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GAPS IN NONSYMMETRIC NUMERICAL SEMIGROUPS

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Abstract

There exist two different types of gaps in the nonsymmetric numerical semigroups $S(d_1, \ldots, d_m)$ finitely generated by a minimal set of positive integers $\{d_1, \ldots, d_m\}$. We give the generating functions for the corresponding sets of gaps. Detailed description of both gap types is given for the 1st nontrivial case $m = 3$.

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1 Introduction

In this paper we study the additive numerical semigroup in $\mathbb{N}\cup\{0\}$ generated by a finite set of positive integers $\{d_1,\ldots,d_m\}$ and the arrangement of the set of the integers which are unrepresentable by $\{d_1,\ldots,d_m\}$. Such integers are also known as the *gaps* of the numerical semigroup. First, recall the main known facts [1]. Semigroup $S(d_1,\ldots,d_m)$,

$$S(d_1,\ldots,d_m)=\left\{s\in\mathbb{N}\cup\{0\}\quad\middle|\quad s=\sum_{i=1}^{m}x_id_i,\quad x_i\in\mathbb{N}\cup\{0\}\right\}\ , \tag{1.1}$$

is said to be generated by a minimal set of m natural numbers $d_1<\ldots<d_m$, $\gcd(d_1,\ldots,d_m)=1$. It is classically known that $d_1\geq m$ [2]. For short we denote the tuple $(d_1,\ldots,d_m)$ by $\mathbf{d}^m$ where m is the *dimension* of $\mathbf{d}^m$. Henceforth $\mathbf{d}^m$ will be a minimal generating set of $\mathsf{S}(\mathbf{d}^m)$. The *conductor* $c(\mathbf{d}^m)$ of $\mathsf{S}(\mathbf{d}^m)$ is defined by $c(\mathbf{d}^m):=\min\{s\in\mathsf{S}(\mathbf{d}^m)\ \mid s+\mathbb{N}\cup\{0\}\subset\mathsf{S}(\mathbf{d}^m)\}$. Denote by $\Delta(\mathbf{d}^m)$ the complement of $\mathsf{S}(\mathbf{d}^m)$ in $\mathbb{N}$, i.e. $\Delta(\mathbf{d}^m)=\mathbb{N}\setminus\mathsf{S}(\mathbf{d}^m)$. It comprises a set of gaps. The cardinality (#) of $\Delta(\mathbf{d}^m)$ is called the *genus* of $\mathsf{S}(\mathbf{d}^m)$,

$$G(\mathbf{d}^m):=\#\Delta(\mathbf{d}^m)\ . \tag{1.2}$$

Introduce the generating function $\Phi(\mathbf{d}^m;z)$ for the set $\Delta(\mathbf{d}^m)$ of unrepresentable integers

$$\Phi(\mathbf{d}^m;z)=\sum_{s\,\in\,\Delta(\mathbf{d}^m)}z^s\ , \tag{1.3}$$

which determines the Frobenius number and the genus as follows

$$F(\mathbf{d}^m)=c(\mathbf{d}^m)-1=\deg\Phi(\mathbf{d}^m;z)\ ,\quad G(\mathbf{d}^m)=\Phi(\mathbf{d}^m;1)\ . \tag{1.4}$$

The semigroup $\mathsf{S}(\mathbf{d}^m)$ is called *symmetric* iff for any integer s the following condition holds

$$s\in\mathsf{S}(\mathbf{d}^m)\quad\Longleftrightarrow\quad F(\mathbf{d}^m)-s\notin\mathsf{S}(\mathbf{d}^m)\ . \tag{1.5}$$

Otherwise $\mathsf{S}(\mathbf{d}^m)$ is called *non–symmetric*. $F(\mathbf{d}^m)$ and $G(\mathbf{d}^m)$ are related as follows

$$2G(\mathbf{d}^m)=c(\mathbf{d}^m)\ \text{ if }\ \mathsf{S}(\mathbf{d}^m)\ \text{ is symmetric, and }\ 2G(\mathbf{d}^m)>c(\mathbf{d}^m)\quad\text{otherwise.} \tag{1.6}$$

2 Two Types of Gaps in Numerical Semigroups

Due to (1.5) the set $\Delta(\mathbf{d}^m)$ can be decomposed into two types,

$$\Delta(\mathbf{d}^m)=\Delta_g(\mathbf{d}^m)\cup\Delta_h(\mathbf{d}^m)\ ,\quad \Delta_g(\mathbf{d}^m)\cap\Delta_h(\mathbf{d}^m)=\emptyset\ , \tag{2.1}$$

where the sets of the g–gaps and the h–gaps [1] are defined, respectively,

$$\Delta_g(\mathbf{d}^m)=\{g_i\notin\mathsf{S}(\mathbf{d}^m)\ \mid F(\mathbf{d}^m)-g_i\in\mathsf{S}(\mathbf{d}^m)\}\ ,\quad \#\Delta_g(\mathbf{d}^m)=c(\mathbf{d}^m)-G(\mathbf{d}^m)\ , \tag{2.2}$$

$$\Delta_h(\mathbf{d}^m)=\{h_i\notin\mathsf{S}(\mathbf{d}^m)\ \mid F(\mathbf{d}^m)-h_i\notin\mathsf{S}(\mathbf{d}^m)\}\ ,\quad \#\Delta_h(\mathbf{d}^m)=2G(\mathbf{d}^m)-c(\mathbf{d}^m)\ . \tag{2.3}$$

Figure 1: Initial part (black) of the semigroup generated by (5,11,13). The arrows show the pairs of h-gaps. More details on this semigroup will be given later, in Example 3.2.

