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THE DEGREE OF C^0-SUFFICIENCY OF ANALYTIC FUNCTION GERMS WITH RESPECT TO AN IDEAL

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Abstract

Let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an analytic function germ of two complex variables and let I be an ideal of $\mathbb{C}\{x, y\}$. We give some formulae for the degree of C^0-sufficiency of f with respect to I. When I is the maximal ideal we retrieve a result of T.C. Kuo and Y.C. Lu.

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1. Introduction

Let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an analytic function germ of two complex variables and let I be an ideal of $\mathbb{C}\{x, y\}$. We say that f is *C^0-sufficient of degree s with respect to I*, if for each $g \in I^{s+1}$ then $f + g$ and f have the same topological type (meaning that there exists a germ of homeomorphism $\varphi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $(f + g) \circ \varphi = f$). We call the *degree of C^0-sufficiency of f with respect to I*, denoted by $\mathrm{suff}_I(f)$, the least $s \in \mathbb{N}$ such that f is C^0-sufficient of degree s with respect to I.

Problem: *How to compute $\mathrm{suff}_I(f)$?*

In the case I is the maximal ideal of $\mathbb{C}\{x, y\}$, T. C. Kuo and Y. C. Lu [3] showed that $\mathrm{suff}_I(f) = [L(f)] + 1$,[1] where $L(f)$ is the Łojasiewicz number of f at the origin in $\mathbb{C}^2$. The authors also gave a formula for $L(f)$ in terms of Puiseux's expansions of f. D. T. Lê and C. Weber [10] computed the Łojasiewicz number via the data of the resolution tree of f.

When $I := \langle g \rangle$ is the principal ideal generated by a germ $g \in \mathbb{C}\{x, y\}$, H. V. Hà [2] showed that $\mathrm{suff}_I(f)$ may be determined in several ways: in terms of the Jacobian quotients of the analytic application germ $(f, g)\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, in terms of the resolution tree of the germ $f \cdot g = 0$ and in terms of intersection multiplicities between the branches of $(f \cdot g)^{-1}(0)$ and the branches of Jacobian curve.

The purpose of this paper is to give some formulae for the degree of C^0-sufficiency of f with respect to I. Namely, let $g_1, g_2, \ldots, g_k$ be generators of I. We shall show that

$$\mathrm{suff}_I(f) = \max_{i=1,2,\ldots,k} \mathrm{suff}_{\langle g_i \rangle}(f),$$

and for generic $(c_1, c_2, \ldots, c_k) \in \mathbb{C}^k$,

$$\mathrm{suff}_I(f) = \mathrm{suff}_{\langle c_1 g_1 + c_2 g_2 + \cdots + c_k g_k \rangle}(f).$$

As a consequence, the degree of C^0-sufficiency of f with respect to I can be computed effectively based on the results of H. V. Hà [2] and H. Maugendre [12], [13], [14] (see also [16]). Moreover, in the case where I is the maximal ideal, we obtain a result of T. C. Kuo and Y. C. Lu in [3].

The paper is organized as follows. We shall formulate the results in Section 2. The notion about the Newton polygon relative to an arc (see [4]), which plays an important role in the proofs of the results, is recalled in Section 3. Finally, the proofs are given in Section 4.

2. Statement of results

Throughout this paper let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an analytic function germ of two complex variables with at most isolated singularity at the origin in $\mathbb{C}^2$, and let $I := \langle g_1, g_2, \ldots, g_k \rangle \subset \mathbb{C}\{x, y\}$ be the ideal generated by analytic function germs

$$g_1, g_2, \ldots, g_k\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0).$$

Our main results are the following.

[1] Here $[a]$ denotes the greatest integer $\leq a$.

Theorem 2.1. *We have*

$$\mathit{suff}_I(f) = \max_{i=1,2,\ldots,k} \mathit{suff}_{\langle g_i \rangle}(f).$$

Theorem 2.2. *There exists an algebraic set $\Omega \subsetneq \mathbb{C}^k$, such that for $(c_1, c_2, \ldots, c_k)$ chosen from $\mathbb{C}^k - \Omega$, we have*

$$\mathit{suff}_I(f) = \mathit{suff}_{\langle c_1 g_1 + c_2 g_2 + \cdots + c_k g_k \rangle}(f).$$

Remark 2.3. (i) Theorem 2.2 was communicated to the author by H. V. Hà as a conjecture.

