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ON THE GALOIS COHOMOLOGY OF UNIPOTENT GROUPS AND EXTENSIONS OF NON-PERFECT FIELDS

Nguyễn Duy Tân[1] and Nguyễn Quốc Thắng[2]
Institute of Mathematics, 18 Hoang Quoc Viet, CauGiay, 10307, Hanoi, Vietnam
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

In this note we discuss, in the case of unipotent groups over non-perfect fields, an analog of Serre's conjectures for unipotent algerbraic group schemes, which relates properties of Galois (or flat) cohomology of unipotent group schemes to finite extensions of non-perfect fields, and Russel's defining equations of one-dimensional unipotent groups.

MIRAMARE – TRIESTE
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[1]duytan@math.ac.vn
[2]Regular Associate of ICTP. Corresponding author: nqthang@math.ac.vn

Introduction

If G is a smooth (i.e., absolutely reduced) affine group scheme defined over a perfect field k then one may define its first Galois cohomology $\mathrm{H}^1(k, G) := \mathrm{H}^1(\mathrm{Gal}(k_s/k), G(k_s))$, where $\mathrm{Gal}(k_s/k)$ denotes the absolute Galois group of k. In [Se1] (Chap. III, Sec. 2.2, Sec. 2.3 and Sec. 3.1) Serre formulated his famous conjectures I and II (see also [Se2], Sec. 4 and 5.). Recall them briefly in the form of questions ([Se2], pp. 236, 237), as follows.

(I) *Let k be a perfect field. Then*

$$cd(k) \leq 1 \Leftrightarrow \mathrm{H}^1(k, G) = 0$$

for all connected smooth k-groups G.

(This is now Steinberg's Theorem [St]; it was extended by Borel - Springer [BS] to the case of arbitrary (non-necessary perfect) fields while restricting to connected reductive groups only.)

(II) *Let k be a perfect field. Then*

$$cd(k) \leq 2 \overset{?}{\Leftrightarrow} \mathrm{H}^1(k, G) = 0$$

for all semisimple simply connected k-groups G.

We refer to [Se2], [BP1], [BP2] for more recent results in the direction of Serre's conjecture (II). In (I) if one drops the condition of perfectness of the field, one needs to restrict oneself to the case of connected reductive groups. It is due to the fact (see, e.g. [Se1]), that if G is a smooth connected (resp. and unipotent) group defined over a perfect field k then its unipotent radical is defined over k (resp. its first Galois cohomology $\mathrm{H}^1(k, G)$ is trivial), but these facts are no longer true if we drop the perfectness condition on k. In fact, even over some fields, such as global (resp. local) function fields every unipotent groups of dimension one which is not isomorphic to $\mathbb{G}_a$ (the additive group) (resp. if $char.k$ is not 2) has infinite Galois cohomology (see, [TT]).

In the first section of this paper, for any field k and for any unipotent k-group scheme G, we prove the existence of a normal composition series $G = G_0 > G_1 > \cdots > G_n = (1)$ of k-subgroup schemes of G, such that G_{i+1} is normal and of codimension 1 in G_i, for all $i \geq 0$. Then, by using Whaples' methods [W1], [W2], [W3], we propose a necessary and sufficient condition on k which ensures the triviality of the first Galois cohomology set for all smooth unipotent groups defined over k. One of the characterizations is the following statement, which is an analog of Serre's characterization of cohomological dimension of the base field, via the triviality of the first Galois cohomology of algebraic groups (see (I), (II) above). Denote by $\mathrm{H}^1_{fl}(k, G)$ the flat cohomology of G. We have (see Theorems 4 and 9)

(III) (Analog of Serre's conjectures for unipotent groups)
Let k be a field of characteristic $p > 0$ and let $cd_p(k) := cd_p(Gal(k_s/k))$ be the cohomological p-dimension of k. Then
a) $cd_p(k) = 0 \Leftrightarrow \mathrm{H}^1_{fl}(k, G) = 0$ for all smooth unipotent k-groups G.
b) k is perfect and $cd_p(k) = 0 \Leftrightarrow \mathrm{H}^1_{fl}(k, G) = 0$ for all unipotent k-group schemes G.

Then using this, we describe relations between various statements regarding finite extensions of degree p or divisible by p of a given field k.

In the second section, we study the equations defining smooth connected unipotent groups of dimension one. In [Ru], P. Russell shows that every smooth connected unipotent k-groups of dimension 1 is k-isomorphic to a k-subgroup of $\mathbb{G}_a^2$ defined by a p-polynomial of the form

(1) $F(x, y) := y^{p^n} - (x + a_1 x^p + \cdots + a_r x^{p^r}),$

where some $a_i \notin k^p$. This explicit equation proves to be of great importance in the study of arithmetic of unipotent group schemes over fields and rings (cf. e.g. [KMT], [Oe], [WW]). In [KMT], many results in [Ru] have been generalized. For example, it has been shown that the number n in (1) is uniquely determined by G (see Corollary of Theorem 2.4.3, [KMT]). Using results of [KMT], we show further that the set $\{i : a_i \notin k^p\}$ also depends only on G. In fact, we prove a slightly more general result (see Proposition 7).

We recall some basic definitions about the theory of unipotent groups over fields (see, e.g [KMT], [Oe], [Ti]). The smooth affine algebraic groups considered here are the same as linear algebraic groups in the sense of [Bo]. Let k be a field. An affine algebraic group scheme defined over k is called unipotent if it is k-isomorphic to a closed k-subgroup scheme of the matrix group consisting of all upper triangular matrices with all 1 on their diagonal (See [SGA3, Exp. XVII]). For simplicity, we call smooth unipotent k-group schemes just k-groups. We recall after Tits that a unipotent k-group G is called k-wound if every k-homomorphism (or even, k-morphism as in [KMT]) $\mathbb{G}_a \to G$ is constant. A polynomial $P := P(x_1, \ldots, x_n)$ in n variable $x_1, \ldots, x_n$ with coefficient in k is said to be universal if $P(k^n) = k$. We say that P is additive if $P(x + y) = P(x) + P(y)$, for any two elements $x \in k^n, y \in k^n$. If this is the case, P is the so-called p-polynomial, i.e, a k-linear combination of $x_i^{p^{m_{i,j}}}$. Denote by p^{m_i} the highest degree of x_i appearing in P with coefficient $c_i \in k \setminus \{0\}$. Then the sum $\sum_{i=1}^{n} c_i x_i^{p^{m_i}}$ is called the principal part of the p-polynomial P.

