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ON OPTIMAL (NON-TROJAN) SEMI-LATIN SQUARES WITH SIDE n AND BLOCK SIZE n: CONSTRUCTION PROCEDURE AND ADMISSIBLE PERMUTATIONS

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Abstract

There is a special family of the $(n \times n)/k$ semi-Latin squares called the Trojan squares which are optimal among semi-Latin squares of equivalent sizes. Unfortunately, Trojan squares do not exist for all k; for instance, there is no Trojan square for $k \geq n$. However, the need usually arises for constructing optimal semi-Latin squares where no Trojan squares exist. Bailey [2] made a conjecture on optimal semi-Latin squares for $k \geq n$ and based on this conjecture, optimal non-Trojan semi-Latin squares are here constructed for $k = n$, considering the inherent Trojan squares for $k < n$. A lemma substantiating this conjecture for $k = n$ is given and proved. In addition, the properties for the admissible permutation sets used in constructing these optimal squares are made evident based on the systematic-group-theoretic algorithm of Bailey and Chigbu [3]. Algorithms for identifying the admissible permutations as well as constructing the optimal non-Trojan $(n \times n)/k = n$ semi-Latin squares for odd n and $n = 4$ are given.

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1. Introduction

An $(n \times n)/k$ semi-Latin square is a square array with n rows and n columns in which nk letters are placed in such a way that:

(1) there are k letters in each cell; and
(2) each letter occurs once in each row and once in each column while the order of occurrence of the letters in each cell is not important.

Bailey and Chigbu [3] also regarded the square as a family of nk permutations of n objects subject to certain restrictions.

A typical $(4 \times 4)/3$ semi-Latin square with integer entries (from 1 to 12) is shown in Figure 1.

1 2 3	4 5 6	7 8 9	10 11 12
6 8 11	1 7 10	3 5 12	2 4 9
5 9 10	2 8 12	1 4 11	3 6 7
4 7 12	3 9 11	2 6 10	1 5 8

Figure 1. A Typical $(4 \times 4)/3$ semi-Latin square

Darby and Gilbert [9] defined the $(n \times n)/k$ Trojan square which is a special type of semi-Latin square as an arrangement obtained by the superposition of k mutually orthogonal $(n \times n)$ Latin squares (where such squares exist) involving k disjoint sets of n varieties so that the resulting square has kn varieties, each occurring in each of the n rows and n columns with each row intersecting each column in a cell or block of k experimental units.

Bailey [1] calculated the efficiency factors for various semi-Latin squares while Bailey [2] gave a lemma for the derivation of the efficiency factors of Trojan squares and some other semi-Latin squares and thereby established the optimality of Trojan squares among all binary incomplete-block designs of equivalent sizes.

Trojan squares have been applied in agricultural field trials as reported by Darby and Gilbert [9] and Rojas and White [11]. However, certain experimental situations arise where Trojan squares do not exist. These problem situations necessitate the search for suitable optimal non-Trojan $(n \times n)/k \geq n$ semi-Latin squares. Chigbu [4] found the optimal $(4 \times 4)/4$ semi-Latin squares using a group-theoretic approach which involved firstly constructing all possible squares of the same size before the enumeration proper. The procedure of Chigbu [4] was subsequently automated by Chigbu and Eze [8].

As an illustrative example of these kind of problems and drawing analogy from the Consumer testing example of Bailey [2], suppose there are twenty five Vacuum cleaners (labelled in alphabetical order from a to y) available for comparison during a five-week period and there are also five housewives available to test them, each housewife using five Vacuum cleaners in her home each week. This experiment could be presented using the semi-Latin square's layout in Figure 2, which consists of five rows and five columns, each row-column intersection (block) containing five Vacuum cleaners.

	Housewives				
Week	1	2	3	4	5
1	afkpu	bglqv	chmrw	dinsx	ejoty
2	bjltv	cfmpw	dgnqx	ehory	aiksu
3	cimsw	djntx	efopy	agkqu	bhlrv
4	dhnrx	eiosy	ajktu	bflpn	cgmqw
5	egoqy	ahkru	bilsv	cjmtw	dfnpx

FIGURE 2. A semi-Latin square for $n = 5$ and $k = 5$

Bailey [2] recommended the correct randomization of the semi-Latin square where rows and columns are regarded as nuisance factors. This involves randomizing independently, the rows, the columns and the experimental units within each block. If the above randomization procedure is applied, we obtain a block structure with three components: the rows, the columns and the blocks within rows and columns. Since in Figure 2, the treatments (Vacuum cleaners) are orthogonal to rows and columns, the $(5 \times 5)/5$ semi-Latin square can be assessed for efficiency just like any incomplete-block design. The given $(5 \times 5)/5$ semi-Latin square (Figure 2) may not produce the optimal canonical efficiency factors and hence, may not necessarily be the optimal square among squares of the same size.

