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THE LIE-ALGEBRAIC STRUCTURES AND INTEGRABILITY OF DIFFERENTIAL AND DIFFERENTIAL-DIFFERENCE NONLINEAR DYNAMICAL SYSTEMS

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Abstract

The infinite-dimensional operator Lie algebras of the related integrable nonlocal differential-difference dynamical systems are treated as their hidden symmetries. As a result of their dimerization the Lax type representations for both local differential-difference equations and nonlocal ones are obtained. An alternative approach to the Lie-algebraic interpretation of the integrable local differential-difference systems [1, 5, 6, 8, 10, 11, 12] is also proposed. The Hamiltonian representation for a hierarchy of Lax type equations on a dual space to the centrally extended Lie algebra of integro-differential operators with matrix-valued coefficients coupled with suitable eigenfunctions and adjoint eigenfunctions evolutions of associated spectral problems is obtained by means of a specially constructed Bäcklund transformation. The Hamiltonian description for the corresponding set of additional symmetry hierarchies is represented. The relation of these hierarchies with Lax type integrable (3+1)-dimensional nonlinear dynamical systems and their triple Lax type linearizations is analyzed. The Lie-algebraic structures, related with centrally extended current operator Lie algebras are discussed with respect to constructing new nonlinear integrable dynamical systems on functional manifolds and super-manifolds. Special Poisson structures and related with them factorized integrable operator dynamical systems having interesting applications in modern mathematical physics, quantum computing mathematics and other fields are constructed. The previous purely computational results of [4, 5] are explained within the approach developed.

1 The Lie-algebraic structure of integrable differential-difference equations

1.1 Introduction

Let $\mathcal{A}$ be an arbitrary continuous associative algebra over $\mathbb{C}$, and $T:\mathcal{A}\rightarrow\mathcal{A}$ be its automorphism:

$$T((k_1a_1+k_2a_2)=k_1T(a_1)+k_2T(a_2),\quad T(ab)=T(a)T(b) \tag{1.1}$$

for all $a_1,\ a_2\in\mathcal{A}$ and $k_1,\ k_2\in\mathbb{C}$. Assume also that on the algebra $\mathcal{A}$ there is defined a linear mapping $\tau:\mathcal{A}\rightarrow\mathbb{C}$ such that:

$$\tau(ab)=\tau(ba),\tau(Ta)=\tau(a) \tag{1.2}$$

for $a,b\in\mathcal{A}$. Define now the algebra $\mathcal{G}$ of all linear homomorphisms $A(T):\mathcal{A}\rightarrow\mathcal{A}$, having such a structure: $A(T)\in\mathcal{G}$ iff

$$A(T)=\sum_{i<<\infty}a_iT^i, \tag{1.3}$$

where $a_i\in\mathcal{A}$ and $i\in\mathbb{Z}$. On the associative operator algebra $\mathcal{G}$ one can define [1] the following linear trace-operation:

$$trA(T):=\tau(a_0), \tag{1.4}$$

satisfying the additional important symmetry property:

$$tr(A(T)B(T))=tr(B(T)A(T)) \tag{1.5}$$

for all $A(T),\ B(T)\in\mathcal{G}$.

It is a standard procedure of converting the associative algebra $\mathcal{G}$ into a Lie algebra $\mathcal{G}$ with the Lie product chosen as the commutator with respect to the point-wise multiplication in $\mathcal{G}$:

$$[A(T),B(T)]:=A(T)B(T)-B(T)A(T) \tag{1.6}$$

for $A(T)$ and $B(T)\in\mathcal{G}$. In addition, we also assume here that the scalar product

$$\langle A(T),B(T)\rangle:=tr(A(T)B(T)) \tag{1.7}$$

is non-degenerate on $\mathcal{G}$, that is from $\langle A(T),B(T)\rangle=0$ for a fixed $A(T)\in\mathcal{G}$ and all $B(T)\in\mathcal{G}$ there follows that $A(T)\equiv 0$.

Thus we have constructed above the operator Lie algebra $\mathcal{G}$ (1.3) with Lie product (1.6), which is endowed with the non-degenerate scalar product (1.7), and is symmetric and *ad*-invariant owing to (1.5), that is

$$\langle A(T),[B(T),C(T)]\rangle=\langle[A(T),B(T)],C(T)\rangle \tag{1.8}$$

for all $A(T),\ B(T)$ and $C(T)\in\mathcal{G}$.

1.2 Central extension

Assume now that the basic associative algebra $\mathcal{A}$ depends on an independent parameter $x \in \mathbb{S}^1$, that makes it possible to define the new associative current algebra $C_{\mathbb{S}^1}(\mathcal{A}) \in C^\infty(\mathbb{S}^1;\mathcal{A})$ naturally generalizing [22] via the construction above, the corresponding current operator Lie algebra $C_{\mathbb{S}^1}(\mathcal{G}) \in C^\infty(\mathbb{S}^1;\mathcal{G})$ with the following modified Tr-operation:

$$TrA(T) := \int_{\mathbb{S}^1} dx\, \tau(a_0)(x) \tag{1.9}$$

for any $A(T) \in C_{\mathbb{S}^1}(\mathcal{G})$. Clearly, the current Lie algebra $C_{\mathbb{S}^1}(\mathcal{G})$ can be endowed with the following scalar product

$$(A(T), B(T)) := Tr(A(T)B(T)) \tag{1.10}$$

for any $A(T)$ and $B(T) \in C_{\mathbb{S}^1}(\mathcal{G})$, naturally related with that of (1.7).

The current Lie algebra $C_{\mathbb{S}^1}(\mathcal{G})$ can now naturally be extended via the usual central extension procedure: $C_{\mathbb{S}^1}(\mathcal{G}) :\to C_{\mathbb{S}^1}(\hat{\mathcal{G}}) = (C_{\mathbb{S}^1}(\mathcal{G}) \oplus \mathbb{C})$ with the Lie product:

$$[(A(T),\alpha),(B(T),\beta)] := ([A(T),B(T)], \omega_2(A(T),B(T))), \tag{1.11}$$

where by definition, $\omega_2 : C_{\mathbb{S}^1}(\mathcal{G}) \times C_{\mathbb{S}^1}(\mathcal{G}) \to C$ is the standard Maurer-Cartan 2-cocycle on $C_{\mathbb{S}^1}(\mathcal{G})$, $\alpha, \beta \in \mathbb{C}$ and for any $A(T),\ B(T) \in C_{\mathbb{S}^1}(\mathcal{G})$

$$\omega_2(A(T),B(T)) := \int_{\mathbb{S}^1} dx\, \langle A(T), dB(T)/dx \rangle. \tag{1.12}$$

Now assume, in addition, that an automorphism $T : \mathcal{A} \to \mathcal{A}$ is naturally extended to an automorphism $T : C_{\mathbb{S}^1}(\mathcal{A}) \to C_{\mathbb{S}^1}(\mathcal{A})$. As a result we have constructed the centrally extended current operator Lie algebra $C_{\mathbb{S}^1}(\hat{\mathcal{G}})$, which we shall interpret further as a symmetry Lie algebra of specially constructed integrable nonlocal differential-difference dynamical systems.

1.3 $\mathcal{R}$-matrix method

Consider a smooth nonlinear dynamical system of the form

$$u_t = K[u] \tag{1.13}$$

on a 2π-periodic functional Poisson manifold $M \subset C^\infty(\mathbb{S}^1;\mathbb{R}^m)$, where $K : M \to T(M)$ – the corresponding vector field on M. Assuming that dynamical system (1.13) possesses a symmetry subalgebra isomorphic to some current Lie subalgebra $C_{\mathbb{S}^1}(\hat{\mathcal{G}}_0)$, define also *a priori* some momentum mapping $\hat{l} : M \to C_{\mathbb{S}^1}(\hat{\mathcal{G}}^*)$ associated with the corresponding current Lie algebra action of $C_{\mathbb{S}^1}(\hat{\mathcal{G}})$ on M, which is Poissonian [4], that is the Poisson structure $\{.,.\}$ on M is invariant with respect to this action and the diagram:

$$\begin{array}{ccc} M & \hat{l}\longrightarrow & C_{\mathbb{S}^1}(\hat{\mathcal{G}}^*) \\ A\downarrow & & \downarrow ad^*A \\ M & \hat{l}\longrightarrow & C_{\mathbb{S}^1}(\hat{\mathcal{G}}^*) \end{array} \tag{1.14}$$

is commuting for any $A \in C_{\mathbb{S}^1}(\hat{\mathcal{G}})$. As a result of the construction above, one can represent the Poisson structure on the manifold M via the standard reduction procedure of the canonical Lie-Poisson structure upon the adjoint space $C_{\mathbb{S}^1}(\hat{\mathcal{G}}^*)$:

$$\{\gamma, \mu\} := (\hat{l}, [\nabla\gamma(\hat{l}), \nabla\mu(\hat{l})])|_{\hat{l}:=\hat{l}[u]} \tag{1.15}$$

for $u \in M$ and any smooth functionals $\gamma, \mu \in D(C_{\mathbb{S}^1}(\hat{\mathcal{G}}^*))$. Concerning the integrable Poissonian flows (1.13) on M, one needs to construct a hierarchy of Poisson commuting functionals $h_j \in \mathfrak{D}(M)$, $j \in \mathbb{Z}_+$, which we shall produce here via the standard $\mathcal{R}$-matrix [22] approach. Namely, consider the space $I(C_{\mathbb{S}^1}(\hat{\mathcal{G}}^*))$ of Casimir functionals $\gamma \in D(C_{\mathbb{S}^1}(\hat{\mathcal{G}}^*))$ satisfying, due to (1.15) and (1.11), a functional equation of the type

$$d\nabla\gamma(l)/dx = [l,\, \nabla\gamma(l)] \tag{1.16}$$

for all $\hat{l} := (l, 1) \in C_{\mathbb{S}^1}(\hat{\mathcal{G}}^*)$. Assume also now that the current Lie algebra $C_{\mathbb{S}^1}(\hat{\mathcal{G}})$ admits a standard $\mathcal{R}$-structure [22]; that is, the new bracket

$$[A, B]_{\mathcal{R}} := [A, \mathcal{R}B] + [\mathcal{R}A, B], \tag{1.17}$$

defined by means of a linear homomorphism $\mathcal{R} : C_{\mathbb{S}^1}(\hat{\mathcal{G}}) \to C_{\mathbb{S}^1}(\hat{\mathcal{G}})$, is a Lie bracket too for all $A, B \in C_{\mathbb{S}^1}(\hat{\mathcal{G}})$. This is the case [22] if a homomorphism $\mathcal{R} : C_{\mathbb{S}^1}(\hat{\mathcal{G}}) \to C_{\mathbb{S}^1}(\hat{\mathcal{G}})$ satisfies the Yang-Baxter condition:

$$\mathcal{R}[A, B]_{\mathcal{R}} = 1/4[A, B] + [\mathcal{R}A, \mathcal{R}B] \tag{1.18}$$

for any $A, B \in C_{\mathbb{S}^1}(\hat{\mathcal{G}})$. In particular, if $\mathcal{R} \in Hom(C_{\mathbb{S}^1}(\mathcal{G}))$, then the corresponding bracket on $C_{\mathbb{S}^1}(\hat{\mathcal{G}})$ naturally reads as

$$[(A, \alpha), (B, \beta)]_{\mathcal{R}} := ([A, B], \omega_2(\mathcal{R}A, B) + \omega_2(A, \mathcal{R}B)) \tag{1.19}$$

for any $A, B \in C_{\mathbb{S}^1}(\mathcal{G})$ and $\alpha, \beta \in \mathbb{C}$. Now we are in a position to formulate the following theorem.