1 Triviality of the first Galois cohomology group

In this section, we first show by using Tits results [Ti], that for all unipotent k-group schemes G of dimension n, there exists a composition series of normal k-subgroup schemes $G_n = G \supset$

$G_{n-1} \supset \cdots \supset G_1$ such that $\dim G_i = i$. In fact, this fact is already implicitly contained in [Ti], (and in [Ke] one may find another proof in the case of smooth connected group schemes). Then, we give equivalent conditions for a non-perfect field k which are sufficient and necessary for the triviality of the first Galois cohomology of an arbitrary smooth unipotent group defined over k.

Proposition 1. *Let G be a unipotent k-group scheme of dimension ≥ 1. Then there exists a normal k-subgroup scheme G' of codimension 1 in G. If, moreover, G is smooth (resp. connected, resp. connected and smooth), G' can be chosen smooth (resp. connected, resp. connected and smooth), too.*

Proof. a) First assume that G is connected. We prove the assertion in three steps:
1) Assume first that G is smooth, k-wound, commutative and of exponent p. Let $\dim G = n$. Then by [Ti], it is known that G is k-isomorphic to a closed k-subgroup H of $\mathbb{G}_a^{n+1}$ given by

$$H = \{(x_1, \ldots, x_n) | P(x_1, \ldots, x_{n+1}) = 0\},$$

where $P = P(T_1, \ldots, T_{n+1})$ is a separable p-polynomial. Write

$$P = \sum_{i=1}^{n+1} a_i T_i^{p^{m_i}} + (\cdots) + \sum_{i=1}^{n+1} b_i T_i,$$

where (by a result of Tits) the principal part $P_{\text{princ}} = \sum_{i=1}^{n+1} a_i T_i^{p^{m_i}}$ vanishes nowhere on $k^{n+1} - \{0\}$ (since G is k-wound). Since P is separable, there exists i such that $b_i \neq 0$, and we can assume that $i = n+1$. Let G' be the connected component of

$$H \cap \{T_{n+1} = 0\} \simeq \{(x_1, \ldots, x_n) | \hat{P}(x_1, \ldots, x_n) = 0\},$$

where $\hat{P}(T_1, \ldots, T_n) = P(T_1, \ldots, T_n, 0)$. Then G' is of codimension 1 in H, and is k-wound since $\hat{P}$ is a separable p-polynomial and its principal part vanishes nowhere on $k^n - \{0\}$.

2) Assume that G is smooth, k-wound. Then by [Ti], there exists a normal subgroups G_1 of G such that G/G_1 is commutative, k-wound and of exponent p. Let $\pi : G \to G/G_1$ be the canonical projection. By Step 1, there exists a normal k-subgroup H of codimension 1 of G/G_1. Let $G' = (\pi^{-1}(H))^o$, the connected component of $\pi^{-1}(H)$. Then G' is a smooth subgroup defined over k of G and of codimension 1 since π is a separable morphism.

3) Assume only that G is smooth. There exists a normal k-subgroup k-split G^d such that G/G^d is k-wound. By Step 2, there exists a normal k-subgroup H of codimension 1 in G/G_1. By the same argument as in Step 2, there exists a normal k-subgroup G' of G of codimension 1.

4) G is not smooth. Then by [SGA 3], (proof of Théorème 3.5, Exp. XVII) we have an exact sequence

$$1 \to \alpha \to G \xrightarrow{p} G' \to 1,$$

where α is an infinitesimal k-subgroup scheme of G and G' is smooth. By Step 3) there exists a normal smooth k-subgroup H' of codimension 1 in G'. The inverse image $H := p^{-1}(H')$ of H' in G is the required one. (Notice that in this case, the inverse image of H' in G needs not be smooth.)

b) Now we assume that G is not connected. Denote by G° the connected component of G. We use induction on the nilpotency length l of G. If $l = 1$, i.e., G is commutative, then any closed k-subgroup scheme $H \subset G^\circ$ of codimension 1 satisfies the assertion. Assume that $l > 1$. Let $C^1G := DG := [G, G]$ the derived subgroup scheme of G. Then G/DG is commutative, hence if $dim(G/DG) > 0$ then there exists a closed normal k-subgroup scheme H' of codimension 1 in G/DG. By taking the preimage of H' in G we get the k-subgroup scheme of G as desired.

Now we assume that $dim(G/DG) = 0$, i.e., $dim(G) = dim(DG)$. We divide the proof into two steps. First we assume that G is smooth and not connected. We proceed by induction on $n = dim(G)$. If $n = 1$, the assertion is trivial. Assume that $n > 1$. We see that G is not commutative. From above it follows that $G^\circ \subset DG \subset G$. Let $H := [G, G^\circ]$. Then H is a closed k-subgroup of G°, generated by the sets $\varphi_g(G^\circ), g \in G$, where $\varphi_g : G^\circ \to H$, $x \mapsto [x, g]$. Thus H is also connected. We claim that $H \neq (1)$. For, if it were so, then $G^\circ \subset Cent(G)$, the center of G, hence by Baer's lemma (see e.g. [Bo], Chap. III), DG is finite, which contrdicts our assumptions that $n = dim(DG) > 1$. Also if $H = G^\circ$, then we have $C^2G := [G, DG] \supset [G, G^\circ] = G^\circ$, ..., $C^{r+1}G := [G, C^rG] \supset [G, G^\circ] = G^\circ$, for all $r \geq 1$, which contradicts the fact that G is nilpotent. Therefore H is a non-trivial proper connected closed k-subgroup of G°. Since H is a normal k-subgroup of G of positive dimension, we may consider the quotient group G/H and the separable projection $p : G \to G/H$. By induction hypothesis ($dim(G) > dim(G/H)$), there exists a normal closed k-subgroup H' of codimension 1 in G/H. The preimage $p^{-1}(H')$ is a normal closed k-subgroup of codimension 1 in G as desired.

Now we assume that G is not smooth. Then as above, we consider the exact sequence

$$1 \to \alpha \to G \xrightarrow{p} G' \to 1,$$

where α is an infinitesimal normal k-subgroup scheme of G, and G' is a smooth k-group scheme. If H' is a closed k-subgroup scheme of codimension 1 in G' then $p^{-1}(H')$ is a normal closed k-subgroup scheme of codimension 1 in G as required. □

Remark. We can choose G' even connected. Indeed, if G' was chosen, then it is clear that its connected component $(G')^\circ$ is also normal in G and has codimension 1 there.

