2 \\ . \\ . \\ . \\ r_\ell \end{array} \right\} \tag{20}$$

we get from (16), (17) and (18)

$$\underline{v}^T\ \underline{P}\ \underline{dv} = \underline{q}^T(\underline{A}\ \underline{dv} + \underline{B}\ \underline{dz}) + \underline{r}^T(\underline{C}\ \underline{dz}) = 0 \tag{21}$$

The equation (21) is valid for any value of $\underline{q}$ and $\underline{r}$ when (13) and (14) and hence (17) and (18) are granted. Equation (21) rewritten becomes

$$(\underline{v}^T\ \underline{P} - \underline{q}^T\ \underline{A})\ \underline{dv} - (\underline{q}^T\ \underline{B} + \underline{r}^T\ \underline{C})\ \underline{dz} = 0 \tag{22}$$

From (22) it is evident that for

$$\underline{v}^T\ \underline{P} - \underline{q}^T\ \underline{A} = 0 \tag{23}$$

and

$$\underline{q}^T \underline{B} + \underline{r}^T \underline{C} = 0 \qquad (24)$$

the minimum condition is granted automatically. Thus (23), (24), (13) and (14) forms a set of $k + \ell + m + n$ equations with just as many unknowns and it can be written in a single notation

$$\begin{array}{l}(23)\\(13)\\(24)\\(14)\end{array} \left\{\begin{array}{cccc} -\underline{P} & \underline{A}^T & \underline{0} & \underline{0} \\ \underline{A} & \underline{0} & \underline{B} & \underline{0} \\ \underline{0} & \underline{B}^T & \underline{0} & \underline{C}^T \\ \underline{0} & \underline{0} & \underline{C} & \underline{0} \end{array}\right\} \left\{\begin{array}{c} \underline{v} \\ \underline{q} \\ \underline{z} \\ \underline{r} \end{array}\right\} = \left\{\begin{array}{c} \underline{0} \\ -\underline{d} \\ \underline{0} \\ -\underline{e} \end{array}\right\} \qquad (25)$$

The order of the equations and the unknowns is chosen so as to make a symmetrical matrix of (25) which is the starting position for the numerical evaluation of $\underline{v}$ and $\underline{z}$ using the least squares method.

II. DISCUSSION OF THE LEAST SQUARES METHOD

The equations (25) represent the least squares method in a general form. In most practical cases it can be simplified by elimination of non-interesting parameters or divided into smaller independent systems which are more easily accessible for computation. When using a computer for the solution of (25) we will always get a solution because the probability of obtaining a determinant whose value is exactly zero is so small that it can be disregarded for all practical purposes. The question which arises is whether the solution obtained by means of the least squares method is representative for the proper solution (assumed to exist) or not. By a representative solution we mean a solution which in a satisfactory manner can be used for a certain purpose instead of the proper solution (which we don't know). Generally formulated the definition seems vague, but in specific cases it becomes more

rigorous. Consider for example the two vectors $\{0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\}$ and $\{0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\}$. To which degree the first is a good representation of the other depends on the use to be made of them. For some purposes they can be considered as almost equal and for others they may be regarded as having nothing to do with each other. The question whether a given solution is representative or not must be judged in each specific case according to specific criteria.

As a basis for the discussion of using the least squares criterion for the judgement we will use a specific case commonly occurring within physics. Suppose we want to determine n parameters y_1 through y_n. At our disposal we have m measurements x_{o1} through x_{om} and m linear or linearised equations giving the connection between the wanted parameters and the measurements. We assume that this connection takes the following simple form

$$\underline{B}\,\underline{y} = \underline{x}_o + \underline{v} \qquad (26)$$

where $\underline{v}$ is the error vector to be added to $\underline{x}_o$ in order to obtain the proper value of the observed $\underline{x}$.

