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**DIFFERENTIAL OPERATORS IN A CLIFFORD
ANALYSIS ASSOCIATED TO DIFFERENTIAL EQUATIONS
WITH ANTI-MONOGENIC RIGHT-HAND SIDES**

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Abstract

This paper deals with the initial value problem of the type

$$\frac{\partial w}{\partial t} = L \left(t, x, w, \frac{\partial w}{\partial x_i} \right) \quad (1)$$

$$w(0, x) = \varphi(x) \quad (2)$$

where t is the time, L is a linear first order operator in a Clifford Analysis and φ is a generalized monogenic function. We give sufficient conditions on the coefficients of operator L under which L is associated to differential equations with anti-monogenic right-hand sides. For such operator L the initial problem (1),(2) is solvable for an arbitrary generalized monogenic initial function φ and the solution is also generalized monogenic for each t .

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1. INTRODUCTION

In this article we consider the first order partial differential equation of the type

$$\frac{\partial u}{\partial t} = \bar{D}u + G := Lu \quad (3)$$

where $u(t, x)$ are unknown function taking values in a Clifford Analysis, t means the time and x run in a bounded domain Ω of \mathbb{R}^{m+1} ($m = 2$ and $m = 3$), \bar{D} is the conjugate operator of the Cauchy-Riemann operator in Clifford analysis and G is linear in the components of u . The coefficients are supposed to depend at least continuous on t, x . The present paper formulates sufficient conditions such that the problem (1),(2) is solvable for each initial function φ which is a generalized monogenic function satisfying a differential equation with anti-monogenic right-hand sides.

A pair (L, l) of differential operators is said to be *associated* if $l(u)=0$ implies $l(Lu)=0$ (see [9]).

The initial value problem of the type

$$\frac{\partial w}{\partial t} = Lw$$

has solutions belonging an associated space for each t provided the initial function belongs to the associated space and an interior estimate is true in the associated space.

2. PRELIMINARIES AND NOTATIONS

Let \mathcal{A} be a Clifford Algebra with the basis

$$\mathcal{B} = \{e_0, e_1, \dots, e_m, e_{12}, e_{13}, \dots, e_{123\dots m}\},$$

where $e_0 = 1$. Denoting $e_{12} = e_{m+1}$, $e_{13} = e_{m+2}$, \dots , $e_{12\dots m} = e_n$, $n = 2^m - 1$, then \mathcal{B} passes into

$$\mathcal{B} = \{e_0, e_1, e_2, \dots, e_m, \dots, e_n\}.$$

Suppose that Ω is a bounded domain of \mathbb{R}^{m+1} . A function u defines in Ω and takes values in the Clifford algebra \mathcal{A} can be presented as

$$u = \sum_{i=0}^n u_i e_i,$$

where $u_i(x)$ are real-valued functions.

Further, we introduce the Cauchy-Riemann operator

$$D = \sum_{j=0}^n e_j \frac{\partial}{\partial x_j}. \quad (4)$$

The conjugate operator of D is given by

$$\bar{D} = \frac{\partial}{\partial x_0} - \sum_{j=1}^n e_j \frac{\partial}{\partial x_j}. \quad (5)$$

Now we consider differential equations of the type

$$Du = F(x, u), \quad (6)$$

where $F(x, u)$ is a linear combinations of the components of u .

Definition 1. A function $u \in C^1(\Omega, \mathcal{A})$ is said to be generalized monogenic in Ω if u satisfies the differential equation (6) (see [11]).

Definition 2. The right-hand side $F(x, u)$ is called anti-monogenic if $\overline{D}F(x, u)$ is linear in the components of u (see [11]).

Considering a pair of differential operators as follow

$$lu = Du - F(x, u) \quad (7)$$

$$Lu = \overline{D}u + G(x, u), \quad (8)$$

where F is anti-monogenic and G is linear in the components of u .

Definition 3. The operator L is said to be associated to the differential equations with anti-monogenic right-hand sides if L is associated to l .

3. SUFFICIENT CONDITIONS FOR ASSOCIATED PAIRS

3.1. **The case $m = 2$.** If $m=2$, then we have 4 basis elements $1, e_1, e_2, e_3$, where $e_3 = e_1e_2$.

Let $F = \sum_{i=0}^3 F_i u_i$, where F_i are given by

$$\begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} 1 & e_1 & e_2 & e_3 \\ -e_1 & 1 & -e_3 & e_2 \\ -e_2 & e_3 & 1 & -e_1 \\ e_3 & e_2 & -e_1 & -1 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}, \quad (9)$$

with $s_j \in C^1(\Omega, \mathbb{R})$, $j=0, 1, 2, 3$. Then F is anti-monogenic (see [11]).