Corollary 2. *Let G be a unipotent k-group scheme of dimension n. Then there is a composition series of normal k-subgroup schemes $G_n = G \supset G_{n-1} \supset \cdots \supset G_1$ such that $\dim G_i = i$. If moreover G is smooth (resp. connected, resp. connected and smooth), the k-subgroup schemes G_i can be chosen smooth (resp. connected, resp. connected and smooth), too.*

Proof. Use induction on dimension of G. □

Remark. If G is k-wound, one cannot expect G to have a composition series such that all factors are also k-wound. In fact, assume that over any non-perfect field of characteristic $p > 0$ every k-wound unipotent group G had a composition series such that all factors were also k-wound. Then it also holds for global function fields. By an induction argument and using [Oe], Theorem 1, we can show that for all k-wound unipotent groups G defined over the global function field k, if the above assumption were true, it would imply that $G(k)$ were finite. But it would contradict Example 3.4, p. 68 of Oesterlé [Oe], which shows that there exists a k-wound unipotent group G of dimension $p - 1$ such that $G(k)$ is infinite, where k can be any global function field of characteristic $p > 2$.

Corollary 3. *Let n be a positive integer.*
a) The following statements are equivalent:
1) $\mathrm{H}^1_{fl}(k, G) = 0$ for all unipotent k-group schemes G of dimension 1;
2) $\mathrm{H}^1_{fl}(k, G) = 0$ for all unipotent k-group schemes G of dimension n;
3) $\mathrm{H}^1_{fl}(k, G) = 0$ for all unipotent k-group schemes G.
b) The following statements are equivalent:
1) $\mathrm{H}^1(k, G) = 0$ for all smooth unipotent k-groups G of dimension 1;
2) $\mathrm{H}^1(k, G) = 0$ for all smooth unipotent k-groups G of dimension n;
3) $\mathrm{H}^1(k, G) = 0$ for all smooth unipotent k-groups G.
c) The following statements are equivalent:
1) $\mathrm{H}^1(k, G) = 0$ for all smooth and connected unipotent k-groups G of dimension 1;
2) $\mathrm{H}^1(k, G) = 0$ for all smooth and connected unipotent k-groups G of dimension n;
3) $\mathrm{H}^1(k, G) = 0$ for all smooth and connected unipotent k-groups G.

Proof. Follows from the proposition by using induction and dévissage. □

Theorem 4. *Let k be an arbitrary field of characteristic $p > 0$. The following statements are equivalent:*
1) k has no Galois extensions of degree divisible by p;
2) k has no separable extensions of degree divisible by p;
3) Every separable p-polynomial in one variable is universal;
4) $\mathrm{H}^1(k, G) = 0$ for all smooth unipotent k-groups G.

Proof. First we observe that if G is an étale finite unipotent group scheme defined over a field k of characteristic $p > 0$, which is a k-form of $\mathbf{F}_p^r$ then there exists a separable p-polynomial $f(T) \in k[T]$ such that $G = Ker(f)$, where f is considered as a k-morphism $\mathbb{G}_a \to \mathbb{G}_a$. Indeed, by Tits [Ti], we know that G can be embedded as a closed k-subgroup of exponent p into $\mathbb{G}_a$.

The exact sequence

$$0 \to G \to \mathbb{G}_a \xrightarrow{f} \mathbb{G}_a \to 0$$

allows us to consider G as the kernel (hence also as zero set) of a separable p-polynomial $f = f(T) \in k[T]$.

We prove firstly that 3) implies 4). First we show that 4) holds for finite étale unipotent k-groups G. We know by [SGA 3] Exp. XVII, Lemme 3.9, that G has a characteristic composition series consisting of k-subgroups G_i:

$$G = G_0 > G_1 > \cdots > G_s = \{1\},$$

where each factor $G'_i := G_{i-1}/G_i$ is a k-form of $(\mathbf{F}_p)^r$. By above observation, $G'_i = Ker(F_i)$, where F_i is a separable p-polynomial in one variable with coefficients in k. One has $\mathrm{H}^1(k, G'_i) \simeq k^+/F_i(k^+)$, hence is trivial by assumption. It follows by induction (i.e., by dévissage) that the same holds for G.

Next we show that 4) holds for smooth connected groups. By Corollary 3, we have only to prove the that $\mathrm{H}^1(k, G) = 0$ in case $\dim G = 1$. Since $\mathrm{H}^1(k, \mathbb{G}_a) = 0$, we consider only the case when G is a nontrivial form of $\mathbb{G}_a$. Then by [Ru], we know that G is k-isomorphic to a closed subgroup of $\mathbb{G}_a^2$ of the form

$$\{(x, y) \in \mathbb{G}_a^2 \,|y^{p^m} = x + a_1 x^p + \cdots + a_r x^{p^r}\},$$

where a_i are not all in k^p. Let $P(x, y) = y^{p^m} - (x + a_1 x^p + \cdots + a_r x^{p^r})$ and consider P as a homomorphism from $\mathbb{G}_a^2$ to $\mathbb{G}_a$. Then we deduce that $\mathrm{H}^1(k, G) = k/ImP$.

Since $f(x) = x + a_1 x^p + \cdots + a_r x^{p^r}$ is a separable p-polynomial, f is universal by assumption and $ImP = k$. Thus, $\mathrm{H}^1(k, G) = 0$.

Now for any smooth unipotent k-group G, let G° denote the connected component of G. Then it is well-known that G/G° is an étale finite k-group scheme, for which and for G° (see above) the assertion 4) holds. From the exact sequence

$$1 \to G^\circ \to G \to G/G^\circ \to 1$$

the assertion 4) follows.

Next, the equivalence between 1) and 2) is trivial.

Now, we prove that 1) is equivalent to 3). The idea of the proof is already given in that of Theorem 1 in [W2]. Since it is short, we present it here for the convenience of readers.