(ii) The above results imply that to compute $\mathrm{suff}_I(f)$ one has only to determine the degree of C^0-sufficiency of f with respect to principal ideals.

Let $g\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an analytic function germ. Suppose that f and g have no common branches (that is, if $f = \prod_i f_i^{\alpha_i}$ and $g = \prod_j g_j^{\beta_j}$ are decompositions of f and g into their distinct irreducible factors, then $f_i \neq g_j$, modulo units, for all i and j). Let us consider the analytic application germ Φ defined as follows:

$$\Phi := (f, g)\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0), \quad (x, y) \mapsto (f(x, y), g(x, y)).$$

The Jacobian germ of Φ is the product of the irreducible components of

$$J(f, g) := \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}$$

which do not divide $f \cdot g$; its reduced zero set is the Jacobian locus $\Gamma(f, g)$ of Φ. The image by Φ of $\Gamma(f, g)$ is the discriminant curve $\Delta(f, g)$ of Φ. If (u, v) are the canonical complex coordinates of $\Phi((\mathbb{C}^2, 0))$, then by definition, $\{u = 0\} = \Phi(\{f = 0\})$ is not a branch of $\Delta(f, g)$. So, if δ represents a branch of $\Delta(f, g)$, there exists a positive rational number q_δ and a Puiseux expansion of δ given by (see [1], [21]):

$$u = a v^{q_\delta} + \text{ higher order terms in } v,$$

where $a \in \mathbb{C} - \{0\}$. Following H. Maugendre [12], the numbers q_δ will be called *Jacobian quotients* of Φ. It is worth noting that if g is a general linear form, then the Jacobian quotients of (f, g) are exactly the *polar quotients* of f (see [7], [17]).

We denote by $V(f)$ (respectively, $V(I)$) the zero set germ at the origin of f (respectively, I) in $\mathbb{C}^2$.

Corollary 2.4. *Suppose that $V(f) \cap V(I) = \{(0, 0)\}$. Then*

$$\mathit{suff}_I(f) = \max_{i=1,2,\ldots,k} \max \left\{ [q_\delta] \;\middle|\; q_\delta \textit{ a Jacobian quotient of } (f, g_i) \right\}.$$

Remark 2.5. Let $g \in \mathbb{C}\{x, y\}$ be a regular germ (i.e., the order of g at the origin in $\mathbb{C}^2$ is equal to 1). In [16], A. Płoski gave an explicit formula for the maximal Jacobian quotient of (f, g) by means of the maximal Jacobian quotients of the branches and some intersection multiplicities.

Example 2.6. Let $f(x,y) := x^3 + y^5$ and $I := \langle g_1(x,y) := x^2, g_2(x,y) := y^2 \rangle$. Then it is easy to see that $V(f) \cap V(I) = \{(0,0)\}$.

In this example, the Jacobian of (f, g_1) (respectively, (f, g_2)) is $J(f, g_1) = 10xy^4$ (respectively, $J(f, g_2) = 6x^2y$). So $\Gamma(f, g_1)$ (respectively, $\Gamma(f, g_2)$) has only one branch $\gamma_1 \colon y = 0$ (respectively, $\gamma_2 \colon x = 0$).

For γ_1, we obtain $f(x,0) = x^3, g_1(x,0) = x^2$; then (u,v)-Puiseux expansion of $\Delta(f, g_1) = \delta_1$ begins by $u = v^{\frac{3}{2}}$. Consequently, $\frac{3}{2}$ is only the Jacobian quotient of (f, g_1).

For γ_2, we have $f(0,y) = y^5, g_2(0,y) = y^2$; then (u,v)-Puiseux expansion of $\Delta(f, g_2) = \delta_2$ begins by $u = v^{\frac{5}{2}}$. Hence, there exists a unique Jacobian quotient of (f, g_2), which is equal to $\frac{5}{2}$.