The following theorem provides formulas for the generating functions of the sets $\Delta_h(\mathbf{d}^m)$ and $\Delta_g(\mathbf{d}^m)$. Recall the relation [3] between $\Phi(\mathbf{d}^m;z)$ and the Hilbert series $H(\mathbf{d}^m;z)$ of the graded subring $\mathsf{k}\left[z^{d_1},\ldots,z^{d_m}\right]$ associated with semigroup $\mathsf{S}(\mathbf{d}^m)$,

$$\Phi(\mathbf{d}^m;z)=\frac{1}{1-z}-H(\mathbf{d}^m;z)\ ,\quad H(\mathbf{d}^m;z)=\frac{Q(\mathbf{d}^m;z)}{\prod_{j=1}^m\left(1-z^{d_j}\right)}\ , \tag{2.4}$$

where $H(\mathbf{d}^m;z)$ has a pole $z=1$ of order 1. The numerator $Q(\mathbf{d}^m;z)$ is a polynomial in z of the form

$$Q(\mathbf{d}^m;z)=1-\sum_{j=1}^{\beta_1(\mathbf{d}^m)}z^{C_{j,1}}+\sum_{j=1}^{\beta_2(\mathbf{d}^m)}z^{C_{j,2}}-\ldots\pm\sum_{j=1}^{\beta_{m-1}(\mathbf{d}^m)}z^{C_{j,m-1}}\ . \tag{2.5}$$

The summands of the powers $z^{C_{j,i}}$ in the last formula stand for the syzygies of different kinds and $C_{j,i}$, $C_{j+1,i}>C_{j,i}$, are the degrees of homogeneous basic invariants for the syzygies of the ith kind. The numbers of the terms $z^{C_{j,i}}$ in the summands are determined by the Betti numbers $\beta_i(\mathbf{d}^m)$ which satisfy the relation

$$1-\beta_1(\mathbf{d}^m)+\beta_2(\mathbf{d}^m)-\ldots\pm\beta_{m-1}(\mathbf{d}^m)=0\ . \tag{2.6}$$

In accordance with (1.4) and (2.4) we get for the Frobenius number,

$$F(\mathbf{d}^m)=\deg Q(\mathbf{d}^m;z)-\sum_{j=1}^m d_j\ ,\qquad \deg Q(\mathbf{d}^m;z)=C_{\beta_{m-1},m-1}\ . \tag{2.7}$$

Theorem 2.1 *The generating functions for the sets $\Delta_h(\mathbf{d}^m)$ and $\Delta_g(\mathbf{d}^m)$ are given by*

$$\sum_{h\,\in\,\Delta_h(\mathbf{d}^m)}z^h \;=\; -H(\mathbf{d}^m;z)-z^{F(\mathbf{d}^m)}H\left(\mathbf{d}^m;z^{-1}\right)\ , \tag{2.8}$$

$$\sum_{g\,\in\,\Delta_g(\mathbf{d}^m)}z^g \;=\; \frac{1}{1-z}+z^{F(\mathbf{d}^m)}H\left(\mathbf{d}^m;z^{-1}\right)\ . \tag{2.9}$$

Proof We have by 2.1

$$\Phi(\mathbf{d}^m;z)=\sum_{g\,\in\,\Delta_g(\mathbf{d}^m)}z^g+\sum_{h\,\in\,\Delta_h(\mathbf{d}^m)}z^h\ . \tag{2.10}$$

[1]The names of gaps reflect their ability (g) or disability (h) to couple with elements of $\mathsf{S}(\mathbf{d}^m)$ in a way given in (2.2) and (2.3).