Assume that k has no Galois extension of degree divisible by p. Let f be an arbitrary separable p-polynomial. We denote $\hat{\mathrm{H}}(.,.)$ the Tate Galois cohomology, then, for every Galois extension K/k with Galois group $\mathcal{G}$, we have

$$\begin{aligned}\hat{\mathrm{H}}^0(\mathcal{G}, f(K^+)) &= (f(K^+) \cap k^+)/\operatorname{Tr}_{K/k}(f(K^+)) \\ &= (f(K^+) \cap k^+)/f(k^+),\end{aligned}$$

since f and $\mathrm{Tr}_{K/k}$ commute. Since $(p, |\mathcal{G}|) = 1$ and $char(k) = p$, so we have $\hat{\mathrm{H}}^0(\mathcal{G}, f(K^+)) = 0$. Then $f(K^+) \cap k^+ = f(k^+)$ for all Galois extension K/k. Since $k_s = \varinjlim K$, and $k_s = f(k_s)$, we have

$$\begin{aligned} k^+ &= k_s \cap k^+ \\ &= f(\varinjlim K) \cap k^+ \\ &= \bigcup (f(K^+) \cap k^+) \\ &= f(k^+). \end{aligned}$$

Conversely, assume that there is a Galois extension K/k with Galois group $\mathcal{G}$ of degree n divisible by p. Then $\mathcal{G}$ acts trivially on $\mathbb{F}_p^+$ so

$$\hat{\mathrm{H}}^0(\mathcal{G}, \mathbb{F}_p^+) \simeq \mathbb{F}_p^+/n\mathbb{F}_p^+ = \mathbb{F}_p^+.$$

Let a be an element of K whose conjugates form a normal basis for K/k and we may assume $\mathrm{Tr}_{K/k}(a) = 1$. Let A be the subgroup of K^+ generated over $\mathbb{F}_p$ by the conjugates of a, and let B be the set of elements of A whose trace is 0. Then $0 \to B \to A \to \mathbb{F}_p^+ \to 0$ is exact, and since A is $\mathcal{G}$-induced, so $\hat{\mathrm{H}}^0(\mathcal{G}, \mathbb{F}_p^+) = \hat{\mathrm{H}}^1(\mathcal{G}, B)$. Let $f(x)$ be a polynomial of degree $|B| = p^{n-1}$ with elements of B as its zero. Then (cf. [W1], Theorem 2) f is known to be a separable p-polynomial. We note that B is invariant under the action of $\mathcal{G}$, hence so are the coefficients of f, i.e., $f(x)$ is a polynomial with coefficients in k. Since $0 \to B \to K^+ \to f(K^+) \to 0$ is exact, we have

$$\begin{aligned} (f(K^+) \cap k^+)/f(k^+) &= \hat{\mathrm{H}}^0(\mathcal{G}, f(K^+)) \\ &\simeq \hat{\mathrm{H}}^1(\mathcal{G}, B) \\ &= \hat{\mathrm{H}}^0(\mathcal{G}, \mathbb{F}_p^+) \\ &= \mathbb{F}_p^+. \end{aligned}$$

So $f(k^+) \neq k^+$.

Finally, we prove that 4) $\Rightarrow$ 1). Assume that f is an arbitrary separable p-polynomial in $k[X]$. Then it defines a separable k-homomorphism $f : \mathbb{G}_a \to \mathbb{G}_a$, and $G := Ker f$ is a smooth unipotent k-group. By assumption we have $\mathrm{H}^1(k, G) = 0$, thus f is surjective, i.e., f is universal.

□

Remarks. 1) Recall that a field k is called Kaplansky field if every p-polynomial is universal (see [Va], [W2]). By Theorem 1 of [W2], k is Kaplansky field if and only if it has no finite extensions of degree divisible by p. So this theorem can be considered as an analog of Theorem 1 of Whaples [W2].

2) In [Ru], P. Russell remarks that if k has no normal extension of degree $p = char(k)$ then $\mathrm{H}^1(k, G)$ is trivial for all smooth connected unipotent k-group G of dimension 1 (and then, as one sees below, the same is true for smooth connected unipotent groups of arbitrary dimension). In fact, the conclusion is trivially true since any non-perfect field k always has a normal extension

of degree $p = char(k)$, thus, if k has no normal extension of degree p, then k is perfect. The argument given there is a bit obscured since there ([Ru], p. 538), it was argued that if a field k has no normal extensions of degree p then every separable p-polynomial in one variable is universal (condition 3 in Theorem 4). However, this is not true in general. For, if this were true then the condition that "k has no normal extensions of degree p" would imply the condition 1) in Theorem 4, but it would contradict a Whaples' result which is stated as follows

Theorem 5. *(Whaples [W1]) Let n be any positive integer. There exists a field K which has algebraic extensions of degrees divisible by n but has no extensions of degree $\leq n$.*

In fact, the proof of the above theorem even shows that, we can choose K such that K has Galois extensions of degrees divisible by n, but no extensions of degree $\leq n$.
3) We give one example of non-perfect fields k satisfying 4) of Theorem 4. In particular, we have the following

Proposition 6. *There are non-perfect fields k such that for all smooth unipotent groups G over k we have $\mathrm{H}^1(k, G) = 0$.*

Proof. Indeed, let k_0 be an arbitrary non-perfect field of characteristic p. Denote by $\mathcal{P}(k_0)$ the set of all separable p-polynomials in one variable T from $k_0[T]$, which are considered as k_0-morphisms $\mathbb{G}_a \to \mathbb{G}_a$ and consider the following sequence of fields inside $\bar{k}_0$

$$k_1 := k_0(P^{-1}(k_0)|P \in \mathcal{P}(k_0)),$$

$$k_2 := k_1(P^{-1}(k_1)|P \in \mathcal{P}(k_1)),$$

$$\vdots$$

$$k_n := k_{n-1}(P^{-1}(k_{n-1})|P \in \mathcal{P}(k_{n-1})),$$

$$k := \cup k_i \text{ (the union is taken in } \bar{k}_0).$$

Then $k = \cup k_i$ is a desirable field. For, it is clear that every separable p-polynomial in one variable with coefficients in k is universal over k. Now we prove that k is non-perfect by showing that $t \notin k^p$, for any $t \in k_0 - k_0^p$. Otherwise, let $t = \alpha^p \in k^p$. In the tower of extension fields $k_0 \subset k_0(\alpha) \subset k$, $k_0(\alpha)/k_0$ is purely inseparable extension and also separable since k/k_0 is separable. Thus $k_0(\alpha) = k_0$, and $t = \alpha^p \in k_0^p$, a contradiction. □

4) Let $\mathcal{G}$ be a profinite group, p a prime number. We recall that (see [Se], Chapter I, Sec.3) *p-cohomological dimension* of $\mathcal{G}$, denoted by $cd_p(\mathcal{G})$, is the lower bound of the integers n such that for every discrete torsion $\mathcal{G}$-module A, and for every $q > n$, the p-primary component of $\mathrm{H}^q(\mathcal{G}, A)$ is null.

We have the following proposition.