Then using Corollary 2.4, we obtain

$$\text{suff}_I(f) = \max\left\{ \left[\frac{3}{2}\right], \left[\frac{5}{2}\right] \right\} = \left[\frac{5}{2}\right] = 2.$$

We now suppose that $(c_1, c_2) \in (\mathbb{C} - \{0\})^2$. The Jacobian of $(f, c_1g_1 + c_2g_2)$ is $J(f, c_1g_1 + c_2g_2) = xy(6c_2x - 10c_1y^3)$. It follows that the Jacobian locus $\Gamma(f, c_1g_1 + c_2g_2)$ is constituted of the three branches $\gamma_1 \colon y = 0, \gamma_2 \colon x = 0$ and $\gamma_3 \colon 6c_2x - 10c_1y^3 = 0$. By a direct computation, the Jacobian quotients of $(f, c_1g_1 + c_2g_2)$ are $\frac{3}{2}$ and $\frac{5}{2}$. Then, from Theorem 2.2 and the works of H. V. Hà in [2] we also obtain $\text{suff}_I(f) = 2$.

Corollary 2.7. *With the assumption of Corollary 2.4, we have*

$$\textit{suff}_I(f) = \max_{i=1,2,\dots,k} \max\left\{ \left[\frac{(\gamma, f)_0}{(\gamma, g_i)_0}\right] \mid \gamma \text{ a irreducible component of } \Gamma(f, g_i) \right\},$$

where $(\cdot,\cdot)_0$ is the intersection multiplicity at the origin between two plane curve germs.

Let $\pi \colon (X, E) \to (\mathbb{C}^2, 0)$ be the minimal resolution of $\{f \cdot g = 0\}$. For each irreducible component E_α of the exceptional divisor E of π, let $m_f(E_\alpha)$ (respectively, $m_g(E_\alpha)$) be the multiplicity of the function $f \circ \pi \colon X \to \mathbb{C}$ along E_α (respectively, $g \circ \pi$). We will say that E_α is a *rupture divisor* if it intersects at least three different components of the total transform of $\{f \cdot g = 0\}$; i.e. if $\overline{\pi^{-1}(\{f \cdot g = 0\}) - E_\alpha}$ has at least three connected components (see [12], [13] for more details).

Corollary 2.8. *Under the hypothesis of Corollary 2.4, we have*

$$\textit{suff}_I(f) = \max_{i=1,2,\dots,k} \max\left\{ \left[\frac{m_f(E_\alpha)}{m_{g_i}(E_\alpha)}\right] \mid E_\alpha \text{ a rupture divisor} \right\}.$$

Let $\mathbb{S}^3_\epsilon$ be the sphere centered at the origin of $\mathbb{C}^2$, with radius $\epsilon > 0$ small enough. The linking number between the links K_1 and K_2 in $\mathbb{S}^3_\epsilon$ is denoted by $l(K_1, K_2)$. Consider the minimal Waldhausen decomposition of $\mathbb{S}^3_\epsilon$ for $K_{f \cdot g}$ (see [20], [11]). Let V be a Seifert manifold of this decomposition and ν any Seifert leaf of V. The *linking quotient* of (f, g) associated to V is the rational number $l_\nu := l(K_f, \nu)/l(K_g, \nu)$. For more details see [14].

Corollary 2.9. *With the hypothesis of Corollary 2.4, we have*

$$\textit{suff}_I(f) = \max_{i=1,2,\dots,k} \max\left\{ [l_\nu] \mid l_\nu \text{ a linking quotient of } (f, g_i) \right\}.$$

The *Lojasiewicz number* $L(f)$ of f at the origin 0 in $\mathbb{C}^2$ is defined as the smallest $\theta > 0$ such that

$$\|\text{gradf } f(x,y)\| \geq constant \ \|(x,y)\|^\theta \quad \text{in a neighbourhood of } 0.$$

It is well-known that $L(f) < \infty$ if and only if f has at most isolated singularity at the origin.

The following result was proved by T. C. Kuo and Y. C. Lu [3] (see also [18]).

Corollary 2.10. *Let $I := \langle x, y\rangle$ be the maximal ideal of $\mathbb{C}\{x,y\}$. Then*

$$suff_I(f) = [L(f)] + 1.$$

3. Newton Polygon relative to an arc

Let us first recall the notion about the Newton polygon relative to an arc (see [4], [1], [21] for details).