It is indeed practically cumbersome to search and obtain the optimal $(n \times n)/k = n$ semi-Latin square for large n using existing methods.

The rest of the presentation deals with the direct construction procedure for obtaining optimal non-Trojan $(n \times n)/k = n$ semi-Latin squares and identifies the inherent admissible permutations for the construction, which is easily adaptable for automation.

2. Preliminaries

Definition 2.1. A Latin square of order n is an array on a set of n letters or symbols such that each letter or symbol occurs exactly once in each row and each column. Figure 3 is a Latin square of order $n = 4$ in a (4×4) array on a set of four letters with each letter occurring exactly once in each row and each column.

A	B	C	D
B	A	D	C
C	D	A	B
D	C	B	A

FIGURE 3. A (4×4) Latin square

It is also possible to have sets of more than two Latin squares (all of the same order n), with the same feature that each pair of the set is orthogonal to each other; such set is made up of mutually orthogonal Latin squares (MOLS). Formally, we say that two Latin squares $L_1 = (X_{ij})$ and $L_2 = (Y_{ij})$, each on the symbols $1, 2, \ldots, n$ are orthogonal to each other if every ordered pair of symbols occurs exactly once among the n^2 pairs (X_{ij}, Y_{ij}), $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$. The Latin square in Figure 4 is mutually orthogonal to the Latin square in Figure 3.

A	B	C	D
C	D	A	B
D	C	B	A
B	A	D	C

FIGURE 4. A (4×4) Latin square orthogonal to that of Figure 3

Superimposing the pair of MOLS in Figure 5 on each other gives the Trojan square shown in Figure 6.

A	C	E
C	E	A
E	A	C

B	D	F
F	B	D
D	F	B

FIGURE 5. Mutually Orthogonal Latin squares of order 3

A B	C D	E F
C F	E B	A D
E D	A F	C B

FIGURE 6. $(3 \times 3)/2$ Trojan square

Definition 2.2. An element of a set of permutations, ${\pi_\gamma}^T$, in S_n such that $i{\pi_\gamma}^T = j$ if and only if $\{\gamma \in L\} \in T(i, j)$ for $1 \leq i, j \leq n$, is admissible in the construction of a Trojan square, T, if all permutations, ${\pi_\gamma}^T$, involved in the construction are such that no two of them occur together more than once in any $T(i, j)$, $1 \leq i, j \leq n$.

Thus, given an $(n \times n)/k$ Trojan square, T, with letter set, L, each letter $\gamma \in L$ defines a permutation, ${\pi_\gamma}^T$, in S_n according as Definition 2.2. In this case, each letter of T determines a unique permutation. Conversely, given the set, ${\pi_\gamma}^T$, such that $\gamma \in L$, we can construct an $(n \times n)/k$ Trojan square, T.

Definition 2.3. A symmetric group of order n, S_n, is a set of all possible permutations of the numbers $\{1, 2, 3, \ldots, n\}$, each of which can be written in a two-row formation in this work; the identity permutation, for instance, would be written as

$$\begin{bmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & 2 & 3 & \ldots & n \end{bmatrix}$$

As an illustration, the Trojan square in Figure 6 could be constructed when bordered by integers, 1, 2, 3, and based on Definition 2.2 such that each letter of the letter set, $L = \{A, B, C, D, E, F\}$, of the square is determined by a unique permutation in S_3 written in a two-row formation; thus:

$$\pi_B^T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \pi_A^T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}, \pi_C^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}, \pi_D^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix},$$

$$\pi_E^T = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \pi_F^T = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}.$$

However, writing the above elements of S_3 in cyclic forms, we have: *identity*, $(2 \quad 3)$, $(1 \quad 2)$, $(1 \quad 2 \quad 3)$, $(1 \quad 3)$, $(1 \quad 3 \quad 2)$, respectively.

Definition 2.4. Given an $(n \times n)/s = 1$ Latin square or an $(n \times n)/s$ semi-Latin square with s letters per cell, replacing each letter of each square by r new letters gives an $(n \times n)/(sr)$ Latin or semi-Latin square which is an r-fold inflation of the original square. For example, Figure 8 is a 2-fold inflated semi-Latin square obtained from the Latin square in Figure 7.

