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GAUSSIAN POLYNOMIALS AND CONTENT IDEAL IN TRIVIAL EXTENSIONS

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Abstract

The goal of this paper is to exhibit a class of Gaussian non-coherent rings R (with zero-divisors) such that $wdim(R) = \infty$ and $fPdim(R)$ is always at most one and also exhibits a new class of rings (with zerodivisors) which are neither locally Noetherian nor locally domain where Gaussian polynomials have a locally principal content. For this purpose, we study the possible transfer of the "Gaussian" property and the property "the content ideal of a Gaussian polynomial is locally principal" to various trivial extension contexts. This article includes a brief discussion of the scopes and limits of our result.

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1. Introduction

Let R be a commutative ring. We say that an ideal is regular if it contains a regular element, i.e. a non-zerodivisor. We say that a ring R has locally a property (P) if each localisation of R at a maximal ideal has the property (P).

The content $C(f)$ of a polynomial $f \in R[X]$ is the ideal of R generated by the coefficients of f. One of its properties is that $C(.)$ is semi-multiplicative, that is, $C(fg) \subseteq C(f)C(g)$. A polynomial $f \in R[X]$ is said to be Gaussian over R if for every polynomial $g \in R[X]$, we have $C(fg) = C(f)C(g)$. A polynomial $f \in R[X]$ is Gaussian provided $C(f)$ is locally principal by [9, Remark 1.1].

Let A be a ring, E be an A-module and $R := A \propto E$ be the set of pairs (a, e) with pairwise addition and multiplication given by $(a, e)(b, f) = (ab, af + be)$. R is called the trivial ring extension of A by E. Considerable work has been concerned with trivial ring extensions. Part of it has been summarized in Glaz's book [6] and Huckaba's book (where R is called the idealization of E by A) [11].

A ring R is called a Gaussian ring if every polynomial with coefficients in R is a Gaussian polynomial. The result of Tsang, and Gilmer, (recently given a new proof by the work of Loper and Roitman [14]) stated that a domain R is Gaussian if and only if it is a Prüfer domain. The work of Glaz [7] concern the characterization of Gaussian ring (which is not a domain). Specially, in [7, Theorem 2.2], Glaz provides necessary and sufficient conditions for a Gaussian ring R of $wdim(R) \leq 1$. Also, she investigates the weak global dimension of a Gaussian coherent ring R, and shows that the only values that $wdim(R)$ may take are $0, 1$ and ∞; but that $fPdim(R)$ is always at most one [7, Theorems 3.2 and 3.3].

In [9, Theorem 3.3], Heinzer and Huneke proved that a locally Noetherian ring has the following property: a regular Gaussian polynomial has locally principal content ideal. Also, in [14, Theorem 4], Loper and Roitman proved that a locally domain satisfies the same property since a finitely generated regular ideal is invertible if and only if it is locally principal (see [9, Remark 1.1]). On the other hand, in [9, Remark 1.6], Heinzer and Huneke show that the following property is false in general: a Gaussian polynomial has locally principal content ideal. Finally, in [3, Example 2.4], the authors show that if $R = D \propto K$, where D is a valuation domain and $K = qf(D)$, then R satisfies the property "the content ideal of a Gaussian polynomial is locally principal" and so the authors exhibit a class of rings (with zerodivisors) which are neither locally Noetherian nor locally domain where a regular Gaussian polynomial has a locally principal content ideal.

The goal of this paper is to exhibit a class of Gaussian non-coherent rings R (with zerodivisors) such that $fPdim(R)$ is always at most one and also exhibits a new class of rings (with zerodivisors) which are neither locally Noetherian nor locally domain where Gaussian polynomials have locally principal content. For this purpose, we study the possible transfer of the "Gaussian" property and the property "the content ideal of a Gaussian polynomial is locally principal" to various trivial extension contexts. In particular, we generalize [3, Example 2.4] in any domain D not necessarily a valuation domain (Theorem 3.1). The article includes a brief discussion of the scopes and limits of our result.

The term local ring does not necessarily mean Noetherian ring with only one maximal ideal.

2. Transfer of Gaussian Property in trivial ring extensions

Recall that a ring R is called a Gaussian ring if every polynomial with coefficients in R is a Gaussian polynomial. Recall that a domain R is Gaussian if and only if it is a Prüfer domain.