Consider the transformation,

$$z^{F(\mathbf{d}^m)}\Phi\left(\mathbf{d}^m;z^{-1}\right)=\sum_{g\,\in\,\Delta_g(\mathbf{d}^m)} z^{F(\mathbf{d}^m)-g}+\sum_{h\,\in\,\Delta_h(\mathbf{d}^m)} z^{F(\mathbf{d}^m)-h}\;. \qquad (2.11)$$

However, according to (2.3)

$$F\left(\mathbf{d}^m\right)-g\in\mathsf{S}\left(\mathbf{d}^m\right)\;,\quad F\left(\mathbf{d}^m\right)-h\in\Delta_h\left(\mathbf{d}^m\right)\;. \qquad (2.12)$$

Thence,

$$z^{F(\mathbf{d}^m)}\Phi\left(\mathbf{d}^m;z^{-1}\right)=\sum_{s\,\in\,[0;F(\mathbf{d}^m)]} z^s+\sum_{h\,\in\,\Delta_h(\mathbf{d}^m)} z^h\;. \qquad (2.13)$$

The integers $s\in[0;F\left(\mathbf{d}^m\right)]$ are also known as *the nongaps* of numerical semigroup. Their cardinality coincides with $\#\Delta_g\left(\mathbf{d}^m\right)$. Making summation of (2.10) and (2.13), we get,

$$\Phi\left(\mathbf{d}^m;z\right)+z^{F(\mathbf{d}^m)}\Phi\left(\mathbf{d}^m;z^{-1}\right)=\sum_{s\,\in\,[0;F(\mathbf{d}^m)]} z^s+\sum_{g\,\in\,\Delta_g(\mathbf{d}^m)} z^g+2\sum_{h\,\in\,\Delta_h(\mathbf{d}^m)} z^h\;. \qquad (2.14)$$

Finally, we come to the generating functions of the set $\Delta_h\left(\mathbf{d}^m\right)$,

$$\sum_{h\,\in\,\Delta_h(\mathbf{d}^m)} z^h=\Phi\left(\mathbf{d}^m;z\right)+z^{F(\mathbf{d}^m)}\Phi\left(\mathbf{d}^m;z^{-1}\right)-\sum_{k=0}^{F(\mathbf{d}^m)} z^k\;. \qquad (2.15)$$

Substituting the 1st equality of (2.4) into (2.15) we obtain

$$\begin{aligned}\sum_{h\,\in\,\Delta_h(\mathbf{d}^m)} z^h &= \frac{1}{1-z}-\frac{z^{F(\mathbf{d}^m)+1}}{1-z}-\sum_{k=0}^{F(\mathbf{d}^m)} z^k-H\left(\mathbf{d}^m;z\right)-z^{F(\mathbf{d}^m)}H\left(\mathbf{d}^m;z^{-1}\right)\\ &= -\left\{H\left(\mathbf{d}^m;z\right)+z^{F(\mathbf{d}^m)}H\left(\mathbf{d}^m;z^{-1}\right)\right\}\;. \qquad (2.16)\end{aligned}$$

Substituting now the 2nd equality of (2.4) into (2.16) and making use of (2.7) we arrive at formula (2.8). Finally, combining (2.4), (2.10) and (2.8) we get formula (2.9). □

Eq. (2.8) allows us to formulate two new theorems. Let a semigroup $\mathsf{S}\left(\mathbf{d}^m\right)$ be given, and the Hilbert series $H\left(\mathbf{d}^m;z\right)$ of its graded subring $\mathsf{k}\left[z^{d_1},\ldots,z^{d_m}\right]$ be given by (2.4) and (2.5). Define the polynomial $V\left(\mathbf{d}^m;z\right)$ by

$$V\left(\mathbf{d}^m;z\right)=(-1)^{m-1}z^{\deg Q(\mathbf{d}^m;z)}Q\left(\mathbf{d}^m;z^{-1}\right)-Q\left(\mathbf{d}^m;z\right)\;, \qquad (2.17)$$

Theorem 2.2 *The semigroup* $\mathsf{S}\left(\mathbf{d}^m\right)$ *is symmetric iff the polynomial* $V\left(\mathbf{d}^m;z\right)$ *is vanishing and* $\mathsf{S}\left(\mathbf{d}^m\right)$ *is nonsymmetric iff* $V\left(\mathbf{d}^m;z\right)$ *is divided by* $\prod_{j=1}^m\left(1-z^{d_j}\right)$.

Proof First, let $\mathsf{S}\left(\mathbf{d}^m\right)$ be symmetric. Then set $\Delta_h\left(\mathbf{d}^m\right)$ is empty, and therefore $\sum_{h\,\in\,\Delta_h(\mathbf{d}^m)} z^h$ is vanishing, that implies $V\left(\mathbf{d}^m;z\right)=0$. Vice versa, if $V\left(\mathbf{d}^m;z\right)$ is vanishing then $\sum_{h\,\in\,\Delta_h(\mathbf{d}^m)} z^h=0$ and set $\Delta_h\left(\mathbf{d}^m\right)$ is empty, i.e. $\mathsf{S}\left(\mathbf{d}^m\right)$ is symmetric.