For any vector $\underline{y}$ we can find a corresponding $\underline{v}$ by means of (26) such that $\underline{y}$ and $\underline{v}$ fit into the equation. So, with no expectations whatsoever on $\underline{y}$ or $\underline{v}$ any n dimensional vector $\underline{y}$ is a solution. However, in all practical cases we have some further information (knowledge), not necessarily in a mathematical form, which can be used to limit the range of possible solutions to (26). This information stems from experiences from similar experiments and/or from repetitions of the experiment which lead to (26). Information which is always available is the expected (mean) square value of any v_i which we denote $p_i^{-1}\mu^2$ or

$$\overline{v_i^2} = p_i^{-1}\mu^2 \tag{27}$$

where μ^2 is called the mean square error on the weight unit and p_i the weight on the observation number i and also, in this particular case, on the i'th equations. The knowledge of the mean square errors, or rather the knowledge that it is very unlikely that any v_i is much larger that $(\overline{v_i^2})^{1/2}$ limits the range of possible solutions. If the solutions within this range are almost equal in a sense defined by the user it is reasonable to pick one of them as representative. The use of the least squares method for that purpose saves us from the unpleasant job of choosing between almost equal possibilities, but we might run into other troubles.

The weighted square error sum $\Sigma p_s v_s^2$ is known in the mean to be

$$\overline{\Sigma p_i v_i^2} = \Sigma p_i \overline{v_i^2} = m \cdot \mu^2$$

where m is the number of equations in (26). If m is large the proper weighted square error sum will be close to its mean whereas the minimum value $\Sigma p_i v_i^2$, as found by means of the least squares method (always equal to zero for n = m), might be so small that it is in conflict with the known mean. In this situation it can be more adequate to search for a representative solution amongst those solutions which, when inserted in (26), delivers a value of $\Sigma p_i v_i^2$ being equal to or nearly equal to its known mean value rather than use the solutions which give rise to an unlikely small minimum value of $\Sigma p_i v_i^2$.

The question is simply if all solutions, giving rise to a value of $\Sigma p_i v_i^2$ being approximately equal to its mean value or smaller, are nearly equal in the user's definition. If so, the least squares method is a practical tool to obtain a unique representative solution. If not, the so-

lution is not sufficiently well defined by means of (26) and the knowledge of expected errors and therefore further information is necessary in order to find a representative solution. The best thing to do, if it is possible, is to plan another experiment which in a more direct way measures the wanted $\underline{y}$, but we will assume that we cannot get further in this way. The only way to find the desired $\underline{y}$ is to solve (26). If (26) is solved for $\underline{y}$ according to a predescribed method we get a solution which we might accept or reject as being physically impossible or unlikely. If the solution is accepted there are no problems, but if it is rejected it means that we have reasons for the rejection and that means again that we possess some information which goes beyond what is expressed by (26) and by our knowledge of likely errors. Naturally this information ought to be built into the method used such that we don't get solutions which we have reason to reject.

Before we arrive at the description of a method which to a high degree is able to utilise information which is almost impossible to express in a stringent mathematical manner, we will briefly turn our attention to the errors stemming from the computation itself - truncation errors. Such errors should be kept so small that they have no significant influence on the result of the calculations. If this is not possible we ought at least to try to carry out the calculations in such a way that we can see how and where the influence makes itself perceptible. The following example shows the necessity of being aware of this point. Suppose (26) is given by

$$\begin{aligned} 1.00\, y_1 + 0.90\, y_2 &= 1.00 \\ 0.90\, y_1 + 1.00\, y_2 &= 0.90 \end{aligned} \qquad (28)$$

The system (28) is easily solved with the assumption that the experimental errors are zero. Truncation errors up to 0.01 can be tolerated in any of the elements of $\underline{B}$ and $\underline{x}$ without "significant" influence of the solution. However, if we solve the system by first producing the so called normal equations, that means that we from $\underline{B}\,\underline{y} = \underline{x}$ produce $\underline{B}^T\underline{B}\underline{y} = \underline{B}^T\underline{x}$, we get from (28)

$$\begin{aligned} 1.81\, y_1 + 1.80\, y_2 &= 1.81 \\ 1.80\, y_1 + 1.81\, y_2 &= 1.80 \end{aligned} \tag{29}$$