From (9), we get

$$F = \sum_{i=0}^3 f_i e_i,$$

where

$$\begin{cases} f_0 = s_0 u_0 + s_1 u_1 + s_2 u_2 - s_3 u_3 \\ f_1 = s_1 u_0 - s_0 u_1 - s_3 u_2 - s_2 u_3 \\ f_2 = s_2 u_0 + s_3 u_1 - s_0 u_2 + s_1 u_3 \\ f_3 = s_3 u_0 - s_2 u_1 + s_1 u_2 + s_0 u_3. \end{cases} \quad (10)$$

Suppose that

$$G = \sum_{j=0}^3 \left(\sum_{i=0}^3 a_i^{(j)} e_i \right) u_j.$$

Then

$$G = \sum_{i=0}^3 g_i e_i,$$

where

$$g_i = \sum_{j=0}^3 a_i^{(j)} u_j. \quad (11)$$

Now assume that the coefficients $a_i^{(j)}$ are continuously differentiable with respect to the space-like x_0, x_1, x_2 and u is a solution of the equation $lu = 0$. We get

$$\begin{aligned}
l(Lu) &= D(Lu) - F(x, Lu) \\
&= D(\bar{D}u + G) - F(x, Lu) \\
&= D\bar{D}u + DG - F(x, Lu) \\
&= \bar{D}Du + DG - F(x, Lu) \\
&= \bar{D}F + DG - F(x, u^*),
\end{aligned} \tag{12}$$

where $u^* = Lu = \bar{D}u + G$.

One gets

$$\begin{aligned}
\bar{D}F &= \left(\frac{\partial f_0}{\partial x_0} + \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) + \left(\frac{\partial f_1}{\partial x_0} - \frac{\partial f_0}{\partial x_1} - \frac{\partial f_3}{\partial x_2} \right) e_1 \\
&+ \left(\frac{\partial f_2}{\partial x_0} + \frac{\partial f_3}{\partial x_1} - \frac{\partial f_0}{\partial x_2} \right) e_2 + \left(\frac{\partial f_3}{\partial x_0} - \frac{\partial f_2}{\partial x_1} + \frac{\partial f_1}{\partial x_2} \right) e_3,
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
DG &= \left(\frac{\partial f_0}{\partial x_0} - \frac{\partial g_1}{\partial x_1} - \frac{\partial g_2}{\partial x_2} \right) + \left(\frac{\partial g_1}{\partial x_0} + \frac{\partial g_0}{\partial x_1} + \frac{\partial g_3}{\partial x_2} \right) e_1 \\
&+ \left(\frac{\partial g_2}{\partial x_0} - \frac{\partial g_3}{\partial x_1} + \frac{\partial g_0}{\partial x_2} \right) e_2 + \left(\frac{\partial g_3}{\partial x_0} + \frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_2} \right) e_3.
\end{aligned} \tag{14}$$

Because u is a solution of the equation $lu=0$, then

$$\begin{cases} \frac{\partial u_0}{\partial x_0} = f_0 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \\ \frac{\partial u_1}{\partial x_0} = f_1 - \frac{\partial u_0}{\partial x_1} - \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_2}{\partial x_0} = f_2 + \frac{\partial u_3}{\partial x_1} - \frac{\partial u_0}{\partial x_2} \\ \frac{\partial u_3}{\partial x_0} = f_3 - \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}. \end{cases} \tag{15}$$

Substituting (15) into $u^* = \bar{D}u + G$, it leads to

$$u^* = \bar{D}u + G = \sum_{i=0}^3 u_i^* e_i,$$

with

$$\begin{cases} u_0^* = f_0 + g_0 + 2\frac{\partial u_1}{\partial x_1} + 2\frac{\partial u_2}{\partial x_2} \\ u_1^* = f_1 + g_1 - 2\frac{\partial u_0}{\partial x_1} - 2\frac{\partial u_3}{\partial x_2} \\ u_2^* = f_2 + g_2 + 2\frac{\partial u_3}{\partial x_1} - 2\frac{\partial u_0}{\partial x_2} \\ u_3^* = f_3 + g_3 - 2\frac{\partial u_2}{\partial x_1} + 2\frac{\partial u_1}{\partial x_2}. \end{cases} \tag{16}$$