A	B	C	D
D	A	B	C
C	D	A	B
B	C	D	A

FIGURE 7. A (4×4) Latin square

αa	βb	γc	δd
δd	αa	βb	γc
γc	δd	αa	βb
βb	γc	δd	αa

FIGURE 8. Inflated $(4 \times 4)/2$ semi-Latin square

3. Theoretical Framework

Bailey [2] gave a conjecture on optimal semi-Latin squares for $k \geq n$, which states that if Δ_1, …, Δ_{n-1} is a set of mutually orthogonal $(n \times n)$ Latin squares and $k = a(n-1) + b$ with $a \geq 1$ and $1 \leq b < n - 1$, and Ω is the superposition of the $(a+1)$-fold inflations of Δ_1, …, Δ_b with the a-fold inflation of Δ_{b+1}, …, Δ_{n-1}, then Ω is an optimal semi-Latin square.

By analogy, therefore, we give Lemma 3.1 as a basis for the construction of the optimal non-Trojan $(n \times n)/k$ semi-Latin squares for $k = n$ while some generalization for any $k \geq n$ follows from Conjecture 3.2.

Lemma 3.1. *If Δ_1, …, Δ_{n-1} is a set of $(n \times n)$ mutually orthogonal Latin squares (MOLS) where they exist, such that $\bigcup_{i=1}^{n-1} \Delta_i = \Omega$, which is a Trojan square constructed by a set of unique* *permutations of S_n, $k = (n-1) + a$ where $a = 1$ and $\Omega^* = \Omega \cup \Omega'$ is the superposition of the a-fold inflation of each of the $\Delta_1, \ldots, \Delta_{n-1}$, with the a-fold inflation of any of $\Delta_1, \ldots, \Delta_{n-1}$, Ω', then Ω^* is an optimal non-Trojan $(n \times n)/k = n$ semi-Latin square.*

Proof. Let an $(n \times n)/k'$ $(k' = (n-1))$ Trojan square, denoted by Ω, be constructed by a set of unique permutations. Let f_i be a subset of the admissible permutations for constructing Ω^*, which is used in constructing Ω: the numbers of integers for each f_i, $\#(f_i) = n \ \forall \ i = 1, \ldots, (kn-n)$. Also, let f'_j be a set of additional permutations for constructing an $(n \times n)/\bar{k}$ semi-Latin square, denoted by Ω', where $\bar{k} = a : \#(f'_j) = n \ \forall \ j = 1, \ldots, n$. The f'_j's arise from any of $\Delta_1, \ldots, \Delta_{n-1}$ MOLS. It is therefore trivial to see that Ω^* arises from adjoining $\Delta_n = \Omega'$ to Ω, hence we write;

$$\left\{ \left(\bigcup_{i=1}^{kn-n} f_i \right) \cup \left(\bigcup_{j=1}^{n} f'_j \right) \right\} = \Omega \cup \Omega' = \Omega^*,$$

which is an optimal non-Trojan $(n \times n)/k = k' + \bar{k} = n$ semi-Latin square.

Conversely, if Ω^* is an optimal non-Trojan $(n \times n)/k = n$ semi-Latin square such that $\Omega^* = \Omega \cup \Omega' = \bigcup_{i=1}^{n} \Delta_i$, then the letters associated with Ω arise from the f_i permutations while those of Ω' arise from the f_j permutations. Hence,

$$\Omega^* = \Omega \cup \Omega' = \left(\bigcup_{i=1}^{kn-n} f_i \right) \cup \left(\bigcup_{j=1}^{n} f'_j \right) = \left(\bigcup_{i=1}^{n-1} \Delta_i \right) \cup \Delta_n = \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_n,$$

which is a superposition of the MOLS and the a-fold inflation of any of the MOLS. □

Conjecture 3.2. *If $\Delta_1, \ldots, \Delta_{n-1}$ is a set of $(n \times n)$ mutually orthogonal Latin squares (MOLS) where they exist, and $k = (n-1) + a$ where a is a positive integer such that $1 \leq a \leq n-1$, and Ω^* is the superposition of the $a+1$-fold inflation of any of the $\Delta_1, \ldots, \Delta_{n-a}$ with the a-fold inflation of each of the remaining $\Delta_1, \ldots, \Delta_{n-(a+1)}$, then Ω^* is an optimal non-Trojan $(n \times n)/k$ semi-Latin square.*

3.1. **Algorithm for identifying admissible permutations for constructing optimal $(n \times n) / k = n$ semi-Latin squares (for odd n).** Comments:

(1) Specify n ($n \in N$; set of natural numbers).
(2) The total number of symbols is $= n^2$.
(3) The step length i signifies the number of iterative steps that give a total of n^2 different symbols with different permutations required for the construction of the optimal $(n \times n)/k = n$ semi-Latin square.
Start:
  (a) Choose a natural number, n.
  (b) Specify and assign two symbols for two identity permutations and represent them in a two-row formation.