Proposition 7. ([Se, Chap. I, Sec.3.3]) *In order that $cd_p(\mathcal{G}) = 0$ it is necessary and sufficient that the order of G be prime to p.*

On the other hand, by Galois theory, k has no Galois extension of degree divisible by p if and only if the Galois group $\mathcal{G} = Gal(k_s/k)$ has the order prime to p. Hence, we can restated a part of Theorem 4 in cohomological terms as a complement to Serre's conjectures (I) and (II) as follows

(III) (Analog of Serre's conjectures for unipotent group schemes)
Let k be a field of characteristic $p > 0$ and let $cd_p(k) := cd_p(Gal(k_s/k))$ be the cohomological p-dimension of k. Then
a) $cd_p(k) = 0 \Leftrightarrow \mathrm{H}^1_{fl}(k, G) = 0$ for all smooth unipotent k-groups G.
b) k is perfect and $cd_p(k) = 0 \Leftrightarrow \mathrm{H}^1_{fl}(k, G) = 0$ for all unipotent k-group schemes G.

Proof. We need only prove b). By considering the infinitesimal k-group scheme α_p represented by the k-algebra $k[T]/T^p$, the direction ($\Leftarrow$) is clear. For the direction ($\Rightarrow$), we need only show that $\mathrm{H}^1_{fl}(k, G) = 0$ for all infinitesimal k-group schemes G. Such a group scheme G has a central composition series

$$G = G_0 > G_1 > \cdots > G_n = (1),$$

where each successive quotient $G_i/G_{i+1} \simeq \alpha_p$ for all $0 \le i \le n-1$ (see [SGA 3], Exp. XVII, Théorème 3.5). Since $\mathrm{H}^1_{fl}(k, \alpha_p) = k/k^p = 0$, it follows by dévissage that the same holds for G. □

Next we deduce from above some corollaries.

Corollary 7. *Let k be an arbitrary field of characteristic $p > 0$. Let k' be a finite extension of k. Then, $\mathrm{H}^1(k, G) = 0$ for all smooth unipotent k-groups G if and only if $\mathrm{H}^1(k', G') = 0$ for all smooth unipotent k'-groups G'.*

(The proof follows from (III) and from [Se1], Chap. II, Prop. 10.) From above we derive the following

Theorem 8. *Let k be any field of characteristic $p > 0$. Consider the following statements*
1) k has no extensions of degree p;
2) k has no extensions of degree divisible by p;
3) k has no normal extensions of degree p;
4) k has no normal extensions of degree divisible by p;
5) k has no Galois extensions of degree p;

6) k has no Galois extensions of degree divisible by p;

7) Every separable p-polynomial in one variable is universal;

8) Every p-polynomial in one variable is universal.

9) $\mathrm{H}^1(k, G) = 0$ *for any smooth unipotent k-group G.*

Then we have the following diagram of relations

$$\begin{array}{ccccc} 1) & \Rightarrow & 3) & \Rightarrow & 5) \\ \Uparrow & & \Uparrow & & \Uparrow \\ 2) & \Leftrightarrow & 4) & \Rightarrow & 6) \\ \Updownarrow & & & & \Updownarrow \\ 8) & \Rightarrow & 7) & \Leftrightarrow & 9) \end{array} \qquad \begin{array}{ccccc} 1) & \not\Leftarrow & 3) & \not\Leftarrow & 5) \\ \not\Downarrow & & \not\Downarrow & & \not\Downarrow \\ 2) & \Leftrightarrow & 4) & \not\Leftarrow & 6) \\ \Updownarrow & & & & \Updownarrow \\ 8) & \not\Leftarrow & 7) & \Leftrightarrow & 9) \end{array}$$

All other related implications or non-implications between the statements above follow from these diagrams.

Proof. The statements 1) $\Rightarrow$ 3), 3) $\Rightarrow$ 5), 2) $\Rightarrow$ 4), 4) $\Rightarrow$ 6), 2) $\Rightarrow$ 1), 4) $\Rightarrow$ 3), 6) $\Rightarrow$ 5) and 8) $\Rightarrow$ 7) are trivial. The equivalence between 2) and 8) is the Theorem 1 in [W1] and the equivalence between 6) and 7) is proved in Proposition 4.

Assume that k has an extension K of degree divisible by p. Let $\mathcal{G} = Gal(\bar{k}/k)$, and let $H = \{\sigma \in \mathcal{G} : \sigma|_K = id\}$ be the subgroup of $\mathcal{G}$ corresponding to K. Then p divides $[\mathcal{G} : H]$. It is known that there exists a subgroup H' of H which is normal in $\mathcal{G}$ and has finite index in $\mathcal{G}$. Let K' be the subfield of $\bar{k}$ corresponding to the group H'. Then K'/k is a normal extension of degree divisible by p, since $[\mathcal{G} : H]$ divides $[\mathcal{G} : H']$. This shows that 4) implies 2).

By using the theorem of Whaples stated above, we derive that 1) $\not\Rightarrow$ 2), 3) $\not\Rightarrow$ 4) and 5) $\not\Rightarrow$ 6). By the example given in previous remark (Proposition 5), we see that 7) $\not\Rightarrow$ 8).

Assume that k is non-perfect. Let K be its $\wp$-closure, i.e., the smallest subfield K in an algebraic closure $\bar{k}$ of k such that the homomorphism $\wp : K \to K, x \mapsto \wp(x) := x^p - x$ is surjective. Then K has no Galois extension of degree p and by an analogous argument as in remarks abvoe, we can show that K is still non-perfect, so K has a normal extension of degree p. This shows that 5) does not imply 3). And we can even show that 3) is equivalent to 5) if and only if k is perfect.

Now we prove that 3) does not imply 1) by exhibiting a perfect field of characteristic 3 which satisfies 3) (or, equivalently 5)) but does not satisfy 1).

Let $k = \mathbb{F}_3(t)$, t is a variable, K_0 the perfect closure of k, and let K be the $\wp$-closure of K_0. Then, K has no normal extension of degree 3. Indeed, assume that K has a normal extension L/K of degree 3. Since K_0 is perfect, it is clear that K is perfect and L/K is a Galois extension of degree 3. By Artin - Schreier theorem, $L = K(\alpha)$, for some root $\alpha \in L$ of the polynomial $\wp(x) = x^3 - x$. By definition of K, α is already in K, so $L = K$, a contradiction. Now we show that K has an extension of degree 3 by proving that the polynomial $f(X) = X^3 + X + t$ is irreducible over K. Assume first that $f(X)$ reducible over k. Then $f(X)$ has a root $P(t)/Q(t)$, where $P, Q \in \mathbb{F}_3[t]$ and $(P, Q) = 1$. We have $P^3 + PQ^2 + tQ^3 = 0$. Therefore $Q = 1$ and $P^3 + P + t = 0$, but it is impossible. So, $f(X)$ irreducible over k.