Thus $F(x, u^*) = \sum_{i=0}^3 f_i^* e_i$, where f_i^* are given by

$$\begin{cases} f_0^* = s_0 u_0^* + s_1 u_1^* + s_2 u_2^* - s_3 u_3^* \\ f_1^* = s_1 u_0^* - s_0 u_1^* - s_3 u_2^* - s_2 u_3^* \\ f_2^* = s_2 u_0^* + s_3 u_1^* - s_0 u_2^* + s_1 u_3^* \\ f_3^* = s_3 u_0^* - s_2 u_1^* + s_1 u_2^* + s_0 u_3^*. \end{cases} \tag{17}$$

Using (10)- (17) and after calculations we obtain

$$l(Lu) = \sum_{i=0}^3 A_i e_i, \quad (18)$$

where

$$A_i = \sum_{j=0}^3 \sum_{k=1}^2 A_i^{(jk)} \frac{\partial u_j}{\partial x_k} + \sum_{l=0}^3 A_i^{(l)} u_l, \quad (19)$$

and

$$\left\{ \begin{array}{l} A_0^{(01)} = -a_1^{(0)} - a_0^{(1)} + 2s_1, \quad A_0^{(02)} = -a_2^{(0)} - a_0^{(2)} + 2s_2 \\ A_0^{(11)} = a_0^{(0)} - a_1^{(1)} - 2s_0, \quad A_0^{(12)} = -a_2^{(1)} + a_0^{(3)} + 2s_3 \\ A_0^{(21)} = -a_1^{(2)} - a_0^{(3)} - 2s_3, \quad A_0^{(22)} = a_0^{(0)} - a_2^{(2)} - 2s_0 \\ A_0^{(31)} = a_0^{(2)} - a_1^{(3)} - 2s_2, \quad A_0^{(32)} = -a_0^{(1)} - a_2^{(3)} + 2s_1 \\ A_0^{(0)} = \left(\frac{\partial s_0}{\partial x_0} + \frac{\partial s_1}{\partial x_1} + \frac{\partial s_2}{\partial x_2} \right) + \left(\frac{\partial a_0^{(0)}}{\partial x_0} - \frac{\partial a_1^{(0)}}{\partial x_1} - \frac{\partial a_2^{(0)}}{\partial x_2} \right) \\ \quad + (a_0^{(1)} - a_1^{(0)})s_1 + (a_0^{(2)} - a_2^{(0)})s_2 + (a_0^{(3)} + a_3^{(0)})s_3 \\ A_0^{(1)} = \left(\frac{\partial s_1}{\partial x_0} - \frac{\partial s_0}{\partial x_1} + \frac{\partial s_3}{\partial x_2} \right) + \left(\frac{\partial a_0^{(1)}}{\partial x_0} - \frac{\partial a_1^{(1)}}{\partial x_1} - \frac{\partial a_2^{(1)}}{\partial x_2} \right) \\ \quad - 2a_0^{(1)}s_0 + (a_0^{(0)} - a_1^{(1)})s_1 + (-a_2^{(1)} - a_0^{(3)})s_2 + (+a_3^{(1)} + a_0^{(2)})s_3 \\ A_0^{(2)} = \left(\frac{\partial s_2}{\partial x_0} - \frac{\partial s_3}{\partial x_1} - \frac{\partial s_0}{\partial x_2} \right) + \left(\frac{\partial a_0^{(2)}}{\partial x_0} - \frac{\partial a_1^{(2)}}{\partial x_1} - \frac{\partial a_2^{(2)}}{\partial x_2} \right) \\ \quad - 2a_0^{(2)}s_0 + (-a_1^{(2)} + a_0^{(3)})s_1 + (a_0^{(0)} - a_2^{(2)})s_2 + (-a_0^{(1)} + a_3^{(2)})s_3 \\ A_0^{(3)} = \left(-\frac{\partial s_3}{\partial x_0} - \frac{\partial s_2}{\partial x_1} + \frac{\partial s_1}{\partial x_2} \right) + \left(\frac{\partial a_0^{(3)}}{\partial x_0} - \frac{\partial a_1^{(3)}}{\partial x_1} - \frac{\partial a_2^{(3)}}{\partial x_2} \right) \\ \quad + (a_0^{(2)} - a_1^{(3)})s_1 + (-a_0^{(1)} - a_2^{(3)})s_2 + (-a_0^{(0)} + a_3^{(3)})s_3. \end{array} \right.$$

