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**HIGHER DIMENSIONAL UNITARY BRAID MATRICES:
CONSTRUCTION, ASSOCIATED STRUCTURES
AND ENTANGLEMENTS**

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Abstract

We construct $(2n)^2 \times (2n)^2$ unitary braid matrices \widehat{R} for $n \geq 2$ generalizing the class known for $n = 1$. A set of $(2n) \times (2n)$ matrices (I, J, K, L) are defined. \widehat{R} is expressed in terms of their tensor products (such as $K \otimes J$), leading to a canonical formulation for all n . Complex projectors P_{\pm} provide a basis for our real, unitary \widehat{R} . Baxterization is obtained. Diagonalizations and block-diagonalizations are presented. The loss of braid property when \widehat{R} ($n > 1$) is block-diagonalized in terms of \widehat{R} ($n = 1$) is pointed out and explained. For odd dimension $(2n + 1)^2 \times (2n + 1)^2$, a previously constructed braid matrix is complexified to obtain unitarity. \widehat{RLL} - and \widehat{RTT} -algebras, chain Hamiltonians, potentials for factorizable S -matrices, complex non-commutative spaces are all studied briefly in the context of our unitary braid matrices. Turaev construction of link invariants is formulated for our case. We conclude with comments concerning entanglements.

1 Introduction

The 4×4 unitary braid matrix

$$\widehat{R} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix} \quad (1.1)$$

and the associated SO3 algebra have been studied extensively in our previous papers (refs. [1, 2] provide further sources). Recently this matrix (along with its conjugate by the permutation matrix P) has been studied as the Bell-matrix in the context of quantum entanglements (see [3, 4, 5] and the references therein). Algebraic aspects have also been studied [4, 5] using different approaches. Joint presence of quantum and topological entanglements is the theme of ref. [3]. Here we present direct generalizations of (1.1) to higher dimensions, namely *unitary* matrices of dimensions $(2n)^2 \times (2n)^2$ ($n \geq 1$), which satisfy the braid equation

$$\widehat{R}_{12}\widehat{R}_{23}\widehat{R}_{12} = \widehat{R}_{23}\widehat{R}_{12}\widehat{R}_{23}, \quad (1.2)$$

where $\widehat{R}_{12} = \widehat{R} \otimes I$ and $\widehat{R}_{23} = I \otimes \widehat{R}$. For $n = 1$ one obtains (1.1). Entanglements are commented upon in sec. 11, referring to previous sections⁵.

Different classes of higher dimensional braid matrices were constructed and studied in a series of previous papers [6, 7, 8, 9]. But unitary was not sought before, though for the Baxterized form the constraint

$$\widehat{R}(\theta) \widehat{R}(-\theta) = I \quad (1.3)$$

is sometimes labeled *unitarity*. In the constructions to follow one has always

$$\widehat{R}^+ \widehat{R} = I_{(2n)^2}. \quad (1.4)$$

Moreover, the spectral parameter θ (or $z = \tanh \theta$) is introduced in such a way when Baxterizing that

$$\widehat{R}^+(\theta) = \widehat{R}(-\theta) \quad (1.5)$$

and unitarity, now really coinciding with (1.3), is maintained.

Our construction is presented in a canonical form for all n by introducing a set of $2n \times 2n$ matrices (I, J, K, L) with particularly simple properties and implementing their tensor products ($K \otimes J$ and so on). The verification of the braid equation and its Baxterized form now become transparent (see (2.11), (2.16) and (2.17)). Denoting the $(2n)^2 \times (2n)^2$ braid matrix as $\widehat{R}_{(2n)}$ (so that (1.1) is now $\widehat{R}_{(2)}$) we verify explicitly the non-equivalence of $\widehat{R}_{(2n)}$ with a block-diagonal, direct sum of $\widehat{R}_{(2)}$'s in the following, precise sense: On can indeed construct a matrix, say, V

⁵Notations: Our notations will be systematically defined in the following sections. But here we point out that our \widehat{R} , the braid matrix is the Yang-Baxter (YB)-matrix of ref. [3] and our YB-matrix $R = P\widehat{R}$ is called *algebraic YB-matrix* in ref. [3]. Other references cited also use different notations. This should be noted to avoid confusion.

such that $V\widehat{R}_{(2n)}V^{-1}$ is block-diagonalized into $\widehat{R}_{(2)}$ blocks. But V does not have a tensored structure of the type $W \otimes W$ and does not conserve the braid property (1.2) (see (4.9-12)). The tensored structure displays this negative result also with clarity. In section 5 we point out briefly that a certain type of complexification of the odd dimensional $\widehat{R}(\theta)$'s of ref. [6] leads to unitarity, where however the Baxterized form is essential. Elsewhere, in this paper, only even dimensional \widehat{R} is considered. After presenting our general constructions we study various aspects of our braid matrices in successive sections: Baxterization, \widehat{RLL} - and \widehat{RTT} -algebras, chain Hamiltonians, potentials for factorizable S -matrices, non-commutative spaces and link invariants. Such aspects deserve further study, which is beyond the scope of this paper. Some points are discussed in the concluding remarks, including certain aspects of entanglements.

2 Constructions (Even dimensions)

The $(2n)^2 \times (2n)^2$ ($n \geq 1$) \widehat{R} matrices satisfying

$$\widehat{R}_{12}\widehat{R}_{23}\widehat{R}_{12} = \widehat{R}_{23}\widehat{R}_{12}\widehat{R}_{23}. \quad (2.1)$$

are constructed in terms of the following operators which are $(2n) \times (2n)$ matrices. Define

$$\begin{aligned} I &= \sum_{i=1}^n ((ii) + (\bar{i}\bar{i})), & J &= \sum_{i=1}^n \left((-1)^{\bar{i}}(i\bar{i}) + (-1)^i(\bar{i}i) \right), \\ K &= \sum_{i=1}^n ((i\bar{i}) + (\bar{i}i)), & L &= \sum_{i=1}^n ((\bar{i}\bar{i}) - (\bar{i}i)), \end{aligned} \quad (2.2)$$

where $\bar{i} = 2n - i + 1$ and (ij) is the $(2n) \times (2n)$ matrix with 1 for the element (row i , column j) and zero elsewhere. The dimension will be indicated (writing, say, $J_{(2n)}$ for J) when crucial but not otherwise. For $n = 1$, apart for the identity I

$$J = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}, \quad K = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad L = J. \quad (2.3)$$

The degeneracy is lifted ($J \neq L$) for $n > 1$. For $n = 2$,

$$J = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}, \quad K = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \quad L = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}. \quad (2.4)$$

Noting that $(-1)^{\bar{i}} = (-1)^{2n-i+1} = -(-1)^i$, one obtains

$$\begin{aligned} JK &= -KJ = \sum_{i=1}^n \left((-1)^{\bar{i}}(ii) + (-1)^i(\bar{i}\bar{i}) \right), \\ LK &= -KL = \sum_{i=1}^n ((ii) - (\bar{i}\bar{i})), \\ JL &= LJ = \sum_{i=1}^n \left((-1)^i(ii) - (-1)^{\bar{i}}(\bar{i}\bar{i}) \right) \end{aligned} \quad (2.5)$$

and

$$-J^2 = K^2 = -L^2 = I. \quad (2.6)$$

The \widehat{R} matrix, presented in sec. 1 (for $n = 1$), is generalized (for $n > 1$) as follows:

Class I:

$$\widehat{R}^{\pm 1} = \frac{1}{\sqrt{2}} (I \otimes I \pm K \otimes J) \quad (2.7)$$

satisfying unitarity

$$\widehat{R}^+ \widehat{R} = I \otimes I = I_{(2n)^2} \quad (2.8)$$

and also

$$\widehat{R}^2 = \sqrt{2} \widehat{R} - I \quad \Rightarrow \quad \widehat{R} + \widehat{R}^{-1} = \sqrt{2} I. \quad (2.9)$$

Using (2.5), (2.6) one obtains

$$\begin{aligned} (K \otimes J \otimes I) (I \otimes K \otimes J) (K \otimes J \otimes I) &= (I \otimes K \otimes J), \\ (I \otimes K \otimes J) (K \otimes J \otimes I) (I \otimes K \otimes J) &= (K \otimes J \otimes I) \end{aligned} \quad (2.10)$$

leading to

$$\widehat{R}_{12} \widehat{R}_{23} \widehat{R}_{12} = \frac{1}{\sqrt{2}} (I \otimes K \otimes J + K \otimes J \otimes I) = \widehat{R}_{23} \widehat{R}_{12} \widehat{R}_{23}. \quad (2.11)$$

Thus the braid equation is satisfied.