(c) Choose $n-1$ symbols for the n-cycle permutations and represent them in a two-row formation in accordance with the following:

(i) For the 1^{st} of these symbols, start with 2 and move clockwise (i.e. to the right) through all the integers of the second row of the two-row formation of the identity permutation.

(ii) For the 2^{nd} of these symbols, start with 3 and move clockwise (i.e. to the right) through all the integers of the second row of the two-row formation of the identity permutation.

(iii) For the $(n-1)^{th}$ of these symbols, start with n and move clockwise (i.e. to the right) through all the integers of the second row of the two-row formation of the identity permutation.

(d) Choose $n-1$ new symbols and repeat statements c(i) through c(iii).

(e) As step length i runs from 1 to $n-2$, choose n other distinct symbols for the $(n-1)$-type permutations;

(i) For the 1^{st} of these symbols, start with 1 and move clockwise (i.e. to the right) through the integers as specified in c(i), jumping i step(s) to obtain the elements for the 1^{st} symbol.

(ii) For the 2^{nd} of these symbols, start with 2, jump i step(s); for the 3^{rd}, start with 3, jump i step(s); and for the n^{th} symbol, start with n and jump i step(s) moving as in e(i); step length i is incremented by 1.

(f) Stop.

3.2. **Illustrative Examples.** We present some specific constructions for the optimal non-Trojan semi-Latin squares as follows:

3.2.1. *Optimal* $(3\times3)/3$ *semi-Latin square.* For the $(3\times3)/3$ semi-Latin square, we need $3^2=9$ symbols and n is odd.

The integers of the symmetric group of order 3, S_3, are 1, 2, 3.

- Firstly, we specify n as 3.
- Secondly, we specify two symbols for the identity permutations written in a two-row formation thus;

$$A=\begin{bmatrix}1&2&3\\1&2&3\end{bmatrix},\qquad B=\begin{bmatrix}1&2&3\\1&2&3\end{bmatrix}$$

  – We specify different $3-1$ symbols for the 3-cycle permutation sets; starting with 2 in the second row of the identity permutation, we move clockwise through the integers of the symmetric group to obtain the 3-cycle permutations;

    ∗ For the 1^{st} 3-cycle permutation, we move clockwise starting with 2 to obtain

$$\alpha=\begin{bmatrix}1&2&3\\2&3&1\end{bmatrix};$$

* For the 2^{nd} 3-cycle permutation, we move in a clockwise direction starting with 3, to obtain

$$\beta = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}.$$

- We specify another $3-1$ new symbols for the 3-cycle permutations and obtain permutations in exactly the same way as above;
  - For the 1^{st} 3-cycle permutation, we move clockwise starting with 2 to obtain

$$\theta = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix};$$

  - For the 2^{nd} 3-cycle permutation, we move clockwise starting with 3 to obtain

$$\gamma = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}.$$

- As step length i runs from 1 to $3-2$,
- We specify another 3 new symbols for the $(3-1)$-type permutations, starting with 1, jumping $i = 1$ step, moving clockwise through the integers of the second row of the identity permutation;
  - For the 1^{st}, we move clockwise jumping 1 step and starting with 1 to obtain

$$a = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix};$$

  - For the 2^{nd}, we move clockwise as above jumping 1 step and starting with 2 to obtain

$$b = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix};$$

  - For the 3^{rd}, we move clockwise accordingly jumping 1 step and starting with 3 to obtain

$$c = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}.$$

- At this stage, step length i is $3-2=1$ and this means that we now have $3^2 = 9$ total number of symbols with different permutations which form the admissible permutations for constructing the optimal $(3 \times 3)/3$ non-Trojan semi-Latin square given in Figure 9.