The remaining coefficients $A_i^{(jk)}$ (and $A_i^{(l)}$, respectively), $i = 1, 2, 3$ have similar structures as $A_0^{(jk)}$ (and $A_0^{(l)}$, respectively). Equating all coefficients $A_i^{(jk)}$ and $A_i^{(l)}$ to zero, we get sufficient conditions under which (l, L) is a associated operator. First, the expression A_0 contains 4.2=8 first order derivatives. Equating the coefficients $A_0^{(jk)}$ of these derivatives to zero, one gets a system of 8 equations as follow

$$\left\{ \begin{array}{l} a_1^{(0)} + a_0^{(1)} - 2s_1 = 0, \quad a_2^{(0)} + a_0^{(2)} - 2s_2 = 0 \\ a_0^{(0)} - a_1^{(1)} - 2s_0 = 0, \quad a_2^{(1)} - a_0^{(3)} - 2s_3 = 0 \\ a_1^{(2)} - a_0^{(3)} - 2s_3 = 0, \quad a_0^{(0)} - a_2^{(2)} - 2s_0 = 0 \\ a_0^{(2)} - a_1^{(3)} - 2s_2 = 0, \quad a_0^{(1)} + a_2^{(3)} - 2s_1 = 0. \end{array} \right. \quad (20)$$

Using the same way for $A_i^{(jk)}$, $i = 1, 2, 3$, we get 3 remaining systems which are similar to (20). Thus one obtains a total of 32 equations. These equations coincide pairwise and, therefore, only 16 equations remain. We shall denote the totality of those 16 equations by group (A).

Solving the group (A), it leads to

$$\left\{ \begin{array}{l} a_1^{(0)} = a_2^{(3)} = -a_0^{(1)} + 2s_1 \\ a_1^{(1)} = a_2^{(2)} = a_0^{(0)} - 2s_0 \\ a_1^{(2)} = -a_2^{(1)} = -a_0^{(3)} - 2s_3 \\ a_1^{(3)} = -a_2^{(0)} = a_0^{(2)} - 2s_2 \\ a_3^{(0)} = -a_0^{(3)}, a_3^{(1)} = -a_0^{(2)} \\ a_3^{(2)} = a_0^{(1)}, a_3^{(3)} = a_0^{(0)}. \end{array} \right. \quad (21)$$

Second, there are 4.4=16 the coefficients $A_i^{(l)}$ of the components of u containing in $l(L)$. Let these the coefficients be equal to zero, we have a system of 16 equations. The system is called group (B). Substituting the relation (21) into group (B) and after calculations, it implies

$$\begin{cases} \frac{\partial a_0^{(j)}}{\partial x_i} = \frac{\partial s_j}{\partial x_i}, & i, j = 0, 1, 2 \\ \frac{\partial a_0^{(3)}}{\partial x_0} = -\frac{\partial s_3}{\partial x_0} = 0 \\ \frac{\partial a_0^{(3)}}{\partial x_1} = -\frac{\partial s_3}{\partial x_1} \\ \frac{\partial a_0^{(3)}}{\partial x_2} = -\frac{\partial s_3}{\partial x_2}. \end{cases} \quad (22)$$

From (22), we obtain

$$\begin{cases} a_0^{(0)} = s_0 + \alpha_0 \\ a_0^{(1)} = s_1 + \alpha_1 \\ a_0^{(2)} = s_2 + \alpha_2 \\ a_0^{(3)} = -s_3 + \alpha_3 \\ \frac{\partial s_3}{\partial x_0} = 0, \end{cases} \quad (23)$$

where α_i are arbitrary real constants.

Substituting (23) into group (B), then which reduces to

$$\begin{cases} \frac{\partial s_0}{\partial x_0} + s_1 \alpha_1 + s_2 \alpha_2 = 0 \\ \frac{\partial s_1}{\partial x_0} - s_0 \alpha_1 - s_2 \alpha_3 = 0 \\ \frac{\partial s_2}{\partial x_0} - s_0 \alpha_2 + s_1 \alpha_3 = 0 \\ \frac{\partial s_3}{\partial x_0} = 0. \end{cases} \quad (24)$$

Denote

$$s = \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix},$$

and

$$A = \begin{bmatrix} 0 & -\alpha_1 & -\alpha_2 & 0 \\ \alpha_1 & 0 & \alpha_3 & 0 \\ \alpha_2 & -\alpha_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then the system (24) turns to

$$\frac{\partial s}{\partial x_0}(x_0, x_1, x_2) = As(x_0, x_1, x_2). \quad (25)$$