Theorem 1.1. *Given a hierarchy of Casimir functionals $\gamma_j \in (C_{\mathbb{S}^1}(\hat{\mathcal{G}}^*))$, $j \in \mathbb{Z}_+$, solving the equation (1.16), then all reduced on M functionals $h_j := \gamma_j|_{l=l[u]}$, $u \in M$, $j \in Z_+$, are in involution with respect to the Poisson bracket*

$$\{\gamma, \mu\}_{\mathcal{R}} := (l, [\nabla\gamma(l), \nabla\mu(l)]_{\mathcal{R}})|_{l:=l[u]} \tag{1.20}$$

reduced from $C_{\mathbb{S}^1}(\hat{\mathcal{G}}^*)$ on M with respect to the diagram (1.14). Moreover, the momentum mapping $\hat{l} : M \to C_{\mathbb{S}^1}(\hat{\mathcal{G}}^*)$ satisfies, with respect to the bracket (1.20) with a Hamiltonian function $\gamma \in I(C_{\mathbb{S}^1}(\hat{\mathcal{G}}^*))$ the following evolution equation in $C_{\mathbb{S}^1}(\mathcal{G}^*)$:

$$dl/dt = ad^*_{\mathcal{R}\nabla\gamma(l)}l - d(\mathcal{R}\nabla\gamma(l))/dx, \tag{1.21}$$

where $t \in \mathbb{R}$ –is an evolution parameter. The flow (1.21) is equivalent to the Lax type representation of (1.13):

$$dl/dt = [l - d/dx, \mathcal{R}\nabla\gamma(l)], \tag{1.22}$$

owing to the supposed non-degeneracy of the scalar product (1.10) on $C_{\mathbb{S}^1}(\mathcal{G}^*) \simeq C_{\mathbb{S}^1}(\mathcal{G})$.

The proof of this Theorem follows directly from the standard scheme devised in [5, 6].

1.4 Integrable nonlocal differential-difference dynamical systems

Assume now that the operator algebra $C_{\mathbb{S}^1}(\mathcal{G})$ admits an automorphism $T : C_{\mathbb{S}^1}(\mathcal{G}) \to C_{\mathbb{S}^1}(\mathcal{G})$, which is the simple shift on $i\delta \in i\mathbb{R}$ along the complexified parameter $x \in \mathbb{S}^1$:

$$T \circ a(x) := a(x + \delta i), \tag{1.23}$$

for all $a \in C_{\mathbb{S}^1}(\mathcal{G})$, where $x \in \mathbb{S}^1$, $i^2 = -1$ and we specify, for brevity, that $\mathcal{A} = \mathbb{R}$.

It is evident that the corresponding space $C_{\mathbb{S}^1}(\mathcal{G})$ is an operator current algebra with the non-degenerate scalar product (1.10), where by definition, $\tau(a) = a$ for any $a \in \mathbb{R}$. Observe now that the current Lie algebra $C_{\mathbb{S}^1}(\mathcal{G})$ admits the usual splitting into two subalgebras:

$$C_{\mathbb{S}^1}(\mathcal{G}) = C_{\mathbb{S}^1}(\mathcal{G}_+) \oplus C_{\mathbb{S}^1}(\mathcal{G}_-), \tag{1.24}$$

where by definition,

$$C_{\mathbb{S}^1}(\mathcal{G}_+) := \left\{ \sum_{i \in \mathbb{Z}_+}^{i << \infty} a_i T^i \in C_{\mathbb{S}^1}(\mathcal{G}) : a_i \in C_{\mathbb{S}^1}(\mathbb{R}) \right\},$$

$$C_{\mathbb{S}^1}(\mathcal{G}_-) := \left\{ \sum_{i \in \mathbb{Z}_-} T^i \circ a_i \in C_{\mathbb{S}^1}(\mathcal{G}) : a_i \in C_{\mathbb{S}^1}(\mathbb{R}) \right\}. \tag{1.25}$$

Thus, one can obtain the following partial solution to the Yang-Baxter equation (1.18):

$$\mathcal{R} := 1/2(P_+ - P_-), \tag{1.26}$$

where $P_\pm : C_{\mathbb{S}^1}(\mathcal{G}) \to C_{\mathbb{S}^1}(\mathcal{G}_\pm)$ are the corresponding projectors on the subalgebras (1.25). Given $A(T)$ and $B(T) \in C_{\mathbb{S}^1}(\mathcal{G})$, one can readily calculate that

$$(A(T), B(T)) = \sum_{j \in \mathbb{Z}} \int_{\mathbb{S}^1} a_j(x) b_j(x) dx, \tag{1.27}$$

where we put

$$A(T) := \sum_{i \in \mathbb{Z}}^{i << \infty} a_i T^i, \quad B(T) := \sum_{j \in \mathbb{Z}}^{j << \infty} T^{-j} \circ b_j \tag{1.28}$$

for some periodic coefficients $a_i, b_j \in C_{\mathbb{S}^1}(\mathbb{R};\mathbb{C})$, $i, j \in \mathbb{Z}$. As a result of formula (1.27), one can identify spaces $C_{\mathbb{S}^1}(\mathcal{G}^*)$ and $C_{\mathbb{S}^1}(\mathcal{G})$, via the relationships

$$C_{\mathbb{S}^1}(\mathcal{G}_+^*) \simeq T \circ C_{\mathbb{S}^1}(\mathcal{G}_-), \quad C_{\mathbb{S}^1}(\mathcal{G}_-^*) \simeq C_{\mathbb{S}^1}(\mathcal{G}_+) \circ T,$$

being held on $C_{\mathbb{S}^1}(\mathcal{G}^*)$. Thus, one can now simply generate Lax type Hamiltonian systems via the recipe (1.22), where $\gamma \in I(C_{\mathbb{S}^1}(\hat{\mathcal{G}}^*))$ satisfies equation (1.16).

Example. Let us take an element $l \in C_{\mathbb{S}^1}(\mathcal{G}_+^*)$ as

$$l := l[u] = i\alpha T + i\epsilon^{-1} u, \tag{1.29}$$

where $\alpha \in \mathbb{C}$ and $\epsilon \in \mathbb{R}$ are some constants, $u \in M \subset C^{\infty}(\mathbb{S}^1;\mathbb{R})$.

Equation (1.16) can easily be solved by setting, for each $n \in \mathbb{Z}_+$,

$$\nabla\gamma_n(l) := \sum_{n-j\in\mathbb{Z}_+} a_j T^j, \tag{1.30}$$

where $a_j \in C^{\infty}(\mathbb{S}^1;\mathbb{R})$, $n-j \in \mathbb{Z}_+$, are some unknown functional parameters that can be obtained from (1.16) successively via the recurrent procedure. As a result, one gets that at $n = 2$,

$$\begin{aligned} a_2 = i\epsilon\alpha^2, \quad a_1 &= i\alpha(Tu + u + 2u\epsilon\delta^{-1}), \\ a_0 &= (H + 2i\delta^{-1}\partial_x^{-1})u_x + i\epsilon^{-1}u^2, ..., \end{aligned} \tag{1.31}$$

where

$$H := (T + 1)(T - 1)^{-1} \tag{1.32}$$

is a well-known integral operator acting as

$$H : f \in S(\mathbb{R};\mathbb{C}) = -\frac{i}{\delta}\int_{\mathbb{R}} cth\frac{\pi(x-\xi)}{\delta} f(\xi)d\xi \ \in \ S(\mathbb{R};\mathbb{C}), \tag{1.33}$$

on the Schwartz space $S(\mathbb{R};\mathbb{C})$ of the rapidly decreasing functions. Olso taking into account that the evolution equation (1.22) can be rewritten as

$$dl/dl = [l - d/dx, \nabla\gamma(l)_+], \tag{1.34}$$

where $\gamma \in I(C_{\mathbb{S}^1}(\hat{\mathcal{G}}^*))$, one obtains immediately from (1.29)-(1.31) and (1.34) the following nonlocal differential-difference Korteweg-de Vries type shallow water equation:

$$du/dt = -i\epsilon(H + 2i\delta^{-1}\partial_x^{-1})u_{xx} + 2uu_x, \tag{1.35}$$

first found in [7] and [8] by means of completely different approaches. As $\delta \to 0$, $\epsilon = 2\delta^{-1}$ the equation (1.35) reduces to the usual Korteweg-de Vries equation

$$du/dt = 1/3u_{xxx} + 2uu_x, \tag{1.36}$$

and alternatively, as $\delta \to \infty$, $\epsilon = 1$, (1.35) reduces to the well-known Benjamin-Ono deep water equation:

$$du/dt = 2uu_x + \frac{1}{\pi}\int_{\mathbb{R}} \frac{u_{\xi\xi}(\xi)d\xi}{\xi - x}, \tag{1.37}$$

where we have assumed that the period in $x \in \mathbb{R}$ of the manifold M is infinity. The latter equation (1.37) was proved to be Lax type integrable too [5, 6] by means of a differential Riemann-Hilbert problem in a strip of the complex extension of the variable $x \in \mathbb{R}$ between $Im(x) = 0$ and $Im(x) = \pm 2\delta$, and then periodically extended vertically to infinity.