Baxterization: Define ($z = \tanh \theta$)

$$\begin{aligned} \widehat{R}(z)^{\pm 1} &= \frac{1}{\sqrt{1+z^2}} (I \otimes I \pm zK \otimes J) \\ &= \frac{1}{\sqrt{1+(\tanh \theta)^2}} (I \otimes I \pm \tanh \theta K \otimes J) \equiv \widehat{R}(\theta)^{\pm 1}, \end{aligned} \quad (2.12)$$

$$\widehat{R}(\pm 1) = \widehat{R}^{\pm 1}. \quad (2.13)$$

They satisfy unitarity (for real z)⁶

$$\widehat{R}^+(z) \widehat{R}(z) = I \otimes I \quad (2.14)$$

and

$$\widehat{R}(z) + \widehat{R}^{-1}(z) = \frac{2}{\sqrt{1+z^2}} I \otimes I. \quad (2.15)$$

⁶For real θ and $-\infty < \theta < +\infty$ one has the domain $-1 < z < +1$. The linearity in z of $\sqrt{z^2+1}\widehat{R}(z)$ is particularly helpful.

Using (2.5), (2.6) and (2.10), we obtain

$$\begin{aligned}
(1+z^2)^{3/2} \widehat{R}_{12}(z) \widehat{R}_{23}(z'') \widehat{R}_{12}(z') &= (1-zz')(I \otimes I \otimes I) + \\
&\quad z''(1+zz')(I \otimes K \otimes J) + \\
&\quad (z+z')(K \otimes J \otimes I) + \\
&\quad z''(z'-z)(K \otimes KJ \otimes I), \tag{2.16}
\end{aligned}$$

$$\begin{aligned}
(1+z^2)^{3/2} \widehat{R}_{23}(z') \widehat{R}_{12}(z'') \widehat{R}_{23}(z) &= (1-zz')(I \otimes I \otimes I) + \\
&\quad (z+z')(I \otimes K \otimes J) + \\
&\quad z''(1+zz')(K \otimes J \otimes I) + \\
&\quad z''(z'-z)(K \otimes KJ \otimes I). \tag{2.17}
\end{aligned}$$

Setting

$$z'' = \frac{z+z'}{1+zz'} = \frac{\tanh \theta + \tanh \theta'}{1 + \tanh \theta \tanh \theta'} = \tanh(\theta + \theta'). \tag{2.18}$$

one obtains the baxterized braid equation (spectral parameter dependent generalization of (2.11))

$$\widehat{R}_{12}(z) \widehat{R}_{23}(z'') \widehat{R}_{12}(z') = \widehat{R}_{23}(z') \widehat{R}_{12}(z'') \widehat{R}_{23}(z) \tag{2.19}$$

or

$$\widehat{R}_{12}(\theta) \widehat{R}_{23}(\theta + \theta') \widehat{R}_{12}(\theta') = \widehat{R}_{23}(\theta') \widehat{R}_{12}(\theta + \theta') \widehat{R}_{23}(\theta). \tag{2.20}$$

Class II: We briefly state that $P\widehat{R}P$ of (1.1) is generalized as

$$\begin{aligned}
\widehat{R}^{\pm 1} &= \frac{1}{\sqrt{2}}(I \otimes I \pm J \otimes K), \\
\widehat{R}(z)^{\pm 1} &= \frac{1}{\sqrt{1+z^2}}(I \otimes I \pm zJ \otimes K). \tag{2.21}
\end{aligned}$$

The braid equation and its baxterization are verified following closely the steps for the preceding case. Unitarity is preserved. Defining the $(2n)^2 \times (2n)^2$ permutation matrix

$$P = \sum_{a,b=1}^{2n} (ab) \otimes (ba), \tag{2.22}$$

we obtain

$$\widehat{R}_{(II)}(z) = P\widehat{R}_{(I)}(z)P, \tag{2.23}$$

where $\widehat{R}_{(I)}(z)$, $\widehat{R}_{(II)}(z)$ correspond to $\widehat{R}(z)$ for class (I) and class (II) respectively.

3 Projectors

We start with the spectral resolution for class (I). Define

$$P_{\pm} = \frac{1}{2}(I \otimes I \pm iK \otimes J). \tag{3.1}$$

Using (2.5) and (2.6) its clear to see that $(\epsilon, \epsilon' = \pm 1)$

$$\begin{aligned} P_\epsilon P_{\epsilon'} &= \frac{1}{4} (I \otimes I + i(\epsilon + \epsilon') K \otimes J + \epsilon \epsilon' I \otimes I) \\ &= \frac{1}{4} ((1 + \epsilon \epsilon') I \otimes I + i(\epsilon + \epsilon') K \otimes J) = P_\epsilon \delta_{\epsilon, \epsilon'}. \end{aligned} \quad (3.2)$$

On such a basis

$$\widehat{R}^{\pm 1} = \frac{1}{\sqrt{2}} ((1 \mp i) P_+ + (1 \pm i) P_-), \quad (3.3)$$

$$\widehat{R}(z)^{\pm 1} = \frac{1}{\sqrt{1+z^2}} ((1 \mp iz) P_+ + (1 \pm iz) P_-). \quad (3.4)$$

Such basis with *complex* projectors for *real* R and $R(z)$ was already implemented in our previous study of SO3 for $n = 1$ (see refs. [1, 2]). For $n = 2$, (3.1) implies that

$$P_{\pm 1} = \frac{1}{2} \begin{vmatrix} I & 0 & 0 & \pm iJ \\ 0 & I & \pm iJ & 0 \\ 0 & \pm iJ & I & 0 \\ \pm iJ & 0 & 0 & I \end{vmatrix} = \pm \frac{i}{\sqrt{2}} \left(\widehat{R} - \frac{(1 \pm i)}{\sqrt{2}} I \otimes I \right), \quad (3.5)$$

$$P_+ + P_- = I \otimes I. \quad (3.6)$$

The projectors play basic roles in the construction of non-commutative spaces associated to \widehat{R} . A parallel treatment of projectors for class (II) can evidently be carried through. It will not be presented explicitly.

4 Diagonalization, block-diagonalization and a non-equivalence

Here the term *non-equivalence* refers to non-conservation of the braid equation. Define

$$M^{\pm 1} = \frac{1}{\sqrt{2}} (I \otimes I \pm iL \otimes J) \quad (4.1)$$

giving, say, for $n = 2$

$$M^{\pm 1} = \frac{1}{\sqrt{2}} \begin{vmatrix} I & 0 & 0 & \pm iJ \\ 0 & I & \pm iJ & 0 \\ 0 & \mp iJ & I & 0 \\ \mp iJ & 0 & 0 & I \end{vmatrix}, \quad (4.2)$$

one obtains (for $\epsilon = \pm$)

$$MP_\epsilon M^{-1} = \frac{1}{2} (I + \epsilon LK) \otimes I. \quad (4.3)$$

For $n = 2$, for example,

$$MP_+ M^{-1} = \begin{vmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad MP_- M^{-1} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{vmatrix}, \quad (4.4)$$

$$M\widehat{R}(z)M^{-1} = \frac{1}{\sqrt{1+z^2}} \begin{vmatrix} (1-iz)I & 0 & 0 & 0 \\ 0 & (1-iz)I & 0 & 0 \\ 0 & 0 & (1+iz)I & 0 \\ 0 & 0 & 0 & (1+iz)I \end{vmatrix}. \quad (4.5)$$

For $z = \pm 1$ one obtains the results for $\widehat{R}^{\pm 1}$ respectively. In (4.5), for $n = 2$, $I \equiv I_{(4)} = \begin{vmatrix} I_{(2)} & 0 \\ 0 & I_{(2)} \end{vmatrix}$, I_{2n} being the identity matrix of $(2n) \times (2n)$ dimensions. Similarly in (4.2), $J \equiv J_{(4)} = \begin{vmatrix} 0 & J_{(2)} \\ J_{(2)} & 0 \end{vmatrix}$. Starting with, for $2n = 2$, and setting $z = 1$ for simplicity,

$$\widehat{R} = \frac{1}{\sqrt{2}} \begin{vmatrix} I_{(2)} & J_{(2)} \\ J_{(2)} & I_{(2)} \end{vmatrix} \equiv \widehat{R}_{(2)} \quad (4.6)$$

can be diagonalized by conjugating with

$$M_{(2)}^{\pm 1} = \frac{1}{\sqrt{2}} \begin{vmatrix} I_{(2)} & \pm i J_{(2)} \\ \mp i J_{(2)} & I_{(2)} \end{vmatrix} \quad (4.7)$$

giving

$$M_{(2)} R_{(2)} M_{(2)}^{\pm 1} = \frac{1}{\sqrt{1+z^2}} \begin{vmatrix} (1-iz)I_2 & 0 \\ 0 & (1+iz)I_2 \end{vmatrix}. \quad (4.8)$$

