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A COHOMOLOGICAL CHARACTERIZATION OF LEIBNIZ CENTRAL EXTENSIONS OF LIE ALGEBRAS

Naihong Hu[1]
Department of Mathematics, East China Normal University, Shanghai 200062, People's Republic of China
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy,

Yufeng Pei[2]
Department of Mathematics, East China Normal University, Shanghai 200062, People's Republic of China

and

Dong Liu[3]
Department of Mathematics, Jiaotong University, Shanghai 200030, People's Republic of China.

Abstract

Motivated by Pirashvili's spectral sequences on a Leibniz algebra, some notions such as invariant symmetric bilinear forms, dual space derivations and the Cartan-Koszul homomorphism are connected together to give a description of the second Leibniz cohomology groups with trivial coefficients of Lie algebras (as Leibniz objects), which leads to a concise approach to determining one-dimensional Leibniz central extensions of Lie algebras. As applications, we contain the discussions for some interesting classes of infinite-dimensional Lie algebras. In particular, our results include the cohomological version of Gao's main Theorem in [11] for Kac-Moody algebras and answer a question in [18].

MIRAMARE – TRIESTE

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[1]Corresponding Author. nhhu@euler.math.ecnu.edu.cn
[2]peiyufeng@gmail.com
[3]dliu@sjtu.edu.cn

1. Introduction

Leibniz algebras, introduced by Loday ([19]) in the context of cyclic homology, are non-antisymmetric generalizations of Lie algebras. There is a (co)homology theory for these algebraic objects, whose properties are similar to those of the classical Chevalley-Eilenberg cohomology theory for Lie algebras. Naturally, each Lie algebra is a Leibniz algebra by definition. Therefore it is interesting to study Leibniz (co)homology of Lie algebras, which may produce new invariants for Lie algebras. Lodder ([21]) obtained the Godbillon-Vey invariants for foliations by a Leibniz cohomology computation of a certain Lie algebra and mentioned how a Leibniz algebra arises naturally in vertex (operator) algebras. Besides this, some relationships with classical geometry have recently been discovered which could allow ones to investigate Leibniz (co)homology in noncommutative geometry and its physical interpretations (see [13, 21], etc.).

As we know, the central extension theory plays an important role in structure and representation theory of Lie algebras (see [1, 3, 4, 7, 8, 10, 15, 16, 17, 22, 24], etc.). Regard a Lie algebra as a Leibniz algebra, it is natural to ask how to determine its Leibniz central extensions and compare the differences between Leibniz and Lie central extensions in question. Loday and Pirashvili ([20]) showed that the Virasoro algebra is a universal central extension of the Witt algebra in the setting of Leibniz algebras as well. Partially motivated by this, Gao ([11]) determined the universal Leibniz central extensions for any Kac-Moody algebras and gave an interesting distinction between affine and non-affine Kac-Moody algebras by virtue of the second Leibniz homology group. A similar question for some other classes of infinite dimensional Lie algebras was discussed in [18, 27], etc. where the arguments involved unfortunately relied on some technical and tedious computations.

The aim of this note is to give a direct way to determine the one-dimensional Leibniz central extensions of Lie algebras. Our motivation is the Pirashvili's long exact sequence ([21, 23]), which is an essential technique for the calculation of Leibniz (co)homology of a Lie algebra. We observe the fact that the invariant symmetric bilinear forms and Leibniz central extensions can be well-connected by the Cartan-Koszul homomorphism (see [16]) in the sequence. While in practice, it is much easier to treat the invariant symmetric bilinear forms on a Lie algebra than to directly determine its Leibniz central extensions. On the other hand, we also notice that determining Leibniz central extensions of a Lie algebra is equivalent to treating dual space derivations of this Lie algebra (cf. [8]), which seemed lacking in enough attention for the Leibniz cases in the literature. We thus combine these two observations together to derive a concise and natural approach to studying the Leibniz central extensions of Lie algebras, which avoids complicated computations when applied to some interesting infinite dimensional Lie algebras including the Witt algebra, the twisted Heisenberg-Virasoro algebra, the Lie algebra of differential operators over a Laurant polynomial ring, Lie algebras of degenerate Block type (Virasoro-like algebra and its q-analog) and Kac-Moody algebras, etc. Our method is more conceptual in the cohomological sense.

Some notions on Leibniz algebras in [19] are collected in Section 2. In Section 3, as a variation of the Pirashvili's long exact sequence, we present a short exact sequence in question — around the description of Leibniz central extensions of Lie algebras, in terms of the invariant symmetric bilinear forms and the Cartan-Koszul homomorphism. We obtain a description of the second Leibniz cohomology group with trivial coefficients by dual space derivations (Theorem 3.5). Section 4 contains some applications of our observation. As a result, our cohomological version is related to Gao's homological version ([11]) in Kac-Moody algebras case (Theorem 4.18). In Section 5, we study the Leibniz central extensions of the quadratic Lie algebras, and construct a counterexample to address a question in [18].

2. Prerequisites on Leibniz algebras

2.1. **Leibniz algebra.** Let $\mathbb{K}$ be an algebraically closed field with $\operatorname{char}\mathbb{K} = 0$.

Definition 2.1. *A Leibniz algebra is a $\mathbb{K}$-module L with a bilinear map $[-,-] : L \times L \longrightarrow L$ satisfying the Leibniz identity*

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \quad \forall \quad x,y,z \in L. \tag{1.1}$$

The center of L is defined to be $\{z \in L \,|\, [z,L] = [L,z] = 0\}$. L is called *perfect* if $[L,L] = L$. If, in addition, $[x,x] = 0$, $\forall\, x \in L$, the Leibniz identity is equivalent to the Jacobi identity. In particular, Lie algebras are examples of Leibniz algebras.

Definition 2.2. *Let L be a Leibniz algebra over $\mathbb{K}$. M is called a representation of L if M is a $\mathbb{K}$-vector space equipped with two actions (left and right) of L, i.e., $[-,-] : L \times M \longrightarrow M$ and $[-,-] : M \times L \longrightarrow M$ satisfying*

$$\begin{aligned} &(MLL) \qquad [m,[x,y]] = [[m,x],y] - [[m,y],x], \\ &(LML) \qquad [x,[m,y]] = [[x,m],y] - [[x,y],m], \\ &(LLM) \qquad [x,[y,m]] = [[x,y],m] - [[x,m],y], \end{aligned} \tag{1.2}$$

for any $m \in M$ and $x,y \in L$.

2.2. **Cohomology of Leibniz algebras.** Let L be a Leibniz algebra over $\mathbb{K}$ and M a representation of L. Denote $C^n(L,M) := \operatorname{Hom}_{\mathbb{K}}(L^{\otimes n}, M)$, $n \geq 0$. The Loday coboundary map $d^n : C^n(L,M) \to C^{n+1}(L,M)$ is defined by

$$\begin{aligned} (d^n f)(x_1,\cdots,x_{n+1}) &= [x_1, f(x_2 \otimes \cdots x_{n+1})] \\ &+ \sum_{i=2}^{n+1} (-1)^i [f(x_1,\cdots,\hat{x}_i,\cdots,x_{n+1}), x_i] \\ &+ \sum_{1\leq i<j\leq n+1} (-1)^{j+1} f(x_1,\cdots,x_{i-1},[x_i,x_j],x_{i+1}\cdots,\hat{x}_j,\cdots,x_{n+1}). \end{aligned}$$

Clearly, $d^{n+1}d^n = 0,\ n \geq 0$. $(C^*(L, M), d)$ is a well-defined cochain complex, whose cohomology is called the cohomology of Leibniz algebra L with coefficients in the representation M:

$$\mathrm{HL}^*(L, M) := \mathrm{H}^*((C^*(L, M), d)).$$

Similarly, we have a chain complex $(C_*(L, M), d)$, whose homology is called the homology of Leibniz algebra L with coefficients in the representation M:

$$\mathrm{HL}_*(L, M) := \mathrm{H}_*((C_*(L, M), d)).$$

2.3. **Cohomology of Lie algebras.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and M a $\mathfrak{g}$-module. Denote the Chevalley-Eilenberg cochain complex by

$$(\Omega^*(\mathfrak{g}, M),\ \delta) := (\mathrm{Hom}(\wedge^*\mathfrak{g}, M),\ \delta),$$

where δ is the Chevalley-Eilenberg coboundary map defined by

$$\begin{aligned}(\delta^n f)(x_1, \cdots, x_{n+1}) &= \sum_{i=1}^{n+1}(-1)^{i+1}x_i \cdot f(x_1, \cdots, \hat{x}_i, \cdots, x_{n+1}) \\ &+ \sum_{1\leq i<j\leq n+1}(-1)^{i+j}f([x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{n+1}).\end{aligned}$$

Then,

$$\mathrm{H}^*(\mathfrak{g}, M) := \mathrm{H}^*((\Omega^*(\mathfrak{g}, M), \delta)),$$

is called the cohomology of Lie algebra $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module M.