Here in the system of normal equations (29), a truncation error of 0.01 in some of the elements might completely violate the solution by giving a zero determinant despite the larger elements. The loss of accuracy which is generally associated with the use of normal equations increases with increasing size of the system and the example shown may be taken as a warning for using the normal equations for the solution of larger systems of linear equations. In the next chapter another method is shown which does not suffer from this bad effect while still being able to operate on systems with rectangular matrices, that means that the number of equations needs not be equal to the number of parameters.

III. FORMATION OF THE TRIANGULAR MATRIX

To avoid complexity we will look at the simple system given by (26)

$$\underline{B}\,\underline{y} = \underline{x}_o + \underline{v} \tag{30}$$

where $\underline{y}$ is a wanted n-dimensional vector, $\underline{x}_o$ a measured m-dimensional vector and $\underline{v}$ the unknown errors in $\underline{x}_o$. $\underline{B}$ is an m n dimensional matrix with known elements. It is assumed that we know the expected

(mean) square error $\overline{v_s^2}$ of each equation. We further have some more or less vague expectations about the solution. Using all available information we want to find a representative solution in the sense described previously. These readers who prefer to see the calculation before the theoretical analysis are referred to a description of the practical computation of the triangle matrix in the appendix.

It is natural to multiply each equation in (30) by a constant so as to get a new system of equations which has equal expected square errors μ^2 on each equation. According to (27) the constant becomes $p_i^{-1/2}$. This operation is trivial and we will assume that it has been carried out without changing notation; it is equivalent with the assumption that $\overline{v_i^2} = \mu^2$ for any i, which is used in the following.

First $\underline{B}$ is considered to consist of the column vectors $\underline{b}_i$ where

$$\underline{b}_i = \begin{Bmatrix} b_{1i} \\ b_{2i} \\ . \\ . \\ . \\ b_{mi} \end{Bmatrix} \tag{31}$$

The equations (30) can then be written

$$\sum_{i=1}^{n} y_i \underline{b}_i = \underline{x}_o + \underline{v} = \underline{x} \tag{32}$$

Now each vector $\underline{b}_i$ is split into two vectors, one $t_{1i} \underline{b}_{o1}$ being proportional to $\underline{b}_1$ and one $\underline{b}_{1i}$ being orthogonal to $\underline{b}_1$ or

$$\underline{b}_i = t_{1i} \underline{b}_{o1} + \underline{b}_{1i} \tag{33}$$

By putting

$$\underline{b}_{o1} = \underline{b}_1 \tag{34}$$

we get

$$t_{1i} = \frac{\underline{b}_1^T \underline{b}_i}{\underline{b}_1^T \underline{b}_1} \tag{35}$$

and

$$\underline{b}_{1i} = \underline{b}_i - t_{1i} \underline{b}_1 \tag{36}$$

Also $\underline{x}$ is split into two orthogonal vectors

$$\underline{x} = t_{1x} \underline{b}_{o1} + \underline{b}_{1x} \tag{37}$$

We note that

$$\underline{b}_{11} = \underline{0} \tag{38}$$

and

$$t_{11} = 1 \tag{39}$$

Now (32) can be written as

$$(\sum_{i=1}^{n} t_{1i} y_i) \underline{b}_{o1} + \sum_{i=2}^{n} y_i \underline{b}_{1i} = t_{1x} \underline{b}_{o1} + \underline{b}_{1x} \tag{40}$$

where $\underline{b}_{o1}$ is orthogonal to all other vectors in the equation (40). Now all vectors $\underline{b}_{1i}$ are split into two vectors, one $t_{2i} \underline{b}_{o2}$ being proportional to $\underline{b}_{12}$ and one $\underline{b}_{2i}$ being orthogonal to $\underline{b}_{12}$. t_{2i} and $\underline{b}_{2i}$ are found in analogy with (35) and (36).