Such diagonalizations indicates clearly how $R_{(4)}$, i.e. \widehat{R} (for $n = 2$) can be block diagonalized into a direct sum of 4 successive $R_{(2)}$, namely (suppressing the argument z for simplicity)

$$V R_{(4)} V^{-1} = \begin{vmatrix} R_{(2)} & 0 & 0 & 0 \\ 0 & R_{(2)} & 0 & 0 \\ 0 & 0 & R_{(2)} & 0 \\ 0 & 0 & 0 & R_{(2)} \end{vmatrix} \equiv \widehat{R}'. \quad (4.9)$$

First one permutes the 2×2 blocks of (4.5) by conjugating with

$$U = U^{-1} = \begin{vmatrix} I_{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{(2)} & 0 \\ 0 & 0 & I_{(2)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{(2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{(2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{(2)} & 0 & 0 \\ 0 & I_{(2)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{(2)} \end{vmatrix}. \quad (4.10)$$

Then conjugate this back to (4.9) by blocks $(M_{(2)}^{-1}, M_{(2)})$ in block diagonal form (4 blocks of $M_{(2)}^{\mp 1}$). The combined conjugation gives V and (4.9). But one can easily see that V is not of the form of a tensor product of some matrix Y (of 4×4 dimension) i.e.

$$V \neq Y \otimes Y, \quad (4.11)$$

where conjugation by invertible $Y \otimes Y$ conserves braid property. This has for consequence that (4.9) does not satisfy the braid equation. Direct computation gives (for $n = 2$, for example)

$$\widehat{R}'_{12} \widehat{R}'_{23} \widehat{R}'_{12} - \widehat{R}'_{23} \widehat{R}'_{12} \widehat{R}'_{23} = \left(I_4 \otimes \widehat{R}_{(2)} \otimes I_4 - I_4 \otimes I_4 \otimes \widehat{R}_{(2)} \right) \neq 0. \quad (4.12)$$

(One uses $\widehat{R}'_{12} = I_4 \otimes \widehat{R}_{(2)} \otimes I_4$, $\widehat{R}'_{23} = I_4 \otimes I_4 \otimes \widehat{R}_{(2)}$; $\widehat{R}_{(2)}^2 = \sqrt{2} \widehat{R}_{(2)} - I$.) Such considerations can easily be generalized to $n > 2$. They show quite explicitly that our generalizations (2.7) (and similarly (2.21)) are intrinsically non-equivalent to direct sums of the $n = 1$ (i.e. 4×4) blocks, which do not conserve the braid property. This holds though the forms can be related via a conjugation (by V).

Further possibilities interchanging roles of J and L : From the structure of the algebra (2.5), (2.6), it is evident that one can replace (I, J, K, L) in the preceding developments by (I, L, K, J) respectively retaining the essential results for $n > 1$. For $n = 1$ the two sets coincide (since $J = L$). The two treatments can be related through suitable permutations of rows and columns. We will not present this aspect explicitly. But since L appears in the diagonalizer M of (4.1), the full scope of the operator L is worth noting. Starting with

$$\widehat{R}(z)^{\pm 1} = \frac{1}{\sqrt{1+z^2}} (I \otimes I \pm zK \otimes L) \quad (4.13)$$

or

$$\widehat{R}(z)^{\pm 1} = \frac{1}{\sqrt{1+z^2}} (I \otimes I \pm zL \otimes K) \quad (4.14)$$

one again obtains, analogously to (2.16) the braid equation (with L replacing J). For $n = 1$, (4.13) and (4.14) coincides with (2.12) and (2.21) respectively. The different formulations are related through permutations of appropriate rows and columns. One retains a symmetric diagonal (unity) and a antisymmetric anti-diagonal. Such strong constraints conserve unitarity and braid property.

5 Odd dimensions (A class of complex unitary braid matrices)

In previous papers [6, 8] $(2n+1)^2 \times (2n+1)^2$ dimensional braid matrices have been constructed and studied for $n > 1$. They were obtained by implementing a *nested sequence* of projectors defined already before (see Ref. [6]). In these sources they were studied as real, symmetric braid matrices with multiple parameters and already Baxterized [6]. For all parameters real (as also the spectral parameter θ) they satisfy

$$\widehat{R}^+(\theta) = \widehat{R}(\theta), \quad \widehat{R}(-\theta) = \widehat{R}(\theta)^{-1}. \quad (5.1)$$

Here we note that:

1. for all parameters pure imaginary and θ real, or alternatively
2. for all parameters real and θ pure imaginary

they become unitary, i.e.

$$\widehat{R}^+(\theta) = \widehat{R}(-\theta) = \widehat{R}(\theta)^{-1}. \quad (5.2)$$

This happens due to the special structure of this class. It is sufficient to illustrate this for 9×9 matrix ($n = 1$). This involves six parameters [6] $(m_{11}^{\pm}, m_{12}^{\pm}, m_{21}^{\pm})$. Making them explicitly pure imaginary as $m_{ij}^{\pm} \rightarrow im_{ij}^{\pm}$ with real m 's on the right and defining

$$\begin{aligned} a_{\pm} &= \frac{1}{2} \left(\exp \left(im_{11}^{(+)} \theta \right) \pm \exp \left(im_{11}^{(-)} \theta \right) \right), \\ b_{\pm} &= \frac{1}{2} \left(\exp \left(im_{12}^{(+)} \theta \right) \pm \exp \left(im_{12}^{(-)} \theta \right) \right), \\ c_{\pm} &= \frac{1}{2} \left(\exp \left(im_{21}^{(+)} \theta \right) \pm \exp \left(im_{21}^{(-)} \theta \right) \right) \end{aligned} \quad (5.3)$$

one obtains

$$a_+ \overline{a_+} + a_- \overline{a_-} = 1, \quad a_+ \overline{a_-} + a_- \overline{a_+} = 0 \quad (5.4)$$

and so on. Now it is easy to see that

$$\widehat{R}(\theta) = \begin{pmatrix} a_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_- \\ 0 & b_+ & 0 & 0 & 0 & 0 & 0 & b_- & 0 \\ 0 & 0 & a_+ & 0 & 0 & 0 & a_- & 0 & 0 \\ 0 & 0 & 0 & c_+ & 0 & c_- & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_- & 0 & c_+ & 0 & 0 & 0 \\ 0 & 0 & a_- & 0 & 0 & 0 & a_+ & 0 & 0 \\ 0 & b_- & 0 & 0 & 0 & 0 & 0 & b_+ & 0 \\ a_- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_+ \end{pmatrix} \quad (5.5)$$

satisfies

$$\widehat{R}(\theta)^+ \widehat{R}(\theta) = \widehat{R}(-\theta) \widehat{R}(\theta) = I. \quad (5.6)$$

Evidently, the braid equation is still satisfied since that does not depend on the reality condition. The generalization for $n > 1$ is trivial. Thus we obtain a class of complex, unitary braid matrices for odd dimensions. Such a class is, however, well defined only in the Baxterized form. The limits of *infinite rapidity* ($\theta \rightarrow \pm\infty$ or $z \rightarrow \pm 1$ in the preceding even dim. ones) give here oscillating exponentials. Since

$$\exp(im_{il}^\pm \theta) = \exp \left[im_{il}^\pm \left(\theta + \frac{2\pi}{m_{il}^\pm} \right) \right] \quad (5.7)$$

if all the m 's are commensurate (with rational ratios) there will be an overall common period for all the parameters as θ varies. Then $\widehat{R}(\theta)$ is periodic in θ . But if at least two m 's are incommensurate $\widehat{R}(\theta)$ is quasi-periodic. Such aspects might be worth study.

Further exploration of complex unitary braid matrices is beyond the scope of this paper. In the following sections only real, even dimensional unitary braid matrices are studied. See, however, the remarks in conclusion.