3. Leibniz cohomology of Lie algebras and Leibniz central extensions

3.1. **Central extensions of Leibniz algebras.** A central extension of L is a pair $(\hat{L}, \pi)$, where $\hat{L}$ is a Leibniz algebra and $\pi : \hat{L} \longrightarrow L$ is a surjective homomorphism whose kernel lies in the center of $\hat{L}$. The pair $(\hat{L}, \pi)$ is a universal central extension of L if for every central extension $(\tilde{L}, \tau)$ of L, there is unique homomorphism $\psi : \hat{L} \longrightarrow \tilde{L}$ for which $\tau \circ \psi = \pi$.

The following result is known.

Proposition 3.1. ([20]) *There exists a one-to-one correspondence between the set of equivalent classes of one-dimensional Leibniz central extensions of L by $\mathbb{K}$ and the second Leibniz cohomology group* $\mathrm{HL}^2(L, \mathbb{K})$.

3.2. **Invariant symmetric bilinear forms.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and M a $\mathfrak{g}$-module. Recall that a symmetric bilinear forms ϕ on $\mathfrak{g}$ is called $\mathfrak{g}$-invariant if ϕ satisfies $\phi([x, y], z) = \phi(x, [y, z]),\ \forall\ x, y, z \in \mathfrak{g}$. Let $\mathrm{B}(\mathfrak{g}, \mathbb{K})$ stand for the set of all $\mathbb{K}$-valued symmetric $\mathfrak{g}$-invariant bilinear forms on Lie algebra $\mathfrak{g}$.

The short exact sequence below is a variation of the first 5-terms of the piece of Pirashvili's long exact sequence (see [23]).

Proposition 3.2. *For any Lie algebra* $\mathfrak{g}$*, there holds the following exact sequence*

$$0 \longrightarrow \mathrm{H}^2(\mathfrak{g}, \mathbb{K}) \xrightarrow{f} \mathrm{HL}^2(\mathfrak{g}, \mathbb{K}) \xrightarrow{g} \mathrm{B}(\mathfrak{g}, \mathbb{K}) \xrightarrow{h} \mathrm{H}^3(\mathfrak{g}, \mathbb{K}),$$

where

- f *is the natural embedding map.*
- g *is defined by* $g(\alpha)(x, y) = \alpha(x, y) + \alpha(y, x)$, $\forall\, \alpha \in \mathrm{HL}^2(\mathfrak{g}, \mathbb{K})$, $x,\, y \in \mathfrak{g}$.
- h *is the Cartan-Koszul map* ([16]) *defined by* $h(\alpha)(x, y, z) = \alpha([x, y], z)$, $\forall\, \alpha \in \mathrm{B}(\mathfrak{g}, k)$, $x,\, y,\, z \in \mathfrak{g}$. □

Corollary 3.3.

$$\frac{\mathrm{HL}^2(\mathfrak{g}, \mathbb{K})}{\mathrm{H}^2(\mathfrak{g}, \mathbb{K})} = ker(h).$$

In particular, $\mathrm{HL}^2(\mathfrak{g}, \mathbb{K}) = \mathrm{H}^2(\mathfrak{g}, \mathbb{K}), \iff ker(h) = 0.$

Remark 3.4. *Note that the natural embedding* $\Omega^*(\mathfrak{g}, M) \hookrightarrow C^*(\mathfrak{g}, M)$ *induces a short exact sequence in the category of cochain complexes of Lie algebra* $\mathfrak{g}$*:*

$$0 \longrightarrow \Omega^*(\mathfrak{g}, M) \longrightarrow C^*(\mathfrak{g}, M) \longrightarrow C^*_{rel}(\mathfrak{g}, M)[2] \longrightarrow 0,$$

where

$$C^*_{rel}(\mathfrak{g}, M)[2] := \frac{C^*(\mathfrak{g}, M)}{\Omega^*(\mathfrak{g}, M)}$$

is the quotient cochain complex. One has the Pirashvili's long exact sequence below (see [21], [23]*), which is the main tool to compare the Lie and the Leibniz cohomology of a Lie algebra in higher dimensions*

$$0 \to \mathrm{H}^2(\mathfrak{g}, M) \to \mathrm{HL}^2(\mathfrak{g}, M) \to \mathrm{H}^0_{rel}(\mathfrak{g}, M) \to \mathrm{H}^3(\mathfrak{g}, M) \to \mathrm{HL}^3(\mathfrak{g}, M) \to \cdots$$

and isomorphisms $\mathrm{HL}^i(\mathfrak{g}, M) = \mathrm{H}^i(\mathfrak{g}, M), \quad i = 0, 1.$

Let $M = \mathbb{K}$*. We have*

$$0 \to \mathrm{H}^2(\mathfrak{g}, \mathbb{K}) \to \mathrm{HL}^2(\mathfrak{g}, \mathbb{K}) \to \mathrm{H}^0_{rel}(\mathfrak{g}, \mathbb{K}) \to \mathrm{H}^3(\mathfrak{g}, \mathbb{K}) \to \mathrm{HL}^3(\mathfrak{g}, \mathbb{K}) \to \cdots.$$

As a consequence (which is not clear for us) of the spectral sequences, Pirashvili claimed (with no proof, see [23] *)* : $\mathrm{H}^0_{rel}(\mathfrak{g}, \mathbb{K}) \cong \mathrm{B}(\mathfrak{g}, \mathbb{K})$*, while we contain a direct elementary proof of Proposition 3.2 in the appendix.*

3.3. **Dual space derivations.** Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and M a $\mathfrak{g}$-module. Recall that a linear map $d : \mathfrak{g} \longrightarrow M$ is called a derivation if

$$d([x, y]) = x \cdot d(y) - y \cdot d(x), \quad \forall\, x,\, y \in \mathfrak{g}.$$

The derivations of the form $x \mapsto x \cdot m$ for some $m \in M$ are called inner derivations. $\mathrm{Der}(\mathfrak{g}, M)$ and $\mathrm{Inn}(\mathfrak{g}, M)$ denote the spaces of derivations and inner derivations, respectively. It is clear that $\mathrm{H}^1(\mathfrak{g}, M) = \mathrm{Der}(\mathfrak{g}, M)/\mathrm{Inn}(\mathfrak{g}, M)$ is the first cohomology group of $\mathfrak{g}$ with coefficients in M.

Theorem 3.5. *For any Lie algebra* $\mathfrak{g}$*, there holds*

$$\mathrm{HL}^2(\mathfrak{g},\mathbb{K}) = \mathrm{H}^1(\mathfrak{g},\mathfrak{g}^*) = \mathrm{Der}(\mathfrak{g},\mathfrak{g}^*)\,/\,\mathrm{Inn}(\mathfrak{g},\mathfrak{g}^*),$$

where $\mathfrak{g}^*$ *is the dual* $\mathfrak{g}$*-module.*

Proof. Let the map $\theta : \mathrm{H}^1(\mathfrak{g},\mathfrak{g}^*) \to \mathrm{HL}^2(\mathfrak{g},\mathbb{K})$ defined by $\theta(\alpha)(x,y) = \alpha(y)(x)$ for any $\alpha \in \mathrm{H}^1(\mathfrak{g},\mathfrak{g}^*)$ and $x,\, y \in \mathfrak{g}$. Since

$$\begin{aligned}\theta(\alpha)(x,[y,z]) &= \alpha([y,z])(x) = (y\cdot\alpha(z))(x) - (z\cdot\alpha(y))(x)\\ &= \alpha(z)([x,y]) - \alpha(y)([x,z])\\ &= \theta(\alpha)([x,y],z) + \theta(\alpha)([z,x],y),\end{aligned}$$