In this section, we study the possible transfer of the Gaussian property to various trivial extension contexts. First, we examine the context of trivial ring extensions of a domain by its quotient field.

Theorem 2.1. *Let D be an integral domain which is not a field, $K = qf(D)$, and $R = D \propto K$ be the trivial ring extension of D by K. Then R is a Gaussian ring if and only if D is a Gaussian domain (Prüfer domain).*

Before proving Theorem 2.1, we establish the following Lemma.

Lemma 2.2. *Let A be a ring, E an A-module and $R = A \propto E$ be the trivial ring extension of A by E. Then A is a Gaussian ring provided R is a Gaussian ring.*

Proof. Let $f = \sum_{i=0}^{n} a_i X^i$ and $g = \sum_{i=0}^{m} b_i X^i$ be two polynomials in $A[X]$, where n and m are positive integers and let $f_R = \sum_{i=0}^{n} (a_i, 0) X^i$ and $g_R = \sum_{i=0}^{m} (b_i, 0) X^i \in R[X]$. We wish to show that $C(fg) = C(f)C(g)$.

Since R is a Gaussian ring, $C_R(f_R g_R) = C_R(f_R)C_R(g_R) =$
$\sum_{i=0}^{n} (AE)(a_i, 0) \sum_{i=0}^{m} (AE)(b_i, 0) = (\sum_{i=0}^{n} Aa_i, \sum_{i=0}^{n} Ea_i)(\sum_{i=0}^{m} Ab_i, \sum_{i=0}^{m} Ea_i) =$
$(C_A(f), C_A(f)E)(C_A(g), C_A(g)E)$. Similarly, we obtain that
$C_R(f_R g_R) = (C_A(fg), C_A(fg)E)$. Therefore, $C_A(fg) = C_A(f)C_A(g)$.

Proof of Theorem 2.1. If R is a Gaussian ring, then D is a Gaussian domain by Lemma 2.2. Conversely, assume that D is a Gaussian domain, that is, D is a Prüfer domain. Let $f = \sum_{i=0}^{n}(a_i, e_i)X^i \in R[X]$. Two cases are then possible.

Case 1: $a_i \neq 0$ for some $i = 1, \ldots, n$. Then the non zero ideal $I = \sum_{i=0}^{n} Da_i$ of a Prüfer domain D is invertible and its inverse ideal is finitely generated: $I^{-1} = ((1/d)\sum_{i=0}^{n} Dd_i) = (1/d)I^{'}$ where $d(\neq 0) \in D$ and $I^{'}$ is an ideal of D. But, $(d, 0)$ is a regular element of R (since $(a, e) \in R$ is regular if and only if $a \neq 0$). Therefore, $(d, 0)^{-1}(I^{'} \propto K)C_R(f) = (d, 0)^{-1}(I^{'} \propto K)(I \propto K) = (d, 0)^{-1}(I^{'}I \propto K) = (d, 0)^{-1}(dD \propto K) = (d, 0)^{-1}(d, 0)(D \propto K) = R$ and so $C_R(f)$ is invertible and f is a Gaussian polynomial.

Case 2: $a_i = 0$ for each $i = 1, \ldots, n$. In this case, $f = \sum_{i=0}^{n}(0, e_i)X^i$ and without loss of generality, we may assume that $e_i \in D$ (since if af is a Gaussian polynomial in $R[X]$ and a is a regular element in R, then f is a Gaussian polynomial in $R[X]$). Let $g = \sum_{i=0}^{m}(b_i, f_i)X^i \in R[X]$. We wish to prove that $C(fg) = C(f)C(g)$. Two cases are then possible:

If there exists $b_i \neq 0$, then by Case 1, g is a Gaussian polynomial and so $C(fg) = C(f)C(g)$.

If $b_i = 0$ for each $i = 1, \ldots, m$, then $g = \sum_{i=0}^{m}(0, f_i)X^i$ and so $C(f)C(g) = (0 \propto \sum_{i=0}^{n} De_i)(0 \propto \sum_{i=0}^{m} Df_i) = 0 = C(fg)$.

In both cases, f is a Gaussian polynomial of $R[X]$ and this completes the proof of Theorem 2.1.