1.5 Dimerization of the nonlocal differential-difference equations

Now consider a situation when a functional element $u \in \hat{M} \subset C^{\infty}(\mathbb{R};\mathcal{B}^n)$, where $\mathcal{B}$ is an associative algebra such as the algebra of pseudo-differential operators. An automorphism $T : A \to A$, where $A \subset C^{\infty}(\mathbb{R};\mathcal{B})$, naturally defines the associative subalgebra $\mathcal{G}$ of homomorphism (1.3), which then is transformed into the operator Lie algebra $\mathcal{G}$ with respect to the Lie product (1.6). Introducing the current operator Lie algebra $C_{\mathbb{S}^1}(\mathcal{G})$ and its central extension by means of the standard Maurer-Cartan cocycle (1.12), one can consider integrable operator dynamical systems [9] of the form (1.12), based on solutions of the characteristic operator equation (1.16). Consequently, one can construct new integrable many-dimensional nonlocal differential-difference dynamical systems. For instance, a suitable operator variant of the Example above is given by an element $l \in C_{\mathbb{S}^1}(\mathcal{G}_+)$, where

$$l := l[\hat{u}] = i\alpha T + i\epsilon^{-1}\hat{u}. \tag{1.38}$$

Here $\hat{u} \in \hat{M} \subset C^{\infty}(\mathbb{S}^1;\mathcal{B})$ and

$$\mathcal{B} := \{\sum_{i\in\mathbb{Z}}^{i<<\infty} u_i\xi^i : u_i \in S(\mathbb{R};\bar{M}), \quad i \in \mathbb{Z}\}, \tag{1.39}$$

where $\bar{M} \subset C^{\infty}(\mathbb{S}^1;\mathbb{R})$ is some functional manifold, and by definition,

$$[\xi, a(y)] := \partial a(y)/\partial y, \ \tau(a) := \int_{\mathbb{R}} dy \ res_{\xi} a(y) \tag{1.40}$$

for any $a \in \mathcal{B}$, $y \in \mathbb{R}$. Having assumed now that $\hat{u} := u + i\delta\xi \in \hat{M}$, from (1.34) and (1.16) one gets a new (different from that in [7]) integrable nonlocal differential-difference Kadomtsev-Petviashvily equation, having applications in hydrodynamics and plasma physics. On the other hand, if the automorphism $T : \mathcal{A} \to \mathcal{A}$ is defined as

$$(Ta)(y) := (T_{\delta}a)(y) := a(y + i\delta), \ y \in \mathbb{R}, \tag{1.41}$$

for any $a \in \mathcal{A}$, $\delta \in \mathbb{R}_+$, and the algebra $\mathcal{A}$ is $(\mathbb{R} \ni x)$-independent, then the characteristic equation (1.16) has the simple form:

$$[\nabla\gamma(l), l] = 0. \tag{1.42}$$

It is evident that equation (1.42) admits the following general solution:

$$\gamma_n^{(p)} := Tr : l_{(p)}^{n/p} \in I(\mathcal{G}^*), \tag{1.43}$$

where $n \in \mathbb{Z}_+$, and by definition, for any $p \in \mathbb{Z}_+$

$$Tr : A_{(p)}(T) := \tau(a_0^{(p)}), \ T \Rightarrow T_{\delta/p}, \tag{1.44}$$

with

$$A_{(p)}(T) := \sum_{k\in\mathbb{Z}}^{k<<\infty} a_k^{(p)} T_{\delta/p}^k \in Hom(\mathcal{A}).$$

For instance, in (1.38) the element $l \in \mathcal{G}_+^*$ can be represented equivalently as

$$l = l_{(p)} = i\alpha(T_{\delta/p})^p + i\epsilon^{-1}\hat{u} \tag{1.45}$$

for any $p \in \mathbb{Z}_+$; this entails the existence of the nontrivial p-th root $l_{(p)}^{1/p} \in \mathcal{G}^*$ making it possible to compute nontrivial Casimir functionals (1.43). By now reducing the element (1.38) upon the degenerate subspace

$$\hat{M}_0 = \{\hat{u} \in \hat{M} : \ u_j \neq 0, j \neq \overline{0,1}, u_0 = u, u_1 = i\epsilon\xi\}, \tag{1.46}$$

from (1.34) and (1.43) one can successively retrieve the nonlocal differential-difference Korteweg-de Vries hierarchy constructed before in [8]:

$$\begin{gathered} du/dt_1 = -u_x, \quad du/dt_2 = 2uu_x - Hu_{xx}, \\ du/dt_3 = 1/4u_{xxx} + 3u^2u_x + 3/2H(uu_x)_x + \\ +3/2u(Hu_x)_x + 3/4H^2u_{xxx}, ..., \end{gathered} \tag{1.47}$$

where $t_j \in \mathbb{R}$, $j \in \mathbb{Z}_+$, are the corresponding evolution parameters and $u \in \bar{M}$.

Note. It is evident from the consideration above that our construction contains, as a partial case, the standard local theory of integrable differential-difference equations [5], if one takes an associative algebra $\mathcal{A}$ equal to $C_{\mathbb{Z}}(\mathcal{B})$ with the componentwise multiplication in $\mathcal{A}$, on which there acts an automorphism $T : \mathcal{A} \to \mathcal{A}$ defined as

$$(Ta)_n := a_{n+1}, \ n \in \mathbb{Z}, \tag{1.48}$$

for any $a \in \mathcal{A} := C_{\mathbb{Z}}(\mathcal{B})$. The associated linear mappings $\tau : \mathcal{A} \to \mathbb{C}$ is defined as follows:

$$\tau(a) := \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} res_\xi \ a_n(y;\xi)dy, \tag{1.49}$$

for any $a \in \mathcal{A}$. The corresponding operator Lie algebra $\mathcal{G}$ of homomorphisms of the space $\mathcal{A}$ is endowed with the $tr-$operation (1.4), which can obviously be naturally centrally extended by means of the standard Maurer-Cartan 2-cocycle (1.12) in the variable $x \in \mathbb{S}^1$. The resulting equation on Casimir functionals $\gamma \in I(\hat{\mathcal{G}}^*)$ can now be written as

$$d\nabla\gamma(l)/dx = [l, \nabla\gamma(l)] \tag{1.50}$$

for any $l \in C_{\mathbb{S}^1}(\hat{\mathcal{G}})$, $x \in \mathbb{S}^1$, and the Lax type equation (1.34) as

$$dl/dt = [l - d/dx, \nabla\gamma(l)_+], \tag{1.51}$$

with $t \in \mathbb{R}$ being an evolution parameter. For example, the element

$$l := l[u,v] := T + v + T^{-1}u \in \mathcal{G}^* \tag{1.52}$$

generates, via solving equation (1.50) and substituting a solution into (1.51), a generalized Toda chain. If this chain is independent of the parameter $x \in \mathbb{S}^1$, it reduces to the standard Toda chain on the manifold $M \subset C_{\mathbb{Z}}(\mathbb{R}^2)$:

$$du/dt = u(Tv - v), \quad dv/dt = v(T^{-1}u - u), \tag{1.53}$$

which is well known to be a Lax type integrable [6, 10] bi-Hamiltonian flow on the discrete manifold M.

1.6 Local differential-difference equations: an alternative approach

Here we consider the Lie-algebraic aspects of differential-difference integrable flows associated with the following generalized linear matrix problem:

$$f_{n+1} = l_n[u; \lambda] f_n, \tag{1.54}$$

where $f \in l_\infty(\mathbb{Z};\mathbb{C}^p)$, $l_n := l_n[u; \lambda] \in G := GL_p(\mathbb{C})$ for all $n \in \mathbb{Z}_N$, with $N \in \mathbb{Z}_+$ fixed, $\lambda \in \mathbb{C}$ is a "spectral" parameter and $u \in M \subset C_{\mathbb{Z}_N}(\mathbb{R}^m)$ is an *a priori* given discrete finite-dimensional manifold. To describe locally defined integrable differential-difference equations associated with the linear problem (1.54), it is necessary to study in detail the natural action of the product-group $G^N := \otimes_{i=1}^N G$ on the phase space $M_G = \{l_n \in G : \, n \in \mathbb{Z}_N\}$:

$$G^N \times M_G :\rightarrow M_G,$$

given as follows:

$$\{g_n : \, n \in \overline{1, N}\} \times \{l_n : \, n \in \mathbb{Z}_N\} := \{g_{n+1} l_n g_n^{-1} : \, n \in \mathbb{Z}_N\}. \tag{1.55}$$

A functional $\gamma \in D(M_G)$ is invariant with respect to the action (1.55) iff the discrete equation

$$\nabla\gamma(l_{n+1}) l_{n+1} = l_n \nabla\gamma(l_n), \tag{1.56}$$

where $\nabla\gamma(l_n) \in \mathcal{G}^*$, $\mathcal{G}$ – the Lie algebra of the formal group G, holds for all $n \in \mathbb{Z}_N$. From the relationship (1.56), one finds readily that the quantity $S_n := S_n(l) = \nabla\gamma(l_n) l_n$, $n \in \mathbb{Z}_N$, called the monodromy matrix, satisfies the following difference equation:

$$S_{n+1}(l) l_n = S_n(L) l_n^{-1} \tag{1.57}$$

for $n \in \mathbb{Z}_N$ and any $l \in M_G$. Applying the usual Sp-operation to both sides of (1.57) one immediately concludes that all functionals

$$\gamma_j(S_N) := Sp : S_n^j(l), \tag{1.58}$$

where $j \in \mathbb{Z}_+$, are independent of the discrete parameter $n \in \mathbb{Z}_N$ and are invariant with respect to the standard Lie group G-action on the basic element $S_n \in \mathcal{G}^*$ due to (1.57). It is an easy task to verify that for all $j \in \mathbb{Z}_+$

$$\nabla\gamma_j(S_N)(l_n) = S_n^j(l) l_n^{-1} \tag{1.59}$$

holds for any $n \in \mathbb{Z}_N$. Whence, one has the following important theorem.

Theorem 2.2. *The hierarchy of invariant with respect to the group action (1.55) functionals $\gamma_j \in D(M_G)$, $j \in \mathbb{Z}_+$, is given exactly by the following expression:*

$$\gamma_j := \gamma_j(S_N) = Sp : S_N^j(l), \tag{1.60}$$

where owing to (1.57) for $S_N(l) \in \mathcal{G}^*$, one has the matrix quantity

$$S_N(l) = \prod_{n=1}^{N-1} l_n[u; \lambda], \tag{1.61}$$

as the real monodromy matrix for the linear problem (1.54).

It is necessary to point out here that a statement, similar to the Theorem above, was also formulated without proof in [11], and based on the classical $\mathcal{R}$-matrix approach to differential-difference equations.