Next, assume that $f(X)$ reducible over K_0. Then there exists $\alpha \in K_0$ such that $f(\alpha) = 0$. In the tower of extension fields $k \subset k(\alpha) \subset K_0$, $k(\alpha)/k$ is purely inseparable, and obviously also separable. So, $k(\alpha) = k$, and $\alpha \in k$, which contradicts to irreducibility of $f(X)$ over k.

Let $K_1 = K_0(\wp^{-1}(a) : a \in K_0)$. Since k is countable, K_0 is also countable and we can write

$$K = K_0^1 \cup K_0^2 \cup \cdots \cup K_0^n \cup \cdots ,$$

where for each i, $K_0^{i+1} = K_0^i(a_i)$ is a Galois extension of degree 3 of K_0^i. We prove that $f(X)$ irreducible over K_1. Assuming the contrary, there exists $\alpha \in K_1$ such that $f(\alpha) = 0$. Let n be the least index such that $\alpha \in K_0^n$ but $\alpha \notin K_0^{n-1}$. Since $K_0^{n-1} \subset K_0^{n-1}(\alpha) \subset K_0^n$ and $[K_0^n : K_0^{n-1}]$, we have $K^{n-1}(\alpha) = K_0^n$ and it is Galois over K_0^{n-1}. Therefore two other roots β, γ of $f(X)$ are also in K_0^n. Since

$$0 = \alpha^3 + \alpha + t = \beta^3 + \beta + t,$$

we have $(\beta - \alpha)^2 + 1 = 0$. So, $X^2 + 1$ reducible over K_0^n. It is impossible since we can easily prove that $X^2 + 1$ is irreducible k, and therefore it remains irreducible over any extension of odd degree of k.

By an analogous argument, we can show that $f(X)$ is irreducible over $K_2, K_3, \ldots, K_n, \ldots$, where

$$K_2 = K_1(\wp^{-1}(a) : a \in K_1),$$
$$\vdots$$
$$K_n = K_{n-1}(\wp^{-1}(a) : a \in K_{n-1}),$$
$$\vdots$$

So, $f(X)$ is irreducible over $K = \cup_{n=0}^{\infty} K_n$.

Finally, other relations between these statements follow from above consideration. Namely we have

1) $\not\Leftarrow$ 5) (since 3) $\not\Leftarrow$ 5), but 3) $\Leftarrow$ 1)).
1) $\not\Leftarrow$ 6) and also 1) $\not\Rightarrow$ 6) : follows from Whaples results quoted above.
1) $\not\Rightarrow$ 8) since 1) $\not\Rightarrow$ 2), and 2) $\Leftrightarrow$ 8).
1) $\not\Leftarrow$ 7) and 1) $\not\Leftarrow$ 9): since 6) $\Leftrightarrow$ 7) $\Leftrightarrow$ 9).
2) $\not\Leftarrow$ 3) since 2) $\Rightarrow$ 1) $\Rightarrow$ 3) and 1) $\not\Leftarrow$ 3).
2) $\not\Leftarrow$ 5) since 2) $\Rightarrow$ 1) and 1) $\not\Leftarrow$ 5).
2) $\not\Leftarrow$ 6) $\Leftrightarrow$ 7) $\Leftrightarrow$ 9) since 2) $\Leftrightarrow$ 8), while 7) $\not\Rightarrow$ 8).
3) $\not\Leftarrow$ 6): Take $k = k_s$ of characteristic $p > 0$, then k has no Galois extensions, while it has lots of normal extensions of degree p.
3) $\not\Rightarrow$ 6): it follows from Whaples Theorem above.
3) $\not\Rightarrow$ 8) since 3) $\not\Rightarrow$ 2) $\Leftrightarrow$ 8).
3) $\not\Leftarrow$ 9) since 6) $\Leftrightarrow$ 9) (see above).

5) $\not\Rightarrow$ 8): since 2) $\Leftrightarrow$ 8), 2) $\Rightarrow$ 3), and 5) $\not\Rightarrow$ 3) (see above).

5) $\not\Rightarrow$ 9): since 5) $\not\Rightarrow$ 6) and 6) $\Leftrightarrow$ 9).

6) $\not\Rightarrow$ 8) since 2) $\Leftrightarrow$ 8) and 6) $\not\Rightarrow$ 2). □

Remarks. 1) One might add the 10-th condition, saying that $\mathrm{H}^1(k, G) = 0$ for all unipotent k-group schemes G, which is equivalent to conditions 2) and 8), but it would destroy the squares of relations above.

2) From above (Theorem 4 and Proposition 6) we see that the two conditions in (III) are not the same.

3) We give an example, which shows that the condition

$$\text{"}\mathrm{H}^1(k, G) = 0 \textit{ for all smooth unipotent } k\textit{-groups } G\text{"}$$

and the condition

$$\text{"}\mathrm{H}^1(k, G) = 0 \textit{ for all connected and smooth unipotent } k\textit{-groups } G\text{"}$$

are not the same. Indeed, take any field k of characteristic $p > 0$ such that certain separable p-polynomial $f(T)$ in one variable with coefficients in k (e.g. the Artin - Schreier map $\wp$) is not surjective as a map $k^+ \to k^+$. Take $a \in k^+ \setminus f(k^+)$. We claim that for the perfect closure $K := k^{-p^\infty}$ of k, we have $K^+ \neq f(K^+)$. If not, $K^+ = f(K^+)$, and we have $a \in f(K^+)$, so $a = f(x)$, $x \in K$. By assumption, we have $k \neq k(x)$. Since x is a root of the separable polynomial $f(T) - a$, $k(x)/k$ is a separable extension. But $k(x) \subset K$ and K/k is purely inseparable, hence so is $k(x)/k$. Thus $k = k(x)$, which is impossible. Therefore $f(K^+) \neq K^+$. Let $G := Ker(f)$. Then G is a finite (smooth) étale unipotent k-group scheme with $\mathrm{H}^1(K, G) = K^+/f(K^+) \neq 0$, while $\mathrm{H}^1(K, H) = 0$ for all connected smooth unipotent k-groups H, since K is perfect.