Fixing x_1, x_2 , then the system (25) can be considered as an ordinary system of differential equations for s_j with constant-coefficients

$$\frac{ds}{dx_0}(x_0, x_1, x_2) = As(x_0, x_1, x_2). \quad (26)$$

The system (26) leads to

$$s(x_0, x_1, x_2) = e^{A(x_0 - x_0^{(0)})} s(x_0^{(0)}, x_1, x_2). \quad (27)$$

According to (21) and (23), it follows that

$$\begin{cases} a_1^{(0)} = a_2^{(3)} = s_1 - \alpha_1 \\ a_1^{(1)} = a_2^{(2)} = -s_0 + \alpha_0 \\ a_1^{(2)} = -a_2^{(1)} = -s_3 - \alpha_3 \\ a_1^{(3)} = -a_2^{(0)} = -s_2 + \alpha_2 \\ a_3^{(0)} = -a_0^{(3)} = s_3 - \alpha_3 \\ a_3^{(1)} = -a_0^{(2)} = -s_2 - \alpha_2 \\ a_3^{(2)} = a_0^{(1)} = s_1 + \alpha_1 \\ a_3^{(3)} = a_0^{(0)} = s_0 + \alpha_0. \end{cases} \quad (28)$$

Note that $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are arbitrary real-constants. Summarizing these calculations, the following theorem has been proved

Theorem 1. *Suppose that the coefficients $a_i^{(j)}, s_j$ are continuously differentiable on x_0, x_1, x_2 and differentiable on the time t . If the conditions (27) and (28) are satisfied, then operator L is associated to the operator ℓ .*

3.2. The case $m = 3$. If $m = 3$ then one gets 8 basis elements e_0, e_1, \dots, e_7 , where $e_4 = e_{12}, e_5 = e_{23}, e_6 = e_{31}, e_7 = e_{123}$.

We consider the differential operators l and L having the type of (7) and (8) with

$$F = \sum_{i=0}^7 F_i u_i, \quad (29)$$

where F_i are given by

$$\begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \end{pmatrix} = \begin{pmatrix} 1 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ -e_1 & 1 & -e_4 & e_6 & e_2 & -e_7 & -e_3 & e_5 \\ -e_2 & e_4 & 1 & -e_5 & -e_1 & e_3 & -e_7 & e_6 \\ -e_3 & -e_6 & e_5 & 1 & -e_7 & -e_2 & e_1 & e_4 \\ e_4 & e_2 & -e_1 & e_7 & -1 & e_6 & -e_5 & -e_3 \\ e_5 & e_7 & e_3 & -e_2 & -e_6 & -1 & e_4 & -e_1 \\ e_6 & -e_3 & e_7 & e_1 & e_5 & -e_4 & -1 & -e_2 \\ -e_7 & e_5 & e_6 & e_4 & e_3 & e_1 & e_2 & -1 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \end{pmatrix} \quad (30)$$

and

$$G = \sum_{j=0}^7 \left(\sum_{i=0}^7 a_i^{(j)} e_i \right) u_j, \quad (31)$$

where $s_j, a_i^{(j)} \in C^1(\Omega, \mathbb{R})$. Then F is anti-monogenic (see [11]).

From (29), (30), and (31), it leads to

$$F = \sum_{i=0}^7 f_i e_i,$$

with

$$\begin{cases} f_0 = s_0u_0 + s_1u_1 + s_2u_2 + s_3u_3 - s_4u_4 - s_5u_5 - s_6u_6 - s_7u_7 \\ f_1 = s_1u_0 - s_0u_1 - s_4u_2 + s_6u_3 - s_2u_4 - s_7u_5 + s_3u_6 + s_5u_7 \\ f_2 = s_2u_0 + s_4u_1 - s_0u_2 - s_5u_3 + s_1u_4 - s_3u_5 - s_7u_6 + s_6u_7 \\ f_3 = s_3u_0 - s_6u_1 + s_5u_2 - s_0u_3 - s_7u_4 + s_2u_5 - s_1u_6 + s_4u_7 \\ f_4 = s_4u_0 - s_2u_1 + s_1u_2 + s_7u_3 + s_0u_4 + s_6u_5 - s_5u_6 + s_3u_7 \\ f_5 = s_5u_0 + s_7u_1 - s_3u_2 + s_2u_3 - s_6u_4 + s_0u_5 + s_4u_6 + s_1u_7 \\ f_6 = s_6u_0 + s_3u_1 + s_7u_2 - s_1u_3 + s_5u_4 - s_4u_5 + s_0u_6 + s_2u_7 \\ f_7 = s_7u_0 - s_5u_1 - s_6u_2 - s_4u_3 + s_3u_4 + s_1u_5 + s_2u_6 - s_0u_7, \end{cases}$$

and

$$G = \sum_{i=0}^7 g_i e_i,$$

where

$$g_i = \sum_{j=0}^7 a_i^{(j)} u_j. \quad (32)$$

By the analogous method which is used in the section 3.1, we obtain

$$l(Lu) = \sum_{i=0}^7 A_i e_i,$$

where

$$A_i = \sum_{j=0}^7 \sum_{k=1}^3 A_i^{(jk)} \frac{\partial u_j}{\partial x_k} + \sum_{l=0}^7 A_i^{(l)} u_l.$$