By putting

$$\underline{b}_{o2} = \underline{b}_{12} \tag{41}$$

the equation (32) alias (40) can be written

$$(\sum_{i=1}^{n} t_{1i}\, y_i)\, \underline{b}_{o1} + (\sum_{i=2}^{n} t_{2i}\, y_i)\, \underline{b}_{o2} + \sum_{i=3}^{n} y_i\, \underline{b}_{2i} = t_{1x}\, \underline{b}_{o1} +$$

$$+ t_{2x}\, \underline{b}_{o2} + \underline{b}_{2x} \qquad (42)$$

where both $\underline{b}_{o1}$ and $\underline{b}_{o2}$ are orthogonal to any other vector in the equation (42). Now all vectors $\underline{b}_{2i}$ are split into two vectors $t_{3i}\, \underline{b}_{o3} + \underline{b}_{3i}$ where

$$\underline{b}_{o3} = \underline{b}_{23} \qquad (43)$$

and so on, so that the end result becomes

$$(\sum_{i=1}^{n} t_{1i}\, y_i)\, \underline{b}_{o1} + (\sum_{i=2}^{n} t_{2i}\, y_i)\, \underline{b}_{o2} + (\sum_{i=3}^{n} t_{3i}\, y_i)\, \underline{b}_{o3} + \ldots + y_n\, \underline{b}_{on}$$

$$= t_{1x}\, \underline{b}_{o1} + t_{2x}\, \underline{b}_{o2} + \ldots + t_{nx}\, \underline{b}_{on} + \underline{b}_{nx} \qquad (44)$$

The equation (44) is exactly the same equation as (30) just differently written. As all vectors $\underline{b}_{o1}$ through $\underline{b}_{on}$ and $\underline{b}_{nx}$ are mutually orthogonal we get an equation for each i when we multiply (44) by $\underline{b}_{oi}^{T}$ and afterwards devide by $\underline{b}_{oi}^{T}\, \underline{b}_{oi}$. The system of equations obviously becomes,

$$\begin{aligned} t_{11}\, y_1 + t_{12}\, y_2 + \ldots\ldots t_{1n}\, y_n &= t_{1x} \\ t_{22}\, y_2 + \ldots\ldots t_{2n}\, y_n &= t_{2x} \\ &\vdots \\ t_{nn}\, y_n &= t_{nx} \end{aligned} \qquad (45)$$

The triangular system (45) is normalised which means

$$t_{11} = t_{22} = \ldots = t_{nn} = 1 \tag{46}$$

Any solution to (45) is a solution to (30) with the error vector $\underline{v} = \underline{b}_{nx}$ because (44) is just a rewriting of (30). As $\underline{b}_{nx}$ is orthogonal to all $\underline{b}_{oi}$ we have,

$$\underline{b}_{nx}^T \underline{B} = \underline{0} \quad \text{or} \quad \underline{v}^T \underline{B} = \underline{0} \tag{47}$$

which means that $\underline{v}^T \underline{v}$, the square error sum, is minimum. This is seen as follows

$$d(\underline{v}^T \underline{v}) = 2\underline{v}^T d\underline{v} = 2\underline{v}^T d(\underline{B}\,\underline{y} - \underline{x}) = 2\underline{v}^T \underline{B}\, d\underline{y} \tag{48}$$

and (48) is equal to zero for any $d\underline{y}$ when $\underline{v}^T \underline{B} = \underline{0}$. Thus the solution of (45) which we will denote by

$$\underline{T}\,\underline{y} = \underline{x}_T \tag{49}$$

is equivalent to the least squares solution of (30). The equation (49) is easily solved for $\underline{y}$. However two very interesting features of (49) are that the mean error of each equation is easily calculated and that the errors of the single equations are independent of each other.