2 Equations defining commutative unipotent groups of exponent p

We first recall some notations and results in Part 1 of [KMT]. Let k be a non-perfect field of characteristic $p > 0$. It is known that the endomorphism ring $R := End_{k-gr}(\mathbb{G}_a)$ can be identified with the noncommutative polynomial k-algebra with one indeterminate F subjected to the relation $F\lambda = \lambda^p F$, for all $\lambda \in k$. A pair (n, α) with $n \in \mathbb{N}$ and $\alpha = \sum a_i F^i \in k[F]$ is called admissible if either (i) $n = 0$ or (ii) $a_0 \neq 0$ and $a_i \notin k^p$ for some $i > 0$. For a commutative affine k-group scheme G, denote by

$$M(G) := \mathrm{Hom}_{k-gr}(G, \mathbb{G}_a),$$

which is a left R-module, hence also a left $k[F]$-module in a natural way. On the other hand, any given left $k[F]$-module M can be considered as a (commutative) p-Lie algebra with zero multiplication and p-power given by $m^{[p]} = Fm$, for all $m \in M$. The universal envelopping

k-algebra $U(M)$ of M has a natural Hopf algebra structure, and the affine k-group scheme corresponding to $U(M)$ is denoted by $D(M)$. It has been shown that (cf. [DG], Chap. IV, Sec. 3, no. 6.2, p. 520) there is an anti-equivalence between the category of commutative k-group schemes with the category of left $k[F]$-modules, via

$$G \mapsto M(G)\ ;\ M \mapsto D(M),$$

where the algebraic k-group schemes correspond to finitely generated modules.

Let $M(n,\alpha)$ be the left $k[F]$-module on a set of 2 generators x, y defined by the relation $F^n y = \alpha x$. Then, there is a natural bijective correspondence $G \mapsto M(G)$; $M \mapsto D(M)$ between the unipotent groups G of dimension 1 and left $k[F]$-modules $M = M(n,\alpha)$, where (n,α) are admissible pairs. More precisely, if G is defined by the equation $y^n = a_0 x + a_1 x^p + \cdots + a_r x^{p^r}$, where $a_0 \neq 0$ and $a_i \notin k^p$ for some i, then the number n in $M(n,\alpha)$ is the power of y in the equation and $\alpha = a_0 + a_1 F + \cdots + a_r F^r$. For $\alpha = \sum a_i F^i \in k[F]$, let $\alpha^{(\nu)} = \sum a_i^{p^\nu} F^i$.

Proposition 9. *Let k be a non-perfect fields of characteristic $p > 0$, and let G_1, G_2) be unipotent smooth k-groups of dimension 1, defined by*
$\{(x,y) \in \mathbb{G}_a^2\, |y^{p^m} = x + a_1 x + \cdots + a_r x^{p^r}, \exists i, a_i \notin k^p\}$,
$\{(x,y) \in \mathbb{G}_a^2\, |y^{p^n} = x + b_1 x + \cdots + b_s x^{p^s}, \exists j, b_j \notin k^p\}$
respectively. If $\mathrm{Hom}_{k-gr}(G_1, G_2)$ *and* $\mathrm{Hom}_{k-gr}(G_2, G_1)$ *are both nontrivial then $m = n$ and the following two sets of indices coincide :*

$$\{i : a_i \notin k^p\} \equiv \{i : b_i \notin k^p\}.$$

Proof. We first prove the second statement of Proposition 6 in the particular case when $n = m = 1$.

Lemma 10. *Let k be a non-perfect fields of characteristic $p > 0$, and let G_1, G_2 be unipotent smooth k-groups of dimension 1, defined by*
$\{(x,y) \in \mathbb{G}_a^2\, |y^{p^m} = x + a_1 x + \cdots + a_r x^{p^r}, \exists i, a_i \notin k^p\}$,
$\{(x,y) \in \mathbb{G}_a^2\, |y^{p^n} = x + b_1 x + \cdots + b_s x^{p^s}, \exists j, b_j \notin k^p\}$
respectively. Assume that $\mathrm{Hom}_{k-gr}(G_1, G_2)$ *and* $\mathrm{Hom}_{k-gr}(G_2, G_1)$ *are nontrivial. Then we have*

$$\{i : a_i \notin k^p\} \equiv \{i : b_i \notin k^p\}.$$

Proof. Let $\alpha = 1 + a_1 F + \cdots + a_r F^{p^r} \in k[F], \beta = 1 + b_1 F + \cdots + b_s F^{p^s} \in k[F]$. Then $\mathrm{Hom}_{k[F]}(M(1,\alpha), M(1,\beta)) \neq 0$ by assumption. Thus, by [KMT], Proposition 5.4.1, there exist $\phi \in k[F] \setminus \{0\}, \xi, \eta \in k[F]$ such that

$(*)$ $\alpha\phi = \xi^{(1)} F + \eta^{(1)} \beta.$

Moreover, we can choose $\eta = c \in k$. Let $\phi = c_0 + c_1 F + \cdots + c_t F^t, c_t \neq 0$. Then

$$\alpha\phi = (1 + a_1F + \cdots + a_rF^{p^r})(c_0 + c_1F + \cdots + c_tF^t)$$
$$= c_0 + (c_1 + a_1c_0^p)F + \cdots + a_rc_t^{p^r}F^{r+t};$$

$\eta^{(1)}\beta = c^p + b_1c^pF + \cdots + b_sc^pF^s$. Assume that $r + t > s$. Then by considering the coefficients of F^{r+t} on both sides of $(*)$, we get $a_rc_t^{p^r} \in k^p$, so a_r is in k^p, which contradicts the assumption on G_1. Therefore, $r + t \leq s$ and $r \leq s$.
Similarly, $s \leq r$. Thus, $r = s$ and we also have $t = 0$ and $\phi = c_0 \in k \setminus \{0\}$. Then

$$\xi^{(1)}F = \alpha\phi - c^p\beta$$
$$= (1 + a_1F + \cdots + a_rF^r)c_0 - c^p(1 + b_1F + \cdots + b_rF^r)$$
$$= (c_0 - c^p) + (a_1c_0 - b_1c^p)F + \cdots + (a_rc_0^{p^r} - b_rc^p)F^r.$$

Therefore, $c_0 = c^p$ and $a_ic_0^{p^i} - b_ic^p \in k^p$ for all i. So, a_i is in k^p if only if b_i is in k^p. And this completes the proof of our lemma.