Next, let $\mathsf{S}\left(\mathbf{d}^m\right)$ be nonsymmetric. Then set $\Delta_h\left(\mathbf{d}^m\right)$ is nonempty, and therefore $\sum_{h\,\in\,\Delta_h(\mathbf{d}^m)} z^h$ is nonvanishing. Being the latter a polynomial with positive unit coefficients, it requires, due to equality (2.8), the divisibility of $V(\mathbf{d}^m; z)$ by $\prod_{j=1}^m \left(1-z^{d_j}\right)$. Vice versa, if $V(\mathbf{d}^m; z)$ is divisible by $\prod_{j=1}^m \left(1-z^{d_j}\right)$ then in accordance with (2.8) the sum $\sum_{h\,\in\,\Delta_h(\mathbf{d}^m)} z^h$ is nonvanishing and therefore the set $\Delta_h(\mathbf{d}^m)$ is nonempty. $\square$

Theorem 2.3 *Let a nonsymmetric semigroup* $\mathsf{S}(\mathbf{d}^m)$ *be given, and the Hilbert series* $H(\mathbf{d}^m; z)$ *of its graded subring* $\mathsf{k}\left[z^{d_1},\ldots,z^{d_m}\right]$ *be given by (2.4), (2.5). Then the least h-gap, denoted by* $\min\Delta_h(\mathbf{d}^m)$*, is given by the minimal degree among the terms which contribute to* $V(\mathbf{d}^m; z)$*. Moreover, this term has coefficient equal to* $+1$*.*

Proof Combining (2.5) and (2.17) we obtain

$$V(\mathbf{d}^m; z) = \sum_{k=1}^{m-1}(-1)^{m-k+1}\sum_{j=1}^{\beta_k(\mathbf{d}^m)}\left\{z^{\deg Q(\mathbf{d}^m;z)-C_{j,k}} + (-1)^m z^{C_{j,k}}\right\} - \left(1+(-1)^m z^{\deg Q(\mathbf{d}^m;z)}\right), \tag{2.18}$$

where the two last terms are cancelled with the similar terms existing in the left sums for $k=m-1,\ j=\beta_{m-1}(\mathbf{d}^m)$. Since, by (2.8)

$$V(\mathbf{d}^m; z) = \prod_{j=1}^m\left(1-z^{d_j}\right)\sum_{h\,\in\,\Delta_h(\mathbf{d}^m)} z^h\ , \tag{2.19}$$

the term with minimal degree of the sum $\sum_{h\,\in\,\Delta_h(\mathbf{d}^m)} z^h$ has to coincide with the corresponding term in the left hand side of (2.19). Thus, it is the term with minimal degree in $V(\mathbf{d}^m; z)$, in particular, among those having coefficient $+1$. $\square$

Corollary 2.4 *Let a nonsymmetric semigroup* $\mathsf{S}(\mathbf{d}^m)$ *be given, and the Hilbert series* $H(\mathbf{d}^m; z)$ *of its graded subring* $\mathsf{k}\left[z^{d_1},\ldots,z^{d_m}\right]$ *be given by (2.4), (2.5). Then the largest h-gap is given by*

$$\max\Delta_h(\mathbf{d}^m) = \deg Q(\mathbf{d}^m; z) - \min\Delta_h(\mathbf{d}^m) - \sum_{j=1}^m d_j\ . \tag{2.20}$$

Proof According to (2.3), for every $s\in\Delta_h(\mathbf{d}^m)$ the integer $F(\mathbf{d}^m)-s\in\Delta_h(\mathbf{d}^m)$. Hence, $\max\Delta_h(\mathbf{d}^m) = F(\mathbf{d}^m) - \min\Delta_h(\mathbf{d}^m)$. $\square$

We finish this Section by the estimation of the upper bound for $\min\Delta_h(\mathbf{d}^m)$. Since all the h-gaps are inside Δ_h, its number satisfies,

$$\#\Delta_h(\mathbf{d}^m) \le \max\Delta_h(\mathbf{d}^m) - \min\Delta_h(\mathbf{d}^m) + 1\ . \tag{2.21}$$

Hence, by (2.3), (2.21) and Corollary 2.4 we obtain the inequality,

$$2G(\mathbf{d}^m) - c(\mathbf{d}^m) \le F(\mathbf{d}^m) - \min\Delta_h(\mathbf{d}^m) - \min\Delta_h(\mathbf{d}^m) + 1 = c(\mathbf{d}^m) - 2\min\Delta_h(\mathbf{d}^m)\,. \tag{2.22}$$

Taking into account (2.2) we get finally,

$$\min\Delta_h(\mathbf{d}^m) \le \#\Delta_g(\mathbf{d}^m)\ . \tag{2.23}$$

Thus, $\min\Delta_h(\mathbf{d}^m)$ cannot exceed the number of nongaps of numerical semigroup $\mathsf{S}(\mathbf{d}^m)$.

3 Low – Dimensional Cases

In this Section we apply Theorem 2.2, Theorem 2.3 and Corollary 2.4 to numerical semigroups generated by two and three positive integers, respectively.