To arrive at the equation number i we have to multiply (30) by $\underline{b}_{oi}^T$ and afterwards divide by $\underline{b}_{oi}^T \underline{b}_{oi}$. The error δ_i on equation number i in (49) thus becomes

$$\delta_i = \frac{\underline{b}_{oi}^T \underline{v}}{\underline{b}_{oi}^T \underline{b}_{oi}} \tag{50}$$

As we don't know $\underline{v}$ we cannot find δ_i, but we know the expected (mean)

value of v_i^2 and we assume that the single v_i's are mutually independent. Thus we have

$$\overline{v_i v_j} = \begin{cases} \mu^2 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \tag{51}$$

and we can then find the expected (mean) value of δ_i^2

$$\overline{\delta_i^2} = \theta_i^2 = \overline{\left(\frac{\underline{b}_{oi}^T \underline{v}}{\underline{b}_{oi}^T \underline{b}_{oi}} \right)^2} = \frac{\mu^2}{\underline{b}_{oi}^T \underline{b}_{oi}} \tag{52}$$

The standard error on the equation number i in the equations (49) thus becomes

$$\theta_i = \mu \, (\underline{b}_{oi}^T \underline{b}_{oi})^{-1/2} \tag{53}$$

That the errors δ_i are mutually independent is shown by

$$\overline{\delta_i \delta_j} = \frac{\overline{(\underline{b}_{oi}^T \underline{v})(\underline{b}_{oj}^T \underline{v})}}{(\underline{b}_{oi}^T \underline{b}_{oi})(\underline{b}_{oj}^T \underline{b}_{oj})} = 0 \quad , \; i \neq j \tag{54}$$

because $\underline{b}_{oi}$ and $\underline{b}_{oj}$ are orthogonal for $i \neq j$ or

$$\underline{b}_{oi}^T \underline{b}_{oj} = 0 \quad , \; i \neq j \tag{55}$$

Thus, instead of using (49) for finding $\underline{y}$ we ought to use

$$\underline{T} \, \underline{y} = \underline{x}_T \pm \underline{\theta} \tag{56}$$

where $\underline{T}$, $\underline{x}_T$ and $\underline{\theta}$, the standard error are known quantities.

IV. SOLUTION OF THE TRIANGLE MATRIX

A mathematician who is presented with the problem of solving (56) and who knows nothing at all about its origin would naturally put $y_n = x_{Tn}$ as the representative value for y_n. Solving the problem by acting as if any $\theta_i = 0$ is in fact using the least squares criterion as was previously proven. However, the person who presents the problem always possesses some information concerning the solution which goes beyond the information contained in (56) and at least he has some purpose with the solution (or it would not be worth the trouble to look for one). He is aware of the fact that his measurements, due to errors, do not define a unique solution, only a range of possible solutions. So, the best he can do is to search for a possible solution which within the error limits fulfills in a satisfactory manner his requirements and perhaps also other criteria which are not expressed by means of (56). How to make use of ones intentions and other vaguely formulated demands must obviously depend on each specific case, but the following two specific examples might serve to illustrate a general line.

Example 1. The curve fitting problem

The equations (30) and hence (56) stem from a curve fitting problem. The parameters y_j are coefficients to polynomials (or other functions) and the condition equations have the form $x_i = \sum_{j=1}^{n} y_j \cdot f_j(i)$ $(i = 1 \ldots m)$ with for instance $f_j(i) = i^{j-1}$. It is desirable that the solution should be as simple as possible which means that the larger j is the more desirable it is that y_j become numerically small.

We start from the last equation in (56)

$$y_n = x_{Tn} \pm \theta_n \qquad (57)$$

If zero lies between $x_{Tn} - \theta_n$ and $x_{Tn} + \theta_n$ we put $y_n = 0$. If zero lies outside the interval we put y_n equal to $x_{Tn} + \theta_n$ or $x_{Tn} - \theta_n$ which ever comes closest to zero. The value adopted for y_n is inserted in equation No. (n-1) in (56) and y_{n-1} is found in the same way with $\pm \theta_{n-1}$ as the limits. In this way a unique solution $\underline{y}$ is obtained. It satisfies the original equations within the error limits and possesses as far as possible the desired properties. Compared to the least squares method the present method has the advantage that it gives the same result if we assume "too many" functions f_j. By judging the accuracy of y_j by means of θ_j we choose a convenient value of y_j within the error limits rather than using a value determined solely by random errors.