Next, we explore a different context, namely, the trivial ring extension R of a local ring (A, M) by an A-module E such that $ME = 0$. Remark that this ring is a total ring by the proof of [13, Theorem 2.6 (1)].

Theorem 2.3. *Let (A, M) be a local ring and E be an A-module with $ME = 0$. Let $R = A \propto E$ be the trivial ring extension of A by E. Then R is a Gaussian ring if and only if A is a Gaussian ring.*

Proof. If R is a Gaussian ring, then A is a Gaussian ring by Lemma 2.2. Conversely, assume that A is a Gaussian ring and let $f = \sum_{i=0}^{n}(a_i, e_i)X^i$ be a polynomial in $R[X]$.

Case 1: $a_i \notin M$ for some $i = 1, \ldots, n$. In this case, a_i is invertible in A and then (a_i, e_i) is invertible in R. Hence, $C_R(f) = R$ and f is a Gaussian polynomial in $R[X]$.

Case 2: $a_i \in M$ for each $i = 1, \ldots, n$. Let $g = \sum_{i=0}^{m}(b_i, f_i)X^i \in R[X]$. We wish to prove that $C(fg) = C(f)C(g)$.

If there exists $b_i \notin M$ for some $i = 1, \ldots, m$, then by Case 1, g is a Gaussian polynomial and then $C(fg) = C(f)C(g)$.

If $b_i \in M$ for each $i = 1, \ldots, m$, we let the two polynomials of $A[X]$: $f_A = \sum_{i=0}^{n} a_i X^i$ and $g_A = \sum_{i=0}^{m} b_i X^i$. Since $ME = 0$ and since $a_i, b_j \in M$ for each $i = 1, \ldots, n$ and for each $j = 1, \ldots, m$, we obtain that $C_R(fg) = (C_A(f_A g_A) \propto (C_A(f_A g_A))E) = ((C_A(f_A g_A), 0) = ((C_A(f_A)C_A(g_A), 0)$ (since A is a Gaussian ring) $= C_R(f)C_R(g)$ and this completes the proof of Theorem 2.3.

Let $R = A \propto E$ be the trivial ring extension of a ring A by an A-module E. The following result states that, in general, R is not a Gaussian ring even if A is a Gaussian ring and E is a free A-module.

Proposition 2.4. *Let D be a local ring such that 2 is not invertible and D contains a regular element which is not invertible. Let $R = D \propto D$ be the trivial ring extension of D by D. Then R is not Gaussian.*

In particular, if D is a valuation domain such that 2 is not invertible (for example Z_{2Z}, where Z is the set of integer numbers), then R is not Gaussian.

Proof. Let (D, M) be a local ring, where M is its maximal ideal and let $a \in M$ be a regular element. Let $R := D \propto D$ be the trivial ring extension of D by D and let $f = (a, 0) + (a, 1)X$, $g = (a, 0) - (a, 1)X$ be two polynomials of $R[X]$. We claim that $C(fg) \neq C(f)C(g)$.

We have $fg = (a, 0)^2 - (a, 1)^2 X^2 = (a^2, 0) - (a^2, 2a)X^2$. Hence, $C(fg) = R(a^2, 0) + R(a^2, 2a)$. On the other hand, $C(f) = R(a, 0) + R(a, 1) = C(g)$. Thus, $C(f)C(g) = R(a^2, 0) + R(a^2, 2a) + R(a^2, a)$. We claim that $(a^2, a)(\in C(f)C(g)) \notin C(fg)$. Deny. There exists $(c, d), (e, f) \in R$ such that $(a^2, a) = (c, d)(a^2, 0) + (e, f)(a^2, 2a) = (ca^2 + ea^2, da^2 + 2ae + fa^2)$. Hence, $a = da^2 + 2ae + fa^2$ and $1 = da + 2e + fa$ since a is a regular element. Thus, $1 \in M$ since $a, 2 \in M$, a contradiction. Therefore, $C(fg) \neq C(f)C(g)$ which means that R is not Gaussian.

3. Transfer of "the content ideal of a Gaussian polynomial is locally principal" property in trivial ring extensions

In this section, we study the possible transfer of the "the content ideal of a Gaussian polynomial is locally principal" property to various trivial extension contexts. First, we examine the context of trivial ring extensions of a domain by its quotient field. This result generalizes [3, Example 2.4 (2)].