6 \widehat{R} LL- and \widehat{R} TT-algebras

For classes I and II the LL- and TT-algebras are simply interchanged due to the relation

$$P \widehat{R}_{(I)}(z) P = \frac{1}{\sqrt{1+z^2}} P (I \otimes I + zK \otimes J) P = \frac{1}{\sqrt{1+z^2}} (I \otimes I + zJ \otimes K) = \widehat{R}_{(II)}(z) \quad (6.1)$$

and the fact that the fundamental blocks of L and T are obtained from

$$L(z) = \widehat{R}(z) P, \quad T(z) = P \widehat{R}(z) \quad (6.2)$$

leading to

$$L_{(II)}(z) = \widehat{R}_{(II)}(z) P = P \widehat{R}_{(I)}(z) = T_{(I)}(z), \quad (6.3)$$

$$T_{(II)}(z) = P \widehat{R}_{(II)}(z) = \widehat{R}_{(I)}(z) P = L_{(I)}(z). \quad (6.4)$$

To obtain higher order representations one implements the same coproduct prescriptions for L and T . Hence the correspondence is maintained. We study below only the case I, suppressing the index ($\widehat{R}_{(I)} \rightarrow \widehat{R}$ and so on). Corresponding to

$$\widehat{R}(z) = \frac{1}{\sqrt{2(1+z^2)}} \left((1+z)\widehat{R} + (1-z)\widehat{R}^{-1} \right) \quad (6.5)$$

one can define (simplifying the external factor irrelevant for our purposes)

$$L(z) = \frac{1}{2} \left((1+z)L^+ + (1-z)L^- \right) = \frac{e^\theta L^+ + e^{-\theta} L^-}{e^\theta + e^{-\theta}}, \quad (6.6)$$

$$L(\pm 1) = L^{\pm 1}. \quad (6.7)$$

($z = \tanh \theta$). The single constraint (with $L_1 = L \otimes I$, $L_2 = I \otimes L$)

$$\widehat{R}(\theta - \theta') L_2(\theta) L_1(\theta') = L_2(\theta') L_1(\theta) \widehat{R}(\theta - \theta') \quad (6.8)$$

can be shown to imply all the three FRT relations [2]

$$\widehat{R} L_2^\epsilon L_1^{\epsilon'} = L_2^{\epsilon'} L_1^\epsilon \widehat{R}, \quad (6.9)$$

where $(\epsilon, \epsilon') = (+, +), (-, -), (+, -)$ respectively.

Constructions such as (6.6) are only possible when $\widehat{R}(z)$ satisfies a quadratic constraint (i.e. $\widehat{R}(z), \widehat{R}(z)^{-1}$ a linear one). In all our constructions (2.15) and (6.5) are guaranteed and hence also (6.8). A quadratic constraint and hence (6.6) can be shown to permit two distinct type of coproducts which coincide for $z = \pm 1$ but are inequivalent for $(-1 < z < 1)$ [2]. In this paper we will, for brevity, restrict our study to the standard prescription,

$$L_{ij}^{(r+1)}(z) = \sum_k L_{ik}^{(1)}(z) \otimes L_{kj}^{(r)}(z), \quad (6.10)$$

where $L_{ik}^{(1)}(z)$ are obtained from (6.2) as follows:

$$L^{(1)}(z) \equiv L(z) = \widehat{R}(z) P = \frac{1}{\sqrt{z^2+1}} \sum_{i,j=1}^{2n} \left((ij) \otimes (ji) + z(-1)^{\bar{j}} (i\bar{j}) \otimes (j\bar{i}) \right) \\ \equiv \begin{vmatrix} L_{11}^{(1)}(z) & L_{12}^{(1)}(z) & \cdots & L_{1,2n}^{(1)}(z) \\ L_{21}^{(1)}(z) & L_{22}^{(1)}(z) & \cdots & L_{2,2n}^{(1)}(z) \\ \vdots & \vdots & \ddots & \vdots \\ L_{2n,1}^{(1)}(z) & L_{2n,2}^{(1)}(z) & \cdots & L_{2n,2n}^{(1)}(z) \end{vmatrix}. \quad (6.11)$$

For $n = 2$, for example, (6.10) gives (suppressing for simplicity the argument z for each L_{ij} and

dropping the overall factor)

$$\begin{aligned}
\mathbf{L}_{1j}^{(r+1)} &= \begin{vmatrix} \mathbf{L}_{1j}^{(r)} & 0 & 0 & z\mathbf{L}_{4j}^{(r)} \\ \mathbf{L}_{2j}^{(r)} & 0 & 0 & -z\mathbf{L}_{3j}^{(r)} \\ \mathbf{L}_{3j}^{(r)} & 0 & 0 & z\mathbf{L}_{2j}^{(r)} \\ \mathbf{L}_{4j}^{(r)} & 0 & 0 & -z\mathbf{L}_{1j}^{(r)} \end{vmatrix}, & \mathbf{L}_{2j}^{(r+1)} &= \begin{vmatrix} 0 & \mathbf{L}_{1j}^{(r)} & z\mathbf{L}_{4j}^{(r)} & 0 \\ 0 & \mathbf{L}_{2j}^{(r)} & -z\mathbf{L}_{3j}^{(r)} & 0 \\ 0 & \mathbf{L}_{3j}^{(r)} & z\mathbf{L}_{2j}^{(r)} & 0 \\ 0 & \mathbf{L}_{4j}^{(r)} & -z\mathbf{L}_{1j}^{(r)} & 0 \end{vmatrix}, \\
\mathbf{L}_{3j}^{(r+1)} &= \begin{vmatrix} 0 & z\mathbf{L}_{4j}^{(r)} & \mathbf{L}_{1j}^{(r)} & 0 \\ 0 & -z\mathbf{L}_{3j}^{(r)} & \mathbf{L}_{2j}^{(r)} & 0 \\ 0 & z\mathbf{L}_{2j}^{(r)} & \mathbf{L}_{3j}^{(r)} & 0 \\ 0 & -z\mathbf{L}_{1j}^{(r)} & \mathbf{L}_{4j}^{(r)} & 0 \end{vmatrix}, & \mathbf{L}_{4j}^{(r+1)} &= \begin{vmatrix} z\mathbf{L}_{4j}^{(r)} & 0 & 0 & \mathbf{L}_{1j}^{(r)} \\ -z\mathbf{L}_{3j}^{(r)} & 0 & 0 & \mathbf{L}_{2j}^{(r)} \\ z\mathbf{L}_{2j}^{(r)} & 0 & 0 & \mathbf{L}_{3j}^{(r)} \\ -z\mathbf{L}_{1j}^{(r)} & 0 & 0 & \mathbf{L}_{4j}^{(r)} \end{vmatrix}, \tag{6.12}
\end{aligned}$$

where $j = 1, 2, 3, 4$. Setting now $\mathbf{L}_{ij}^{(0)} = \delta_{ij}$, we obtain

$$\mathbf{L}_{11}^{(1)} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z \end{vmatrix}, \quad \mathbf{L}_{12}^{(1)} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad \text{etc.} \tag{6.13}$$

Define

$$\mathbf{L}^{(r+1)} = \sum_{i=1}^{2n} \mathbf{L}_{ii}^{(r+1)}. \tag{6.14}$$

When $n = 2$, for example, we have

$$\text{Tr} \left(\mathbf{L}^{(r+1)} \right) = \sum_{i=1}^4 \text{Tr} \left(\mathbf{L}_{ii}^{(r+1)} \right) = \text{Tr} \left((1-z) \left(\mathbf{L}_{11}^{(r)} + \mathbf{L}_{33}^{(r)} \right) + (1+z) \left(\mathbf{L}_{22}^{(r)} + \mathbf{L}_{44}^{(r)} \right) \right). \tag{6.15}$$

Observing that

$$\begin{aligned}
\text{Tr} \left(\mathbf{L}_{11}^{(r)} \right) &= (1-z) \text{Tr} \left(\mathbf{L}_{11}^{(r-1)} \right), & \text{Tr} \left(\mathbf{L}_{22}^{(r)} \right) &= (1+z) \text{Tr} \left(\mathbf{L}_{22}^{(r-1)} \right), \\
\text{Tr} \left(\mathbf{L}_{33}^{(r)} \right) &= (1-z) \text{Tr} \left(\mathbf{L}_{33}^{(r-1)} \right), & \text{Tr} \left(\mathbf{L}_{44}^{(r)} \right) &= (1+z) \text{Tr} \left(\mathbf{L}_{44}^{(r-1)} \right), \tag{6.16}
\end{aligned}$$

we deduce

$$\text{Tr} \left(\mathbf{L}_{ii}^{(r)} \right) = (1 + (-1)^i z)^r, \quad i = 1, 2, 3, 4. \tag{6.17}$$