3.1 Semigroup $\mathsf{S}\left(\mathbf{d}^2\right)$

Semigroup $\mathsf{S}\left(\mathbf{d}^2\right)$ is always symmetric [4], and its Hilbert series reads

$$H\left(\mathbf{d}^2;z\right)=\frac{1-z^{d_1d_2}}{\left(1-z^{d_1}\right)\left(1-z^{d_2}\right)}\;. \tag{3.1}$$

Simple calculation of (2.8) in accordance with Theorem 2.2 yields,

$$\sum_{h\,\in\,\Delta_h(\mathbf{d}^2)} z^h=0\;. \tag{3.2}$$

3.2 Nonsymmetric semigroup $\mathsf{S}\left(\mathbf{d}^3\right)$

Nonsymmetric semigroup $\mathsf{S}\left(\mathbf{d}^3\right)$ was studied recently in [5]. Recall its main results.
Let $\mathsf{S}\left(d_1,d_2,d_3\right)\subset\mathbb{Z}_+\cup\{0\}$ be the additive nonsymmetric numerical semigroup finitely generated by a minimal set of positive integers $d_1<d_2<d_3$ such that $\gcd(d_1,d_2,d_3)=1$. Following Johnson [6] define *the minimal relation* for the triple $\mathbf{d}^3=(d_1,d_2,d_3)$,

$$a_{11}d_1=a_{12}d_2+a_{13}d_3\;,\quad a_{22}d_2=a_{21}d_1+a_{23}d_3\;,\quad a_{33}d_3=a_{31}d_1+a_{32}d_2\;, \tag{3.3}$$

where

$$\begin{aligned}&a_{jj}=\min\left\{v_{jj}\mid v_{jj}\ge 2,\ v_{jj}d_j=v_{jk}d_k+v_{jl}d_l,\ v_{jk},v_{jl}\in\mathbb{Z}_+\right\}\;, \\ &\gcd(a_{jj},a_{jk},a_{jl})=1\;,\quad\text{and}\quad (j,k,l)=(1,2,3),\ (2,3,1),\ (3,1,2)\;.\end{aligned} \tag{3.4}$$

The uniquely defined values of $v_{ij}, i\neq j$ which give a_{ii} will be denoted by $a_{ij},\ i\neq j$. The degeneracy of the matrix $((a_{ij}))$ together with (3.4) results in strong equalities [5], [6] relating the matrix elements a_{ij} and the generators d_k: for any permutation of indices (i,j,k), $i,j,k=1,2,3$ the following identities hold:

$$a_{ii}=a_{ji}+a_{ki}\;,\qquad a_{ii}a_{jj}=d_k+a_{ij}a_{ji}\;. \tag{3.5}$$

The Hilbert series $H\left(\mathbf{d}^3;z\right)$, the Frobenius number $F(\mathbf{d}^3;z)$ and the genus $G\left(\mathbf{d}^3\right)$ read [5],

$$H\left(\mathbf{d}^3;z\right)=\frac{Q(\mathbf{d}^3;z)}{\left(1-z^{d_1}\right)\left(1-z^{d_2}\right)\left(1-z^{d_3}\right)}\;, \tag{3.6}$$

where

$$Q(\mathbf{d}^3;z)=1-\sum_{i=1}^{3}z^{a_{ii}d_i}+z^{1/2\left[\langle\mathbf{a},\mathbf{d}\rangle-J\left(\mathbf{d}^3\right)\right]}+z^{1/2\left[\langle\mathbf{a},\mathbf{d}\rangle+J\left(\mathbf{d}^3\right)\right]}\;, \tag{3.7}$$

$$F\left(\mathbf{d}^3\right)=\frac{1}{2}\left[\langle\mathbf{a},\mathbf{d}\rangle+J\left(\mathbf{d}^3\right)\right]-\sum_{i=1}^{3}d_i\;,\quad G\left(\mathbf{d}^3\right)=\frac{1}{2}\left(1+\langle\mathbf{a},\mathbf{d}\rangle-\prod_{i=1}^{3}a_{ii}-\sum_{i=1}^{3}d_i\right), \tag{3.8}$$

$$J^2\left(\mathbf{d}^3\right)=\langle\mathbf{a},\mathbf{d}\rangle^2-4\sum_{i>j}a_{ii}a_{jj}d_id_j+4d_1d_2d_3\;,\quad \langle\mathbf{a},\mathbf{d}\rangle=\sum_{i=1}a_{ii}d_i\;. \tag{3.9}$$

We prove the following Proposition.

Proposition 3.1 *The minimal and the maximal values in* $\Delta_h\left(\mathbf{d}^3\right)$ *read*

$$\min\Delta_h\left(\mathbf{d}^3\right) = J\left(\mathbf{d}^3\right)\ ,\qquad \max\Delta_h\left(\mathbf{d}^3\right) = F\left(\mathbf{d}^3\right) - J\left(\mathbf{d}^3\right)\ . \tag{3.10}$$