$A \; B \; a$	$\theta \; \alpha \; b$	$\gamma \; \beta \; c$
$\gamma \; b \; \beta$	$A \; B \; c$	$\theta \; \alpha \; a$
$\theta \; \alpha \; c$	$\gamma \; \beta \; a$	$A \; B \; b$

FIGURE 9. Optimal non-Trojan $(3 \times 3)/3$ semi-Latin square

Table 1 shows the summary of the admissible permutations generated according to types, using the algorithm.

	Identity	$(n-1)$-type permutations	n-cycle permutation
Required Number	2	3	4
Associated Letters for Figure 9	A, B	a, b, c	α, β, θ, γ

TABLE 1. Summary of the admissible permutations for the optimal $(3 \times 3)/3$ semi-Latin square

3.2.2. *Optimal* $(5 \times 5)/5$ *semi-Latin square.* In a similar manner, we identify the admissible permutations for the optimal non-Trojan $(5 \times 5)/5$ semi-Latin square. Using the algorithm of section 3.1, thus;

The integers of the symmetric group of order 5, S_5, are 1, 2, 3, 4 and 5.

(1) Firstly, we specify n as 5.

(2) Secondly, we specify two symbols for the identity permutations written in a two-row formation thus;

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix};$$

(3) We specify $5-1$ symbols for the 5-cycle permutation sets; starting with 2, 3, 4 and 5; on each occasion we move clockwise through the integers of the second row of the two-row formation of the identity permutation to generate the 5-cycle permutations, thus;

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{bmatrix}, D = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{bmatrix}, E = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix}.$$

(4) We repeat the construction with $5-1$ distinct symbols, thus;

$$G = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}, H = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{bmatrix}, I = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{bmatrix}, J = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix}.$$

(5) As step length i runs from 1 to $(5-2)$;

(6) We specify another 5 new symbols for the $(5-1)$-type permutations. Starting with 1, jumping $i = 1$ step, and by moving clockwise (i.e. to the right) through the integers of the second row of the two-row formation of the identity permutation, we generate the following:

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{bmatrix}, \beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{bmatrix}, \gamma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 4 & 1 \end{bmatrix},$$

$$\delta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 5 & 2 \end{bmatrix}, \theta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 1 & 3 \end{bmatrix}.$$

(7) We increment step length i by 1 (i.e. $1+1=2$) and go back to (6) and continue until $i = 5-2 = 3$, then we stop.

(8) At this stage, we have generated the following permutations with their respective attached symbols:

$$a = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{bmatrix}, b = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 1 & 4 \end{bmatrix}, c = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 2 & 5 \end{bmatrix}, d = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{bmatrix},$$

$$e = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 4 & 2 \end{bmatrix}, f = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{bmatrix}, g = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{bmatrix}, h = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{bmatrix},$$

$$i = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 & 5 \end{bmatrix}, j = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}.$$

Thus, the permutations so generated, called the admissible permutations, are then used accordingly to construct the semi-Latin square in Figure 10.

$A\ \alpha\ a\ f\ F$	$B\ \beta\ b\ g\ G$	$C\ \gamma\ c\ h\ H$	$D\ \delta\ d\ i\ I$	$E\ \theta\ e\ j\ J$
$E\ \delta\ c\ g\ J$	$A\ \theta\ d\ h\ F$	$B\ \alpha\ e\ i\ G$	$C\ \beta\ a\ j\ H$	$D\ \gamma\ b\ f\ I$
$D\ \beta\ e\ h\ I$	$E\ \gamma\ a\ i\ J$	$A\ \delta\ b\ j\ F$	$B\ \theta\ c\ f\ G$	$C\ \alpha\ d\ g\ H$
$C\ \theta\ b\ i\ H$	$D\ \alpha\ c\ j\ I$	$E\ \beta\ d\ f\ J$	$A\ \gamma\ e\ g\ F$	$B\ \delta\ a\ h\ G$
$B\ \gamma\ d\ j\ G$	$C\ \delta\ e\ f\ H$	$D\ \theta\ a\ g\ I$	$E\ \alpha\ b\ h\ J$	$A\ \beta\ c\ i\ F$

FIGURE 10. Optimal non-Trojan $(5 \times 5)/5$ semi-Latin square

Table 2 summarizes the admissible permutations for the constructed optimal $(5 \times 5)/5$ semi-Latin square according to the types of permutations.

	Identity	$(5-1)$-type permutations	5-cycle permutation
Required Number	2	15	8
Associated Letters for Figure 10	A, F	α, β, γ, δ, θ, a, b, c, d, e, f, g, h, i, j	B, C, D, E, G, H, I, J

TABLE 2. Summary of the admissible permutations for the optimal $(5 \times 5)/5$ semi-Latin square

3.3. **Optimal** $(4 \times 4)/4$ **semi-Latin square.** The algorithm given in section 3.1 can only be used for construction when n is odd.