Using (6.17), we finally obtain

$$\text{Tr} \left(\mathbf{L}^{(r)} \right) = 2 \left((1+z)^r + (1-z)^r \right). \tag{6.18}$$

The \widehat{R} TT constraints are conveniently written as

$$\widehat{R}(\theta - \theta') \mathbf{T}(\theta) \otimes \mathbf{T}(\theta') = \mathbf{T}(\theta') \otimes \mathbf{T}(\theta) \widehat{R}(\theta - \theta'). \tag{6.19}$$

Starting from (6.2) and a coproduct prescription parallel to (6.10) one obtains analogously, for

example, when $n = 2$ (suppressing again arguments z)

$$\begin{aligned}
\mathbb{T}_{1j}^{(r+1)} &= \begin{vmatrix} \mathbb{T}_{1j}^{(r)} & 0 & 0 & z\mathbb{T}_{4j}^{(r)} \\ \mathbb{T}_{2j}^{(r)} & 0 & 0 & z\mathbb{T}_{3j}^{(r)} \\ \mathbb{T}_{3j}^{(r)} & 0 & 0 & z\mathbb{T}_{2j}^{(r)} \\ \mathbb{T}_{4j}^{(r)} & 0 & 0 & z\mathbb{T}_{1j}^{(r)} \end{vmatrix}, & \mathbb{T}_{2j}^{(r+1)} &= \begin{vmatrix} 0 & \mathbb{T}_{1j}^{(r)} & -z\mathbb{T}_{4j}^{(r)} & 0 \\ 0 & \mathbb{T}_{2j}^{(r)} & -z\mathbb{T}_{3j}^{(r)} & 0 \\ 0 & \mathbb{T}_{3j}^{(r)} & -z\mathbb{T}_{2j}^{(r)} & 0 \\ 0 & \mathbb{T}_{4j}^{(r)} & -z\mathbb{T}_{1j}^{(r)} & 0 \end{vmatrix}, \\
\mathbb{T}_{3j}^{(r+1)} &= \begin{vmatrix} 0 & z\mathbb{T}_{4j}^{(r)} & \mathbb{T}_{1j}^{(r)} & 0 \\ 0 & z\mathbb{T}_{3j}^{(r)} & \mathbb{T}_{2j}^{(r)} & 0 \\ 0 & z\mathbb{T}_{2j}^{(r)} & \mathbb{T}_{3j}^{(r)} & 0 \\ 0 & z\mathbb{T}_{1j}^{(r)} & \mathbb{T}_{4j}^{(r)} & 0 \end{vmatrix}, & \mathbb{T}_{4j}^{(r+1)} &= \begin{vmatrix} -z\mathbb{T}_{4j}^{(r)} & 0 & 0 & \mathbb{T}_{1j}^{(r)} \\ -z\mathbb{T}_{3j}^{(r)} & 0 & 0 & \mathbb{T}_{2j}^{(r)} \\ -z\mathbb{T}_{2j}^{(r)} & 0 & 0 & \mathbb{T}_{3j}^{(r)} \\ -z\mathbb{T}_{1j}^{(r)} & 0 & 0 & \mathbb{T}_{4j}^{(r)} \end{vmatrix}. & (6.20)
\end{aligned}$$

By setting again $\mathbb{T}_{ij}^{(0)} = \delta_{ij}$, we obtain

$$\mathbb{T}_{11} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z \end{vmatrix}, \quad \mathbb{T}_{12} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad \text{etc.} \quad (6.21)$$

Similarly, we obtain, for $n = 2$

$$\begin{aligned}
\text{Tr} \left(\mathbb{T}^{(r+1)} \right) &= \sum_{i=1}^4 \text{Tr} \left(\mathbb{T}_{ii}^{(r+1)} \right) \\
&= \text{Tr} \left((1+z) \left(\mathbb{T}_{11}^{(r)} + \mathbb{T}_{33}^{(r)} \right) + (1-z) \left(\mathbb{T}_{22}^{(r)} + \mathbb{T}_{44}^{(r)} \right) \right), \\
&= \text{Tr} \left((1+z)^2 \left(\mathbb{T}_{11}^{(r-1)} + \mathbb{T}_{33}^{(r-1)} \right) + (1-z)^2 \left(\mathbb{T}_{22}^{(r-1)} + \mathbb{T}_{44}^{(r-1)} \right) \right), \\
&\quad \vdots \\
\text{Tr} \left(\mathbb{T}^{(r+1)} \right) &= 2 \left((1+z)^r + (1-z)^r \right). & (6.22)
\end{aligned}$$

The L-algebra, for $n = 1$, has been studied extensively in a previous paper [2] which provides further references. Here we have presented briefly certain basic features of L_{ij}^{\pm} and \mathbb{T}_{ij} for all orders (r) . Their further study concerning representations will not be undertaken in this paper. One application of the T-matrices concerns the Hamiltonians encoding the evolution of the states in base space, whether the states are entangled or not. Hamiltonians are briefly presented in the next section.

Denoting the diagonal matrix (sec. 4) $(M\widehat{R}(z'')M^{-1})$ as $D(z'')$ for all n and $z'' = \tanh(z - z')$ one reduces (6.19) to

$$D(z'') \left(M\mathbb{T}(z) \otimes \mathbb{T}(z') M^{-1} \right) = \left(M\mathbb{T}(z') \otimes \mathbb{T}(z) M^{-1} \right) D(z''). \quad (6.23)$$

This expresses the TT'-algebra in terms of sums of terms such that permutation of z and z' has extremely simple consequences. The LL'-algebra can be treated similarly.

The TT- and LL-algebras and their representations will be studied in detail in our follow-up paper(s) [10] along the lines of [1, 2, 11], namely, presentation of the TT-algebras as matrix bialgebras, construction of their dual bialgebras, presentation, of the LL-algebras as FRT-duals of the TT-algebras, development of the representation theory of all mentioned algebras.

7 Hamiltonians

Sections on Hamiltonians in previous papers [8, 9] cite various references. Here we briefly present some generic features arising from the structure of our unitary braid matrices. The Hamiltonian, of order r , is defined in terms of the transfer matrix $\mathbb{T}^{(r)}(\theta)$ of order r as,

$$H = \left(\mathbb{T}^{(r)}(\theta) \right)_{\theta=0}^{-1} \left(\partial_{\theta} \left(\mathbb{T}^{(r)}(\theta) \right) \right)_{\theta=0} \quad (7.1)$$

$$= \sum_{k=1}^r I \otimes I \otimes \cdots \otimes \hat{R}_{k,k+1}(0) \otimes I \otimes \cdots \otimes I, \quad (7.2)$$

where

$$\hat{R}_{k,k+1}(0) = \left(\partial_{\theta} \hat{R}_{k,k+1}(\theta) \right)_{\theta=0}. \quad (7.3)$$

For cyclic boundary condition (and order r) one imposes $k+1 = r+1 \approx 1$). Thus for $r=2$

$$H = \hat{R}_{12}(0) + \hat{R}_{21}(0) = \hat{R}(0) + P\hat{R}(0)P. \quad (7.4)$$

For $\hat{R}(\theta) = \frac{1}{\sqrt{1+\tanh^2\theta}} (I \otimes I + \tanh\theta K \otimes J)$, we obtain

$$\hat{R}(0) = K \otimes J. \quad (7.5)$$

For the second class, i.e. when $\hat{R}(\theta) = \frac{1}{\sqrt{1+\tanh^2\theta}} (I \otimes I + \tanh\theta J \otimes K)$, we deduce

$$\hat{R}(0) = J \otimes K = P(K \otimes J)P. \quad (7.6)$$

Thus, in particular, for $r=2$, both classes lead to the same result

$$H = K \otimes J + J \otimes K = \sum_{i,j} \left((-1)^{\bar{j}} + (-1)^{\bar{i}} \right) ((i\bar{i}) \otimes (j\bar{j})). \quad (7.7)$$