A regular procedure of generating the integrable differential-difference flows associated naturally with the hierarchy of invariant functionals (1.60) on the discrete manifold M_G shall now be developed. Consider an associative algebra $\mathcal{G}_\lambda$ of $N \times N$ matrices, which is converted into the Lie algebra $\mathcal{G}_\lambda$ with the standard Lie product (1.6), allowing an $\mathcal{R}$-structure with the Lie product $[.,.]_\mathcal{R}$. Then, evidently, all functionals (1.60) are Casimir on the space $\mathcal{G}_\lambda^*$, generating on M_G the following of the Lax type Poissonnian flows:

$$dS_N/d\tau_j = [S_N, \mathcal{R}\nabla\gamma_j(S_N)] \tag{1.62}$$

for all $j \in \mathbb{Z}_+$. But the flows (1.62), as it is easy to verify, also induce the corresponding flows on an element $l \in M_G$:

$$dl_n/d\tau_j = \{\mathcal{R}(\Psi_{n+1}^{-1}(l)\nabla\gamma_j(S_N)\Psi_{n+1}(l))\}l_n -$$

$$-l_n\{\mathcal{R}(\Psi_n^{-1}(l)\nabla\gamma_j(S_N)\Psi_n(l))\}, \tag{1.63}$$

with

$$\Psi_n(l) := \prod_{j=1}^{n} l_j[u; \lambda],$$

for all $n \in \mathbb{Z}$, which nevertheless, do not conserve the invariant functionals $\gamma_k \in D(M_G)$ for all $j, k \in \mathbb{Z}_+$. To remedy this problem, let us generate hierarchy of flows on $\mathcal{G}_\lambda^*$, making use of the well-known [10, 11] second nonlinear in $S_N \in \mathcal{G}_\lambda^*$ $\mathcal{R}$-structure on the Lie algebra $\mathcal{G}_\lambda$:

$$\mathcal{R}(S_N)a := \mathcal{R}^*[S_N, a]S_N + [S_N, \mathcal{R}(aS_N)] \tag{1.64}$$

for any $a \in \mathcal{G}_\lambda$. Using (1.64), one easily obtains the following new hierarchy of Lax type flows on M_G :

$$dS_N/dt_j = [S_N, \mathcal{R}(\nabla\gamma_j(S_N)S_N)] \tag{1.65}$$

for all $j \in \mathbb{Z}_+$. The flows (1.65) are clearly also naturally associated with the next following flows on M_G for any $j \in \mathbb{Z}_+$

$$dl_n/dt_j = \mathcal{P}_{n+1}(l)l_n - l_n\mathcal{P}_n(l), \tag{1.66}$$

where for all $n \in \mathbb{Z}_N$ we defined

$$\mathcal{P}_n(l) := \mathcal{R}(\Psi_n^{-1}(l)\nabla\gamma_j(S_N)S_N\Psi_n(l)). \tag{1.67}$$

As the following identity

$$\nabla\gamma_j(l_n) = \Psi_n^{-1}(l)\nabla\gamma_j(S_N)S_N\Psi_n(l) \tag{1.68}$$

holds for all $j \in \mathbb{Z}_+, n \in \mathbb{Z}$, the evolutions (1.66) can be rewritten down as flows

$$dl_n/dt_j = (\mathcal{R}\nabla\gamma_j(l_{n+1}))l_n - l_n(\mathcal{R}\nabla\gamma_j(l_n)) \tag{1.69}$$

on M_G for all $j \in \mathbb{Z}_+$, $n{\in}\mathbb{Z}$. It is straightforward to verify that for all $k, j \in \mathbb{Z}_+$

$$\begin{aligned} d\gamma_k/dt_j = (\nabla\gamma_k(l_n), dl_n/dt_j) = (\nabla\gamma_k(l_n), (\mathcal{R}\nabla\gamma_j(l_{n+1}))l_n - l_n\nabla\gamma_j(l_n)) = \\ = (\mathcal{R}^*(l_n\nabla\gamma_k(l_n)), \nabla\gamma_j(l_{n+1}) - (\mathcal{R}^*(\nabla\gamma_k(l_n))l_n), \nabla\gamma_j(l_n)) = \\ = (\mathcal{R}^*(l_{n-1}\nabla\gamma_k(l_{n-1})), \nabla\gamma_j(l_n)) - (\mathcal{R}^*(\nabla\gamma_k(l_n)l_n), \nabla\gamma_j(l_n)) = \\ = (l_{n-1}\nabla\gamma_k(l_{n-1}) - \nabla\gamma_k(l_n)l_n), \mathcal{R}\nabla\gamma_j(l_n) \equiv 0, \end{aligned} \tag{1.70}$$

since the invariant functionals $\gamma_k \in D(M_G)$, $k \in \mathbb{Z}_+$, satisfy, in virtue of (1.56), $\nabla\gamma_k(l_n)l_n = l_{n-1}\nabla\gamma_k(l_{n-1})$ for all $n \in \mathbb{Z}$. Thus the above built evolution Lax type equations (1.66), (1.67) possess the hierarchy of invariant functionals (1.60) that are in involution with respect to the corresponding, reduced on the finite-dimensional manifold M, the Poisson bracket (1.65) on the adjoint space $\mathcal{G}_\lambda^*$. This concludes the self-contained analysis of the Lie-algebraic integrability of the local differential-difference Lax type flows on the group manifold M_G. Some applications of the construction above are presented in [12].

2 Integrable three-dimensional coupled nonlinear dynamical systems related to centrally extended operator Lie algebras

2.1 Short introduction

Lax representations [24] for integrable (1+1)-dimensional nonlinear dynamical system hierarchies [20, 30, 1] on functional manifolds were first interpreted as Hamiltonian flows on the dual space to the Lie algebra of integro-differential operators in [18]. An algebraic method for constructing Lax integrable (2+1)-dimensional nonlinear dynamical systems by means of two commuting flows from the hierarchy on the suitable co-adjoint action orbit of an integro-differential operator with an infinite integral part was proposed in [21, 43]. The relationship among some Lax integrable (1+1)- and (2+1)-dimensional systems with the corresponding hierarchies of Hamiltonian flows on dual spaces to centrally extended, by means of the standard Maurer-Cartan two-cocycle, Lie algebras was intensively investigated, in particular, in [22, 35, 39, 38].

Every Hamiltonian flow of such a type on the dual space, either to the operator Lie algebra or to its central extension, can be written as the compatibility condition of the spectral relationship for the corresponding integro-differential operator and the suitable eigenfunction evolution. If the above spectral relationship admits a finite set of eigenvalues an important problem of finding the Hamiltonian representation for the Lax type hierarchy coupled with the evolutions of eigenfunctions and appropriate adjoint eigenfunctions naturally arises. It was partly solved in the papers [23, 33, 34, 36, 41] for the Lie algebra of integro-differential operators and its super-generalization by means of a property of variational Casimir functionals under certain Lie-Bäcklund transformation.

Part 2.2 deals with a general Lie-algebraic scheme for constructing a hierarchy of Lax type integrable flows as Hamiltonian ones on the dual space to the centrally extended Lie algebra of integro-differential operators with matrix-valued coefficients.

In Part 2.3 the Hamiltonian structure for the related coupled Lax type hierarchy is obtained by means of the Bäcklund transformation technique developed in [23, 34, 36, 41].

In Part 2.4 the corresponding hierarchies of additional or so-called "ghost" symmetries [23, 28, 29] for the coupled Lax type flows are also shown to be Hamiltonian. It is established that an additional hierarchy of Hamiltonian flows is generated by the Poisson structure, equal to the tensor product of the $\mathcal{R}$-deformed canonical Lie-Poisson bracket [20, 31, 34, 36, 41, 11] with the standard Poisson bracket on related eigenfunctions and adjoint eigenfunctions spaces [19, 34, 36, 41], and the corresponding natural powers of suitable eigenvalue are their Hamiltonian functions. The method for introducing one more variable into $(2+1)$-dimensional nonlinear dynamical systems by use of the additional symmetries, which preserves their Lax integrability, is proposed and an integrable $(3|1+1)$-dimensional analog of the Davey-Stewartson system [42, 44] is constructed.

2.2 The Lie-algebraic structure of Lax type integrable (2+1)-dimensional dynamical systems

Let $\tilde{\mathcal{G}} := C^{\infty}(\mathbb{S}\times\mathbb{S};\mathcal{G})$ be a current Lie algebra of mappings taking values in a semisimple matrix Lie algebra $\mathcal{G}$. Using $\tilde{\mathcal{G}}$, one constructs the Lie algebra $\hat{\mathcal{G}}$ of matrix integro-differential operators

$$a := \mathbf{I}\xi^m + \sum_{j<m} a_j \xi^j,$$

where $a_j \in \hat{\mathcal{G}}$, $j < m$, $j \in \mathbb{Z}$, $m \in \mathbb{N}$, and the symbol $\xi := \partial/\partial x$ denotes the differentiation with respect to the independent variable $x \in \mathbb{R}/2\pi\mathbb{Z} \simeq \mathbb{S}$. The Lie structure in $\hat{\mathcal{G}}$ is defined as

$$[a,b] := a\circ b - b\circ a$$

for all $a,\ b \in \hat{\mathcal{G}}$, where "$\circ$" is the composition of integro-differential operators taking the form:

$$a \circ b := \sum_{\alpha \in \mathbb{Z}_+} \frac{1}{\alpha !} \frac{\partial^\alpha a}{\partial \xi^\alpha} \frac{\partial^\alpha b}{\partial x^\alpha}.$$

On the Lie algebra $\hat{\mathcal{G}}$ there exists the ad-invariant non-degenerate symmetric bilinear form:

$$(a, b) := \int_0^{2\pi} \int_0^{2\pi} Tr\,(a \circ b)\ dxdy\,, \tag{2.1}$$

where Tr-operation for all $a \in \tilde{\mathcal{G}}$ is defined as

$$Tr\, a := res_\xi\, tr\, a = tr\, a_{-1},$$

and tr is the usual matrix trace-operation. Employing the scalar product (2.1) one transforms the Lie algebra $\hat{\mathcal{G}}$ into a metrizable Lie algebra. As a consequence, its dual linear space of matrix integro-differential operators $\hat{\mathcal{G}}^*$ is identified with the Lie algebra, that is $\hat{\mathcal{G}}^* \simeq \hat{\mathcal{G}}$.