We now give another algorithm used specifically to construct the optimal non-Trojan $(4 \times 4)/4$ semi-Latin square. The algorithm in section 3.3.1 follows the same principle and technique as that of odd n in section 3.1.

3.3.1. *Algorithm for identifying admissible permutations for constructing optimal* $(4 \times 4)/4$ *semi-Latin square.* The integers in S_4 are 1, 2, 3 and 4.

(1) The total number of symbols is given as 4^2.
(2) Specify two symbols for two identity permutations, thus;

$$a = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix};$$

(3) Specify $4-1$ different symbols and move anti-clockwise (i.e. a step to the left) and clockwise (i.e. a step to the right) from one symbol to another in alternation through the integers of the identity permutation;

(a) For the 1^{st} symbol, start with 2 and move anti-clockwise to obtain, thus;

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix};$$

(b) For the 2^{nd} symbol, start with 3 and move clockwise through the integers of the identity permutation to obtain

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}.$$

(c) For the 3^{rd} symbol, start with 4 and move clockwise through the integers of the identity permutation to obtain

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

(4) Repeat statement 3 but with $4-1$ new and distinct symbols, thus;

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}, \quad \theta = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

(5) Specify 4 new distinct symbols again and for the 1^{st}, jump 1 step from the right; for the 2^{nd}, jump another 1 step from the left; and continue in that order until the rest of the integers are exhausted;

(a) For the 1^{st}, start with 1, jump 1 step (digit) to the right to pick an element, move right and exhaust the rest of the integers, thus;

$$c = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{bmatrix};$$

(b) For the 2^{nd}, start with 2, jump 1 step (digit) to the left to pick an element, move left and exhaust the rest of the integers, thus;

$$d = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{bmatrix};$$

(c) For the 3^{rd}, start with 3, jump 1 step (digit) to the right to pick an element, move right and exhaust the rest of the integers, thus;

$$e = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{bmatrix};$$

(d) For the 4^{th}, start with 4, jump 1 step (digit) to the left to pick an element, move left and exhaust the rest of the integers, thus;

$$f = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{bmatrix}.$$

(6) Specify another 4 new distinct symbols and for the 1^{st}, jump 2 steps (digits) to the right; for the 2^{nd}, jump another 2 steps to the left and continue in that order until the rest of the integers are exhausted;

(a) For the 1^{st}, start with 1, jump 2 steps (digits) to the right to pick an element, move right and exhaust the rest of the integers, thus;

$$D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{bmatrix};$$

(b) For the 2^{nd}, start with 2, jump 2 steps (digits) to the left to pick an element, move left and exhaust the rest of the integers, thus;

$$E = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{bmatrix};$$

(c) For the 3^{rd}, start with 3, jump 2 steps (digits) to the right to pick an element, move right and exhaust the rest of the integers, thus;

$$F = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{bmatrix}.$$

(d) For the 4^{th}, start with 4, jump 2 steps (digits) to the left to pick an element, move left and exhaust the rest of the integers, thus;

$$G = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{bmatrix}.$$

(7) At this stage, there are exactly $4^2 = 16$ different symbols arising from different permutations and/or permutation sets, so we stop.

The admissible permutations are hence combined accordingly to construct the optimal $(4\times4)/4$ semi-Latin square in Figure 11.

a b c D	A α d E	B β e F	C θ f G
A α e G	a b f F	C θ c E	B β d D
B β f E	C θ e D	a b d G	A α c F
C θ d F	B β c G	A α f D	a b e E

FIGURE 11. Optimal non-Trojan $(4 \times 4)/4$ semi-Latin square

The optimal non-Trojan $(4\times4)/4$ semi-Latin square so constructed here is optimally equivalent to those of Bailey [2] and Chigbu [5, 6].