Proof   Write the numerator $Q\left(\mathbf{d}^3; z\right)$ in the following form,

$$Q\left(\mathbf{d}^3; z\right) = 1 - \sum_{i+1}^{3} z^{a_{ii}d_i} + z^{F\left(\mathbf{d}^3\right)-J\left(\mathbf{d}^3\right)+\Sigma} + z^{F\left(\mathbf{d}^3\right)+\Sigma}\ ,\quad \Sigma = \sum_{i=1}^{3} d_i\ , \tag{3.11}$$

that in accordance with (2.17) gives polynomial $V\left(\mathbf{d}^3; z\right)$

$$V\left(\mathbf{d}^3; z\right) = -\sum_{i=1}^{3} z^{F\left(\mathbf{d}^3\right)+\Sigma-a_{ii}d_i} + z^{J\left(\mathbf{d}^3\right)} + \sum_{i=1}^{3} z^{a_{ii}d_i} - z^{F\left(\mathbf{d}^3\right)+\Sigma}\ . \tag{3.12}$$

Write equation (2.8) for the 3-dim case

$$\sum_{h\,\in\,\Delta_h\left(\mathbf{d}^3\right)} z^h = \frac{V\left(\mathbf{d}^3; z\right)}{(1-z^{d_1})(1-z^{d_2})(1-z^{d_3})}\ , \tag{3.13}$$

and notice that every term of the generating polynomial at the left hand side of (3.13) has coefficient equal to 1. Moreover, the term of minimal degree of this polynomial has to coincide with the term having minimal degree in $V\left(\mathbf{d}^3; z\right)$, and therefore such term has also coefficient equal to 1. By (3.12), polynomial $V\left(\mathbf{d}^3; z\right)$ has four candidates to have the minimal degree: $z^{J\left(\mathbf{d}^3\right)}$ and $z^{a_{ii}d_i}$, $i = 1, 2, 3$.
We prove that the degrees of the last three terms always exceed $J\left(\mathbf{d}^3\right)$. Making use of formula (3.9) and relations (3.3) and (3.5), we obtain the following representation for $J\left(\mathbf{d}^3\right)$, holding for any permutation (i, j, k) of $(1, 2, 3)$:

$$J\left(\mathbf{d}^3\right) = |a_{ij}d_j - a_{ji}d_i|\ . \tag{3.14}$$

One can always suppose that $a_{ij} > a_{ji}$ up to permutation of indices i, j, k. Since all a_{ij} are positive in the nonsymmetric case, we have by (3.14),

$$J\left(\mathbf{d}^3\right) < a_{ij}d_j\ . \tag{3.15}$$

However, by (3.5) we have $a_{ij}d_j = a_{jj}d_j - a_{kj}d_j$, or, in other words,

$$a_{ij}d_j < a_{jj}d_j\ . \tag{3.16}$$

Combining the last inequality with (3.15) we get for every $i = 1, 2, 3$,

$$J\left(\mathbf{d}^3\right) < a_{ii}d_i\ . \tag{3.17}$$

So, $J\left(\mathbf{d}^3\right)$ is the minimal degree of the terms entering polynomial $V(z)$ and therefore, by comparing the left and right hand sides of (3.13), we have $\min\Delta_h\left(\mathbf{d}^3\right) = J\left(\mathbf{d}^3\right)$. The second part of (3.10) follows by Corollary 2.4. □
As example, consider the semigroup of Figure 1.

Example 3.2 $\{d_1, d_2, d_3\} = \{5, 11, 13\}$

$$((a_{ij})) = \left(\begin{array}{ccc} 7 & 2 & 1 \\ 4 & 3 & 1 \\ 3 & 1 & 2 \end{array}\right), \quad \left\{\begin{array}{l} a_{11}d_1 = 35 \\ a_{22}d_2 = 33 \\ a_{33}d_3 = 26 \end{array}\right., \quad \left\{\begin{array}{l} F(5,11,13) = 19 \\ G(5,11,13) = 12 \\ J(5,11,13) = 2 \end{array}\right., \quad \left\{\begin{array}{l} \Delta_g(5,11,13) = \{1,3,4,6,8,9,14,19\} \\ \Delta_h(5,11,13) = \{2,7,12,17\} \end{array}\right.,$$

$$H\left(5,11,13;z\right) = \frac{Q\left(5,12,13;z\right)}{(1-z^5)(1-z^{11})(1-z^{13}))}, \quad Q\left(5,11,13;z\right) = 1 - z^{26} - z^{33} - z^{35} + z^{46} + z^{48},$$

$$\sum_{h\in\Delta_h} z^h = \frac{V\left(5,11,13;z\right)}{(1-z^5)(1-z^{11})(1-z^{13}))}, \quad V\left(5,11,13;z\right) = z^2 - z^{13} - z^{15} - z^{22} + z^{26} + z^{33} + z^{35} - z^{46}.$$

One more statement can be proved for $\min \Delta_h\left(\mathbf{d}^3\right)$.

Proposition 3.3

$$J\left(\mathbf{d}^3\right) \leq 1 + \prod_{i=1}^{3} a_{ii} - \sum_{i=1}^{3} d_i \ . \tag{3.18}$$

Proof   Combining (2.23) with (3.8) and (3.10) we immediately obtain (3.18). □

4 High – Dimensional Cases

In this Section we extend our results to numerical semigroups generated by four and more positive integers. The following theorem indicates that for $m > 3$ the situation is much more difficult.