Theorem 3.1. *Let D be an integral domain, $K = qf(D)$, and $R = D \propto K$ be the trivial ring extension of D by K. Then R satisfies the property "the content ideal of a Gaussian polynomial is locally principal".*

Proof. Let $T = K \propto K$ be the trivial ring extension of K by K. Then T satisfies the property "the content ideal of a Gaussian polynomial is locally principal" since T has $M = 0 \propto K = R(0,1)$ as a unique proper ideal. Without loss of generality, we may assume that D is not a field. Let $h : T \to T/M(\cong K)$ be the canonical surjection, then $R := D \propto K = h^{-1}(D)$.

Let $f = \sum_{i=0}^{n}(a_i, e_i)X^i$ be a Gaussian polynomial of $R[X]$, where $a_i \in D$ and $e_i \in K$ for each $i = 1, \ldots, n$.

Case 1. $a_i \neq 0$ for some $i = 0, \ldots, n$. In this case, $C(f) = \sum_{i=0}^{n} R(a_i, e_i)$ is a regular ideal (since $(a, e) \in R$ is regular if and only if $a \neq 0$) and f is a regular polynomial since it is Gaussian. Then, by [3, Theorem 2.1 (1)], $C(f)$ is locally principal.

Case 2. $a_i = 0$ for each $i = 1, \ldots, n$. Our aim is to show that $C(f) = \sum_{i=0}^{n} R(0, e_i)$ is locally principal. Without loss of generality, we may assume that $e_i \in D$ since $(d, 0)C(f) = \sum_{i=0}^{n} R(0, de_i)$ and $(d, 0)$ is a regular element of R for each $d \in D - \{0\}$. Let $f_D = \sum_{i=0}^{n} e_iX^i \in D[X]$. To show that f_D is a Gaussian polynomial of $D[X]$, consider $g_D = \sum_{i=0}^{n} d_iX^i \in D[X]$ and set $g = \sum_{i=0}^{n}(d_i, 0)X^i \in R[X]$. Then $C(fg) = C(f)C(g)$ since f is a Gaussian polynomial of $R[X]$, that is $0 \propto C_D(f_Dg_D) = 0 \propto [C_D(f_D)C_D(g_D)]$. Hence, $C_D(f_Dg_D) = C_D(f_D)C_D(g_D)$ and then f_D is a Gaussian polynomial of $D[X]$.

Therefore, $C(f_D)$ is locally principal by [14, Theorem 4] since D is a domain and f_D is a Gaussian polynomial of $D[X]$. To end the proof, we have to show that $C(f) := 0 \propto \sum_{i=0}^{n} De_i$ is locally principal.

Let $P = p \propto K \in max(R)$, where $p \in max(D)$. Set $S := R - P$ and $S_1 := (D - p) \propto 0$. Then $S_1 \subseteq S$ and S, S_1 are two multiplicatives sets of R. Then, by [17, Corollary 3.79, p. 104] $C(f)_P = S^{-1}C(f) = S^{-1}[S_1^{-1}(0 \propto C_D(f_D)] = S^{-1}[0 \propto (C_D(f_D))_p]$. But, $C_D(f_D)_p = (De)_p$ for some $e \in D$ since $C_D(f_D)$ is locally principal. Therefore, $C(f)_P = S^{-1}[0 \propto (De)_p] = S^{-1}[S_1^{-1}R(0, e)] = S^{-1}R(0, e) = R(0, e)_P$ which is a principal ideal of R_P and this completes the proof of Theorem 3.1.

Remark 3.2. The hypothesis "$qf(D) = K$" is necessary in Theorem 3.1 even if D is a field (see Example 4.8).

Now, we explore the trivial ring extension R of a local ring (A, M) by an A-module E such that $ME = 0$. Remark that this ring is a total ring by the proof of [13, Theorem 2.6 (1)].