$\theta(\alpha) \in \mathrm{HL}^2(\mathfrak{g},\mathbb{K})$. θ is well-defined. If $\theta(\alpha) = \bar{0}$ in $\mathrm{HL}^2(\mathfrak{g},\mathbb{K})$, there exists an element β in $C^1(\mathfrak{g},\mathbb{K})$ such that, for any $x, y \in \mathfrak{g}$, $d^1(\beta)(x,y) = \theta(\alpha)(x,y)$. Then

$$-\beta([x,y]) = \alpha(y)(x),$$

which implies α is a 1-coboundary in $\Omega^1(\mathfrak{g},\mathfrak{g}^*)$, i.e. $\alpha = \bar{0} \in \mathrm{H}^1(\mathfrak{g},\mathfrak{g}^*)$. Hence, θ is injective. On the other hand, for any $\beta \in \mathrm{HL}^2(\mathfrak{g},\mathbb{K})$, we define a map $\alpha \in \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{g}^*)$ by $\alpha(x)(y) := \beta(y,x)$, for any $x, y \in \mathfrak{g}$. Since

$$\begin{aligned}\alpha([x,y])(z) &= \beta(z,[x,y]) = \beta([z,x],y) - \beta([z,y],x) = \alpha(y)([z,x]) - \alpha(x)([z,y]),\\ &= (x\cdot\alpha(x))(z) - (y\cdot\alpha(x))(z),\end{aligned}$$

for any $z \in \mathfrak{g}$, we get $\alpha([x,y]) = x\cdot\alpha(x) - y\cdot\alpha(x)$ for any $x, y \in \mathfrak{g}$, which means $\alpha \in \mathrm{H}^1(\mathfrak{g},\mathfrak{g}^*)$. Hence, θ is surjective. □

Denote $\mathrm{SDer}(\mathfrak{g},\mathfrak{g}^*) := \{\phi \in \mathrm{Der}(\mathfrak{g},\mathfrak{g}^*) \mid \phi(x)(y) + \phi(y)(x) = 0,\ \forall\ x,\, y \in \mathfrak{g}\}$. By Theorem 3.5 above and Proposition 1.3 (2) in [7], we have

Corollary 3.6. $\mathrm{HL}^2(\mathfrak{g},\mathbb{K})/\,\mathrm{H}^2(\mathfrak{g},\mathbb{K}) = \mathrm{Der}(\mathfrak{g},\mathfrak{g}^*)\,/\,\mathrm{SDer}(\mathfrak{g},\mathfrak{g}^*)\big(\subset H^0_{rel}(\mathfrak{g},\mathbb{K})\big)$.

Corollary 3.7. *If* $\mathfrak{g}$ *is a finite dimensional Lie algebra over* $\mathbb{K}$ *with a nondegenerate invariant symmetric bilinear form* ψ*, then* $\mathrm{HL}^2(\mathfrak{g},\mathbb{K}) = \mathrm{Der}(\mathfrak{g},\mathfrak{g})\,/\,\mathrm{Inn}(\mathfrak{g},\mathfrak{g})$. *If, in addition,* $\mathfrak{g}$ *is simple, then* $\mathrm{HL}^2(\mathfrak{g},\mathbb{K}) = 0$.

Proof. It follows from $\mathfrak{g} \simeq \mathfrak{g}^*$ as $\mathfrak{g}$-modules. □

More generally, due to Farnsteiner ([9], Theorem 3.1), we have

Corollary 3.8. *Let* $\mathfrak{g}$ *be a Lie algebra with a nondegenerate invariant symmetric bilinear form* ψ*. Assume that* $\mathfrak{h} \subset \mathfrak{g}$ *a finite dimensional diagonalizable subalgebra,* $\mathfrak{g} = \bigoplus_\alpha \mathfrak{g}_\alpha$ *the root space decomposition with* $\dim \mathfrak{g}_\alpha < \infty$*. Then* $\mathrm{HL}^2(\mathfrak{g},\mathbb{K}) = \mathrm{H}^1(\mathfrak{g},\mathfrak{g})$. □

4. Applications: Leibniz central extensions of some Lie algebras

4.1. **Lie algebras of Virasoro type.** Let $\mathbb{K}[t,t^{-1}]$ be the Laurent polynomial algebra over $\mathbb{K}$ and $\frac{d}{dt}$ be the differential operator on $\mathbb{K}[t,t^{-1}]$.

Definition 4.1. *Set $L_n = -t^{n+1}\frac{d}{dt}, n \in \mathbb{Z}$. The Witt algebra $\mathcal{W} = \bigoplus_{n\in\mathbb{Z}} \mathbb{K}L_n$ is defined by $[L_m, L_n] = (m-n)L_{m+n},\ m,\, n \in \mathbb{Z}$.*

It is well-known that $\mathrm{H}^2(\mathcal{W},\mathbb{K}) = \mathrm{H}_2(\mathcal{W},\mathbb{K}) = \mathbb{K}\alpha$, where

$$\alpha(L_m, L_n) = \delta_{m+n},0\frac{m^3-m}{12}.$$

Definition 4.2. ([1]) *Set $L_n = -t^{n+1}\frac{d}{dt}, I_n = t^n, n \in \mathbb{Z}$. The Lie algebra*

$$\mathcal{H} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \bigoplus \bigoplus_{m\in\mathbb{Z}} \mathbb{C}I_m$$

is defined by $[L_m, L_n] = (m-n)L_{m+n},\ [I_m, I_n] = 0,\ [L_m, I_n] = -nI_{m+n},\ m,\, n \in \mathbb{Z}$.

In [1], the authors proved that $\dim \mathrm{H}^2(\mathcal{H},\mathbb{K}) = \dim \mathrm{H}_2(\mathcal{H},\mathbb{K}) = 3$. In fact, as the universal central extension of $\mathcal{H}$, the twisted Heisenberg-Virasoro algebra H_{Vir} has a basis $\{I_m, L_m, C_I, C_L, C_{LI} \mid m \in \mathbb{Z}\}$ satisfying the following relations:

$$\begin{aligned}
[I_m, I_n] &= n\delta_{m+n,0}C_I,\\
[L_m, L_n] &= (m-n)L_{m+n} + \delta_{m+n,0}\frac{1}{12}(m^3-m)C_L,\\
[L_m, I_n] &= -nI_{m+n} + \delta_{m+n,0}(m^2-m)C_{LI},\\
[H_{Vir}, C_L] &= [H_{Vir}, C_I] = [H_{Vir}, C_{LI}] = 0.
\end{aligned}$$

Definition 4.3. *Set $D = t\frac{d}{dt}$. The Lie algebra of differential operators*

$$\mathcal{D} = \bigoplus_{m\in\mathbb{Z}, n\in\mathbb{Z}_+} \mathbb{K}t^mD^n$$

is defined by $[\,t^mD^r, t^nD^s\,] = t^{m+n}((D+n)^r - (D+m)^s),\quad m,\, n \in \mathbb{Z},\ r,\ s \in \mathbb{Z}_+$.

It is known [17] that $\dim \mathrm{H}^2(\mathcal{D},\mathbb{K}) = \dim \mathrm{H}_2(\mathcal{D},\mathbb{K}) = 1$ and the universal central extension of $\mathcal{D}$ is a Lie algebra

$$\mathcal{W}_{1+\infty} = \bigoplus_{m\in\mathbb{Z}, n\in\mathbb{Z}_+} \mathbb{K}t^mD^n \bigoplus \mathbb{K}C$$

with relations: $[t^mD^r, t^nD^s] = t^{m+n}((D+n)^r - (D+m)^s) + \psi(t^mD^r, t^nD^s)C$, where $\psi(t^{m+r}D^r, t^{n+s}D^s) = \delta_{m+n,0}(-1)^r r!s!\binom{m+r}{r+s+1}$, for $m,\, n \in \mathbb{Z},\ r,\ s \in \mathbb{Z}_+$.

Remark 4.4. *It is clear that $\mathcal{W}$ and $\mathcal{H}$ are Lie subalgebras of $\mathcal{D}$.*

Proposition 4.5. *There is no non-trivial invariant symmetric bilinear form on Lie algebra $\mathfrak{g}$, where $\mathfrak{g} = \mathcal{W}, \mathcal{H},$ or $\mathcal{D}$.*

Proof. Assume that f is an invariant symmetric bilinear form on $\mathfrak{g}$.

(1) $\mathfrak{g} = \mathcal{W}$.