Let f denote a germ of analytic function with Taylor expansion

$$f(x,y) = f_m(x,y) + f_{m+1}(x,y) + \cdots .$$

By a linear transformation, we may assume that f is *regular in x of order m* in the sense that $f_m(1,0) \neq 0$.

By a fractional (convergent) power series we mean a series of the form

$$\lambda : x = \lambda(y) := a_1 y^{n_1/N} + a_2 y^{n_2/N} + \cdots , \quad a_i \in \mathbb{C},$$

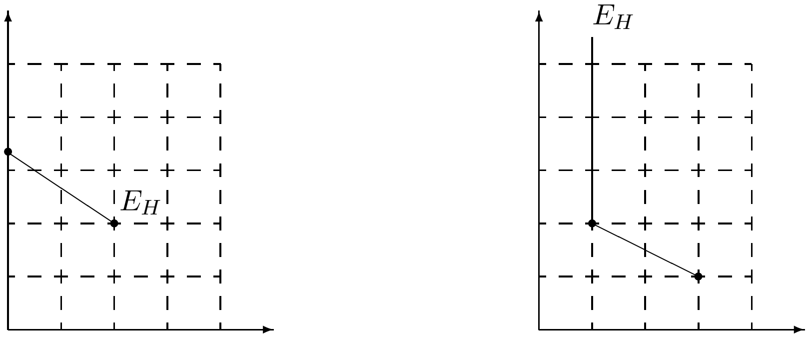
where $N \leq n_1 < n_2 \cdots$ are positive integers, having no common divisor, such that $\lambda(\tau^N)$ has positive radius of convergence. Let us apply the change of variables

$$X = x - \lambda(y), \quad Y = y,$$

to $f(x,y)$, yielding

$$f(X + \lambda(Y), Y) := \sum c_{ij} X^i Y^{j/N}.$$

For each $c_{ij} \neq 0$, let us plot a dot at $(i, j/N)$, called a *Newton dot*. The *Newton polyhedron of f relative to λ*, denoted by $\mathbb{P}_+(f,\lambda)$, is the convex hull in $\mathbb{R}^2_+$ of the set $\{(i,j/N) + v \mid c_{ij} \neq 0, v \in \mathbb{R}^2_+\}$. We shall denote the union of all compact faces of $\mathbb{P}_+(f,\lambda)$ by $\mathbb{P}(f,\lambda)$. Following T. C. Kuo and A. Parusiński [4], $\mathbb{P}_+(f,\lambda)$ will be called the *Newton polygon of f relative to λ*. The "highest Newton edge", often denoted by E_H, means the following: If the highest vertex is on the y-axis, E_H is the compact edge to its right; otherwise, E_H is the vertical edge on this vertex, as illustrated below.



Let λ be a Newton-Puiseux root of $\frac{\partial f}{\partial x} = 0$:

$$\lambda : x = \lambda(y) := a_1 y^{n_1/N} + a_2 y^{n_2/N} + \cdots, \quad 1 \leq N \leq n_1 < n_2 < \cdots .$$

We can identify λ with the analytic arc

$$\lambda\colon x = a_1\tau^{n_1} + a_2\tau^{n_2} + \cdots, \quad y = \tau^N, \quad |\tau| \ll 1.$$

Following [21], λ is called a "branch" of the polar curve, or simply a "polar branch".

Remark 3.1. If f has only isolated singularity at the origin, then for any polar arc λ of f the Newton polygon $\mathbb{P}(f, \lambda)$ meets the y-axis. In fact, suppose that this is not the case. Then we have

$$\begin{aligned} f(\lambda(y), y) &= 0, \\ \frac{\partial f}{\partial x}(\lambda(y), y) &= 0. \end{aligned}$$

These imply that the following relation is satisfied on the arc λ :

$$0 = \frac{\partial f}{\partial x}\frac{\partial \lambda}{\partial y} + \frac{\partial f}{\partial y} = \frac{\partial f}{\partial y},$$

which is a contradiction.