4. PROPERTIES OF THE ADMISSIBLE PERMUTATIONS FOR CONSTRUCTING OPTIMAL NON-TROJAN (SEMI-LATIN) SQUARES

Chigbu [7] classified the elements of S_n used in constructing the $(n \times n)/2$ Trojan squares for n odd-prime all of which are admissible for construction as follows: the *identity* permutation, the $\left(\frac{n-1}{2}\right) \times 2$-cycle permutations and the n-cycle permutations. In general, the admissible permutations for constructing the optimal non-Trojan $(n \times n)/k = n$ semi-Latin squares can be classified into three categories: the *identity* permutations, the $(n-1)$-type permutations fixing each of 1 to n at a time, and the n-cycle permutations, adopting the usual terminologies in the literature. On the whole, the properties of the admissible permutations for constructing the optimal $(n \times n)/k = n$ non-Trojan squares are summarized as follows:

- For every optimal $(n \times n)/k = n$ non-Trojan square there is a total of n^2 symbols arranged according to the permutations of S_n which are grouped into three as follows:
  - Identity permutations;
  - $(n-1)$-type permutations which fixes each of 1 to n at a time;

  - n-cycle permutations;
- For every optimal non-Trojan semi-Latin square, the number of identity permutation involved in construction is two;
- Pre- or post-multiplying any of the $(n-1)$-type permutation with itself gives either another $(n-1)$-type permutation or the identity permutation;
- Pre- or post-multiplying any of the n-cycle permutations with itself gives another n-cycle permutation;
- Multiplying any pair of the $(n-1)$-type permutations gives either an n-cycle permutation or another $(n-1)$-type permutation;
- Multiplying any of the n-cycle permutations with an $(n-1)$-type permutation gives the $(n-1)$-type permutation or another n-cycle permutation.

The interesting thing to note is that multiplying any member of any of the three groups of admissible permutations with another gives a result which is still within the set of the admissible permutations.

5. Optimality Criteria and Comparison

In making comparisons between constructed squares, we use three main criteria which are based on their canonical efficiency factors. The efficiency of any design can be measured by the following popular optimality criteria: A-, D- and E-.

A-criterion: maximizes the harmonic mean of the canonical efficiency factors; equivalently, it minimizes the average variance of the estimators of simple contrasts.

D-criterion: maximizes the geometric mean of the canonical efficiency factors; equivalently, it minimizes the volume of the ellipsoid of the confidence around the estimates of the treatment effects.

E-criterion: maximizes the minimum of the canonical efficiency factors.

It is well known that an $(n \times n)/k$, semi-Latin square, Γ, can be assessed for efficiency as an incomplete-block design with n^2 blocks of size k and nk treatments each occurring n times. The incidence matrix, N, of Γ is the $nk \times n^2$ treatment-by-block matrix whose entry in row t and column b is the number of times that treatment t occurs in block b. Hence, the information matrix, A^*, is given as $A^* = I - (nk)^{-1}NN'$, where the canonical efficiency factors of Γ are the eigenvalues of A^*, excluding the zero eigenvalue for all-one vector.

5.1. Optimality Conditions.

As given by Chigbu [4, 6], we state as follows:

(1) If the harmonic mean of the canonical efficiency factors of a design is at least as large as that of any other design with the same values for the number of treatments (t), number of blocks (b), number of replications of each treatment (r) and size of each block (k),

then the design is said to be A-optimal. We therefore state thus:

$$A = (t-1)\left(\sum_{i=1}^{t-1}\frac{1}{e_i}\right)^{-1};$$

(2) If the geometric mean of the canonical efficiency factors of a design is at least as large as that of any other design with the same values of t, b, r and k, then the design is said to be D-optimal. Symbolically, we state, thus:

$$D = \left(\prod_{i=1}^{t-1} e_i\right);$$

(3) A design whose smallest canonical efficiency factor is at least as large as that of any other design with the same values of t, b, r and k is said to be E-optimal. In symbols, we state thus;

$$E = min(e_1, e_2, \ldots, e_{t-1}).$$

5.2. **Hypothetical Trojan squares.** Trojan squares are known to be A-, D- and E-optimal among semi-Latin squares of equivalent sizes. As given by Bailey [2], the hypothetical Trojan squares have exactly the same features as Trojan squares if they were to exist. This implies that if there exist a Trojan square for the $(n \times n)/k$ semi-Latin square, the properties of such square should correspond to that of the hypothetical Trojan square. The properties of the hypothetical Trojan square are used here as bases for comparison with any other non-Trojan square of the same size and especially the optimal ones, such that closeness of computed values for the constructed squares to those of the hypothetical ones is most desired. Thus, computational results on optimality which are closest to those obtainable from (idealistic) hypothetical Trojan squares are preferred.

According to Bailey [2], any $(n \times n)/k$ hypothetical Trojan square has canonical efficiency factors given as: $1 - k^{-1}$ with multiplicity $k(n-1)$, and 1 with multiplicity $k-1$.