Note that $\mathcal{W}$ is generated as Lie algebra by the elements L_3, L_{-2} with

$$f(L_3, L_3) = \frac{1}{3} f(L_3, [L_3, L_0]) = 0,$$
$$f(L_3, L_{-2}) = \frac{1}{8} f(L_3, [L_3, L_{-5}] = 0,$$
$$f(L_{-2}, L_{-2}) = -\frac{1}{2} f(L_{-2}, [L_{-2}, L_0]) = 0.$$

(2) $\mathfrak{g} = \mathcal{H}$.

Note that $\mathcal{H}$ is generated as Lie algebra by the elements L_3, L_{-2} and I_1 with

$$f(L_m, L_n) = 0, \quad n, m \in \mathbb{Z},$$
$$f(I_1, I_1) = f(I_1, [I_1, L_0]) = f([I_1, I_1], L_0) = 0,$$
$$f(I_1, L_3) = -\frac{1}{2} f([I_{-2}, L_3], L_3) = -\frac{1}{2} f(I_{-2}, [L_3, L_3]) = 0,$$
$$f(I_1, L_{-2}) = \frac{1}{3} f([I_3, L_{-2}], L_{-2}) = \frac{1}{3} f(I_3, [L_{-2}, L_{-2}]) = 0.$$

(3) $\mathfrak{g} = \mathcal{D}$.

Note that $\mathcal{D}$ can be generated as Lie algebra by t, t^{-1} and D^2 (see [25]).

By a direct computation, we have

$$t^2 (\frac{d}{dt})^2 = D^2 - D,$$
$$t^m (\frac{d}{dt})^n = \frac{1}{n+1} \Big[t^m (\frac{d}{dt})^{n+1}, t \Big], \quad m \in \mathbb{Z}, n \in \mathbb{Z}_+.$$

By (1) and (2), $f(t^{\pm 1}, t^{\pm 1}) = f(t^{\pm 1}, D) = f(D, D) = 0$.

$$\begin{aligned} f\Big(t(\frac{d}{dt})^2, t^2(\frac{d}{dt})^2\Big) &= f\Big(t(\frac{d}{dt})^2, -\frac{1}{3}[t, t^2(\frac{d}{dt})^3]\Big) = -\frac{1}{3} f\Big([t(\frac{d}{dt})^2, t], t^2(\frac{d}{dt})^3\Big) \\ &= -\frac{2}{3} f\Big(t\frac{d}{dt}, t^2(\frac{d}{dt})^3\Big) = -\frac{2}{3} f\Big(t\frac{d}{dt}, -\frac{1}{4}[t, t^2(\frac{d}{dt})^4]\Big) \\ &= \frac{1}{6} f\Big([t\frac{d}{dt}, t], t^2(\frac{d}{dt})^4\Big) = \frac{1}{6} f\Big(t, t^2(\frac{d}{dt})^4\Big) \\ &= \frac{1}{6} f\Big(t, -\frac{1}{5}[t, t^2(\frac{d}{dt})^5]\Big) = -\frac{1}{30} f\Big([t, t], t^2(\frac{d}{dt})^5\Big) \\ &= 0. \end{aligned}$$

Similarly, we have $f(D, t^2(\frac{d}{dt})^2) = 0$. Then,

$$\begin{aligned} f(D^2, D^2) &= f\Big(D + t^2(\frac{d}{dt})^2, D + t^2(\frac{d}{dt})^2\Big) \\ &= f(D, D) + 2f\Big(D, t^2(\frac{d}{dt})^2\Big) + f\Big(t^2(\frac{d}{dt})^2, t^2(\frac{d}{dt})^2\Big) \\ &= 2f\Big(D, t^2(\frac{d}{dt})^2\Big) + f\Big(t^2(\frac{d}{dt})^2, t^2(\frac{d}{dt})^2\Big) \\ &= 0. \end{aligned}$$

In summary, $B(\mathfrak{g}, \mathbb{K}) = 0$ for $\mathfrak{g} = \mathcal{W}$, $\mathcal{H}$, or $\mathcal{D}$. □

Corollary 4.6. $\mathrm{HL}^2(\mathfrak{g}, \mathbb{K}) = \mathrm{H}^2(\mathfrak{g}, \mathbb{K})$, *for* $\mathfrak{g} = \mathcal{W}$, $\mathcal{H}$, *or* $\mathcal{D}$.

Remark 4.7. *In a different way, for* $\mathfrak{g} = \mathcal{W}$, *Corollary 4.6 was obtained in* [20]. *For* $\mathfrak{g} = \mathcal{D}$, *Corollary 4.6 was obtained in* [18].

4.2. **Lie algebras of Block type.**

Definition 4.8. ([6]) *Let* A *be a torsion-free abelian group.* $\phi : A \times A \longrightarrow \mathbb{K}$ *is a non-degenerate, skew-symmetric,* $\mathbb{Z}$*-bilinear function. Then we have the degenerate Block algebra*

$$\mathcal{L}(A,\phi) = \bigoplus_{x \in A - \{0\}} \mathbb{K}e_x$$

with Lie bracket $[e_x, e_y] = \phi(x,y)e_{x+y}$.

Definition 4.9. *Let* $A = \mathbb{Z} \times \mathbb{Z}$ *and* $\phi((m,n),(m_1,n_1)) = nm_1 - mn_1$. *It is clear that* ϕ *is a skew-symmetric bi-additive function. Then we have the Virasoro-like algebra* (*see* [15])

$$\mathcal{V} = \bigoplus_{(m,n)\in \mathbb{Z}\times\mathbb{Z} - \{(0,0)\}} \mathbb{K}e_{m,n}$$

with Lie bracket $[e_{m,n}, e_{m_1,n_1}] = (nm_1 - mn_1)e_{m+n,m_1+n_1}$.

Definition 4.10. *Let* $A = \mathbb{Z} \times \mathbb{Z}$, $\phi((m,n),(m_1,n_1)) = q^{nm_1} - q^{mn_1}$ *with a fixed* $q \in \mathbb{K}^*$ *being non-root of unity. It is clear that* ϕ *is a skew-symmetric bi-additive function. Then we have the* q*-analogue Virasoro-like algebra* (*see* [15])

$$\mathcal{V}_q = \bigoplus_{(m,n)\in \mathbb{Z}\times\mathbb{Z} - \{(0,0)\}} \mathbb{K}e_{m,n}$$

with Lie bracket $[e_{m,n}, e_{m_1,n_1}] = (q^{nm_1} - q^{mn_1})e_{m+n,m_1+n_1}$.

Lemma 4.11. ([27]) $\mathrm{B}(\mathcal{L}(A,\phi),\mathbb{K}) = \mathbb{K}\alpha$, *where* $\alpha(e_x, e_y) = \delta_{x+y,0}$.

Proposition 4.12. $\mathrm{HL}^2(\mathcal{L}(A,\phi),\mathbb{K}) = \mathrm{H}^2(\mathcal{L}(A,\phi),\mathbb{K})$.

Proof. By Corollary 3.3 and Lemma 4.11, it suffices to prove the image of α under the Cartan-Koszul homomorphism h is non-zero. If $h(\alpha) = \bar{0} \in \mathrm{H}^3(\mathfrak{g},\mathbb{K})$, there exists a skew-symmetric bilinear form ψ on $\mathfrak{g}$ such that

$$\begin{aligned} h(\alpha)(e_x,e_y,e_z) &= \alpha([e_x,e_y],e_z) = d(\psi)(e_x,e_y,e_z) \\ &= \psi(e_x,[e_y,e_z]) + \psi(e_y,[e_z,e_x]) + \psi(e_z,[e_x,e_y]) \end{aligned}$$

for all $e_x, e_y, e_z \in \mathcal{L}(A,\phi)$. Let $x + y + z = 0$ and $\phi(x,y) \neq 0$. Then

$$\phi(x,y)\psi(e_x,e_{-x}) + \phi(x,y)\psi(e_y,e_{-y}) + \phi(x,y)\psi(e_z,e_{-z}) = \phi(x,y),$$
$$\psi(e_x,e_{-x}) + \psi(e_y,e_{-y}) + \psi(e_z,e_{-z}) = 1.$$

On the other hand, since $-x - y - z = 0$, the above identity holds for $-x, -y, -z$ instead of x, y, z. Using the skew-symmetry of ψ, we have

$$\psi(e_x,e_{-x}) + \psi(e_y,e_{-y}) + \psi(e_z,e_{-z}) = -1.$$

However, this is impossible. Therefore, $h(\alpha) \neq \bar{0}$. □

From [6], it follows

Corollary 4.13.