The linear subspaces $\hat{\mathcal{G}}_+^* \subset \hat{\mathcal{G}}^*$ and $\hat{\mathcal{G}}_-^* \subset \hat{\mathcal{G}}$ defined as

$$\begin{aligned} \hat{\mathcal{G}}_+ \quad &: \quad = \left\{ a := \xi^{n(\hat{a})} + \sum_{j=0}^{n(\hat{a})-1} a_j \xi^j :\ a_j \in \hat{\mathcal{G}}, \quad j = \overline{0, n(\hat{a})} \right\}\ , \\ \hat{\mathcal{G}}_- \quad &: \quad = \left\{ b := \sum_{j=0}^{\infty} \xi^{-(j+1)} b_j :\ b_j \in \hat{\mathcal{G}}, \quad j \in \mathbb{Z}_+ \right\}\ , \end{aligned} \tag{2.2}$$

are Lie subalgebras in $\hat{\mathcal{G}}$ and $\hat{\mathcal{G}} = \hat{\mathcal{G}}_+ \oplus \hat{\mathcal{G}}_-$. Owing to splitting $\hat{\mathcal{G}}$ into the direct sum of its Lie subalgebras (2.2) one can construct a so-called Lie-Poisson structure on $\hat{\mathcal{G}}^*$ by use of the special linear endomorphism $\mathcal{R}$ of $\hat{\mathcal{G}}$ [20, 31, 11]:

$$\mathcal{R} := (P_+ - P_-)/2, \quad P_\pm \hat{\mathcal{G}} := \hat{\mathcal{G}}_\pm, \quad P_+ P_- = 0.$$

The centrally extended Lie commutator on $\hat{\mathcal{G}}_c := \hat{\mathcal{G}} \oplus \mathbb{C}$ is given as [22, 35, 39]:

$$[(a, \alpha), (b, \beta)] := ([a, b], \omega(\hat{a}, \hat{b}))\ , \tag{2.3}$$

where $\alpha, \beta \in \mathbb{C}$, is generated by means of the standard Maurer-Cartan two-cocycle on $\hat{\mathcal{G}}$:

$$\omega(a, b) := (a, [\partial/\partial y, b])\ ,$$

where $\partial/\partial y$ is the differentiation with respect to the independent variable $y \in S$ and $[\partial/\partial y, b] := \partial b/\partial y$. The commutator (2.3) can be deformed by means of the above endomorphism $\mathcal{R}$ of $\hat{\mathcal{G}}$:

$$[(a, \alpha), (b, \beta)]_{\mathcal{R}} := ([a, b]_{\mathcal{R}}, \omega_{\mathcal{R}}(a, b))\ , \tag{2.4}$$

where the R-commutator takes the form:

$$[a, b]_{\mathcal{R}} := [\mathcal{R}a, b] + [a, \mathcal{R}b]\ ,$$

and the R-deformed two-cocycle is determined in the following way:

$$\omega(a,b)_{\mathcal{R}} := \omega(\mathcal{R}a,b) + \omega(a,\mathcal{R}b) \ .$$

For any Fréchet smooth functionals $\gamma, \mu \in D(\hat{\mathcal{G}}_c^*)$ the Lie-Poisson bracket on $\hat{\mathcal{G}}_c^*$ related with the commutator (2.4) and the extended scalar product:

$$((a,\alpha),(b,\beta)) := (a,b) + \alpha\beta \ ,$$

where $a, b \in \hat{\mathcal{G}}$ and $\alpha, \beta \in \mathbb{C}$, is given as

$$\{\gamma,\mu\}_{\mathcal{R}}(l) = (l, [\nabla\gamma(l), \nabla\mu(l)]_{\mathcal{R}}) + c\omega_{\mathcal{R}}(\nabla\gamma(l), \nabla\mu(l)) \ , \qquad (2.5)$$

where $l \in \hat{\mathcal{G}}^*$ and $c \in \mathbb{C}$. Owing to the scalar product (2.1) the gradient $\nabla\gamma(l) \in \hat{\mathcal{G}}$ of some functional $\gamma \in D(\hat{\mathcal{G}}_c^*)$ at the point $l \in \hat{\mathcal{G}}^*$ is naturally defined as

$$\delta\gamma(l) := (\nabla\gamma(l), \delta l) \ .$$

Consider the Casimir functionals $\gamma_n \in I(\hat{\mathcal{G}}_c^*)$, $n \in N$, as

$$\gamma_n(l) := \int_0^{2\pi}\int_0^{2\pi} Tr\,(\xi^n \hat{l}_0) dxdy \ , \qquad (2.6)$$

which are invariant with respect to the Ad^*-action of the abstract Lie group $\hat{\mathcal{G}}_c$ corresponding to $\hat{\mathcal{G}}_c^*$ and satisfying the following condition [38]

$$(l - c\partial/\partial y) \circ \Phi = \Phi \circ (l_0 - c\partial/\partial y) \qquad (2.7)$$

at a point $l \in \hat{\mathcal{G}}^*$. In (2.7)

$$\hat{l}_0 := \xi^m + \sum_{j<m} c_j \xi^j \in \hat{\mathcal{G}}^*,$$

where $c_j \in \tilde{\mathcal{G}}$, $[\xi, c_j] = 0$, $j < m$, $j \in \mathbb{Z}$ and $m \in \mathbb{N}$,

$$\Phi = 1 + \sum_{r>0} \Phi_r \xi^{-r} \in \hat{\mathcal{G}}_-,$$

where $\Phi_r \in \tilde{\mathcal{G}}$, $r \in \mathbb{N}$, and $\hat{\mathcal{G}}_-$ means the suitable abstract Lie group [21, 22, 38], generated by the Lie subalgebra $\hat{\mathcal{G}}_-$. Similar to [38], one can show that the condition (2.7) is equivalent to the following relationship

$$[l - c\partial/\partial y, \nabla\gamma_n(l)] = 0 \ , \qquad (2.8)$$

for all $n \in \mathbb{N}$. In the case of $c = 0$ the Casimir functionals take the usual Adler's form [18, 34].

The Lie-Poisson bracket (2.5) generates the hierarchy of Hamiltonian dynamical systems on $\hat{\mathcal{G}}_c^*$ with Casimir funtionals $\gamma_n \in I(\hat{\mathcal{G}}_c^*)$, $n \in N$, as the corresponding Hamiltonian functions, taking the form:

$$d\hat{l}/dt_n := [\mathcal{R}\nabla\gamma_n(l),\ l - c\partial/\partial y] = [(\nabla\gamma_n(l))_+,\ l - c\partial/\partial y] \ . \qquad (2.9)$$

where the lower index "+" signs the differential part of the corresponding integro-differential operator. The latter equation is equivalent to the usual commutator Lax type representation. It is easy to verify that for every $n \in \mathbb{N}$ the relationship above is the compatibility condition of a system of linear integro-differential equations of the form:

$$(l - c\partial/\partial y)f = \lambda f \ , \tag{2.10}$$

and

$$df/dt_n = (\nabla\gamma_n(l))_+ f \ , \tag{2.11}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $f \in W := W(\mathbb{S}\times\mathbb{S}; H)$ and H is a matrix representation space of the Lie algebra $\mathcal{G}$. The dynamical system related to (2.11) on the adjoint function space $W^* := W^*(\mathbb{S}\times\mathbb{S}; H)$ is

$$df^*/dt_n = -(\nabla\gamma_n(l))_+^* f^*, \tag{2.12}$$

where $f^* \in W^*$ is a solution of the adjoint spectral equation

$$(l^* + c\partial/\partial y)f^* = \nu f^*, \tag{2.13}$$

with a spectral parameter $\nu \in \mathbb{C}$.

In the sequel, it shall be further assumed that the spectral equation (2.10) admits $N \in \mathbb{N}$ different eigenvalues $\lambda_i \in \mathbb{C}$, $i = \overline{1,N}$, in which case one studies the algebraic properties of equation (2.9) combined with $N \in \mathbb{N}$ copies of (2.11):

$$df_i/dt_n = (\nabla\gamma_n(\hat{l}))_+ f_i \tag{2.14}$$

for the corresponding eigenfunctions $f_i \in W(\mathbb{S}\times\mathbb{S}; H)$, $i = \overline{1,N}$, and the same number of copies of (2.12):

$$df_i^*/dt_n = -(\nabla\gamma_n(\hat{l}))_+^* f_i^* \ , \tag{2.15}$$

for the suitable adjoint eigenfunctions $f_i^* \in W^*(\mathbb{S}\times\mathbb{S}; H)$ for N different eigenvalues $\nu_i \in \mathbb{C}$, $i = \overline{1,N}$ of (2.13), being considered as a coupled evolution system on the space $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$. The same problem for $c = 0$ and $N = 1$ has been studied in the papers [33, 34].

2.3 The Poisson bracket on the extended phase space

To make the exposition here more concise, the following notation shall be used for the gradient vector:

$$\nabla\gamma(\tilde{l}, \tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*) := (\delta\gamma/\delta\tilde{l},\ \delta\gamma/\delta\tilde{\mathrm{f}},\ \delta\gamma/\delta\tilde{\mathrm{f}}^*)^\top,$$

where $\tilde{\mathrm{f}} := (\tilde{f}_1, \ldots, \tilde{f}_N)$, $\tilde{\mathrm{f}}^* := (\tilde{f}_1^*, \ldots, \tilde{f}_N^*)$ and $\delta\gamma/\delta\tilde{\mathrm{f}} := (\delta\gamma/\delta\tilde{f}_1, \ldots, \delta\gamma/\delta\tilde{f}_N)$, $\delta\gamma/\delta\tilde{\mathrm{f}}^* := (\delta\gamma/\delta f_1^*, \ldots, \delta\gamma/\delta\tilde{f}_N^*)$, at a point $(\tilde{l}, \tilde{\mathrm{f}}, f^*)^\top \in \hat{\mathcal{G}}^* \oplus W^N \oplus W^{*N}$ for any smooth functional $\gamma \in D(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$.