Theorem 4.1 (H. Bresinsky, [7])
For given $m \geq 4$ there exists an m-dim semigroup with associated ring $\mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^m\right)\right]$ requiring an arbitrary large number of generators for their defining ideals I_m.

This means that the 1st Betti number $\beta_1(\mathbf{d}^m)$ can be arbitrary large. Moreover, the next statement makes the progress quite doleful.

Theorem 4.2 (L. Székely and N. Wormald, [8])
The number $\#\{Q(\mathbf{d}^m; z)\}$ of non–zero coefficients in the polynomials $Q(\mathbf{d}^m; z)$ is not bounded by any function of m for $m \geq 4$, although it is finite for every choice of the generators d_i.

However, the following theorem shows how the multiplicity d_1 and the dimension m of numerical semigroups give an upper bound for $\#\{Q(\mathbf{d}^m; z)\}$.

Theorem 4.3 (L. Fel, [5])
The number of non–zero coefficients in the polynomial $Q(\mathbf{d}^m; z), m \geq 4$ is bounded and satisfies the inequality

$$\#\{Q(\mathbf{d}^m; z)\} \leq d_1 2^{m-1} - 2(m-1) \ . \tag{4.1}$$

Unfortunately, the Frobenius problem for nonsymmetric numerical semigroups $\mathsf{S}\left(\mathbf{d}^m\right),\ m\geq 4$, is still open nowadays, although many numerical algorithms are developed [9]. However, only the symmetric semigroups $\mathsf{S}\left(\mathbf{d}^4\right)$ are to our knowledge studied [10], [11] by means of commutative algebra.

Performing dozens of examples, we formulate here a conjecture, which should imply the generalization to any m of a statement true when $m=3$ (see Proposition 3.1).

Denote by ξ_k the degrees of the terms $z^{C_{j,i}}$ in polynomial $Q\left(\mathbf{d}^m;z\right)$, and by $\Xi\left(\mathbf{d}^m\right)$ the set of ξ_k excluding their minimal ($\xi=0$) and maximal ($\xi=\deg Q\left(\mathbf{d}^m;z\right)$) values. Define also the complementary set $\overline{\Xi}\left(\mathbf{d}^m\right)$,

$$\overline{\Xi}\left(\mathbf{d}^m\right):=\left\{\deg Q\left(\mathbf{d}^m;z\right)-\xi_k,\ \mid \xi_k\in\Xi\left(\mathbf{d}^m\right)\right\}\ . \tag{4.2}$$

Since the degrees $C_{j,1}$, $j=1,\ldots,\beta_1\left(\mathbf{d}^m\right)$, in (2.5) are enumerated in ascending order, we have

$$\min\Xi\left(\mathbf{d}^m\right)=C_{1,1}\ . \tag{4.3}$$

By Theorem 2.3 the minimal degree entering polynomial $V\left(\mathbf{d}^m;z\right)$ in (2.17) coincides with $\min\Delta_h\left(\mathbf{d}^m\right)$. Moreover, the set of degrees of all terms entering $V\left(\mathbf{d}^m;z\right)$ is contained in the union $\Xi\left(\mathbf{d}^m\right)\cup\overline{\Xi}\left(\mathbf{d}^m\right)$. So, there are three possible cases,

1. $\min\Delta_h\left(\mathbf{d}^m\right)=\min\Xi\left(\mathbf{d}^m\right)$,
2. $\min\Delta_h\left(\mathbf{d}^m\right)=\min\overline{\Xi}\left(\mathbf{d}^m\right)$,
3. $\min\Delta_h\left(\mathbf{d}^m\right)\neq\min\Xi\left(\mathbf{d}^m\right)$ and $\min\Delta_h\left(\mathbf{d}^m\right)\neq\min\overline{\Xi}\left(\mathbf{d}^m\right)$.

The 1st case is excluded by the following argument. The Hilbert series $H\left(\mathbf{d}^m;z\right)$ of the semigroup and the polynomial $Q\left(\mathbf{d}^m;z\right)$ are related by equality $H\left(\mathbf{d}^m;z\right)\prod_{i=1}^m\left(1-z^{d_i}\right)=Q\left(\mathbf{d}^m;z\right)$. Thus, since the degrees of the terms at the left hand side are all elements of the semigroup, no elements of $\Xi\left(\mathbf{d}^m\right)$ belong to $\Delta\left(\mathbf{d}^m\right)$, whereas evidently $\min\Delta_h\left(\mathbf{d}^m\right)\in\Delta\left(\mathbf{d}^m\right)$.