$$\mathrm{HL}^2(\mathcal{L}(A,\phi),\mathbb{K}) = \{[\alpha_\mu] \mid \alpha_\mu(x,y) = \delta_{x+y,0}\mu(x), \forall\ x,\ y \in A,\ \ \mu \in \mathrm{Hom}_{\mathbb{K}}(A,\mathbb{K})\}.$$

Remark 4.14. *In a different way, Corollary 4.13 was obtained by* [27]*. For the (q-analogue) Virasoro-like algebras, Corollary 4.13 was obtained by* [18]*.*

4.3. **Kac-Moody algebras.**

Lemma 4.15. *For a Lie algebra* $\mathfrak{g}$ *with* $\dim \mathrm{B}(\mathfrak{g},\mathbb{K}) \le 1$*, in both cases below:*

- *if* $\mathrm{B}(\mathfrak{g},\mathbb{K}) = 0$; *or*
- *if* $\mathrm{B}(\mathfrak{g},\mathbb{K}) = \mathbb{K}\phi$ $(\phi \neq 0)$*, and there exists a subalgebra* $\mathfrak{a} \cong \mathfrak{sl}(2,\mathbb{K})$ *such that* $\phi|_{\mathfrak{a}} \neq 0$*,*

then $\mathrm{HL}^2(\mathfrak{g},\mathbb{K}) = \mathrm{H}^2(\mathfrak{g},\mathbb{K})$.

Proof. The first case is clear by Corollary 3.3. Now for nonzero $\phi \in \mathrm{B}(\mathfrak{g},\mathbb{K})$, if there is a subalgebra $\mathfrak{a}$ as a $\mathfrak{sl}(2,\mathbb{K})$-copy such that $\phi|_{\mathfrak{a}} \neq 0$, then $\phi|_{\mathfrak{a}}$ is a nonzero scalar multiple of the Killing form on $\mathfrak{a}$. Let h be the Cartan-Koszul homomorphism: $h(\phi)(a,b,c) = \phi([a,b],c)$ for any $a,b,c \in \mathfrak{g}$. If $h(\phi) = \bar{0}$, then by definition of $H^3(\mathfrak{g},\mathbb{K})$, there exists a skew-symmetric bilinear form θ on $\mathfrak{g}$ such that $\delta^2(\theta) = h(\phi)$, i.e., $\theta(a,[b,c]) + \theta(b,[c,a]) + \theta(c,[a,b]) = \phi([a,b],c)$, for $a,\ b,\ c \in \mathfrak{g}$. Take $x,\ y,\ h \in \mathfrak{a}$ satisfying $[x,y] = h,\ [h,y] = -2y,\ [h,x] = 2x$. Then

$$\theta(x,[y,h]) + \theta(y,[h,x]) + \theta(h,[x,y]) = \phi([x,y],h),$$
$$2\theta(x,y) + 2\theta(y,x) + \theta(h,h) = \phi(h,h).$$

Therefore, $\phi(h,h) = 0$. It is contradict to the property of the Killing form. This fact means that $\ker(h) = 0$, which gives rise to the required result. □

Lemma 4.16. $\mathrm{HL}^2(\mathfrak{g},\mathbb{K}) = \mathrm{Hom}(\mathrm{HL}_2(\mathfrak{g},\mathbb{K}),\mathbb{K})$*, for any perfect Leibniz algebra* $\mathfrak{g}$*.*

Proof. Following the universal coefficient theorem for Leibniz algebras in [5], one has the following short exact sequence

$$0 \longrightarrow \mathrm{Ext}(\mathrm{HL}_1(\mathfrak{g},\mathbb{K}),\mathbb{K}) \longrightarrow \mathrm{HL}^2(\mathfrak{g},\mathbb{K}) \longrightarrow \mathrm{Hom}(\mathrm{HL}_2(\mathfrak{g},\mathbb{K}),\mathbb{K}) \longrightarrow 0,$$

where $\mathrm{Ext}(\mathrm{HL}_1(\mathfrak{g},\mathbb{K}),\mathbb{K})$ denotes the abelian extension of $\mathrm{HL}_1(\mathfrak{g},\mathbb{K})$ by $\mathbb{K}$. As a result, it is clear that $\mathrm{Ext}(\mathrm{HL}_1(\mathfrak{g},\mathbb{K}),\mathbb{K}) = 0$ since $\mathrm{HL}_1(\mathfrak{g},\mathbb{K}) = \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] = 0$. □

Let $\mathfrak{g}$ be a perfect Lie algebra ($[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$) over $\mathbb{K}$. Then, by [12], there exist a universal central extension $\pi : \hat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ in the category of Leibniz algebras, and a universal central extension $\tilde{\pi} : \tilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ in the category of Lie algebras. And we have the following

Lemma 4.17. $\mathrm{HL}_2(\tilde{\mathfrak{g}},\mathbb{K}) = \ker\{\mathrm{HL}_2(\mathfrak{g},\mathbb{K}) \twoheadrightarrow \mathrm{H}_2(\mathfrak{g},\mathbb{K})\}$.

Proof. See 4.6 in [20] or Corollary 2.7 in [12]. □

Let R be a unital, commutative and associative algebra over $\mathbb{K}$. The R-module of Kähler differentials $\Omega^1_{R|\mathbb{K}}$ is generated by $\mathbb{K}$-linear symbols $\mathrm{d}\,a$ for $a \in R$ with the relation $\mathrm{d}(ab) = a\,\mathrm{d}\,b + b\,\mathrm{d}\,a$, for any $a, b \in R$. In particular, if $R = \mathbb{K}[t, t^{-1}]$, then $\Omega^1_{R|\mathbb{K}} = \bigoplus_{m\in\mathbb{Z}} \mathbb{K}t^m\,\mathrm{d}\,t$. For any positive integer r, let

$$\Omega^1_{R|\mathbb{K}}(r) = \bigoplus_{i\in\mathbb{Z}} \mathbb{K}t^{ir-1}\,\mathrm{d}\,t, \quad \Omega'^1_{R|\mathbb{K}}(r) = \bigoplus_{i\in\mathbb{Z}-\{0\}} \mathbb{K}t^{ir-1}\,\mathrm{d}\,t$$

be two $\mathbb{K}$-subspaces of $\Omega^1_{R|\mathbb{K}}$, and $\Omega'^1_{R|\mathbb{K}}(r)$ is a subspace of $\Omega^1_{R|\mathbb{K}}(r)$ of codimension of 1.

Theorem 4.18. *Let $A = (a_{ij})_{1\le i,j\le n}$ denote an $n\times n$-matrix of rank ℓ with entries in $\mathbb{K}$. Using the notations in* [14]*, denote by $\mathfrak{g}(A)$ the Kac-Moody Lie algebra associated to A. Let $\mathfrak{g}'(A)$ be the derived algebra of $\mathfrak{g}(A)$, $\mathfrak{c}$ the center of $\mathfrak{g}'(A)$, and $\bar{\mathfrak{g}}(A) = \mathfrak{g}'(A)/\mathfrak{c}$. Thus,*

(1) *If $a_{ii} \neq 0,\ 1 \le i \le n$, then $\dim \mathrm{HL}^2(\mathfrak{g}(A), \mathbb{K}) = (n-\ell)^2$;*

(2) *If A is an indecomposable generalized Cartan matrix of affine $X_n^{(r)}$ type, then*

$$\mathrm{HL}^2(\bar{\mathfrak{g}}(A), \mathbb{K}) = (\Omega^1_{R|\mathbb{K}}(r))^*, \qquad \mathrm{HL}^2(\mathfrak{g}'(A), \mathbb{K}) = (\Omega'^1_{R|\mathbb{K}}(r))^*.$$

(3) *If A is an indecomposable generalized Cartan matrix of non-affine type, then*

$$\mathrm{HL}^2(\bar{\mathfrak{g}}(A), \mathbb{K}) = \mathfrak{c}^*, \qquad \mathrm{HL}^2(\mathfrak{g}'(A), \mathbb{K}) = 0.$$