On the spaces $\hat{\mathcal{G}}_c^*$ and $W^N \oplus W^{*N}$ there exist canonical Poisson structures such as

$$\delta\gamma/\delta\tilde{l} :\xrightarrow{\tilde{\theta}} [\tilde{l} - c\partial/\partial y, (\delta\gamma/\delta\tilde{l})_+] - [\tilde{l} - c\partial/\partial y, \delta\gamma/\delta\tilde{l}]_+ \ , \tag{2.16}$$

where $\tilde{\theta} : T^*(\hat{\mathcal{G}}_c^*) \to T(\hat{\mathcal{G}}_c^*)$ is an implectic operator corresponding to (2.5) at a point $\tilde{l} \in \hat{\mathcal{G}}^*$ and

$$(\delta\gamma/\delta\tilde{\mathrm{f}},\, \delta\gamma/\delta\tilde{\mathrm{f}}^*)^\top :\xrightarrow{\tilde{J}} (-\delta\gamma/\delta\tilde{\mathrm{f}}^*,\, \delta\gamma/\delta\tilde{\mathrm{f}})^\top \ , \tag{2.17}$$

where $\tilde{J} : T^*(W^N \oplus W^{*N}) \to T(W^N \oplus W^{*N})$ is an implectic operator corresponding to the symplectic form $\omega^{(2)} = \sum_{i=1}^N d\tilde{f}_i^* \wedge d\tilde{f}_i$ at a point $(\tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*) \in W^N \oplus W^{*N}$. It should be noted here that the Poisson structure (2.16) generates equation (2.9) for any Casimir functional $\gamma \in I(\hat{\mathcal{G}}_c^*)$. Thus, on the extended phase space $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$ one can obtain a Poisson structure as the tensor product $\tilde{\Theta} := \tilde{\theta} \otimes \tilde{J}$ of (2.16) and (2.17).

Consider the following Bäcklund transformation:

$$(\tilde{l}, \tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*)^\top :\xrightarrow{B} (l(\tilde{l}, \tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*),\, \mathrm{f} = \tilde{\mathrm{f}},\, \mathrm{f}^* = \tilde{\mathrm{f}}^*)^\top \ , \tag{2.18}$$

generating on $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$ a Poisson structure $\Theta : T^*(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}) \to T(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$. The main condition imposed on the mapping (2.18) is the coincidence of the resulting dynamical system

$$(dl/dt_n,\, d\mathrm{f}/dt_n,\, d\mathrm{f}^*/dt_n)^\top := -\Theta\, \nabla\overline{\gamma}_n(l, \mathrm{f}, \mathrm{f}^*) \tag{2.19}$$

with equations (2.9), (2.14) and (2.15) in the case when functionals $\overline{\gamma}_n \in I(\hat{\mathcal{G}}_c^*)$, $n \in \mathbb{N}$, are independent of the variables $(f, f^*) \in W^N \oplus W^{*N}$.

To satisfy that condition, a variation of a Casimir functional $\overline{\gamma}_n := \gamma_n|_{l = l(\tilde{l}, \mathrm{f}, \mathrm{f}^*)} \in D(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$, $n \in \mathbb{N}$, will be found, under the constraint $\delta\tilde{l} = 0$, taking into account the evolutions (2.14), (2.15) and the Bäcklund transformation definition (2.18). One has

$$\begin{aligned}
\delta\overline{\gamma}_n(\tilde{l}, \tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*)\Big|_{\delta\tilde{l}=0} &= \sum_{i=1}^N \Big(< \delta\overline{\gamma}_n/\delta\tilde{f}_i, \delta\tilde{f}_i > + < \delta\overline{\gamma}_n/\delta\tilde{f}_i^*, \delta\tilde{f}_i^* > \Big) \\
&= \sum_{i=1}^N \Big(< -d\tilde{f}_i^*/dt_n, \delta\tilde{f}_i > + < d\tilde{f}_i/dt_n, \delta\tilde{f}_i^* > \Big)\Big|_{\tilde{\mathrm{f}}=\mathrm{f},\, \tilde{\mathrm{f}}^*=\mathrm{f}^*} \\
&= \sum_{i=1}^N \big(< (\delta\gamma_n/\delta l)_+^* f_i^*, \delta f_i > + < (\delta\gamma_n/\delta l)_+ f_i, \delta f_i^* > \big) \\
&= \sum_{i=1}^N (< f_i^*, (\delta\gamma_n/\delta l)_+ \delta f_i > + < (\delta\gamma_n/\delta l)_+ f_i, \delta f_i^* >) \\
&= \sum_{i=1}^N \big((\delta\gamma_n/\delta l, (\delta f_i)\xi^{-1} \otimes f_i^*) + (\delta\gamma_n/\delta l, f_i\xi^{-1} \otimes \delta f_i^*) \big) \\
&= \left(\delta\gamma_n/\delta l, \delta \sum_{i=1}^N f_i \xi^{-1} \otimes f_i^* \right) := (\delta\gamma_n/\delta l, \delta l) \ ,
\end{aligned} \tag{2.20}$$

where $\gamma_n \in I(\hat{\mathcal{G}}_c^*)$, $n \in \mathbb{N}$, and the brackets $< .,. >$ denotes a pairing of the spaces W^* and W.

As a result of expression (2.20), one obtains the relationship:

$$\delta l|_{\delta\tilde{l}=0} = \sum_{i=1}^{N} \delta(f_i\xi^{-1} \otimes f_i^*) \ . \tag{2.21}$$

Having assumed the linear dependence of l on $\tilde{l} \in \hat{\mathcal{G}}^*$ one finds right away from (2.21) that

$$l = \tilde{l} + \sum_{i=1}^{N} f_i\xi^{-1} \otimes f_i^*. \tag{2.22}$$

Thus, the Bäcklund transformation (2.18) can be written as

$$(\tilde{l}, \tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*)^\top :\xrightarrow{B} (l = \tilde{l} + \sum_{i=1}^{N} f_i\xi^{-1} \otimes f_i^*, \mathrm{f}, \mathrm{f}^*)^\top. \tag{2.23}$$

Expression (2.23) generalizes results obtained both for the scalar form of the Lie algebra of integro-differential operators in [36] and for the matrix one in [34]. The existence of the Bäcklund transformation (2.23) makes it possible to formulate the following result.

Theorem 2.1. *The dynamical system (2.19) on $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$ is equivalent to the following system of evolution equations:*

$$d\tilde{l}/dt_n = [(\nabla\overline{\gamma}_n(\tilde{l}))_+, \tilde{l}] - [\nabla\overline{\gamma}_n(\tilde{l}), \tilde{l}]_+ \ ,$$
$$d\tilde{\mathrm{f}}/dt_n = \delta\overline{\gamma}_n/\delta\tilde{\mathrm{f}}^* \ , \quad d\tilde{\mathrm{f}}^*/dt_n = -\delta\overline{\gamma}_n/\delta\tilde{\mathrm{f}} \ ,$$

where $\overline{\gamma}_n := \gamma_n|_{l=l(\tilde{l},\mathrm{f},\mathrm{f}^*)} \in D(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$ and $\gamma_n \in I(\hat{\mathcal{G}}_c^*)$ is a Casimir functional at a point $l \in G^*$ for every $n \in N$, under the Bklund transformation (2.23).

Now by means of simple calculations using the formula:

$$\Theta = B'\tilde{\Theta}B'^*,$$

where $B' : T(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}) \to T(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$ is a Fréchet derivative of (2.23), one easily finds the following form of the Bäcklund transformed Poisson structure Θ on $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$:

$$\nabla\gamma(l, \mathrm{f}, \mathrm{f}^*) :\xrightarrow{\Theta} \begin{pmatrix} [l - c\partial/\partial y, (\delta\gamma/\delta l)_+] - [l - c\partial/\partial y, \delta\gamma/\delta l]_+ + \\ \sum_{i=1}^{N}(f_i\xi^{-1} \otimes (\delta\gamma/\delta f_i) - (\delta\gamma/\delta f_i^*)\xi^{-1} \otimes f_i^*) \\ -\delta\gamma/\delta\mathrm{f}^* - (\delta\gamma/\delta l)_+\mathrm{f} \\ \delta\gamma/\delta\mathrm{f} + (\delta\gamma/\delta l)_+^*\mathrm{f}^* \end{pmatrix} , \tag{2.24}$$

where $\gamma \in D(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$ is an arbitrary smooth functional. Hence, one has the following theorem.

Theorem 2.2. *The hierarchy of dynamical systems (2.9), (2.14) and (2.15) is Hamiltonianian with respect to the Poisson structure Θ in the form (2.24) that has the functionals $\overline{\gamma}_n := \gamma_n \in I(\hat{\mathcal{G}}_c^*)$, $n \in N$, as Casimir invariants on $\hat{\mathcal{G}}_c^*$.*

Based on expression (2.19) one can construct a new hierarchy of Hamiltonian evolution equations describing commutative flows generated by Casimir invariants $\gamma_n \in I(\hat{\mathcal{G}}_c^*)$, $n \in \mathbb{N}$, which are in involution with respect to the Poisson bracket (2.5) on the extended phase space $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$.

2.4 The hierarchies of additional symmetries

The hierarchy (2.9), (2.14) and (2.15) of evolution equations possesses another natural set of invariants including all higher powers of the eigenvalues λ_k, $k=\overline{1,N}$. The latter can be considered as Fréchet smooth functionals on the extended phase space $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$ owing to the obvious representation:

$$\lambda_k^s =< f_k^*, (l - c\partial/\partial y)^s f_k > , \tag{2.25}$$

where $s \in N$, holding under the normalizing constraints

$$< f_k^*, f_k >= 1 .$$

In the case of the Bäcklund transformation (2.22), where

$$l := l_+ + \sum_{i=1}^{N} f_i \xi^{-1} \otimes f_i^* \tag{2.26}$$

formula (2.25) gives rise to the following variation of the functionals $\lambda_k^s \in D(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$, $k=\overline{1,N}$:

$$\begin{aligned} \delta\lambda_k^s &=< \delta f_k^*, (l - c\partial/\partial y)^s f_k > \\ &+ < (f_k^*, \delta(l - c\partial/\partial y)^s) f_k > + < f_k^*, (l - c\partial/\partial y)^s (\delta f_k) > \\ = \;& (M_k^s, \delta l_+) + \sum_{i=1}^{N} \left(< (-M_k^s + \delta_k^i (l - c\partial/\partial y)^s)^* f_i^*, \delta f_i >\right) \\ &+ \sum_{i=1}^{N} \left(< (-M_k^s + \delta_k^i (l - c\partial/\partial y)^s) f_i, \delta f_i^* >\right) , \end{aligned}$$

where δ_k^i is the Kronecker symbol and the operators M_k^s, $s \in \mathbb{N}$, are determined as

$$M_k^s := \sum_{p=0}^{s-1} ((l - c\partial/\partial y)^p f_k) \xi^{-1} \otimes ((l^* + c\partial/\partial y)^{s-1-p} f_k^*) .$$