Theorem 3.3. *Let (A, M) be a local ring which is not a field and $E(\neq 0)$ an A-module with $ME = 0$. Let $R = A \propto E$ be the trivial ring extension of A by E. Then R does not satisfy the property "the content ideal of a Gaussian polynomial is locally principal".*

Proof. Let (A, M) be a local ring which is not a field, $E \neq 0$ an A-module with $ME = 0$ and let $R = A \propto E$ be the trivial ring extension of A by E. Let $a(\neq 0) \in M$, $e(\neq 0) \in E$, and let $f = (a, 0) + (0, e)X \in R[X]$. Our aim is to show that f is a Gaussian polynomial and $C(f)$ is not principal (since R is a local ring).

To show that f is a Gaussian polynomial, let $g \in R[X]$. Without loss of generality, we may assume that $C(g) \in M \propto E$ since R is a local ring (since if $C(g) = R$, then g is a Gaussian polynomial and $C(fg) = C(f)C(g)$ as desired). Then $C(f)C(g) = [R(a, 0) + R(0, e)]C(g) = R(a, 0)C(g) = C((a, 0)g) = C(fg)$ since $g \in (M \propto E)[X]$ and $(0, e)(M \propto E) = 0_R$. Hence f is a Gaussian polynomial of $R[X]$.

It remains to show that $C(f)$ is not principal. Deny. Assume that $C(f) := R(a, 0) + R(0, e) = R(b, h)$, where $(b, h) \in M \propto E$. We claim that $b \neq 0$.
Indeed, if $b = 0$, then $(a, 0) \in C(f) = R(0, h)$ implies that $a = 0$, a contradiction. Hence $b \neq 0$.

But $(0, e) \in C(f) = R(b, h)$. Hence, $(0, e) = (c, l)(b, h) = (cb, ch)$ for some $(c, l) \in R$ (since $b \in M$). Then, $cb = 0$ and $c \in M$ (since $c \notin M$ implies that c is invertible and $b = 0$, a contradiction). Therefore, $e = ch = 0$, a contradiction. Then $C(f)$ is not a principal ideal of R and this completes the proof of Theorem 3.3.

Remark 3.4. Under the hypothesis of Theorem 3.3, let $f = \sum_{i=0}^{n} a_i X^i \in A[X]$ and let $f_R = \sum_{i=0}^{n} (a_i, e_i) X^i \in R[X]$, where $a_i \in A$ and $e_i \in E$. Since $ME = 0$, we may easily show that f is a Gaussian polynomial in $A[X]$ if and only if f_R is a Gaussian polynomial in $R[X]$ for some (respectively, for each) $e_i \in E$.

If $A \subseteq E$ are two rings and $R = A \propto E$, we have the following result.

Proposition 3.5. *Let $A \subseteq E$ be two rings and $R = A \propto E$ be the trivial ring extension of A by E. Then, R satisfies the property "the content ideal of a Gaussian polynomial is locally principal" implies that so is A.*

Before proving Proposition 3.5, we establish the following Lemma.

Lemma 3.6. *Under the hypotheses of Proposition 3.5, let $f_A = \sum_{i=0}^{n} a_i X^i \in A[X]$ be a Gaussian polynomial of $A[X]$. Then $f = \sum_{i=0}^{n} (0, a_i)X^i$ is a Gaussian polynomial of $R[X]$.*

Proof. Let $g = \sum_{i=0}^{m} (b_i, h_i)X^i \in R[X]$ and set $g_A = \sum_{i=0}^{m} b_i X^i \in A[X]$. Our aim is to show that $C_R(fg) = C_R(f)C_R(g)$. But, $C_R(f) = \sum_{i=0}^{n} (A \propto E)(0, a_i) = 0 \propto C_A(f_A)$. Hence, $C_R(f)C_R(g) = 0 \propto (C_A(f_A)C_A(g_A)) = 0 \propto C_A(f_A g_A)$ since f_A is a Gaussian polynomial of $A[X]$. On the other hand, one can see that $C_R(fg) = 0 \propto C(f_A g_A)$. Therefore, $C_R(fg) = C_R(f)C_R(g)$ and f is a Gaussian polynomial in $R[X]$.

Proof of Proposition 3.5. Let $f_A = \sum_{i=0}^{n} a_i X^i \in A[X]$ be a polynomial Gaussian of $A[X]$ and set $f = \sum_{i=0}^{n} (0, a_i)X^i \in R[X]$. By Lemma 3.6, f is a Gaussian polynomial of $R[X]$ and then $C_R(f)(= 0 \propto C_A(f_A))$ is locally principal. Our aim is to show that $C_A(f_A)$ is locally principal.