Take $r > 0$. Let $\mathcal{M}_r$ denote the application

$$\mathcal{M}_r\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (i, j) \mapsto (ri, rj).$$

Let $g \in \mathbb{C}\{x, y\}$. Take any arc λ. We define a positive number $r(g, \lambda)$ as follows: The number $r(g, \lambda)$ is just the smallest rational such that $\mathcal{M}_r(E_H)$ contains at least one Newton dot of $g(X + \lambda(Y), Y)$. By the definition, all Newton dots of g lie on or above $\mathcal{M}_{r(g,\lambda)}(E_H)$. We say that E_H is *disturbed* by g if $r(g, \lambda) \leq 1$. (This definition is slightly different from the one in [4].)

Lemma 3.2. ([4, Theorem 4.1]) *Let $f, g\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be two analytic function germs. Then following conditions are equivalent:*

(i) For all $t \in \mathbb{C}$,

$$|g(x, y)| \ll \|grad_{(x,y)}[f(x, y) + tg(x, y)]\| \quad as \quad (x, y) \to (0, 0).$$

(ii) Take any polar branch λ of f. The highest Newton edge of $\mathbb{P}(f, \lambda)$ is not disturbed by $g(x, y)$.

Proof. The claim follows very closely the lines of the proof of [4, Theorem 4.1]. We will leave to the reader to verify this fact. □

We will need the following lemma which is an important ingredient in the proofs of Theorem 2.1 and Theorem 2.2.

Lemma 3.3. *Let λ be an arc. Let $g, h\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be two germs of complex analytic functions. Then the following statements hold:*
(i) $r(g+h,\lambda) \geq \min\{r(g,\lambda), r(h,\lambda)\}$.
(ii) $r(gh,\lambda) = r(g,\lambda) + r(h,\lambda)$.
(iii) $r(g^s,\lambda) = sr(g,\lambda)$ *for all s positive integer.*

Proof. We first need some notations. Let $v := (v_x, v_y) \in \mathbb{R}^2_+$ be a normal vector of the line that contains the highest Newton edge E_H of $\mathbb{P}(f,\lambda)$. In view of Remark 3.1 above, $v_x > 0$ and $v_y > 0$. We define

$$\nu := \min\{v_x i + v_y \frac{j}{N} \mid (i, \frac{j}{N}) \in \mathbb{P}_+(f,\lambda)\}.$$

This means that ν is the least value attained by the linear function

$$(i,j) \mapsto v_x i + v_y \frac{j}{N}$$

on the Newton polygon $\mathbb{P}_+(f,\lambda)$. Then, for any $s > 0$, a Newton dot $(i, j/N)$ lies on or above $\mathcal{M}_s(E_H)$ if and only if

$$v_x i + v_y \frac{j}{N} \geq s\nu.$$

Now we can give a proof of the lemma.

It is obvious that $(i, j/N)$ is a Newton dot of $(g+h)$ then so is g (or h). Then, by definition, we obtain the statement (i).

To show (ii) let us write

$$\begin{aligned} g(X+\lambda(Y),Y) &= \textstyle\sum_{i,j} a_{ij} X^i Y^{j/N}, \\ h(X+\lambda(Y),Y) &= \textstyle\sum_{m,n} b_{mn} X^m Y^{n/N}. \end{aligned}$$

By definition, the following inequalities hold

$$\begin{aligned} v_x i + v_y \frac{j}{N} &\geq r(g,\lambda)\nu, \\ v_x m + v_y \frac{n}{N} &\geq r(h,\lambda)\nu. \end{aligned}$$

These give

$$v_x(i+m) + v_y \frac{j+n}{N} \geq [r(g,\lambda) + r(h,\lambda)]\nu.$$

We conclude that all Newton dots of $g \cdot h$ lie on or above $\mathcal{M}_{r(g,\lambda)+r(h,\lambda)}(E_H)$. Moreover, it is easy to see that there exists at least one Newton dot of $g \cdot h$ that lies on $\mathcal{M}_{r(g,\lambda)+r(h,\lambda)}(E_H)$. These two facts obviously imply (ii).