Example 2. Unfolding of a spectrum

A spectrum $y(u)$ is given by a known folding function $a(u,\tau)$ and a measured spectrum $x(\tau)$ by

$$x(\tau) = \int_{u_1}^{u_2} y(u)a(u,\tau)du \qquad (\tau_1 \leq \tau \leq \tau_2) \tag{58}$$

A numerical evaluation of the wanted $y(u)$ demands a representation in a finite number of points. We define

$$a_{ij} = a(u_1 + (i-1)\Delta u,\ \tau_1 + (j-1)\Delta\tau) \tag{59}$$

with

$$\Delta u = \frac{u_2 - u_1}{m - 1} \tag{60}$$

and

$$\Delta\tau = \frac{\tau_2 - \tau_1}{n - 1} \tag{61}$$

Further

$$x_i = x(\tau_1 + (i-1)\Delta\tau) \qquad (i = 1, 2 \ldots m) \tag{62}$$

and

$$y_j = y(u_1 + (j-1)\Delta u) \cdot \Delta u \qquad (j = 1, 2 \ldots n) \tag{63}$$

Now we can write

$$\underline{A}\,\underline{y} = \underline{x} + \underline{v} \tag{64}$$

which shows that the numerical unfolding process is equivalent to solving a system of m linear equations with n unknowns. When writing the system in practice it is necessary to decide the size of m and n. If the spectrum $x(\tau)$ is measured by means of a multi-channel analyser m is naturally taken as the number of channels and if $x(\tau)$ is recorded as a continous spectrum it is natural to take the representative points x_i so closely spaced that any point of $x(\tau)$ can (within the error limits) be obtained by linear interpolation between the two neighbouring points. The choice of m is thus a matter of common sense. Similar considerations lead to the choice of n. The spectrum y(u) is represented by $\underline{y}$ and it is desirable that any value of y(u) can be obtained by linear interpolation between its two neighbouring points so n should be taken so large that any three columns of $\underline{A}$ are (nearly) linear dependent. This leads immediately to singularity of $\underline{A}$, but this is obviously a necessary condition in order to get $\underline{y}$ as a representative for y(u) in the desired way. We might very well end up with a system which has more unknowns than equations, i.e. $n > m$.

Before the evaluation of (64) it is necessary to discuss ones expectations about the solution. In this example we will assume that we

don't know what the solution "ought" to look like and content ourselves with using some general considerations to arrive at an acceptable solution.

The first thing to be done is to multiply each equation in (64) by a factor such that the expected error becomes μ on each equation. This operation changes the $\underline{A}$ matrix. We assume that this operation has been carried out without changing the notation. For reasons, which will be explained later, we put

$$\underline{y} = \underline{D}\,\underline{w} \tag{65}$$

where $\underline{D}$ is a $n \cdot n$ dimensional diagonal matrix with the known diagonal elements d_{11} through d_{nn} and $\underline{w}$ a new unknown n dimensional vector. The equation (64) then becomes

$$\underline{A}\,\underline{D}\,\underline{w} = \underline{x} + \underline{v} \tag{66}$$

which can be solved for $\underline{w}$ and $\underline{y}$ is then found from (65).