Now, we proceed to prove the proposition. Let

$$\alpha = 1 + a_1F + \cdots + a_rF^r, \beta = 1 + b_1F + \cdots + b_sF^s.$$

Then we have

$$G_1 \simeq D(M(m, \alpha)), G_2 \simeq D(M(n, \beta)),$$

where $(m, \alpha), (n, \beta)$ are admissible pairs and by assumption, $\mathrm{Hom}_{k[F]}(M(m, \alpha), M(n, \beta))$ and $\mathrm{Hom}_{k[F]}(M(n, \beta), M(m, \alpha))$ are nontrivial. Assume that $n > m$. Then by [KMT], Theorem 5.3.1, we have $\mathrm{Hom}_{k[F]}(M(m, \alpha), M(n, \beta)) = 0$. So $n \leq m$. Similarly, $m \leq n$ and we get $n = m$. Let $G'_1 = G_1^{(p^{n-1})}$ and $G'_2 = G_2^{(p^{n-1})}$. Then from [KMT], Proposition 5.4.1 it follows that $\mathrm{Hom}_{k-gr}(G'_1, G'_2)$ and $\mathrm{Hom}_{k-gr}(G'_2, G'_1)$ are nontrivial. Let

$$r' = \max\{i : a_i \notin k^p\}, s' = \max\{j : b_j \notin k^p\}.$$

Then we have

$$G'_1 \simeq D(M(1, \alpha')), G'_2 \simeq D(M(1, \beta')),$$

where $\alpha = 1 + a_1F + \cdots + a_{r'}F^{r'}$, $\beta = 1 + b_1F + \cdots + b_{s'}F^{s'}$. By above lemma, we get $\{i : a_i \notin k^p\} \equiv \{i : b_i \notin k^p\}$. □

Corollary 11. *With notations and assumptions as in the lemma, we have*

$$Card(\mathrm{Hom}_{k-gr}(G_1, G_2)) = Card(\mathrm{Hom}_{k-gr}(G_2, G_1))$$

and also equal to

$$Card(\mathrm{End}_{k-gr}(G_1)) = Card(\mathrm{End}_{k-gr}(G_2)).$$

Proof. Let $f : G_2 \to G_1$ be a nonzero k-homomorphism. Then it is clearly an epimorphism and for all $g \in \mathrm{Hom}_{k-gr}(G_1, G_2)$ the k-homomorphisms $g \circ f \in \mathrm{End}_{k-gr}(G_2)$ are mutually distinct. So,

$$Card(\mathrm{Hom}_{k-gr}(G_1, G_2)) \leq Card(\mathrm{End}_{k-gr}(G_2)).$$

Similarly,

$$Card(\mathrm{End}_{k-gr}(G_2)) \leq Card(\mathrm{Hom}_{k-gr}(G_1, G_2))$$

and then

$$Card(\mathrm{Hom}_{k-gr}(G_1, G_2)) = Card(\mathrm{End}_{k-gr}(G_2)).$$

We have also

$$Card(\mathrm{Hom}_{k-gr}(G_2, G_1)) = Card(\mathrm{End}_{k-gr}(G_1)).$$

The second statement follows from the first one.

As the proof of the lemma shows, the elements in $\mathrm{Hom}_{k-gr}(G_2, G_1)$ bijectively correspond to elements $c \in k$ such that

(2) $a_i c^{p^{i+1}} - b_i c^p \in k^p$, for all i.

Similarly, elements in $\mathrm{Hom}_{k-gr}(G_1, G_2)$ bijectively correspond to elements $d \in k$ such that

(3) $b_i d^{p^{i+1}} - a_i d^p \in k^p$, for all i.

Now we observe that if $c \in k\backslash\{0\}$ satisfies (2) then $d = c^{-1}$ satisfies (3), therefore $Card(\mathrm{Hom}_{k-gr}(G_2, G_1)) \leq Card(\mathrm{Hom}_{k-gr}(G_1, G_2))$, and because of the symmetry, we have

$$Card(\mathrm{Hom}_{k-gr}(G_2, G_1)) = Card(\mathrm{Hom}_{k-gr}(G_1, G_2)).$$

□

Corollary 12. *With notation and assumptions as above, assume further that G_1, G_2 are k-wound. Then we have isomorphisms of finite abelian groups*

$$\mathrm{Hom}_{k-gr}(G_2, G_1) \simeq \mathrm{Hom}_{k-gr}(G_1, G_2)$$

$$\simeq \mathrm{End}_{k-gr}(G_1) \simeq \mathrm{End}_{k-gr}(G_2).$$

Proof. Since the groups $G_i, i = 1, 2$, are k-wound, $\mathrm{End}_{k-gr}(G_i)$ are finite by [Ru], or Proposition 5.6.1 of [KMT], which says that there is a number d_i which is a power of p, such that

$$\mathrm{Aut}_{k-gr}(G_i) \simeq \{x \in k | x^{d_i - 1} = 1\},$$

$$\mathrm{End}_{k-gr}(G_i) \simeq \{x \in k | x^{d_i} = x\}.$$

Since the groups involved have the same cardinality by above corollary, it follows that

$$\mathrm{Hom}_{k-gr}(G_1, G_2) \simeq \mathrm{End}_{k-gr}(G_2),$$

$$\mathrm{Hom}_{k-gr}(G_2, G_1) \simeq \mathrm{End}_{k-gr}(G_1).$$

Again, the structure of the endomorphisms groups above implies that

$$\mathrm{Hom}_{k-gr}(G_2, G_1) \simeq \mathrm{Hom}_{k-gr}(G_1, G_2)$$

$$\simeq \mathrm{End}_{k-gr}(G_1) \simeq \mathrm{End}_{k-gr}(G_2)$$

as claimed. □

Corollary 13. *Let k be a non-perfect field of characteristic $p > 0$, G an unipotent k-group of dimension 1. If there are k-isomorphisms of G with the following k-groups*
$\{(x, y) \in \mathbb{G}_a^2 \mid y^{p^m} = x + a_1 x^p + \cdots + a_r x^{p^r}, \exists i, a_i \notin k^p\}$
and
$\{(x, y) \in \mathbb{G}_a^2 \mid y^{p^n} = x + b_1 x^p + \cdots + b_s x^{p^s}, \exists j, b_j \notin k^p\}$.
then $m = n$ and $\{i : a_i \notin k^p\} \equiv \{i : b_i \notin k^p\}$.

Proof. The proof follows from Proposition 10. □

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