Proof. (1) Theorem 3.2 in [8] states that if $a_{ii} \neq 0,\ 1 \le i \le n$, then

$$\dim \mathrm{H}^1(\mathfrak{g}(A), (\mathfrak{g}(A))^*) = (n-\ell)^2.$$

By Theorem 3.5, we know $\mathrm{HL}^2(\mathfrak{g}(A), \mathbb{K}) = \mathrm{H}^1(\mathfrak{g}(A), (\mathfrak{g}(A))^*)$. Hence

$$\dim \mathrm{HL}^2(\mathfrak{g}(A), \mathbb{K}) = (n-\ell)^2.$$

(2) We assume that A is an indecomposable generalized Cartan matrix of affine $X_n^{(r)}$ type. By Gabber-Kac's radical Theorem (Theorem 9.11 and the following remarks in [14], pp. 159), we know that $\mathfrak{g}'(A)$ and $\bar{\mathfrak{g}}(A)$ can be presented in term of $3n$ generators f_i, h_i, e_i $(1 \le i \le n)$ and the Chevalley-Serre relations. By Theorem 3.17 in [11][4], which states that $\mathrm{HL}_2(\bar{\mathfrak{g}}(A), \mathbb{K}) = \Omega^1_{R|\mathbb{K}}(r)$, we have by Lemma 4.16

$$\mathrm{HL}^2(\bar{\mathfrak{g}}(A), \mathbb{K}) = (\Omega^1_{R|\mathbb{K}}(r))^*.$$

It is well known that $\mathfrak{g}'(A)$ is the universal covering of $\bar{\mathfrak{g}}(A)$ in the category of Lie algebras (see [24]), that is, we have the following exact sequence:

$$0 \longrightarrow \mathrm{H}_2(\bar{\mathfrak{g}}(A), \mathbb{K}) \longrightarrow \mathfrak{g}'(A) \longrightarrow \bar{\mathfrak{g}}(A) \longrightarrow 0,$$

where $\mathrm{H}_2(\bar{\mathfrak{g}}(A), \mathbb{K}) = \Omega^1_{R|\mathbb{K}}(r)/\Omega'^1_{R|\mathbb{K}}(r) = t^{-1}\,\mathrm{d}\,t$. Therefore,

$$\ker\{\mathrm{HL}_2(\bar{\mathfrak{g}}(A), \mathbb{K}) \twoheadrightarrow \mathrm{H}_2(\bar{\mathfrak{g}}(A), \mathbb{K})\} = \Omega'^1_{R|\mathbb{K}}(r).$$

By Lemma 4.17, we have $\mathrm{HL}_2(\mathfrak{g}'(A), \mathbb{K}) = \Omega'^1_{R|\mathbb{K}}(r)$. Hence, by Lemma 4.16 again,

$$\mathrm{HL}^2(\mathfrak{g}'(A), \mathbb{K}) = (\Omega'^1_{R|\mathbb{K}}(r))^*.$$

[4]Note that author in [11] used the different notations.

(3) We assume that A is an indecomposable generalized Cartan matrix of non-affine type. Recall Theorem 3.1 [3], which says that $\bar{\mathfrak{g}}(A)$ possesses a non-degenerate invariant symmetric bilinear form ϕ if and only if A is symmetrizable. Moreover, such forms on $\bar{\mathfrak{g}}(A)$ are unique up to scalars.

If A is symmetrizable, then Berman's above result means $\dim \mathrm{B}(\bar{\mathfrak{g}}(A), \mathbb{K}) \geq 1$. On the other hand, for any $0 \neq \psi \in \mathrm{B}(\bar{\mathfrak{g}}(A), \mathbb{K})$, ψ is necessarily non-degenerate because $\bar{\mathfrak{g}}(A)$ is simple which is a consequence of Theorem 4.3 [14] and Exercise 1.4 [14]. So Berman's above result insures $\mathrm{B}(\bar{\mathfrak{g}}(A), \mathbb{K}) = \mathbb{K}\phi$. The non-degeneracy of ϕ naturally satisfies the nontrivial property of its restriction to any $\mathfrak{sl}(2, \mathbb{K})$-copy. Lemma 4.15 shows

$$\mathrm{HL}^2(\bar{\mathfrak{g}}(A), \mathbb{K}) = \mathrm{H}^2(\bar{\mathfrak{g}}(A), \mathbb{K}) = \mathfrak{c}^*,$$

where the second "=" was proved by Theorem 2.3 [3].

If A is non-symmetrizable, there is no non-degenerate invariant symmetric bilinear form on $\bar{\mathfrak{g}}(A)$. Since $\bar{\mathfrak{g}}(A)$ is simple (by Theorem 4.3 [14] and Exercise 1.4 [14]), there is no invariant symmetric bilinear form on $\bar{\mathfrak{g}}(A)$, i.e., $\mathrm{B}(\bar{\mathfrak{g}}(A), \mathbb{K}) = 0$. Then by Corollary 3.3 or Lemma 4.15, we have

$$\mathrm{HL}^2(\bar{\mathfrak{g}}(A), \mathbb{K}) = \mathrm{H}^2(\bar{\mathfrak{g}}(A), \mathbb{K}) = \mathfrak{c}^*,$$

where the second "=" was proved by Theorem 2.3 [3].

Furthermore, since $\mathfrak{g}'(A)$ is the universal Leibniz central extension of $\bar{\mathfrak{g}}(A) = \mathfrak{g}'(A)/\mathfrak{c}$. Consequently, $\mathrm{HL}^2(\mathfrak{g}'(A), \mathbb{K}) = 0$.

We complete the proof. □

Remark 4.19. *Theorem 4.18* (2), (3) *give a criterion to distinguish affine or nonaffine Kac-Moody algebras by means of vanishing property of the second Leibniz cohomology groups of* $\mathfrak{g}'(A)$ *with trivial coefficients, where the homological versions of Theorem 4.18* (2) *and the second statement of* (3) *were due to Gao* ([11]). *However, strictly speaking, owing to Gabber-Kac's Theorem* (*see* [14]), *the definition of* $\mathfrak{g}'(A)$ *for non-symmetrizable cases adopted by Gao is different from ours used here since our* $\mathfrak{g}'(A)$ *in Kac's normal notation sense is the quotient of the former. So in this sense, we get the same result for the quotient object.*

5. Quadratic Leibniz algebras and their central extensions

5.1. **Quadratic Leibniz algebra.**

Definition 5.1. $(\mathfrak{g}, \phi)$ *is called a quadratic Leibniz algebra if* ϕ *is a symmetric invariant bilinear form on the Leibniz algebra* $\mathfrak{g}$.

Lemma 5.2. *Let* $\mathfrak{g}$ *be a Lie algebra. If* $(\mathfrak{g}, \phi)$ *is a quadratic Leibniz algebra and* d *is a derivation of* $\mathfrak{g}$, *then* $f(x, y) := \phi(x, dy)$ *is a Leibniz* 2*-cocycle on* $\mathfrak{g}$.

Proof. For any $x, y, z \in \mathfrak{g}$, one has

$$f(x,[y,z]) = \phi(x,d[y,z]) = \phi(x,[dy,z]+[y,dz])$$
$$= \phi([x,y],dz) - \phi([x,z],dy) = f([x,y],z) - f([x,z],y).$$

So f is a Leibniz 2-cocycle on $\mathfrak{g}$. $\square$

Corollary 5.3. *If d is a skew-derivation,* i. e., $\phi(dx,y)+\phi(x,dy)=0$, *then* $f(x,y):=\phi(x,dy)$ *is a Lie 2-cocycle on* $\mathfrak{g}$.