8 Potentials for factorizable S -matrices

Such potentials have been studied in our previous papers [8, 9] where basic sources are cited. They are given by inverse Cayley transforms

$$-iV(z) = (R(z) - \lambda(z)I)^{-1} (R(z) + \lambda(z)I), \quad (8.1)$$

where $R(z) = P\hat{R}(z)$ is the YB-matrix and the parameter $\lambda(z)$ has been introduced to guarantee the existence of the inverse [8, 9]. To absorb the normalization factor of our unitary $R(z)$ we define

$$\mu(z) = \sqrt{1+z^2}\lambda(z), \quad (8.2)$$

$$\begin{aligned} -iV(z) &= \left(\sqrt{1+z^2}R(z) - \mu(z)I \right)^{-1} \left(\sqrt{1+z^2}R(z) + \mu(z)I \right) \\ &= I_{(2n)^2} + 2\mu(z) \left(\sqrt{1+z^2}R(z) - \mu(z)I \right)^{-1} \\ &\equiv I_{(2n)^2} + 2\mu(z)X(z). \end{aligned} \quad (8.3)$$

We consider YB-matrices for our class I. For $n = 1$,

$$\sqrt{1+z^2}R(z) = \begin{vmatrix} 1 & 0 & 0 & z \\ 0 & z & 1 & 0 \\ 0 & 1 & -z & 0 \\ -z & 0 & 0 & 1 \end{vmatrix}. \quad (8.4)$$

Hence (suppressing arguments z)

$$X(z) = \begin{vmatrix} K_1(1-\mu) & 0 & 0 & -K_1z \\ 0 & K_2(z+\mu) & K_2 & 0 \\ 0 & K_2 & -K_2(z-\mu) & 0 \\ K_1z & 0 & 0 & K_1(1-\mu) \end{vmatrix}, \quad (8.5)$$

where

$$K_1 = \frac{1}{(1-\mu)^2 + z^2}, \quad K_2 = \frac{1}{(z^2 - \mu^2) + 1}. \quad (8.6)$$

From (8.2) and (8.6) it follows that the inverse $X(z)$ is well defined for $\lambda(z) \neq \pm 1$, $\left(\frac{1 \pm iz}{1 \mp iz}\right)^{1/2}$. With a V satisfying (8.1) where $R(z)$ is a YB-matrix and the parameter $\mu(z)$ and hence $\lambda(z)$ suitably chosen so that K_1, K_2 are well defined the Lagrangian is constructed in terms of the elements $V_{(ab,cd)}$ [12, 13] where

$$V = \sum_{ab,cd} V_{(ab,cd)} (ab) \otimes (cd). \quad (8.7)$$

The fermionic Lagrangian is of the form

$$\mathcal{L} = \int dx (i\bar{\psi}_a \gamma_\nu \partial_\nu \psi_a - g (\bar{\psi}_a \gamma_\nu \psi_c) V_{ab,cd} (\bar{\psi}_b \gamma_\nu \psi_d)). \quad (8.8)$$

The scalar Lagrangian has an interaction term of the form $(\bar{\varphi}_a \bar{\varphi}_c) V_{ab,cd} (\varphi_b \varphi_d)$. For $n = 2$, using the same (K_1, K_2) one obtains the non-zero elements of X as

$$\begin{aligned} X(1j, 1k) &= K_1 ((1-\mu) \delta_{1j} \delta_{1k} - z \delta_{4j} \delta_{4k}), \\ X(4j, 4k) &= K_1 ((1-\mu) \delta_{4j} \delta_{4k} + z \delta_{1j} \delta_{1k}), \\ X(1j, 4k) &= K_2 ((\mu+z) \delta_{1j} \delta_{4k} + \delta_{4j} \delta_{1k}), \\ X(4j, 1k) &= K_2 ((\mu-z) \delta_{4j} \delta_{1k} + \delta_{1j} \delta_{4k}), \\ X(2j, 2k) &= K_1 ((1-\mu) \delta_{2j} \delta_{2k} + z \delta_{3j} \delta_{3k}), \\ X(3j, 3k) &= K_1 ((1-\mu) \delta_{3j} \delta_{3k} - z \delta_{2j} \delta_{2k}), \\ X(2j, 3k) &= K_2 ((\mu-z) \delta_{2j} \delta_{3k} + \delta_{3j} \delta_{2k}), \\ X(3j, 2k) &= K_2 ((\mu+z) \delta_{3j} \delta_{2k} + \delta_{2j} \delta_{3k}), \\ X(1j, 2k) &= K_2 (\mu \delta_{1j} \delta_{2k} + \delta_{2j} \delta_{1k} + z \delta_{3j} \delta_{4k}), \\ X(2j, 1k) &= K_2 (\delta_{1j} \delta_{2k} + \mu \delta_{2j} \delta_{1k} - z \delta_{4j} \delta_{3k}), \\ X(3j, 4k) &= K_2 (z \delta_{1j} \delta_{2k} + \mu \delta_{3j} \delta_{4k} + \delta_{4j} \delta_{3k}), \\ X(4j, 3k) &= K_2 (-z \delta_{2j} \delta_{1k} + \delta_{3j} \delta_{4k} + \mu \delta_{4j} \delta_{3k}). \end{aligned} \quad (8.9)$$

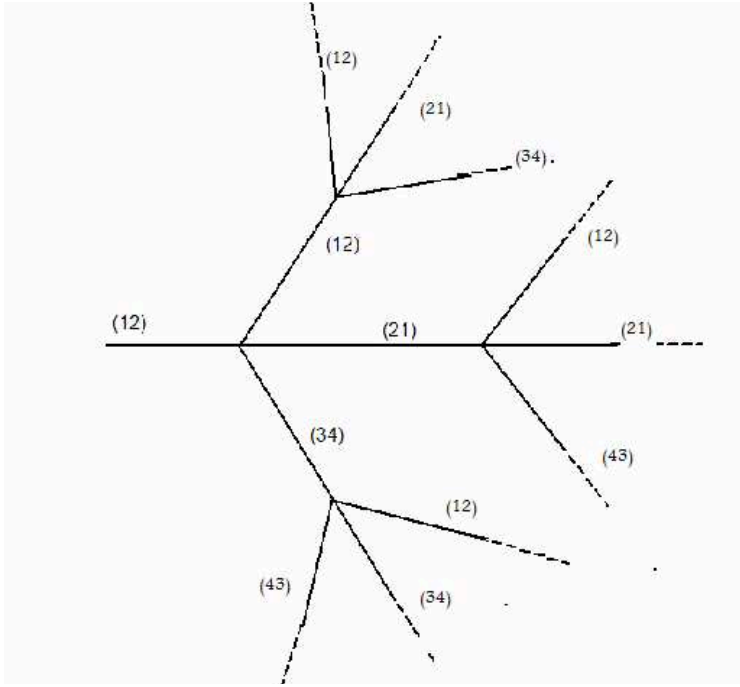
Finally, defining

$$\begin{aligned}
C_1 &= (\mu\delta_{1j}\delta_{3k} + \delta_{3j}\delta_{1k} - z\delta_{2j}\delta_{4k}), \\
C_2 &= (\mu\delta_{2j}\delta_{4k} + \delta_{4j}\delta_{2k} + z\delta_{1j}\delta_{3k}), \\
C_3 &= (\mu\delta_{3j}\delta_{1k} + \delta_{1j}\delta_{3k} - z\delta_{4j}\delta_{2k}), \\
C_4 &= (\mu\delta_{4j}\delta_{2k} + \delta_{2j}\delta_{4k} + z\delta_{3j}\delta_{1k}),
\end{aligned}$$

we obtain

$$\begin{aligned}
X(1j, 3k) &= K_1 K_2 (K_2^{-1} C_1 - 2\mu z C_2), \\
X(2j, 4k) &= K_1 K_2 (K_2^{-1} C_2 + 2\mu z C_1), \\
X(3j, 1k) &= K_1 K_2 (K_2^{-1} C_3 - 2\mu z C_4), \\
X(4j, 2k) &= K_1 K_2 (K_2^{-1} C_4 + 2\mu z C_3).
\end{aligned} \tag{8.10}$$

Branching through successive scatterings can be two- or three-fold at each stage. Thus, for example, schematically,



9 Non-commutative spaces

Use of the projectors to construct covariant calculus for non-commutative spaces has been studied in previous paper [1, 7] where basic sources [14, 15] are cited. We present below briefly the generalization of the formalism of sec. 2.3 of ref. [1] to cases $n > 1$.

The projectors (3.1), $(2n)^2 \times (2n)^2$ matrices, $P_{\pm} = \frac{1}{2} (I \otimes I \pm iK \otimes J)$ being complex, though \hat{R} is real, special features arise. Setting, for example, X being the column vector of the coordi-

nates,

$$P_- X \otimes X = 0 \quad (9.1)$$

one obtains

$$X_i X_j = i(-1)^{\bar{j}} X_{\bar{i}} X_{\bar{j}}. \quad (9.2)$$

Since $i(-1)^{\bar{j}} = -i(-1)^j$ one half of the constraints repeats, in consistent fashion, the other half. Thus, for $n = 1$

$$X_1 X_1 = i X_2 X_2, \quad X_1 X_2 = -i X_2 X_1 \quad (9.3)$$

are sufficient.