As $\underline{y}$ is representative for a smooth function y(u) we expect the higher terms of a Fourier series giving $\underline{y}$ to be small. This is also true for $\underline{w}$ when $d_{ii} = 1$ for all i, because then $\underline{w} = \underline{y}$. Now $\underline{w}$ is expanded in a Fourier series by putting

$$\underline{w} = \underline{F}\,\underline{\eta} \tag{67}$$

with $\underline{F} = \{f_{ij}\}$ and

$$f_{ij} = \sin\left[\frac{2\pi(i-1)}{n} \cdot \frac{j}{2}\right] \quad \text{for j even}$$

$$\cos\left[\frac{2\pi(i-1)}{n} \cdot \frac{(j-1)}{2}\right] \quad \text{for j odd} \tag{68}$$

$$i = (1 \ldots n) \qquad j = (1 \ldots \min(n, m))$$

and $\underline{\eta}$ is a vector with the dimension min(n, m).

Now (66) becomes

$$\underline{A}\,\underline{D}\,\underline{F}\,\underline{\eta} = \underline{x} + \underline{v} \tag{69}$$

The matrix $\underline{B} = \underline{A}\,\underline{D}\,\underline{F}$ is now converted into a triangular matrix as described in Chapter III and in the Appendix and $\underline{\eta}$ is calculated as $\underline{y}$ was calculated in Example 1. By using (67) and (65) we get the solution $\underline{y}_o$ which is smooth (in the sense described above) and representative. However, the $\underline{y}$ vector found in this way might contain negative elements. If so, we consider the found $\underline{y}_o$ as a first approximation to the desired solution. To find an, as far as possible smooth, solution without negative elements we make use of $\underline{D}$. If we for the next iteration put $d_{ii} = y_{oi}$ for all i then the "smoothest" $\underline{w}$ which fits (66) obviously becomes $w_j = 1$ for all j. Inserting this result in (65) would yield again $\underline{y}_o$. However, by putting $d_{ii} = y_{io}^2$, solving (69) for $\underline{\eta}$ and, finding $\underline{w}$ from (67) we get a new representative $\underline{y}$, from (65) and the iteration can proceed. Any representative value found in this way is evidently a solution to (64), but the solutions are rejected as such and used in $\underline{D}$ if they contain negative elements. The iteration is rapidly convergent (3 iterations suffice in general) provided the μ value used is not too small compared to the "real" experimental mean error. It is preferable to use a value which is somewhat larger (for instance three times) than the experimental mean error in order to compensate for excess statistical errors and errors which are due to the fact that n (the number of elements in $\underline{y}$) is a finite number so that any three succeeding columns in $\underline{B}$ are not exactly linearly dependent. The $\underline{y}$ value obtained

by this method is the "smoothest" representative solution with non-negative elements which can be obtained within the given error limts. Thus the unfolded spectrum contains only such details which are absolutely necessary to satisfy the original equations. No "oscillations" are present.

V. DISCUSSION

The uncertainty which is associated with the presence of experimental errors in systems of linear equations cannot be eliminated by any mathematical method. The knowledge of the error probability distribution function determines a range of likely solutions to the system. When one wants a unique representative solution it is advisable, as far as possible, to make up one's mind in order to formulate one's demands on a representative solution rather than a priori specifying a mathematical method for limiting the range of possible solutions. The present paper shows a method of transforming the original system of equations into a triangular system where the range of possible solutions makes itself visible in a very clear way. By means of two examples it is shown that even vaguely formulated demands can be sufficient to enable us to get a uniquely defined and acceptable solution also in cases where the number of unknowns is larger than the number of equations as shown in the last example. The problem of truncation errors occurring during the computation has been touched, and it was an investigation of this problem which lead to the procedure of forming the triangular system in order to minimise truncation errors. The clear exposition of the range of possible solutions which was a result hereof was found more interesting, however, and therefore the emphasis was put on that point. If the present method of form-

ing the triangular matrix is used truncation errors are in general without practical importance if the relative error of the largest elements in each column of $\underline{B}$ is smaller than the relative experimental error of the largest $\underline{x}$ elements and further if in the computation at least one extra digit is used to define the most accurate $\underline{B}$ element.

By solving physical problems we are accustomed to place weight only on such information which can rigorously be expressed in a mathematical formula. As shown in the present paper other kinds of information can be just as important.