5.2. **A negative answer to a question in** [18]**.**

Proposition 5.4. *Assume that $\mathfrak{g}$ is a finite dimensional simple Lie algebra over $\mathbb{K}$. Construct the Lie algebra $\mathfrak{g}\otimes\mathbb{K}((t))$ with bracket*

$$[x\otimes r, y\otimes s]' = [x,y]\otimes rs, \quad x,\, y\in\mathfrak{g},\, r,\, s\in\mathbb{K}((t)).$$

Then $\mathfrak{g}\otimes\mathbb{K}((t))$ is a simple Lie algebra over $\mathbb{K}$ and

$$\mathrm{H}^2(\mathfrak{g}\otimes\mathbb{K}((t)),\mathbb{K}) \subsetneq \mathrm{HL}^2(\mathfrak{g}\otimes\mathbb{K}((t)),\mathbb{K}).$$

Proof. Define $\phi(x\otimes r, y\otimes s) = (x,y)\,\mathrm{Res}(rs)$, where Res is a linear function on $\mathbb{K}((t))$ and takes the coefficient of t^{-1} for every series, $(\,,)$ is the Killing form on $\mathfrak{g}$. Then ϕ is an invariant symmetric bilinear form on $\mathfrak{g}\otimes\mathbb{K}((t))$ and $t^k\frac{d}{dt}, k\in\mathbb{Z}-\{0\}$ is a derivation of $\mathbb{K}((t))$. By Lemma 5.2, we get a non-trivial Leibniz 2-cocycle of the Lie algebra $\mathfrak{g}\otimes\mathbb{K}((t))$:

$$f(x\otimes\sum_{m\geq N}a_m t^m,\, y\otimes\sum_{n\geq N}b_n t^n) = (x,y)\sum_{m,n\geq N} n a_m b_n \delta_{m+n+k,0}, \tag{5.1}$$

where f is well-defined since the summation in (5.1) is finite, and f is not skew symmetric, that is, $\mathrm{H}^2(\mathfrak{g}\otimes\mathbb{K}((t)),\mathbb{K}) \subsetneq \mathrm{HL}^2(\mathfrak{g}\otimes\mathbb{K}((t)),\mathbb{K})$. $\square$

Remark 5.5. $\mathfrak{g}\otimes\mathbb{K}((t))$ *is a simple Lie algebra* (*see* [26]), *and* $\dim\mathrm{H}^2(\mathfrak{g}\otimes\mathbb{K}((t)),\mathbb{K})=1$, *but* $\dim\mathrm{HL}^2(\mathfrak{g}\otimes\mathbb{K}((t)),\mathbb{K})=\infty$, *which then leads to a negative answer to a question in* [18].

6. Appendix: Some Notes

6.1. **An elementary proof of the exact sequence in Proposition 3.2.** We shall prove the following

(1) The map g is well-defined, that is, $g(\alpha)=g(\alpha+d^1\beta)\in\mathrm{B}(\mathfrak{g},\mathbb{K})$ for any $\alpha\in\mathrm{HL}^2(\mathfrak{g},\mathbb{K})$ and $\beta\in\mathrm{HL}^1(\mathfrak{g},\mathbb{K})$ since $g(d^1\beta)\equiv 0$ by definition. Clearly, $g(\alpha)$ is symmetric. Since, for any $x,y,z\in\mathfrak{g}$,

$$g(\alpha)([x,y],z) = \alpha([x,y],z)+\alpha(z,[x,y])$$
$$= \alpha(x,[y,z])+\alpha([x,z],y)+\alpha([z,x],y]) - \alpha([z,y],x)$$
$$= \alpha(x,[y,z])+\alpha([y,z],x)$$
$$= g(\alpha)(x,[y,z]),$$

$g(\alpha)$ is $\mathfrak{g}$-invariant.

(2) $\mathrm{Im}(f)=\ker(g)$. For any $\alpha\in \mathrm{H}^2(\mathfrak{g},\mathbb{K})$ and any $x,y\in\mathfrak{g}$, we have

$$g(f(\alpha))(x,y)=g(\alpha)(x,y)=\alpha(x,y)+\alpha(y,x)=0,$$

since α is skew symmetric. Then $\mathrm{Im}(f)\subseteq\ker(g)$. On the other hand, for any $\beta\in\ker(g)$, we have $g(\beta)(x,y)=0$, which implies that $\alpha(x,y)+\alpha(y,x)=0$ for any $x,y\in\mathfrak{g}$. Then α is a skew symmetric Leibniz 2-cocycle, namely, a Lie 2-cocycle in $\mathrm{H}^2(\mathfrak{g},k)$. Hence, we have $\ker(g)\subseteq\mathrm{Im}(f)$.

(3) $\mathrm{Im}(g)=\ker(h)$. Now, $\forall\,\alpha\in\mathrm{HL}^2(\mathfrak{g},\mathbb{K})$, define a bilinear form α' on $\mathfrak{g}$ by

$$\alpha'(x,y)=\alpha(y,x),$$

for any $x,y\in\mathfrak{g}$. Denote $\beta:=\frac{\alpha+\alpha'}{2}$ and $\gamma:=\frac{\alpha-\alpha'}{2}$. Then β is symmetric and γ is skew symmetric. Also, we have $\alpha=\beta+\gamma$.

Now, for any $x,y,z\in\mathfrak{g}$,

$$\begin{aligned}(h(g(\alpha)))(x,y,z)&=g(\alpha)([x,y],z)=\alpha([x,y],z)+\alpha(z,[x,y])\\&=\alpha([x,y],z)+\alpha([z,x],y)+\alpha([y,z],x)\\&=\alpha(z,[x,y])+\alpha(y,[z,x])+\alpha(x,[y,z])\\&\quad-\alpha([x,y],z)-\alpha([z,x],y)-\alpha([y,z],x)\\&=2\gamma(z,[x,y])+2\gamma(y,[z,x])+2\gamma(x,[y,z])\\&=\delta^2(2\gamma)(x,y,z),\end{aligned}$$

which implies that $h(g(\alpha))$ is a 3-coboundary in $\Omega^3(\mathfrak{g},\mathbb{K})$. Then $\mathrm{Im}(g)\subseteq\ker(h)$. Conversely, for any $\omega\in\ker(h)$, there exists an element $\eta\in\Omega^2(\mathfrak{g},\mathbb{K})$ such that $\delta^2(\eta)=h(\omega)$. Namely, for any $x,y,z\in\mathfrak{g}$, there holds

$$\eta(z,[x,y])+\eta(y,[z,x])+\eta(x,[y,z])=\omega([x,y],z).$$

Then

$$\omega([x,y],z)+\eta([x,y],z)=\omega(x,[y,z])+\eta(x,[y,z])+\omega([x,z],y)+\eta([x,z],y),$$

since $\omega(x,[y,z])+\omega([x,z],y)=0$. Let $\rho=\frac{1}{2}(\omega+\eta)$. Then $g(\rho)=\omega$ and

$$\rho([x,y],z)=\rho(x,[y,z])+\rho([x,z],y),$$

which means that $\rho\in\mathrm{HL}^2(\mathfrak{g},\mathbb{K})$. Hence $\ker(h)\subseteq\mathrm{Im}(g)$.

We complete the proof. □

6.2. **An explicit interpretation of construction of Thm 4.18 (2).** We give an explicit construction for Theorem 4.18 for the affine case (in simply-laced types).

Proposition 6.1. *Let A be a generalized Cartan matrix of affine $X_n^{(1)}$ type. Consider the loop realizations of $\mathfrak{g}(A)$, $\mathfrak{g}'(A)$, and $\bar{\mathfrak{g}}(A)$:*

$$\begin{aligned}\bar{\mathfrak{g}}(A) &= \mathfrak{g}\otimes\mathbb{K}[t,t^{-1}],\\ \mathfrak{g}'(A) &= \mathfrak{g}\otimes\mathbb{K}[t,t^{-1}]\oplus\mathbb{K}c,\\ \mathfrak{g}(A) &= \mathfrak{g}\otimes\mathbb{K}[t,t^{-1}]\oplus\mathbb{K}c\oplus\mathbb{K}d,\end{aligned}$$

where $\mathfrak{g}$ is a finite simple Lie algebra over $\mathbb{K}$, $\mathbb{K}[t,t^{-1}]$ is the algebra of Laurent polynomials in t, c is a central element of $\mathfrak{g}'(A)$, and d is a degree operator defined by $d(x\otimes t^n)=n(x\otimes t^n)$, $d(c)=0$.

Then we have

$$\dim H^2(\bar{\mathfrak{g}}(A),\mathbb{K})=1,\qquad \dim HL^2(\bar{\mathfrak{g}}(A),\mathbb{K})=\infty; \tag{i}$$

$$\dim H^2(\mathfrak{g}'(A),\mathbb{K})=0,\qquad \dim HL^2(\mathfrak{g}'(A),\mathbb{K})=\infty; \tag{ii}$$

$$\dim H^2(\mathfrak{g}(A),\mathbb{K})=0,\qquad \dim HL^2(\mathfrak{g}(A),\mathbb{K})=1. \tag{iii}$$

Proof. Due to Benkart and Moody [2],

$$\mathrm{H}^1(\mathfrak{g}'(A),\mathfrak{g}'(A))=\mathrm{H}^1(\bar{\mathfrak{g}}(A),\bar{\mathfrak{g}}(A))=\bigoplus_{m\in\mathbb{Z}}\mathbb{K}d_m,$$

where $d_m(x\otimes t^n)=m(x\otimes t^{m+n})$, and $d_0=d$. Due to Farnsteiner [8],

$$\mathrm{H}^1(\mathfrak{g}(A),\mathfrak{g}(A))=\mathbb{K}D,$$

where $D(d)=c$, $D(\mathfrak{g}'(A))=0$.