Let $p \in max(A)$. To show that $C_A(f_A)_p$ is a principal ideal in A, set $P = p \propto E \in max(R)$. Then, $C_R(f)_P = R(0, e)_P$ for some $e \in C_A(f_A)$ (since $C_R(f) = 0 \propto C_A(f_A)$) since $C_R(f)$ is locally principal. Set $S := R - P$ and $S_1 := (A - p) \propto 0$. Then $S_1 \subseteq S$ are two multiplicative sets of R. Then, by [17, Corollary 3.79, p. 104] $S^{-1}R(0, e) = C_R(f)_P = S^{-1}C_R(f) = S^{-1}[S_1^{-1}(0 \propto C_A(f_A)] = S^{-1}[0 \propto (C_A(f_A))_p]$. It remains to show that $(C_A(f_A))_p = (Ae)_p$.

Since $e \in C_A(f_A)$, then $(Ae)_p \subseteq C_A(f_A))_p$. Conversely, let $(x/1) \in C_A(f_A))_p$. Then $(0, (x/1))/(1, 0) \in S^{-1}R(0, e)$ and $(0, (x/1))/(1, 0) = (a, l)(0, e)/(b, h) = (0, ae)/(b, h)$, where $(b, h) \in S$, that is $b \notin p$. Hence, there exists $(c, k) \in P$ (that is $c \notin p$) such that $(c, k)(b, h)(0, (x/1)) = (c, k)(0, ae)$ and so $(0, cb(x/1)) = (0, cae)$. Thus, $cb(x/1) = cae$ and so $(x/1) = (cae)/(cb) = (ae)/b = (a/b)(e/1) \in (Ae)_p$. Therefore, $(C_A(f_A))_p = (Ae)_p$ and this completes the proof of Proposition 3.5.

Remark 3.7. We known that a Gaussian domain R satisfy "the content ideal of a Gaussian polynomial is locally principal" Property. This results is false, in general, if R a ring with zerodivisors (see Example 3.9).

4. Examples

Recall that the small finitistic projective dimension of a ring R, denoted by $fPdim(R)$, is equal to the supremum of the projective dimensions of R-modules E, which satisfy $pd_R(E) < \infty$, and E admits a finite resolution consisting of finitely generated projective modules. In [7, Theorem 3.2], Glaz shows that a coherent Gaussian ring has $fPdim(R) \leq 1$. Now, we exhibit a class of non-coherent Gaussian rings such that $wdim(R) = \infty$ and $fPdim(R) = 0$ (respectively, $wdim(R) \geq 2$ and $fPdim(R) = 1$) (See Examples 4.1, 4.3, and 4.4).

Recall that, for non-negative integers n and d, a ring R is called an (n, d)-ring if each n-presented module of R has a projective dimension at most d. An $(n, 0)$-ring is called an n-Von Neumann regular ring. Hence, the 1-Von Neumann regular ring is the Von Neumann regular ring. It is clear that each n-Von Neumann regular ring has $fPdim(R) = 0$. For instance, see [12, 13, 15].

Example 4.1. Let (A, M) be a valuation domain which is not a field and M is its maximal ideal, E is an A-module with $ME = 0$ and let $R = A \propto E$ be the trivial ring extension of A by E. Assume that E has an infinite dimension as (A/M)-vector space. Then:

1) R is a total Gaussian ring.

2 R is not coherent.

3) $wdim(R) = \infty$.

4) $fPdim(R) = 0$.

Before proving Example 4.1, we establish the following Lemma.

Lemma 4.2. *Let A be a ring, let I ($\neq A$) be an ideal of A and let R be the trivial ring extension of A by a nonzero free (A/I)-module E. Then $fd_R(I \propto E) = \infty$. In particular, $wdim(R) = \infty$.*

Proof. Note that this proof is inspired by the proof of [15, Lemma 2.2 (1)].