Introducing the $(2n) \times (2n)$ projectors $\frac{1}{2}(I \pm iJ)$ and defining

$$x_{\pm} = \frac{1}{2}(I \pm iJ) X, \quad (9.4)$$

$$\begin{aligned} 2P_-(X \otimes X) &= X \otimes X - (KX) \otimes (iJX) \\ &= (x_+ + x_-) \otimes (x_+ + x_-) - (\bar{x}_+ + \bar{x}_-) \otimes (x_+ - x_-), \end{aligned} \quad (9.5)$$

where

$$KX = \bar{X} = (\bar{x}_+ + \bar{x}_-) \quad (9.6)$$

has been implemented. Thus, in terms of x_{\pm} the constraints are real.

Define

$$Q = \nu P_+ - I \otimes I, \quad (9.7)$$

where $\nu \neq 0, 1$, is otherwise a free parameter to start with. Then

$$Q^{-1} = \frac{1}{\nu - 1} (I \otimes I - \nu P_-). \quad (9.8)$$

A covariant prescription for the differentials Z and the modular structure is

$$Q(Z \otimes X) = X \otimes Z, \quad P_+(Z \otimes Z) = 0. \quad (9.9)$$

One can define a mobile frame [7, 16, 17] starting with

$$\Theta = \sum_i \Theta_i Z_i, \quad (9.10)$$

where the coefficients Θ_i are to be so constructed that

$$[\Theta, X_i] = 0. \quad (9.11)$$

A computation leads to

$$X_i \Theta_j = \sum_{k,l} \Theta_k (Q^{-1}P)_{kj,il} X_l, \quad (9.12)$$

where Q^{-1} is given by (9.8) and the permutation matrix P by (2.22). We now show how to relate $Q^{-1}P$ to L^{\pm} as in the references cited above. This involves particular choices of ν . From

(3.3), its easy to see that $\widehat{R}^{\pm 1} = e^{\mp i\pi/4} (I \otimes I - (1 \mp i) P_-)$. By choosing $\nu = 1 \mp i$, we can write, using (6.2), that

$$\mathcal{Q}^{-1}P = e^{\mp i\pi/4} \left(\widehat{R}^{\pm 1}P \right) = e^{\mp i\pi/4} \mathbf{L}^{\pm}. \quad (9.13)$$

Thus, for example, one can set with $\nu = 1 + i$,

$$X_i \Theta_j = \sum_{k,l} \Theta_k \left(e^{i\pi/4} \mathbf{L}^- \right)_{kj,il} X_l. \quad (9.14)$$

We will not undertake here any explicit construction of Θ_i .

10 Link invariants (Turaev constructions)

We now construct an *enhanced system* [7, 18, 19] starting with our unitary braid matrices. This implies explicit construction of a $(2n) \otimes (2n)$ matrix \mathcal{F} satisfying

$$\widehat{R}^{\pm 1} (\mathcal{F} \otimes \mathcal{F}) = (\mathcal{F} \otimes \mathcal{F}) \widehat{R}^{\pm 1}, \quad (10.1)$$

$$\text{Tr}_2 \left(\widehat{R}^{\pm 1} (\mathcal{F} \otimes \mathcal{F}) \right) = a^{\pm 1} b \mathcal{F}, \quad (10.2)$$

where (a, b) are invertible parameters and $\text{Tr}_2 \left(\sum_{i,j,k,l} c_{ij,kl} (ij) \otimes (kl) \right) = \sum_{i,j} \left(\sum_k c_{ij,kk} \right) (ij)$.

It is sufficient to consider our class I, i.e. $\widehat{R}^{\pm 1} = \frac{1}{\sqrt{2}} (I \otimes I \pm K \otimes J)$. Define

$$\mathcal{F} = \sum_{j=1}^n d_j \left((jj) + (\bar{j}\bar{j}) \right). \quad (10.3)$$

Its follows that

$$J\mathcal{F} = \mathcal{F}J = \sum_{j=1}^n (-1)^{\bar{j}} \left((j\bar{j}) - (\bar{j}j) \right) d_j, \quad K\mathcal{F} = \mathcal{F}K = \sum_{j=1}^n \left((j\bar{j}) + (\bar{j}j) \right) d_j. \quad (10.4)$$

Hence

$$\widehat{R}^{\pm 1} (\mathcal{F} \otimes \mathcal{F}) = \frac{1}{\sqrt{2}} (\mathcal{F} \otimes \mathcal{F} \pm K\mathcal{F} \otimes J\mathcal{F}) = \frac{1}{\sqrt{2}} (\mathcal{F} \otimes \mathcal{F} \pm \mathcal{F}K \otimes \mathcal{F}J) = (\mathcal{F} \otimes \mathcal{F}) \widehat{R}^{\pm 1}. \quad (10.5)$$

Using the results $\text{Tr}_2 (\mathcal{F} \otimes \mathcal{F}) = \mathcal{F} \text{Tr} (\mathcal{F}) = 2 \left(\sum_{j=1}^n d_j \right) \mathcal{F}$ and $\text{Tr}_2 ((K \otimes J) (\mathcal{F} \otimes \mathcal{F})) = (K\mathcal{F}) \text{Tr} (J\mathcal{F}) = 0$, we obtain

$$\text{Tr}_2 \left(\widehat{R}^{\pm 1} \mathcal{F} \otimes \mathcal{F} \right) = \frac{1}{\sqrt{2}} \text{Tr}_2 (\mathcal{F} \otimes \mathcal{F} \pm K\mathcal{F} \otimes J\mathcal{F}) = \frac{1}{\sqrt{2}} \text{Tr}_2 (\mathcal{F} \otimes \mathcal{F}) = \sqrt{2} \left(\sum_{j=1}^n d_j \right) \mathcal{F}. \quad (10.6)$$

Thus (10.2) is also satisfied with

$$a = 1, \quad b = \sqrt{2} \left(\sum_{j=1}^n d_j \right). \quad (10.7)$$

Note that for $n = 1$, \mathcal{F} degenerates to (a factor times) the unit matrix I_2 . From $n = 2$ onwards a structure begins to appear along the diagonal. Thus, for $n = 2$,

$$\mathcal{F} = \begin{vmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_2 & 0 \\ 0 & 0 & 0 & d_1 \end{vmatrix}, \quad (10.8)$$

where $d_1 = d_2$ is not excluded, but in general $d_1 \neq d_2$. The preceding construction can also be carried through (with the same \mathcal{F}) for the baxterized $\widehat{R}^{\pm 1}(z)$. For each n , \mathcal{F} has n free parameters.

With (\mathcal{F}, a, b) thus obtained one can define (since in our case $a = 1$)

$$\wp(\beta) = b^{(-m+1)} \text{Tr}(\rho_m(\beta) \cdot \mathcal{F}^{\otimes m}), \quad (10.9)$$

where $\wp(\beta)$ is the representation of the braid β associated to \widehat{R} and ρ_m is the endomorphism of $V^{\otimes m}$. It can be shown (sec. 15 of ref. [19]) that this provides an invariant of oriented links, Markov invariance being assured. For *unknot* (no crossing)

$$\wp(0) = \text{Tr}(\mathcal{F}) = b/\sqrt{2}. \quad (10.10)$$

For our unitary matrices (I denoting $I_{(2n)}$)

$$\widehat{R} + \widehat{R}^{-1} = \sqrt{2}I \otimes I = \sqrt{2}I_{(2n)^2} \quad (10.11)$$

and hence, for all n ,

$$\widehat{R}^4 = -I_{(2n)^2}, \quad \widehat{R}^8 = I_{(2n)^2}, \quad (10.12)$$

(one obtains (10.12) most directly by writing (3.3) as $\widehat{R} = e^{-i\pi/4}P_+ + e^{i\pi/4}P_-$). Restrictions on the skein relations and the periodicity (eight-fold) implicit in (10.11) and (10.12) are pointed out in sec. 4 of ref. [3].