The hypothetical A-optimal $(3\times3)/3$ Trojan square has the following efficiency factors: 0.6667 with multiplicity 6, and 1 with multiplicity 2. Thus, the A-optimality value, for instance, for a hypothetical $(3\times3)/3$ Trojan square evaluates to 0.7273. Also, the hypothetical $(5\times5)/5$ Trojan square has its canonical efficiency factors as: 0.8 with multiplicity 20, and 1 with multiplicity 4 where its A-optimality evaluates to 0.8276.

5.3. **Results of Comparison.** Tables 3 through 6 show the summary of the different optimality criteria of the constructed $(n \times n)/k = n$ non-Trojan semi-Latin squares for $n = k = 3$, 4, 5 and 7 in comparison with their equivalent hypothetical Trojan squares based on the A-, D- and E-optimality criteria.

Type	A-optimality	D-optimality	E-optimality
Optimal non-Trojan	0.6154	0.6866	0.3333
Hypothetical Trojan	0.7273	0.7378	0.6667

Table 3. Summary of Results for the $(3 \times 3)/3$ semi-Latin square

Type	A-optimality	D-optimality	E-optimality
Optimal non-Trojan	0.7500	0.7759	0.5000
Hypothetical Trojan	0.7895	0.7944	0.5000

Table 4. Summary of Results for the $(4 \times 4)/4$ semi-Latin square

Type	A-optimality	D-optimality	E-optimality
Optimal non-Trojan	0.4577	0.7250	0.0417
Hypothetical Trojan	0.8276	0.8303	0.8000

Table 5. Summary of Results for the $(5 \times 5)/5$ semi-Latin square

Type	A-optimality	D-optimality	E-optimality
Optimal non-Trojan	0.8664	0.8731	0.7143
Hypothetical Trojan	0.8727	0.8738	0.8571

Table 6. Summary of Results for the $(7 \times 7)/7$ semi-Latin square

6. Conclusion

We have constructed the optimal $(n \times n)/k = n$ (non-Trojan) semi-Latin squares using an approach that has some group-theoretic basis in combination with the notion of the superposition of mutually orthogonal Latin square methods for constructing semi-Latin squares. Using the methods in combination, we have been able to establish that the admissible permutations according to which the symbols for constructing an optimal non-Trojan square are arranged have unique properties as highlighted in section 4.

The method of construction primarily depends on the availability of $(n-1)$ mutually orthogonal Latin squares, which can be constructed via any of the methods in the literature, some of which were mentioned in this work.

Also, the existence of special types of semi-Latin squares, known as Trojan squares, gave the impetus for our construction of these optimal non-Trojan squares. Using the methods made evident in this work, it is easy to see that the optimal $(n \times n)/k = n$ (non-Trojan) semi-Latin square cannot exist if there is no $(n \times n)/k = n - 1$ Trojan square. Therefore, for any optimal $(n \times n)/k = n$ non-Trojan square, there must be an $(n \times n)/k = n - 1$ Trojan square with admissible permutations which are also among the admissible permutations of the optimal $(n \times n)/k = n$ non-Trojan square.

The optimal non-Trojan squares constructed for different sizes of n were found to be A-, D- and E-optimal. For instance, for $n = 3$, the optimal non-Trojan semi-Latin square constructed is A-, D- and E-optimal; for $n = 4$, the optimal non-Trojan semi-Latin square gave exactly the same canonical efficiency factors as the optimal $(4\times4)/4$ semi-Latin square reported by Bailey [2] and Chigbu [5, 6]; for $n = 5$, the optimal non-Trojan square is also A-, D- and E-optimal.

The algorithms developed in this work identify the admissible permutations for any $(n \times n)/k = n$ semi-Latin square for odd n and specifically for $n = 4$. Both algorithms which are implementable on QBasic platform identify the admissible permutations used in constructing the optimal non-Trojan squares, which were subsequently tested for optimality via computations using the Matlab software.

Patterson and Williams [10] recommended using block designs whose concurrences, λ_{ij}, of the i^{th} and j^{th} treatments, belong to the set $\{0, 1, 2\}$. However, the optimal non-Trojan squares constructed here have some of their treatment concurrences, λ_{ij}'s, as large as n; our interest here is to construct optimal non-Trojan squares whose canonical efficiency factors will be as close as possible to those of the hypothetical Trojan square as could be seen in the above results. Optimal non-Trojan semi-Latin squares could therefore be considered to be handy for suitable experiments.

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