The 3rd case implies that the terms in $V\left(\mathbf{d}^m;z\right)$ with degree $\min\Xi\left(\mathbf{d}^m\right)$ and $\min\overline{\Xi}\left(\mathbf{d}^m\right)$ are cancelled out, i.e.

$$\min\Xi\left(\mathbf{d}^m\right)=\min\overline{\Xi}\left(\mathbf{d}^m\right)\quad\Longrightarrow\quad C_{1,1}=\min\overline{\Xi}\left(\mathbf{d}^m\right)\ . \tag{4.4}$$

By consequence, in the 3rd case we should have

$$\min\Delta_h\left(\mathbf{d}^m\right)>C_{1,1}\ . \tag{4.5}$$

Since by the same definition (2.3), $\min\Delta_h\left(\mathbf{d}^m\right)$ cannot exceed $\frac{1}{2}\,F\left(\mathbf{d}^m\right)$, the following conjecture, supported by numerical experiments, should imply the impossibility of inequality (4.5), and hence of the 3rd case for nonsymmetric semigroups.

Conjecture 1

$$C_{1,1}>\frac{1}{2}\,F\left(\mathbf{d}^m\right)\ . \tag{4.6}$$

Thus, we should have, like for $m = 3$, that the 2nd case is the solely possible, i.e.,

$$\min \Delta_h (\mathbf{d}^m) = F(\mathbf{d}^m) + \sum_{k=1}^{m} d_k - \max \Xi (\mathbf{d}^m) \ . \tag{4.7}$$

We finish this Section by two numerical Examples for the 4-dim and 5-dim numerical semigroups illustrating Conjecture 1 and formula (4.7). The tuple $\mathbf{d}^4$ in Example 4.4 was studied in [12]. The polynomials $Q(\mathbf{d}^m; z)$ in both Examples were obtained by means of diagrammatic calculation on the set $\Delta(\mathbf{d}^m)$ developed for $m = 3$ and extended for $m \geq 4$ [5].

Example 4.4 $\{d_1, d_2, d_3, d_4\} = \{103, 133, 165, 228\}$

$$\left\{\begin{array}{lcl} a_{11}d_1 & = & 824 \\ a_{22}d_2 & = & 1197 \\ a_{33}d_3 & = & 825 \\ a_{44}d_4 & = & 1368 \end{array}\right. , \quad \left\{\begin{array}{lcl} F(103,133,165,228) & = & 1436 \\ G(103,133,165,228) & = & 840 \\ \Delta_h(103,133,165,228) & = & \{19,\ldots,1417\} \\ \#\{Q(103,133,165,228;z)\} & = & 26 \end{array}\right. , \quad \begin{array}{lcl} C_{1,1} & = & 824 \\ \min \Delta_h & = & 2065-2046 \end{array} ,$$

$$\begin{aligned} H(103,133,165,228;z) &= \frac{Q(103,133,165,228;z)}{(1-z^{103})(1-z^{133})(1-z^{165})(1-z^{228})} , \\ Q(103,133,165,228;z) &= 1 - z^{824} - z^{825} - z^{1077} - z^{1096} - z^{1197} - z^{1216} + z^{1319} + z^{1362} - z^{1368} + z^{1489} + z^{1508} + \\ & z^{1533} + z^{1546} + z^{1609} + z^{1737} + z^{1756} - z^{1774} + z^{1780} + z^{1881} + z^{1900} - z^{1945} - z^{1984} - \\ & z^{2003} - z^{2046} - z^{2065} \end{aligned}$$

Example 4.5 $\{d_1, d_2, d_3, d_4, d_5\} = \{5, 7, 9, 11, 13\}$

$$\left\{\begin{array}{lcl} a_{11}d_1 & = & 20 \\ a_{22}d_2 & = & 14 \\ a_{33}d_3 & = & 18 \\ a_{44}d_4 & = & 22 \\ a_{55}d_5 & = & 28 \end{array}\right. , \quad \left\{\begin{array}{lcl} F(5,7,9,11,13) & = & 8 \\ G(5,7,9,11,13) & = & 6 \\ \Delta_g(5,7,9,11,13) & = & \{1,3,4,8\} \\ \Delta_h(5,7,9,11,13) & = & \{2,6\} \\ \#\{Q(5,7,9,11,13;z)\} & = & 50 \end{array}\right. , \quad \begin{array}{lcl} C_{1,1} & = & 14 \\ \min \Delta_h & = & 53-51 \end{array} ,$$

$$\begin{aligned} H(5,7,9,11,13;z) &= \frac{Q(5,7,9,11,13;z)}{(1-z^5)(1-z^7)(1-z^9)(1-z^{11})(1-z^{13})} , \\ Q(5,7,9,11,13;z) &= 1 - z^{14} - z^{16} - 2z^{18} - 2z^{20} - 2z^{22} + z^{23} - z^{24} + 2z^{25} - z^{26} + 3z^{27} + 4z^{29} + 4z^{31} + 3z^{33} - \\ & z^{34} + 2z^{35} - 2z^{36} + z^{37} - 3z^{38} - 3z^{40} - 3z^{42} - 2z^{44} - z^{46} + z^{47} + z^{49} + z^{51} + z^{53} . \end{aligned}$$

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