In this case, there are $8 \cdot 8 \cdot 3 = 192$ coefficients of the first order derivatives, and $8 \cdot 8 = 64$ coefficients of the components u_0, u_1, \dots, u_7 .

Putting

$$S = \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_7 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & \alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 & 0 & -\alpha_7 \\ -\alpha_1 & 0 & -\alpha_4 & \alpha_6 & 0 & \alpha_7 & 0 & 0 \\ -\alpha_2 & \alpha_4 & 0 & -\alpha_5 & 0 & 0 & \alpha_7 & 0 \\ -\alpha_3 & -\alpha_6 & \alpha_5 & 0 & \alpha_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_7 & 0 & -\alpha_6 & \alpha_5 & \alpha_3 \\ 0 & \alpha_7 & 0 & 0 & \alpha_6 & 0 & -\alpha_4 & \alpha_1 \\ 0 & 0 & \alpha_7 & 0 & -\alpha_5 & \alpha_4 & 0 & \alpha_2 \\ -\alpha_7 & 0 & 0 & 0 & -\alpha_3 & -\alpha_1 & -\alpha_2 & 0 \end{bmatrix}.$$

Using a similar way as in the case $m = 2$, it leads to the following theorem

Theorem 2. *Suppose that the coefficients $a_i^{(j)}$, s_j are continuously differentiable on x_0, x_1, x_2, x_3 and differentiable on the time t . Then operator L is associated to the operator ℓ if the following conditions are satisfied*

i)

$$S(x_0, x_1, x_2, x_3) = e^{B(x_0 - x_0^{(0)})} S(x_0^{(0)}, x_1, x_2, x_3)$$

ii)

$$\left\{ \begin{array}{l} a_0^{(0)} = a_4^{(4)} = a_5^{(5)} = a_6^{(6)} = s_0 + \alpha_0 \\ a_1^{(0)} = a_2^{(4)} = a_7^{(5)} = -a_3^{(6)} = s_1 + \alpha_1 \\ a_2^{(0)} = a_1^{(4)} = a_3^{(5)} = a_7^{(6)} = s_2 + \alpha_2 \\ a_3^{(0)} = a_7^{(4)} = -a_2^{(5)} = a_1^{(6)} = s_3 + \alpha_3 \\ a_4^{(0)} = -a_0^{(4)} = -a_6^{(5)} = a_5^{(6)} = s_4 + \alpha_4 \\ a_5^{(0)} = a_6^{(4)} = -a_0^{(5)} = -a_4^{(6)} = s_5 + \alpha_5 \\ a_6^{(0)} = -a_5^{(4)} = a_4^{(5)} = -a_0^{(6)} = s_6 + \alpha_6 \\ a_7^{(0)} = -a_3^{(4)} = -a_1^{(5)} = -a_2^{(6)} = s_7 + \alpha_7 \\ a_0^{(1)} = a_4^{(2)} = -a_6^{(3)} = a_5^{(7)} = s_1 - \alpha_1 \\ a_1^{(1)} = a_2^{(2)} = a_3^{(3)} = a_7^{(7)} = -s_0 + \alpha_0 \\ a_2^{(1)} = -a_1^{(2)} = -a_7^{(3)} = a_3^{(7)} = s_4 - \alpha_4 \\ a_3^{(1)} = a_7^{(2)} = -a_1^{(3)} = -a_2^{(7)} = -s_6 + \alpha_6 \\ a_4^{(1)} = -a_0^{(2)} = -a_5^{(3)} = -a_6^{(7)} = -s_2 + \alpha_2 \\ a_5^{(1)} = a_6^{(2)} = a_4^{(3)} = -a_0^{(7)} = s_7 - \alpha_7 \\ a_6^{(1)} = -a_5^{(2)} = a_0^{(3)} = a_4^{(7)} = s_3 - \alpha_3 \\ a_7^{(1)} = -a_3^{(2)} = a_2^{(3)} = -a_1^{(7)} = -s_5 + \alpha_5, \end{array} \right.$$

where $\alpha_0, \alpha_1, \dots, \alpha_7$ are arbitrary real-constants.