Finally, (iii) is a direct consequence of (ii). □

4. Proof of the main results

Proof of Theorem 2.1. Since $\langle g_i \rangle \subset I$, it is easy to see that

$$\mathrm{suff}_I(f) \geq \max_{i=1,2,\ldots,k} \mathrm{suff}_{\langle g_i \rangle}(f).$$

Hence, one has only to prove $\mathrm{suff}_I(f) \leq s := \max_{i=1,2,\ldots,k} \mathrm{suff}_{\langle g_i \rangle}(f)$. This means, by definition, that $f+g$ and f have the same topological type for all $g \in I^{s+1}$.

Take any $g \in I^{s+1}$. By the results in [9] and [19] (see also [15]), it suffices to show that the one parameter deformation $f(x,y) + tg(x,y)$ is a μ-constant family of isolated singularities. By [8], this is equivalent to the following

$$|g(x,y)| \ll \|\mathrm{grad}_{(x,y)}[f(x,y) + tg(x,y)]\| \quad \text{as } (x,y) \to (0,0).$$

Since $g \in I^{s+1}$, we may write

$$g(x,y) = \sum_{|\alpha|=s+1} a_\alpha(x,y) g_1^{\alpha_1}(x,y) g_2^{\alpha_2}(x,y) \ldots g_k^{\alpha_k}(x,y),$$

where $a_\alpha, \alpha := (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$, are analytic functions of two complex variables. (Recall that the ideal I is generated by $g_1, g_2, \ldots, g_k$.)

Take any polar branch λ of f :

$$\lambda : x = c_1\tau^{n_1} + c_2\tau^{n_2} + \cdots, \quad y = \tau^N, \quad \text{for } |\tau| \ll 1,$$

where $1 \leq N \leq n_1 < n_2 \cdots$. By Lemma 3.2, one needs to prove that the highest Newton edge E_H of the Newton polygon $\mathbb{P}(f,\lambda)$ of f relative to λ is not disturbed by g. To do this, by Lemma 3.3(i), it suffices to show that E_H is not disturbed by $a_\alpha g_1^{\alpha_1} g_2^{\alpha_2} \ldots g_k^{\alpha_k}$ for all $\alpha \in \mathbb{N}^k$ with $|\alpha| = s+1$. By Lemma 3.3(ii), this is satisfied provided that the next inequality holds

$$r(g_1^{\alpha_1} g_2^{\alpha_2} \ldots g_k^{\alpha_k}, \lambda) > 1 \quad \text{for all } |\alpha| = s+1.$$

Take any $i \in \{1, 2, \ldots, k\}$ and consider the principal ideal $\langle g_i \rangle$ generated by g_i. The following relation

$$s = \max_{i=1,2,\ldots,k} \mathrm{suff}_{\langle g_i \rangle}(f) \geq \mathrm{suff}_{\langle g_i \rangle}(f),$$

yields $f + tg_i^{s+1}$ and f have the same topological type for all $t \in \mathbb{C}$. Then, by a criterion of equisingularity (see [5], [6]), $F_t(x,y) := f(x,y) + tg_i^{s+1}(x,y)$ is a μ-constant family of isolated singularities. It follows from the results of D. T. Lê and K. Saito in [8] that

$$|g_i^{s+1}(x,y)| = |\partial_t F_t(x,y)| \ll \|\mathrm{grad}_{(x,y)} F_t(x,y)\| \quad \text{as } (x,y) \to (0,0).$$

Hence, by Lemma 3.2, the highest Newton edge E_H of $\mathbb{P}(f,\lambda)$ is not disturbed by g_i^{s+1}. Or equivalently,

$$r(g_i^{s+1}, \lambda) > 1.$$

Then, by Lemma 3.3(iii), we get the following inequality

$$(s+1) r(g_i, \lambda) > 1.$$

This gives

$$r(g_i, \lambda) > \frac{1}{s+1}.$$

Therefore, using Lemma 3.3(ii)-(iii), for each $|\alpha| = s+1$ we obtain

$$r(g_1^{\alpha_1} g_2^{\alpha_2} \dots g_k^{\alpha_k}, \lambda) = \sum_{i=1}^{k} r(g_i^{\alpha_i}, \lambda) = \sum_{i=1}^{k} \alpha_i r(g_i, \lambda) > \sum_{i=1}^{k} \frac{\alpha_i}{s+1} = 1.$$