Let $\{x_i \,|\, i \in L\}$ be a generating set for I and let $\{f_j \,|\, j \in H\}$ be a basis of the free (A/I)-module E. Consider the exact sequence of R-modules:

$$0 \to Ker(u) \to R^{(L)} \oplus R^{(H)} \xrightarrow{u} I \propto E \to 0$$

where, for each tuple $((a_i,e_i)_{i\in L},\,(b_j,c_j)_{j\in H}) \in R^{(L)} \oplus R^{(H)}$ (that is, $(a_i,e_i)_{i\in L} \in R^{(L)} = (A \propto E)^{(L)}$ with each $a_i \in A$ and each $e_i \in E$, and $(b_j,c_j)_{j\in H} \in R^{(H)} = (A \propto E)^{(H)}$ with each $b_j \in A$ and each $c_j \in E$), u is defined by $u((a_i,e_i)_{i\in L},(b_j,c_j)_{j\in H}) = \sum_{i\in L}(a_i,e_i)(x_i,0) + \sum_{j\in H}(b_j,c_j)(0,f_j)$
$= (\sum_{i\in L} a_i x_i, \sum_{i\in H} b_j f_j)$. (This last equality follows from the fact that $IE = 0$.) Since $\{f_j \mid j \in H\}$ is a basis of the (A/I)-module E, then $Ker(u) = (U \propto E^{(L)}) \oplus (I \propto E)^{(H)}$, where $U = \{(a_i)_{i\in L} \in A^{(L)} / \sum_{i\in L} a_i x_i = 0\}$. Therefore, $I \propto E \cong (R^{(L)} \oplus (R^{(H)})/((U \propto E^{(L)}) \oplus (I \propto E)^{(H)}) \cong (R^{(L)}/(U \propto E^{(L)})) \oplus (R/(I \propto E))^{(H)}$ as R-modules.

We claim that $I \propto E$ is not a flat R-module. Indeed, assume that $I \propto E$ is a flat R-module. Then, $R/(I \propto E)$ is also a flat R-module, since $I \propto E \cong (R^{(L)}/(U \propto E^{(L)})) \oplus (R/(I \propto E))^{(H)}$ as R-modules. Therefore, the exact sequence of R-modules:

$$0 \to I \propto E \to R \xrightarrow{v} R/(I \propto E) \to 0$$

shows that

$(I \propto E) \cap J = (I \propto E)J$ for each ideal J of R by [6, Theorem 1.2.3 (1), p.8].

Hence for $J := I \propto E$, we obtain that $I \propto E) = (I \propto E)(I \propto E) = I^2 \propto 0$ since $IE = 0$. But this contradicts the fact that $E \neq 0$.

Let $d = fd_R(I \propto E)$. We claim that $d = \infty$. We just proved that $d \neq 0$. Assume that $d \neq \infty$. Then, $fd_R(R/(I \propto E)) \leq d$ (since $I \propto E \cong (R^{(L)}/(U \propto (A/I)^{(L)})) \oplus (R/(I \propto E))^{(H)}$ as R-modules). Thus, by the exact sequence of R-modules:

$$0 \to I \propto E \to R \xrightarrow{v} R/(I \propto E) \to 0,$$

$fd_R(I \propto E) \leq d - 1$, which contradicts $fd_R(I \propto E) = d$.

Proof of Example 4.1

1) R is a Gaussian ring by Theorem 2.3 and R is a total ring by the proof of [13, Theorem 2.6 (1)].

2) R is not coherent since R is a 2-Von Neumann regular ring which is not a Von Neumann regular ring by [15, Theorem 2.1].

3) $wdim(R) = \infty$ by Lemma 4.2.

4) $fPdim(R) = 0$ since R is a $(2,0)$-ring by [15, Theorem 2.1].

Example 4.3. Let (A, M) be a valuation domain which is not a field and M is its maximal ideal and let $R = A \propto (A/M)$ be the trivial ring extension of A by A/M. Assume that M is not finitely generated. Then

1) R is a total Gaussian ring.

2 R is not coherent.

3) $wdim(R) = \infty$.

4) $fPdim(R) = 0$.

Proof. **1)** R is a Gaussian ring by Theorem 2.3 and R is a total ring by the proof of [13, Theorem 2.6 (1)].

2) R is not coherent since R is a 3-Von Neumann regular ring which is not a 2-Von Neumann regular ring by [12, Theorem 1.1].

3) $wdim(R) = \infty$ by Lemma 4.2.

4) $fPdim(R) = 0$ since R is a $(3, 0)$-ring by [12, Theorem 1.1 (1)].