11 Entangled remarks

Though, as we explicitly displayed in sec. 4, \widehat{R} block diagonalized with $\widehat{R}_{(2)}$ (generalization of (4.9) for $n > 2$ being direct) does not satisfy the braid equation, yet one can write (6.8) as

$$\begin{aligned} \left(V\widehat{R}(\theta - \theta')V^{-1} \right) (VL_2(\theta)V^{-1}) (VL_1(\theta')V^{-1}) &= (VL_2(\theta)V^{-1}) \times \\ & (VL_1(\theta')V^{-1}) \left(V\widehat{R}(\theta - \theta')V^{-1} \right). \end{aligned} \quad (11.1)$$

Block diagonal ansatz for $(VL_1(\theta')V^{-1})$, $i = 1, 2$, in terms of the L-functions for SO3 (studied extensively in [2]) will evidently satisfy (11.1). Structures obtained on conjugating back with V^{-1} might be of interest. But explicit verification is necessary. One can treat the \widehat{R} TT-relations analogously. Certain crucial properties of L- and T-functions have been presented in sec. 6. We hope to present a more through study elsewhere.

As already pointed out in [2], statistical models associated to unitary \widehat{R} cannot have only non-negative Boltzmann weights. Negative and complex weights need suitable interpretations. But the simple structure of the Hamiltonians governing evolution of the states is worth noting. Our (7.7) is an example of all n . The complex non-commutative spaces associated to unitary \widehat{R} (sec. 6) deserve further study. So does the complex unitary braid matrix for odd dimensions (sec. 5). *Topological entanglements* have been presented in [3] in terms of link invariants and topological fields. Their possible relations with quantum entanglements have been emphasized. Here we have briefly presented Turaev construction of link invariants for all n . Concerning fields we have shown how our \widehat{R} can be implemented in constructing potentials for factorizable S -matrices. Already, for $n = 2$, the structure is considerably enriched. A canonical formulation for all n would be interesting.

In [3] the unitary matrix (1.1) is presented as a common source of quantum and also of topological entanglements. Acting on the base space of states $(|++\rangle, |+-\rangle, |-+\rangle, |--\rangle)$ \widehat{R} of (1.1) generates entangled Bell-states. But \widehat{R} also has braid property and hence leads to link invariants, links being viewed as topological entanglements. It also leads to S -matrices where one can permute the successive scattering (sec. 8). Our $(2n)^2 \times (2n)^2$ braid matrices generalize the states

$$\frac{1}{\sqrt{2}}(|++\rangle \pm |--\rangle), \quad \frac{1}{\sqrt{2}}(|+-\rangle \pm |-+\rangle) \quad (11.2)$$

for (1.1) to

$$\frac{1}{\sqrt{2}}(I \otimes I + K \otimes J) |\mathbb{V}\rangle \otimes |\mathbb{V}\rangle, \quad (11.3)$$

where one may adopt the notation of *spin- n* components with

$$|\mathbb{V}\rangle \equiv \begin{pmatrix} |n\rangle \\ |n-1\rangle \\ \vdots \\ |1\rangle \\ |-1\rangle \\ \vdots \\ |-n+1\rangle \\ |-n\rangle \end{pmatrix}. \quad (11.4)$$

One obtains as direct generalizations of (11.2) the states

$$\frac{1}{\sqrt{2}}(|n-j\rangle |n-k\rangle \pm |-n+j\rangle |-n+k\rangle), \quad (11.5)$$

where $0 \leq j, k \leq n-1$. Here the subspaces of $|\mathbb{V}\rangle$ can denote any property of the system under consideration with $2n$ orthogonal states, through one may use the term *spin* for convenience. Now the link invariants will correspond to more general constructions of sec. 10. The quantum and topological entanglements are both generalized simultaneously in the sense indicated above. The potential for factorizable S -matrix is now generalized too as indicated in sec. 8. Note that

if instead of using \widehat{R} of (2.7) one implements the Baxterized $\widehat{R}(z)$ of (2.12), one obtains, in place of (11.5) the superpositions

$$\frac{1}{\sqrt{1+z^2}} (|n-j\rangle |n-k\rangle \pm z |-n+j\rangle |-n+k\rangle). \quad (11.6)$$

The matrix $\widehat{R}(z)$ is still unitary, though (2.1) is now replaced by (2.19).

Corresponding to odd dimensional and complex (but unitary) $\widehat{R}(\theta)$ one has superpositions with complex coefficients. For simplicity we consider only the simplest, but typical, case of $n = 1$ corresponding to (5.5). Using a simple, evident, notation

$$\widehat{R}(\theta) \left(\left(\begin{array}{c} |+\rangle \\ |0\rangle \\ |-\rangle \end{array} \right) \otimes \left(\begin{array}{c} |+\rangle \\ |0\rangle \\ |-\rangle \end{array} \right) \right) \quad (11.7)$$

yields superpositions (with $(a_{\pm}, b_{\pm}, c_{\pm})$ of (5.3))

$$a_{\pm} |++\rangle + a_{\mp} |--\rangle, \quad b_{\pm} |+0\rangle + b_{\mp} |-0\rangle, \quad a_{\pm} |+-\rangle + a_{\mp} |-\rangle, \quad c_{\pm} |0+\rangle + c_{\mp} |0-\rangle, \quad |00\rangle \quad (11.8)$$

Note the special status of the central state $|00\rangle$. It was explained in ref. [6] how the structure of $\widehat{R}(\theta)$ depends crucially on the existence of the central element 1. The same feature singles out $|00\rangle$ in (11.8). Setting, say,

$$m_{ij}^{(+)} = -m_{ij}^{(-)} = m_{ij} \quad (11.9)$$

one obtains the simpler superpositions

$$\cos(m_{11}\theta) |\pm\pm\rangle + i \sin(m_{11}\theta) |\mp\mp\rangle \quad (11.10)$$

and so on. The content and significance of parameter dependent, unitary rotations of the base space (11.6), (11.8), (11.10), where the matrix involved satisfies Baxterized braid equation, deserves further study.

ADDENDUM: After having completed this paper we received the preprint of Yong Zhang and Mo-Lin Ge [20]. In their construction, in contrast to ours, there are some phases involving free parameters on the anti-diagonal of the braid matrices. Such phases provide what the authors designate as *generalized* constructions. But *these phases are spurious*. They can be absorbed by conjugations by matrices of the form $Y \otimes Y$, where Y is a $(2n) \times (2n)$ invertible matrix for \widehat{R} of dimension $(2n)^2 \times (2n)^2$. Such conjugations *preserve the braid property*. The required conjugation is simple. It is sufficient to illustrate this for the 4×4 case (eq. (29) [20]). This generalization for $n > 1$ will be evident, recognizing the constraint (28) of ref. [20] as a crucial feature. Define

$$Y = \begin{vmatrix} e^{-i\varphi/4} & 0 \\ 0 & e^{i\varphi/4} \end{vmatrix}. \quad (A.1)$$

Then

$$Y \otimes Y \begin{vmatrix} 0 & 0 & 0 & e^{i\varphi} \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -e^{-i\varphi} & 0 & 0 & 0 \end{vmatrix} Y^{-1} \otimes Y^{-1} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \otimes \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}. \quad (\text{A.2})$$

This is $L \otimes K$ ($=J \otimes K$ for $n = 1$) of our (4.14). (The remaining term of $\sqrt{2}\widehat{R}$ is $I \otimes I$ and invariant.) Removal of the phases makes the tensored structure ($L \otimes K$, for example) evident. Thus one arrives at our formalism, deriving results in canonical forms with ease and power.

One can certainly introduce not only phases but a more complicated parametrization in the tensored structure, also, trivially, by substituting

$$J \otimes K \Rightarrow (X_{(2n)} \otimes X_{(2n)}) (J \otimes K) (X_{(2n)}^{-1} \otimes X_{(2n)}^{-1}) = (X_{(2n)} J X_{(2n)}^{-1}) \otimes (X_{(2n)} K X_{(2n)}^{-1}) \quad (\text{A.3})$$

and so on, $X_{(2n)}$ being invertible $(2n) \times (2n)$ matrix. Such parameters are, evidently, to be removed when present rather than introduced. Unitary $X_{(2n)}$ can preserve unitarity, but even so, should be eliminated.

We would like to add that this is not the first time we *conjugate away* spurious parameters haunting the literature on braid matrices. A much less simple exercise was provided by the so-called *hybrid deformations* [21]. (Another claim was recently disproved in a note [22], using however a different type of argument.) Being very much conscious of the possibility of hidden equivalences we scrupulously displayed the *non-equivalence* embodied in our (4.12). This was, as explained, related to the absence of a tensored structure ($V \neq Y \otimes Y$ for some suitable Y).

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