In other words, all Newton dots of $g_1^{\alpha_1} g_2^{\alpha_2} \dots g_k^{\alpha_k}, |\alpha| = s+1$, lie above $\mathcal{M}_1(E_H) = E_H$. As a consequence, the highest Newton edge E_H is not disturbed by $g_1^{\alpha_1} g_2^{\alpha_2} \dots g_k^{\alpha_k}$. The theorem is proven. □

Remark 4.1. It is not difficult to see that the proof of Theorem 2.1 also show the following relation

$$\mathrm{suff}_I(f) = \max_{i=1,2,\dots,k} \max \left\{ \left[\frac{1}{r(g_i, \lambda)} \right] \mid \lambda \text{ a polar branch of } f \right\}.$$

Proof of Theorem 2.2. For simplicity of notation, we let

$$s := \mathrm{suff}_I(f).$$

Take any polar branch, λ, of f. We define

$$r_*(\lambda) := \min_{\alpha \in \mathbb{N}^k, |\alpha| = s+1} r(g_1^{\alpha_1} g_2^{\alpha_2} \dots g_k^{\alpha_k}, \lambda).$$

Then, by the definitions, it is not difficult to verify that the set of $(a_\alpha)_{|\alpha|=s+1}$ satisfying

$$r\left(\sum_{|\alpha|=s+1} a_\alpha g_1^{\alpha_1} g_2^{\alpha_2} \dots g_k^{\alpha_k}, \lambda \right) > r_*(\lambda)$$

is a proper algebraic set in $\mathbb{C}^D$, where D is the number of "monomials" of degree $s+1$ in $g_1, g_2, \dots, g_k$.

On the other hand, we can write

$$\left[\sum_{i=1}^{k} c_i g_i \right]^{s+1} = \sum_{|\alpha|=s+1} p_\alpha(c_1, c_2, \dots, c_k) g_1^{\alpha_1} g_2^{\alpha_2} \dots g_k^{\alpha_k},$$

where $p_\alpha, |\alpha| = s+1$, are polynomials in the k variables $c_1, c_2, \dots, c_k$ with complex coefficients.

These imply the set Ω_λ of $c := (c_1, c_2, \dots, c_k) \in \mathbb{C}^k$ that satisfy the inequality

$$r\left(\left[\sum_{i=1}^{k} c_i g_i \right]^{s+1}, \lambda \right) > r_*(\lambda)$$

is a proper algebraic set in $\mathbb{C}^k$.

Let $\Omega := \cup_\lambda \Omega_\lambda$ where the (finite) union is taken over by all the polar branches λ of f. Clearly, Ω is a proper algebraic set in $\mathbb{C}^k$. We will prove that for each $c \notin \Omega$,

$$\mathrm{suff}_{\langle c_1 g_1 + c_2 g_2 + \dots + c_k g_k \rangle}(f) = \mathrm{suff}_I(f) =: s.$$

Firstly, by a straightforward verification one can easily see that

$$I^{s+1} \supset \langle c_1 g_1 + c_2 g_2 + \dots + c_k g_k \rangle^{s+1}.$$

In particular, this gives $\mathrm{suff}_{\langle c_1g_1+c_2g_2+\cdots+c_kg_k\rangle}(f) \le s$. So one only needs to show the inequality $\mathrm{suff}_{\langle c_1g_1+c_2g_2+\cdots+c_kg_k\rangle}(f) \ge s$.