Thus, one obtains the exact forms of gradients for the functionals $\lambda_k^s \in D(\hat{\mathcal{G}}_s^* \oplus W^N \oplus W^{*N})$, $k=\overline{1,N}$:

$$\begin{aligned} \nabla\lambda_k^s(l_+, \mathrm{f}, \mathrm{f}^*) \;&=\; (M_k^s, (-M_k^s + \delta_k^i (l - c\partial/\partial y)^s)^* f_i^*, \\ (-M_k^s + \delta_k^i (l - c\partial/\partial y)^s) f_i \;&:\; \quad i=\overline{1,N})^\top . \end{aligned} \tag{2.27}$$

By means of expressions (2.27), (2.16) and (2.17) one finds a new hierarchy of coupled evolution equations on $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$:

$$dl_+/d\tau_{s,k} = -[M_k^s, l_+ - c\partial/\partial y]_+ , \tag{2.28}$$

$$df_i/d\tau_{s,k} = (-M_k^s + \delta_k^i (l - c\partial/\partial y)^s) f_i , \tag{2.29}$$

$$df_i^*/d\tau_{s,k} = (M_k^s - \delta_k^i (l - c\partial/\partial y)^s)^* f_i^* , \tag{2.30}$$

where $i=\overline{1,N}$ and $\tau_{s,k}\in\mathbb{R}$, $s\in\mathbb{N}$, are evolution parameters. Owing to the Bäcklund transformation (2.26) equation (2.28) can be rewritten in the following commutator form:

$$\begin{aligned} &dl/d\tau_{s,k} = -[M_k^s, l - c\partial/\partial y] \\ =\ & -\lambda_k^p \nu_k^{s-1-p}[M_k^1, l-c\partial/\partial y] = \lambda_k^p\nu_k^{s-1-p} dl/d\tau_{1,k}\ , \end{aligned} \tag{2.31}$$

where $p=\overline{0,s-1}$. Thereby, one arrives at the following theorem.

Theorem 2.3. *For $k=\overline{1,N}$ and $s\in N$, the dynamical systems (2.31), (2.29) and (2.30) are Hamiltonian with respect to the Poisson structure Θ in the form (2.24), and the invariant functionals $\overline{\gamma}_s := \lambda_k^s \in D(\hat{\mathcal{G}}_c^*\oplus W^N \oplus W^{*N})$.*

Proof. To prove the theorem it is sufficient to show that

$$[d/dt_n, d/d\tau_{1,k}] = 0\ ,\quad [d/d\tau_{1,k}, d/d\tau_{1,q}] = 0\ , \tag{2.32}$$

where $k,q=\overline{1,N}$ and $n\in N$. The first equality in formula (2.32) follows from the identities:

$$d(\nabla\gamma_n(l))_+/d\tau_{1,k} = [(\nabla\gamma_n(l))_+, M_1^1]_+\ ,\quad dM_1^1/dt_n = [(\nabla\gamma_n(l))_+, M_1^1]_-\ ,$$

and the second one is a consequence of the equation

$$dM_k^1/d\tau_{1,q} - dM_q^1/d\tau_{1,k} = [M_k^1, M_q^1]$$

finishing the proof. □

Thus, for every $k=\overline{1,N}$ and all $s\in N$ the dynamical systems (2.31), (2.29) and (2.30) on $\hat{\mathcal{G}}_c^*\oplus W^N\oplus W^{*N}$ form a hierarchy of additional homogeneous (or so-called "ghost") symmetries for the Lax type flows (2.9), (2.14) and (2.15) on $\hat{\mathcal{G}}_c^*\oplus W^N\oplus W^{*N}$. The additional symmetry hierarchy for Lax type integrable one-dimensional dynamical systems associated with the Lie algebra $\hat{\mathcal{G}}^*$ of integro-differential operators was first described as an infinitely graded algebra in [28]. It has been used for constructing Lax type integrable two-dimensional dynamical systems in [29].

If $N\geq 2$, one can obtain a new class of nontrivial Hamiltonian flows $d/dT_n := d/dt_n + \sum_{k=1}^{N-1} d/d\tau_{n,k}$, $n\in N$, on $\hat{\mathcal{G}}_c^*\oplus W^N\oplus W^{*N}$ in the Lax type form using the invariants considered above for the centrally extended Lie algebra $\hat{\mathcal{G}}_c^*$ of integro-differential operators. These flows act on the eigenfunctions $(f_i, f_i^*)\in W\oplus W^*$, $i=\overline{1,N}$, and generate some integrable $(N+1)$-dimensional nonlinear dynamical systems.

For example, in the case of the element $l := \partial/\partial x + f_1\xi^{-1}\otimes f_1^* + f_2\xi^{-1}\otimes f_2^* \in \hat{\mathcal{G}}^*$ with $(f_1,f_2,f_1^*,f_2^*)\in W^2(\mathbb{S}\times\mathbb{S};H)\times W^{*2}(\mathbb{S}\times\mathbb{S};H)$, the flows $d/d\tau := d/d\tau_{1,1}$ and $d/dT := d/dT_2 = d/dt_2 + d/d\tau_{2,1}$ on $\hat{\mathcal{G}}_c^*\oplus W^2\oplus W^{*2}$ acting on the functions f_i, f_i^*, $i=\overline{1,2}$, give rise to dynamical systems such as

$$\begin{aligned} &f_{1,\tau} = f_{1,x} - cf_{1,y} + f_2 u\ ,\quad f_{1,\tau}^* = f_{1,x}^* - cf_{1,y}^* + f_2^*\bar{u}\ , \\ &f_{2,\tau} = -f_1\bar{u}\ ,\quad f_{2,\tau}^* = -f_1^* u\ , \end{aligned} \tag{2.33}$$

and

$$
\begin{aligned}
f_{1,T} &= f_{1,xx} + f_{1,\tau\tau} + wf_1 + 2f_1 v_\tau \ , \\
f^*_{1,T} &= -f^*_{1,xx} - f^*_{1,\tau\tau} - wf^*_1 - 2f^*_1 v_\tau \ , \\
f_{2,T} &= f_{2,xx} + wf_2 - f_{1,\tau}\bar{u} + f_1 \bar{u}_\tau \ , \\
f^*_{2,T} &= -f^*_{2,xx} - wf^*_2 + f^*_{1,\tau} u - f^*_1 u_\tau \ , \\
cw_y &= w_x - 2(f_1 \otimes f^*_1 + f_2 \otimes f^*_2)_x \ , \\
u_x &= f_1^T f_2^* \ , \quad \bar{u}_x = f_1^{*T} f_2 \ , \quad v_x = f_1^T f_1^* \ ,
\end{aligned}
\tag{2.34}
$$

where one sets $(\nabla\gamma_2(l))_+ := \partial^2/\partial x^2 + w$ for some function $w \in \tilde{\mathcal{G}}$ depending parametrically on the variables $\tau, T \in \mathbb{R}$. The systems (2.33) and (2.34) represent a Lax type integrable (3+1)-dimensional generalization of the (2+1)-dimensional system equivalent to that of Davey-Stewartson [42, 44] with an infinite sequence of conservation laws, that can be found from (2.6) and have the following form

$$
\gamma_n(l) := tr \int_0^{2\pi} \int_0^{2\pi} (f_1 \partial^{n-1} f_1^*/\partial x^{n-1} + f_2 \partial^{n-1} f_2^*/\partial x^{n-1}) dxdy \ ,
$$

where $n \in N$. Its Lax type linearization is given by the spectral problem (2.10) extended by the set of evolution equations:

$$
f_\tau = -M_1^1 f \ , \tag{2.35}
$$

$$
f_T = ((\nabla\gamma_2(l))_+ - M_1^2) f \ , \tag{2.36}
$$

for an arbitrary eigenfunction $f \in W(\mathbb{S} \times \mathbb{S}; H)$. The relationships (2.35) and (2.36) give rise to the additional nonlinear constraint:

$$
w_\tau = 2(f_1 \otimes f_1^*)_x. \tag{2.37}
$$

When $\dim H = 1$, the Lax type representation (2.10), (2.35) and (2.36) for the above (3+1)-dimensional generalization (2.33), (2.34) and (2.37) of the Davey-Stewartson system [42, 44] has the equivalent matrix form:

$$
\frac{dF}{dx} = \begin{pmatrix} 0 & 0 & f_1^* \\ 0 & 0 & f_2^* \\ -f_1 & -f_2 & \lambda + c\partial/\partial y \end{pmatrix} F \ ,
$$

$$
\frac{dF}{d\tau} = \begin{pmatrix} -(\lambda + c\partial/\partial y) & \bar{u} & f_1^* \\ -u & 0 & 0 \\ -f_1 & 0 & 0 \end{pmatrix} F \ ,
$$

$$
\frac{dF}{dT} = CF \ ,
$$

where $F = (F^1, F^2, F^3 = f)^\top \in W(\mathbb{S} \times \mathbb{S}; \mathbb{C}^3)$, $C := \{C_{mn} \in gl(3;\mathbb{C}) : \ m, n = \overline{1,3}\}$, and

$$
\begin{aligned}
C_{11} &= -(\lambda + c\partial/\partial y)^2 - u\bar{u} - 2f_1 f_1^* \ , \\
C_{12} &= -f_1 f_2^* - (\lambda + c\partial/\partial y)\bar{u} - \bar{u}_\tau \ , \\
C_{13} &= 2((\lambda + c\partial/\partial y)f_1^* - f_{1,x}^*) - \bar{u} f_2^* \ , \\
C_{21} &= -(\lambda + c\partial/\partial y)u - u_\tau - f_1 f_2^* \ , \\
C_{22} &= -f_2 f_2^* + u\bar{u} \ , \\
C_{23} &= (\lambda + c\partial/\partial y)f_2^* - f_{2,x}^* + u f_1^* \ , \\
C_{31} &= -(\lambda + c\partial/\partial y)f_1 - f_{1,x} - f_{1,\tau} \ , \\
C_{32} &= -(\lambda + c\partial/\partial y)f_2 - f_{2,x} + \bar{u} f_1 \ , \\
C_{33} &= (\lambda + c\partial/\partial y)^2 + w - f_2 f_2^*,
\end{aligned}
$$

to which one can effectively apply the standard inverse spectral transform method [25, 30].

The results obtained above can also be used for constructing a wide class of integrable (3+1)-dimensional nonlinear dynamical systems with triple Lax type linearizations [34].