Recall that the invariant symmetric bilinear form $\bar{\phi}$ on $\bar{\mathfrak{g}}(A)$ defined by

$$\bar{\phi}(x\otimes t^m,y\otimes t^n)=\delta_{m+n,0}(x,y),\qquad \forall\, x,\, y\in\mathfrak{g},\, m,\, n\in\mathbb{Z},$$

where $(\ ,\ )$ is the Killing form on $\mathfrak{g}$. It is clear that we have the invariant symmetric bilinear forms ϕ' on $\mathfrak{g}'(A)$ and ϕ on $\mathfrak{g}(A)$ defined by

$$\begin{aligned}&\phi'(x\otimes t^m,y\otimes t^n)=\delta_{m+n,0}(x,y),\\ &\phi'(c,c)=0=\phi'(x\otimes t^m,c)=0,\qquad \forall\, x,\, y\in\mathfrak{g},\, m,\, n\in\mathbb{Z},\end{aligned}$$

and

$$\begin{aligned}&\phi(c,c)=\phi(d,d)=0,\quad \phi(x\otimes t^m,c)=\phi(x\otimes t^m,d)=0,\quad \phi(c,d)=1,\\ &\phi(x\otimes t^m,y\otimes t^n)=\delta_{m+n,0}(x,y),\qquad \forall\, x,\, y\in\mathfrak{g},\, m,\, n\in\mathbb{Z}.\end{aligned}$$

(i) Now, we know that

$$\mathrm{H}^2(\bar{\mathfrak{g}}(A),\mathbb{K})=\mathbb{K}\alpha_0,$$

where $\alpha_0(x\otimes t^m,\, y\otimes t^n)=\bar{\phi}(x\otimes t^m,\, d(y\otimes t^n))=n\delta_{m+n,0}(x,y)$ for all $x,\, y\in\mathfrak{g}$, $m,\, n\in\mathbb{Z}$. Then

$$\mathrm{HL}^2(\bar{\mathfrak{g}}(A),\mathbb{K})=\bigoplus_{k\in\mathbb{Z}}\mathbb{K}\alpha_k,$$

where $\alpha_k(x\otimes t^m,y\otimes t^n)=\bar{\phi}(x\otimes t^m,d_k(y\otimes t^n))=n\delta_{m+n+k,0}(x,y)$ for all $x,\, y\in\mathfrak{g}$, $m,\, n\in\mathbb{Z}$. In particular, α_k is not skew-symmetric for $i\in\mathbb{Z}-\{0\}$.

(ii) It is well-known that $\mathrm{H}^2(\mathfrak{g}'(A),\mathbb{K})=0$. However,

$$\mathrm{HL}^2(\mathfrak{g}'(A),\mathbb{K})=\bigoplus_{k\in\mathbb{Z}-\{0\}}\mathbb{K}\beta_k,$$

where $\beta_k(X,Y)=\phi'(X,d_k Y)$, for all $X,\ Y\in\mathfrak{g}'(A),\ k\in\mathbb{Z}-\{0\}$. In particular,

$$\beta_k(x\otimes t^m,\ y\otimes t^n)=\phi'(x\otimes t^m,\ d_k(y\otimes t^n))=n\delta_{m+n+k,0}(x,y)$$

for all $x,\ y\in\mathfrak{g},\ m,\ n\in\mathbb{Z},\ k\in\mathbb{Z}-\{0\}$.

(iii) From [8], we have $\mathrm{H}^2(\mathfrak{g}(A),\mathbb{K})=0$. However, we have a Leibniz 2-cocycle $\bar{0}\neq\gamma\in\mathrm{HL}^2(\mathfrak{g}(A),\mathbb{K})$, which is defined by $\gamma(X,Y)=\phi(X,DY)$ for all $X,\ Y\in\mathfrak{g}(A)$ (Proposition 5.4). In particular,

$$\gamma(d,\ d)=\phi(d,Dd)=\phi(d,c)=1,$$

which implies $\gamma\notin\mathrm{H}^2(\mathfrak{g}(A),\mathbb{K})$, since γ is not skew symmetric.

Since $\dim\mathrm{H}^1(\mathfrak{g}(A),\mathfrak{g}(A))=1$ ([8]) and Corollary 3.7, we can conclude that

$$\mathrm{HL}^2(\mathfrak{g}(A),\mathbb{K})=\mathbb{K}\gamma.$$

□

Remark 6.2. *Lemma 2.8 in* [BGK] : S. Berman, Y. Gao, Y. Krylyuk, Quantum Tori and the Structure of Elliptic Quasi-simple Lie Algebras, Journal of Functional Analysis, **135** (1996), 339 — 389.

tells us: If we let $\mathfrak{g}=sl_{n+1}$ *and* S *be a commutative associative algebra, we have* $\mathrm{B}(\mathfrak{g}\otimes S,\mathbb{K})=S^*$*, which is the dual space of* S*. In fact, this is true for all simple Lie algebras and all commutative associative algebras* [27]*. However, the standard invariant symmetric bilinear form* $\bar{\phi}$ *is the only proper candidate, who can produce nontrivial* (*Lie*) 2*-cocycles by the* (*skew*) *outer derivation of* $\mathfrak{g}$ (*see* [14] *Chapter 7, Exercise*).

6.3. **A broader version of Lemma 4.15 and a remark on affine case.** As a broader version of Lemma 4.15 with the same proof, we have the following

Lemma 6.3. *For a Lie algebra* $\mathfrak{g}$*, in both cases below*:

- *if* $\mathrm{B}(\mathfrak{g},\mathbb{K})=0$; *or*
- *if* $\mathrm{B}(\mathfrak{g},\mathbb{K})\neq 0$*, but for any nontrivial* $\phi\in\mathrm{B}(\mathfrak{g},\mathbb{K})$*, there exists a subalgebra* $\mathfrak{a}\cong\mathfrak{sl}(2,\mathbb{K})$ *such that* $\phi|_{\mathfrak{a}}\neq 0$*,*

then $\mathrm{HL}^2(\mathfrak{g},\mathbb{K})=\mathrm{H}^2(\mathfrak{g},\mathbb{K})$. □

The following example shows a strange phenomenon happened for affine cases why they don't satisfy the second condition of the above Lemma.

Example 6.4. *Consider the following bilinear form* ϕ *on the loop algebra* $sl_2\otimes\mathbb{K}[t,t^{-1}]$*:*

$$\phi(x\otimes t^m,y\otimes t^n)=\kappa(x,y)\delta_{m+n,1}\quad \textit{for all } m,n\in\mathbb{Z},x,y\in\{e,f,h\},$$

where κ *is the Killing form on* sl_2*. It is clear that* ϕ *is a nonzero, invariant and symmetric.*

Let $\mathfrak{a} = \{x \otimes t^m, y \otimes t^n, z \otimes t^k\}$ *a subalgebra of* $sl_2 \otimes \mathbb{K}[t, t^{-1}]$, *which is isomorphic to* sl_2 *as Lie algebas.*

Then

$$[x \otimes t^m, y \otimes t^n] = z \otimes t^k = [x, y] \otimes t^{m+n},$$
$$[z \otimes t^k, x \otimes t^m] = 2x \otimes t^m = [z, x] \otimes t^{k+m}$$
$$[z \otimes t^k, y \otimes t^n] = -2y \otimes t^n = [z, y] \otimes t^{k+n}$$

Then

$$m + n = k = 0, \ z = [x, y], \ [z, x] = 2x, \ [z, y] = -2y.$$

It is easy to see $\phi|_{\mathfrak{a}} = 0$. □

Exercise 7.3 in [14] gives the general definition of any invariant symmetric bilinear forms on loop algebras.

Problem. Based on the relationship given in Corollary 3.3, a direct proof is interesting to show why happened $\dim HL^2(\bar{\mathfrak{g}}(A), \mathbb{K}) = \infty$, while $\dim H^2(\bar{\mathfrak{g}}(A), \mathbb{K}) = 1$ for affine Lie algebras $\bar{\mathfrak{g}}(A)$. □

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