Example 4.4. Let D be a Prüfer domain which is not a field, $K = qf(D)$ and $R := D \propto K$ be the trivial ring extension of D by K. Then

1) R is a Gaussian ring.

2 R is not coherent.

3) $fPdim(R) = 1$.

4) $wdim(R) \geq 2$.

Proof. **1)** R is a Gaussian ring by Theorem 2.1.

2) R is not coherent by [13, Theorem 3.1 (1)].

3) $fPdim(R) = 1$ since R is a $(2, 1)$-ring which is not a $(2, 0)$-ring by [13, Example 3.4].

4) The principal ideal $R(0, 1)$ is not flat since it is not free (since $(0, 1)R(0, 1) = (0, 0)$) and R is local. Hence, $wdim(R) \geq 2$.

The following example shows that there is no connection between the notion of n-Von Neumann regular ring with fixed Krull dimension and the "Gaussian" property.

Example 4.5. Let n be a positive integer, D be a local domain which is not a valuation domain such that $dim(D) = n$ (Krull dimension), $A = D \propto K$ where $K = qf(D)$ and let $R = A \propto E$ be the trivial ring extension of A by E where E is an A/M-vector space with infinite rank, where M is a maximal ideal of a local ring A. Then

1) R is local.

2) R is not coherent.

3) R is not Gaussian.

4) $dim(R) = n$.

5) R is a 2-Von Neumann regular ring. In particular, it is an m-Von Neumann regular ring for each $m \geq 2$.

Proof. 1) R is local by [11, Theorem 25.1].

2) R is not coherent by [15, Theorem 2.1].

3) The ring A is not Gaussian by Theorem 2.1 since D is not Gaussian (Prüfer). Hence, the ring R is not Gaussian by Theorem 2.3.

4) $dim(R) = n$ by [11, Theorem 25.1].

5) R is a 2-Von Neumann regular ring by [15, Theorem 2.1].

Now, we are able to construct a new class of rings (with zerodivisors) which are neither locally Noetherian nor locally domain where Gaussian polynomials have locally principal content.

Example 4.6. Let D be a local domain which is not a field, $K = qf(D)$ and $R := D \propto K$ be the trivial ring extension of D by K. Then

1) A Gaussian polynomials have (locally) principal content.

2 R is a local ring with zerodivisors.

3) R is not coherent. In particular, R is neither (locally) Noetherian nor (locally) domain.

Proof. 1) By Theorem 3.1.

2) R is a local ring by [11, Theorem 25.1] and with zerodivisors since $(0 \propto K)(0 \propto K) = 0_R$.

3) R is not coherent by [13, Theorem 3.1(1)]. In particular, R is neither (locally) Noetherian nor (locally) domain.

The following example shows that there is no connection between the notion of Krull dimension and the property "the content ideal of a Gaussian polynomial is locally principal".

Example 4.7. Let n be a positive integer, (A, M) be a local ring which is not a field such that $dim(A) = n$ (Krull dimension) and M is its maximal ideal, E is an A-module with $ME = 0$ and let $R = A \propto E$ be the trivial ring extension of A by E. Then

1) $dim(R) = n$ by [11, Theorem 25.1].

2) R does not satisfy the property "the content ideal of a Gaussian polynomial is locally principal" by Theorem 3.3.

The following example shows that the hypothesis "$qf(D) = K$" is necessary in Theorem 3.1 even if D is a field.

Example 4.8 [3, Example 2.3]. Let k be a proper subfield of a field K and let $R := k \propto K$ be the trivial ring extension of k by K . Then:

1) R is a local ring.

2) Each polynomial in $R[X]$ is Gaussian.

3) There exists $f \in R[X]$ such that $C(f)$ is not a (locally) principal ideal.

The following example shows a Gaussian ring R does not satisfy "the content ideal of a Gaussian polynomial is locally principal" Property in general.

Example 4.9. Let (A, M) be a valuation domain which is not a field and M is its maximal ideal, $E(\neq 0)$ is an A-module with $ME = 0$ and let $R = A \propto E$ be the trivial ring extension of A by E. Then:

1) R is a Gaussian ring by Theorem 2.3.

2) R does not satisfies the property "the content ideal of a Gaussian polynomial is locally principal" by Theorem 3.3.

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