Suppose, by contradiction, that

$$l := \mathrm{suff}_{\langle c_1g_1+c_2g_2+\cdots+c_kg_k\rangle}(f) < s.$$

Then, by definition, there exists an analytic function germ $g \in I^{l+1}$ such that $f+g$ and f have not the same topological type. By the works of D. T. Lê and C. P. Ramanujam [9] and J. G. Timourian [19] (see also [15]), $f(x,y)+tg(x,y)$ is not a μ-constant family of isolated singularities. Then it follows from [8] and Lemma 3.2 that there exists a polar branch λ of f such that the highest Newton edge of $\mathbb{P}(f,\lambda)$ is disturbed by g. This implies that

$$r(g,\lambda) \le 1.$$

Since $g \in I^{l+1}$, we can write

$$g(x,y) = \sum_{|\alpha|=l+1} a_\alpha(x,y) g_1^{\alpha_1}(x,y) g_2^{\alpha_2}(x,y) \ldots g_k^{\alpha_k}(x,y),$$

where a_α are analytic functions of two complex variables. Then, from the inequality $r(g,\lambda) \le 1$ and using Lemma 3.3(i)-(ii), there is $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$ with $|\alpha| = l+1$ such that

$$r(g_1^{\alpha_1} g_2^{\alpha_2} \ldots g_k^{\alpha_k}, \lambda) \le 1.$$

On the other hand, by the definition and Lemma 3.3(ii)-(iii), it is easy to check that $r_*(\lambda)$ is the least value attained by the linear function

$$\sum_{i=1}^{k} x_i r(g_i, \lambda)$$

on the polytope $\{x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k \mid x_i \ge 0, x_1 + x_2 + \cdots + x_k = s+1\}$. In particular, this gives

$$r_*(\lambda) \le \sum_{i=1}^{k} \frac{s+1}{l+1} \alpha_i r(g_i, \lambda) = \frac{s+1}{l+1} r(g_1^{\alpha_1} g_2^{\alpha_2} \ldots g_k^{\alpha_k}, \lambda).$$

Moreover, it follows from Lemma 3.3(iii) that

$$\begin{aligned} r\left(\left[\sum_{i=1}^{k} c_i g_i\right]^{l+1}, \lambda\right) &= (l+1) r\left(\left[\sum_{i=1}^{k} c_i g_i\right], \lambda\right) \\ &= \frac{l+1}{s+1} r\left(\left[\sum_{i=1}^{k} c_i g_i\right]^{s+1}, \lambda\right) \\ &= \frac{l+1}{s+1} r_*(\lambda). \end{aligned}$$

(The last relation follows from $c \notin \Omega$.)

Therefore

$$r\left(\left[\sum_{i=1}^{k} c_i g_i\right]^{l+1}, \lambda\right) = \frac{l+1}{s+1} r_*(\lambda) \le r(g_1^{\alpha_1} g_2^{\alpha_2} \ldots g_k^{\alpha_k}, \lambda) \le 1.$$

As a consequence, the highest Newton edge of $\mathbb{P}(f,\lambda)$ is disturbed by $\left[\sum_{i=1}^k c_i g_i\right]^{l+1}$. By Lemma 3.2 and then by the works of D. T. Lê and K. Saito [8], $f + t\left[\sum_{i=1}^k c_i g_i\right]^{l+1}$ is not a μ-constant family of isolated singularities. In particular, the germs $f + \left[\sum_{i=1}^k c_i g_i\right]^{l+1}$ and f do not have the same topological type, which contradicts the fact that f is C^0-sufficient of degree l with respect to the principal ideal $\langle \sum_{i=1}^k c_i g_i\rangle$. This contradiction proves the theorem. □

Proof of Corollary 2.4: By the hypothesis $V(f) \cap V(I) = \{(0,0)\}$, we have $V(f) \cap V(g_i) = \{(0,0)\}$ for all $i = 1, 2, \ldots, k$. Then the assertion is an immediate consequence of Theorem 2.1 and [2, Theorem 1].

Proof of Corollary 2.7: It follows from Theorem 2.1 and [2, Corollary 2].

Proof of Corollary 2.8: It is a direct consequence of Theorem 2.1 and [2, Corollary 1].

Proof of Corollary 2.9: The statement follows from Theorem 2.1 and [14, Theorem 1].

Proof of Corollary 2.10: It is well known that if $g = c_1 x + c_2 y$ is a general linear form, then the Jacobian quotients of (f,g) are just the polar quotients of f (see [7], [17]). On the other hand, B. Teissier [18] showed that

$$L(f) + 1 = \max\{q \mid q \text{ a polar quotient of } f\}.$$

Hence the claim follows from Theorem 2.2 and Corollary 2.4.

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