4. CONSTRUCTION THE OPERATOR L

4.1. **The case $m = 2$.** From the theorem 1, we can see that, easily, the operator L has the

form $Lu = \overline{D}u + G$, where $G = \sum_{i=0}^3 g_i e_i$ and

$$\left\{ \begin{array}{l} g_0 = (s_0 + \alpha_0)u_0 + (s_1 + \alpha_1)u_1 + (s_2 + \alpha_2)u_2 + (-s_3 + \alpha_3)u_3 \\ g_1 = (s_1 - \alpha_1)u_0 + (-s_0 + \alpha_0)u_1 - (s_3 + \alpha_3)u_2 + (-s_2 + \alpha_2)u_3 \\ g_2 = (s_2 - \alpha_2)u_0 + (s_3 + \alpha_3)u_1 + (-s_0 + \alpha_0)u_2 + (s_1 - \alpha_1)u_3 \\ g_3 = (s_3 - \alpha_3)u_0 - (s_2 + \alpha_2)u_1 + (s_1 + \alpha_1)u_2 + (s_0 + \alpha_0)u_3. \end{array} \right.$$

Example 1. If we choose $s_0 = s_1 = s_2 = 0$, and s_3 is independent on x_0 , then the condition (24) holds automatically. This means that the relation (27) is satisfied. Using the system (28) we obtain the operator

$$\begin{aligned} Lu = & \overline{D}u + [\alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2 - (s_3 - \alpha_3)u_3] \\ & + [-\alpha_1 u_0 + \alpha_0 u_1 - (s_3 + \alpha_3)u_2 + \alpha_2 u_3] e_1 \\ & + [-\alpha_2 u_0 + (s_3 + \alpha_3)u_1 + \alpha_0 u_2 - \alpha_1 u_3] e_2 \\ & + [(s_3 - \alpha_3)u_0 - \alpha_2 u_1 + \alpha_1 u_2 + \alpha_0 u_3] e_3, \end{aligned}$$

is associated to the operator

$$lu = Du + (u_3 + u_2e_1 - u_1e_2 - u_0e_3)s_3.$$

Note that, if $s_3 = 0$, then the operator l turns to the Cauchy-Riemann of $\mathbb{R}^{(3)}$ which L is associated to.

Example 2. *Similarly, If s_0, s_1, s_2, s_3 are independent on x_0 , and $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then the condition (27) holds. So we have the operator*

$$\begin{aligned} Lu = \overline{D}u &+ [(s_0 + \alpha_0)u_0 + s_1u_1 + s_2u_2 - s_3u_3] \\ &+ [s_1u_0 + (-s_0 + \alpha_0)u_1 - s_3u_2 - s_2u_3]e_1 \\ &+ [s_2u_0 + s_3u_1 + (-s_0 + \alpha_0)u_2 + s_1u_3]e_2 \\ &+ [s_3u_0 - s_2u_1 + s_1u_2 + (s_0 + \alpha_0)u_3]e_3, \end{aligned}$$

is associated to the operator

$$\begin{aligned} lu = Du &- (s_0u_0 + s_1u_1 + s_2u_2 - s_3u_3) \\ &- (s_1u_0 - s_0u_1 - s_3u_2 - s_2u_3)e_1 \\ &- (s_2u_0 + s_3u_1 - s_0u_2 + s_1u_3)e_2 \\ &- (s_3u_0 - s_2u_1 + s_1u_2 + s_0u_3)e_3. \end{aligned}$$

4.2. The case $m = 3$. In this case, by similar way as in the section 4.1, and using the condition (ii) of theorem 2, we can construct the operator L , easily.

Example 3. *If $s_0, s_1, s_2, \dots, s_7$ are independent on x_0 , and $\alpha_1 = \alpha_2 = \dots = \alpha_7 = 0$, then the condition (i) of theorem 2 is satisfied. So the operator L as follow*