ACKNOWLEDGEMENTS

The author would like to thank his colleagues, especially Dr. F. Robben and Dr. E. Larsson, for valuable discussions and criticism. Also thanks are due to Mr. S. Svensson for making computer programmes.

REFERENCE

1. ANDERSEN, E,
Adjustment of observations by the method of least squares.
København, 1955. Geodætisk Instituts Skrifter, 3:22.

APPENDIX

The equations (30) are written without attention to the errors $\underline{v}$

$$
\begin{aligned}
&b_{11}\, y_1 + b_{12}\, y_2 + \dots\ b_{1n}\, y_n = x_{o1}\\
&b_{21}\, y_1 + b_{22}\, y_2 + \dots\ b_{2n}\, y_n = x_{o2}\\
&\cdot\\
&\cdot\\
&\cdot\\
&b_{m1}\, y_1 + b_{m2}\, y_2 + \dots\ b_{mn}\, y_n = x_{om}
\end{aligned}
\qquad \text{(I)}
$$

The b_{ij}'s and the x_{oi}'s are actual numbers, the y_i's are the unknowns. Now the "first" normal equation is generated as

$$
\sum_{i=1}^{m} b_{i1}^{2}\, y_1 + \sum_{1}^{m} b_{i1}\, b_{i2}\, y_2 + \dots \sum_{1}^{m} b_{i1}\, b_{in}\, y_n = \sum_{1}^{m} b_{i1}\, x_{o1} \qquad \text{(II)}
$$

$$
(\underline{b}_1^T\, \underline{b}_1)y_1 + (\underline{b}_1^T\, \underline{b}_2)y_2 + \dots\ (\underline{b}_1^T\, \underline{b}_n)y_n = \underline{b}_1^T\, \underline{x}_o \qquad \text{(IIa)}
$$

The equation (II) is used to eliminate y_1 from each equation in the system (I). Besides, (II) is stored as the first equation of the triangular system. Now (I) has changed to a new system of m equations with (n-1) unknowns. From this system y_2 is eliminated in the same way and the equation used for the elimination is stored as the second equation of the triangular system and so on. When the last unknown y_n is eliminated we have arrived at an n · n dimensional triangular system and its right hand side. It is denoted

$$
\underline{T}'\, \underline{y} = \underline{x}'_T \qquad \text{(III)}
$$

The equations (III) have (as shown in Chapter III) independent statistical errors. Now (III) has to be normalised by dividing each equation No. j by t'_{jj} ($= \underline{b}_{oj}^T \underline{b}_{oj}$), however, when the value of a t'_{jj} is very small it might consist of truncation errors only and the corresponding equation is then worthless for the determination of $\underline{y}$. The normalisation is therefore made as follows. If t'_{jj} has a significant value the corresponding equation is divided by t'_{jj} and the mean error on the right hand side becomes $\theta_j = (t'_{jj})^{-1/2} \cdot \mu$ (from (53)). If t'_{jj} has an insignificant value we put all elements of the normalised equation including the right hand side element equal to zero with the exception of the diagonal element which is put equal to 1. The right hand side mean error is put equal to a very large number. In the latter case the equation is still worthless for the determination of $\underline{y}$, as is evident from the large mean error.

Before we can write the set of normalised equations

$$\underline{T}\,\underline{y} = \underline{x}_T \qquad \text{(IV)}$$

we must establish a criterion for whether t'_{jj} is significant or not. As the error on the right hand side becomes large for a small value of t'_{jj} (it becomes $(t'_{jj})^{-1/2} \cdot \mu$) anyhow, any sensible criterion will do, for instance that t'_{jj} is significant when $t'_{jj} \geq t'_{11} \cdot 10^{-6}$ and insignificant for $t'_{jj} < t'_{11} \cdot 10^{-6}$ with six significant digits in the computation. The solution of (IV) for $\underline{y}$ depends on the right hand side error vector and is shown in Chapter IV.

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