$$\begin{aligned} Lu = \overline{D}u &+ [(s_0 + \alpha_0)u_0 + s_1u_1 + s_2u_2 + s_3u_3 - s_4u_4 - s_5u_5 - s_6u_6 - s_7u_7] \\ &+ [(s_1u_0 - (s_0 - \alpha_0)u_1 - s_4u_2 + s_6u_3 + s_2u_4 - s_7e_5 + s_3e_6 + s_5u_7]e_1 \\ &+ [s_2u_0 + s_4u_1 - (s_0 - \alpha_0)u_2 - s_5u_3 + s_1u_4 - s_3u_5 - s_7u_6 + s_6u_7]e_2 \\ &+ [s_3u_0 - s_6u_1 + s_5u_2 - (s_0 - \alpha_0)u_3 - s_7u_4 + s_2u_5 - s_1u_6 + s_4u_7]e_3 \\ &+ [s_4u_0 - s_2u_1 + s_1u_2 + s_7u_3 + (s_0 + \alpha_0)u_4 + s_6u_5 - s_5u_6 + s_3u_7]e_4 \\ &+ [s_5u_0 + s_7u_1 - s_3u_2 + s_2u_3 - s_6u_4 + (s_0 + \alpha_0)u_5 + s_4u_6 + s_1u_7]e_5 \\ &+ [s_6u_0 + s_3u_1 + s_7u_2 - s_1u_3 + s_5u_4 - s_4u_5 + (s_0 + \alpha_0)u_6 + s_2u_7]e_6 \\ &+ [s_7u_0 - s_5u_1 - s_6u_2 - s_4u_3 + s_3u_4 + s_1u_5 + s_2u_6 - (s_0 - \alpha_0)u_7]e_7. \end{aligned}$$

Example 4. Analogously, if we choose $s_0 = s_1 = \dots = s_7 = 0$, then the condition (i) of theorem 2 holds. Therefore, the operator L as follow

$$\begin{aligned}
Lu &= \overline{D}u + G \\
&= \overline{D}u + [\alpha_0 u_0 - \alpha_1 u_1 - \alpha_2 u_2 - \alpha_3 u_3 - \alpha_4 u_4 - \alpha_5 u_5 - \alpha_6 u_6 + \alpha_7 u_7] \\
&+ [\alpha_1 u_0 + \alpha_0 u_1 + \alpha_4 u_2 - \alpha_6 u_3 + \alpha_2 u_4 - \alpha_7 u_5 + \alpha_3 u_6 - \alpha_5 u_7] e_1 \\
&+ [\alpha_2 u_0 - \alpha_4 u_1 + \alpha_0 u_2 + \alpha_5 u_3 + \alpha_1 u_4 - \alpha_3 u_5 - \alpha_7 u_6 - \alpha_6 u_7] e_2 \\
&+ [\alpha_3 u_0 + \alpha_6 u_1 - \alpha_5 u_2 + \alpha_0 u_3 - \alpha_7 u_4 + \alpha_2 u_5 - \alpha_1 u_6 - \alpha_4 u_7] e_3 \\
&+ [\alpha_4 u_0 + \alpha_2 u_1 - \alpha_1 u_2 - \alpha_7 u_3 + \alpha_0 u_4 + \alpha_6 u_5 - \alpha_5 u_6 - \alpha_3 u_7] e_4 \\
&+ [\alpha_5 u_0 - \alpha_7 u_1 + \alpha_3 u_2 - \alpha_2 u_3 - \alpha_6 u_4 + \alpha_0 u_5 + \alpha_4 u_6 - \alpha_1 u_7] e_5 \\
&+ [\alpha_6 u_0 - \alpha_3 u_1 - \alpha_7 u_2 + \alpha_1 u_3 + \alpha_5 u_4 - \alpha_4 u_5 + \alpha_0 u_6 - \alpha_2 u_7] e_6 \\
&+ [\alpha_7 u_0 + \alpha_5 u_1 + \alpha_6 u_2 + \alpha_4 u_3 + \alpha_3 u_4 + \alpha_1 u_5 + \alpha_2 u_6 + \alpha_0 u_7] e_7,
\end{aligned}$$

and L is associated to the Cauchy-Riemann operator of $\mathbb{R}^{(4)}$

5. INITIAL VALUE PROBLEMS WITH GENERALIZED MONOGENIC INITIAL FUNCTIONS

The H.Lewy example (see [6]) shows that, there are linear first order differential equations with infinitely differentiable coefficients not having any solutions. On the other hand, the theorem 1 and 2 allow us to construct operator L such that the initial value problem (1), (2) is solvable for each generalized monogenic initial function φ , which is a solution of the differential equation with the anti-monogenic right -hand sides. Note that the interior estimate holds in this case (see [12]). To sum up, we get the following theorem.

Theorem 3. *Suppose that the operator L is associated to the differential equations with anti-monogenic right-hand sides. Suppose, further, that the initial function φ is an arbitrary generalized monogenic function. Then the initial value problem (1), (2) is solvable. The solution $u(t, x)$ is generalized monogenic for each t .*

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