2.5 Remarks

Several regular Lie-algebraic approaches [23, 34, 35, 38, 43] for constructing Lax type integrable multi-dimensional (mainly 2+1) nonlinear dynamical systems on functional manifolds and their supersymmetric generalizations are well known. In this paper, a new method is developed for introducing one more variable into Lax type integrable (2+1)-dimensional dynamical systems arising as flows on dual spaces to the centrally extended matrix Lie-algebra of integro-differential operators. It is based on the recent, naturally constructed hierarchy of additional invariants [1, 36]. The resulting integrable (3+1)-dimensional dynamical systems obtained by means of this method possess infinite sequences of conservation laws and related triple Lax type linearizations. Owing to the latter property, their soliton type solutions can be found by means of either the standard inverse spectral transform method [25, 30] or Darboux-Bäcklund transformations [27, 37, 40].

The structure of the constructed Lie-Bäcklund transformation (2.23), being a key point of the devised method, strongly depends on an *ad*-invariant scalar product chosen for an operator Lie algebra $\hat{\mathcal{G}}$ and on a suitable Lie algebra decomposition (see [20, 1]). Since there exist other possibilities of choosing the corresponding *ad*-invariant scalar products on $\hat{\mathcal{G}}$, such decompositions give rise naturally to other Bäcklund transformations.

A new development of the results obtained via the method described above is in progress for the case of some special centrally extended Lie algebras of super-integro-differential operators [26, 32].

3 Tensor Poisson structures, factorized operator dynamical systems, their Hamiltonian analysis and integrability

3.1 Problem setting

Consider a usual Tr-metrizible associative operator algebra $\mathfrak{g}$ endowed with the standard commutator Lie structure and admitting a decomposition into two Lie subalgebras $\mathfrak{g}_+ \oplus \mathfrak{g}_- = \mathfrak{g}$. Then it follows from the standard [1, 22] Lie-algebraic theory of dynamical systems that one can construct on $\mathfrak{g}^*$ a so-called Lax flow as follows:

$$\frac{dl}{dt} = [\nabla\gamma_+(l), l] \tag{3.1}$$

here $l \in \mathfrak{g}^*$, $\gamma \in \mathrm{I}(\mathfrak{g}^*)$ is a Casimir function on $\mathfrak{g}^*$, that is $[\nabla\gamma(l), l] = 0$, with the associated gradient decomposition $\nabla\gamma(l) := \nabla\gamma_+(l) \oplus \nabla\gamma_-(l)$ for all $l \in \mathfrak{g}^*$, with $t \in \mathbb{R}$ being an evolutions parameter.

The flow (3.1) is Lax - type integrable since all Casimir functions on $\mathfrak{g}^*$ generate invariants of (3.1) commuting with each other in view of the well-known Adler-Kostant-Symes theorem. In general, a Casimir function $\gamma \in \mathrm{I}(\mathfrak{g}^*)$ can be constructed as an analytical functional on $\mathfrak{g}^*$ in the following form:

$$\gamma := Tr\ \gamma[l], \tag{3.2}$$

where, by definition, $Tr\ (ab) := (a, b)$ is the above mentioned ad-invariant non-degenerate symmetric $Tr-$ metrics on $\mathfrak{g} \simeq \mathfrak{g}^*$.

The expression (3.1) clearly defines the Hamiltonian vector field d/dt on $\mathfrak{g}^*$ with respect to the usual Lie-Poisson bracket on $\mathfrak{g}^*$, modified with respect to the Lie-bracket $[.,.]_{\mathcal{R}} := [P_+(.), P_+(.)] - [P_-(.), P_-(.)]$ on $\mathfrak{g}$, where $P_\pm\mathfrak{g} := \mathfrak{g}_\pm$ are the corresponding projectors. Now let us take another element $\tilde{l} \in \mathfrak{g}^*$, and construct the flow d/dt on $\mathfrak{g}^*$:

$$d\tilde{l}/dt = \left[\nabla\tilde{\gamma}_+(\tilde{l}), \tilde{l}\right] \tag{3.3}$$

where it is assumed that $\tilde{\gamma} = \gamma \in \mathrm{I}(\mathfrak{g}^*)$. Thus, one has two integrable flows (3.1) and (3.3) subject to the same Casimir function $\gamma \in \mathrm{I}(\mathfrak{g}^*)$, generating the same vector field d/dt on $\mathfrak{g}^*$. Now we pose the following problem : find the relationships between elements l and $\tilde{l} \in \mathfrak{g}^*$ evolving with respect to flows (3.1) and (3.3) and describe their dual Hamiltonian properties. This problem will be treated in detail below.

3.2 Factorization properties

Due to the Lax form of equations (3.1) and (3.3) there exist one - parametric subgroups $a(t)$ and $\tilde{a}(t) \in \exp\mathfrak{g}_+$, $t \in \mathbb{R}$, such that for any $l(0)$ and $\tilde{l}(0) \in \mathfrak{g}^*$

$$\begin{aligned} l(t) &= Ad^*_{a(t)}l(0) = a^{-1}(t)l(0)a(t), \\ \tilde{l}(t) &= Ad^*_{\tilde{a}(t)}l(0) = \tilde{a}^{-1}(t)\tilde{l}(0)\tilde{a}(t), \end{aligned} \tag{3.4}$$

where it is obviously

$$\begin{aligned} da(t)/dt &= -\, a(t)\nabla\gamma_+(l), \\ d\widetilde{a}(t)/dt &= -\, \widetilde{a}(t)\nabla\gamma_+(\widetilde{l}), \end{aligned} \tag{3.5}$$

for all $t \in \mathbb{R}$. From (3.4) it is seen that for all $t \in \mathbb{R}$

$$\begin{aligned} a(t)l(t)a^{-1}(t) &= l(0), \\ \widetilde{a}(t)\widetilde{l}(t)\widetilde{a}^{-1}(t) &= \widetilde{l}(0). \end{aligned} \tag{3.6}$$

Assume now that there exists an element $B(0) \in \exp \mathfrak{g}_+$ such that the expression

$$Ad^*_{B(0)}l(0) = \widetilde{l}(0)$$

holds, or equivalently

$$B^{-1}(0)l(0)B(0) = \widetilde{l}(0). \tag{3.7}$$

Whence the equalities (3.6) give rise to the following relationships :

$$\widetilde{l} = B^{-1}lB, \tag{3.8}$$

where we assume by definition that $B \in \exp \mathfrak{g}_+$ is given as

$$B := B(t) = a(t)^{-1}B(0)\tilde{a}(t) \tag{3.9}$$

for all $t \in \mathbb{R}$.

Let us now assume that an element $A \in \exp \mathfrak{g}_+$ is defined as

$$A := lB. \tag{3.10}$$

It is evident that this is equivalent to the statement that the expression $A(0) = l(0)B(0) \in \exp \mathfrak{g}_+$ holds for the given element $l(0) \in \mathfrak{g}^*$. As a result of the representations (3.10) and (3.9), one can readily find the following evolution equations on $A, B \in \exp \mathfrak{g}_+$, introduced first in [15]:

$$\begin{aligned} dA/dt &= \nabla\gamma_+(l)A - A\nabla\gamma_+(\widetilde{l}), \\ dB/dt &= \nabla\gamma_+(l)B - B\nabla\gamma_+(\widetilde{l}) \end{aligned} \tag{3.11}$$

for all $t \in \mathbb{R}$. Thereby, we stated the following factorizing flows theorem.

Theorem 3.1. *Let an element $l \in \mathfrak{g}^*$ admits the factorization $l=AB^{-1}$ with A, $B \in \exp \mathfrak{g}_+$. Then the Lax type flows (3.1) and (3.3) are also factorized into two flows (3.11) with the element $\tilde{l}=B^{-1}A = A^{-1}lA \in \mathfrak{g}^*$.*

Proof. A proof is needed only for the last representation $\tilde{l} = B^{-1}A = A^{-1}lA \in \mathfrak{g}^*$. Really, owing to (3.8) $\widetilde{l} = B^{-1}lB$, from (3.10) one immediately finds that $\widetilde{l} = B^{-1}A = I \cdot B^{-1}A \equiv A^{-1}lB \cdot B^{-1}A = A^{-1}lA \in \mathfrak{g}^*$, and this ends the proof. $\square$

Thus, we have constructed two factorized equations (3.11) subject to the representations $l = AB^{-1}$, $\widetilde{l} = B^{-1}A \in \mathfrak{g}^*$ with elements $A, B \in \exp \mathfrak{g}_+$ and the common invariant Casimir function $\gamma(l) = \gamma(\widetilde{l}) \in \mathrm{I}(\mathfrak{g}^*)$. Next we proceed to an analysis of Hamiltonian properties of the flows (3.11) obtained above.

3.3 Hamiltonian analysis

Let us consider the flows (3.1) and (3.2) as being Hamiltonian on $\mathfrak{g}^*\otimes\mathfrak{g}^*$ subject to the following tensor doubled standard Poissonian structure, suggested in [36, 14]:

$$\vartheta : \begin{pmatrix} \nabla\gamma(l) \\ \nabla\gamma(l^*) \end{pmatrix} \longrightarrow \begin{pmatrix} [\nabla\gamma_+(l), l] - [\nabla\gamma(l), l]_+ \\ \left[\nabla\gamma_+(\widetilde{l}), \widetilde{l}\right] - \left[\nabla\gamma(\widetilde{l}), \widetilde{l}\right]_+ \end{pmatrix}, \tag{3.12}$$

where $\gamma \in D(\mathfrak{g}^*)$ is any smooth functional on $\mathfrak{g}^* \otimes \mathfrak{g}^*$. Now employing the transformation

$$\Phi(A, B; l, \widetilde{l}) = 0 \Leftrightarrow l - AB^{-1} = 0, \quad \widetilde{l} - B^{-1}A = 0, \tag{3.13}$$

which obviously may be considered as a standard Bäcklund transformation, we can construct a new Poisson structure $\eta : T^*(\mathrm{G}_+\times\mathrm{G}_+) \longrightarrow T(\mathrm{G}_+\times\mathrm{G}_+)$ on the subgroup space $\mathrm{G}_+\times\mathrm{G}_+$ with respect to the phase variables $(A, B) \in \mathrm{G}_+\times\mathrm{G}_+$. Thereby one finds [13] the corresponding to (3.12) and (3.13) transformed Poissonian structure $\eta : T^*(\mathrm{G}_+\times\mathrm{G}_+) \longrightarrow T(\mathrm{G}_+\times\mathrm{G}_+)$ at $(A, B) \in \mathrm{G}_+\times\mathrm{G}_+$, where

$$\begin{aligned} \eta &= \tau\vartheta\tau^*, \\ \tau &= \Phi'_{(l,\widetilde{l})}\Phi'^{-1}_{(A,B)}. \end{aligned} \tag{3.14}$$

Making use of the expressions

$$\Phi'_{(A,B)} = \begin{pmatrix} -(.)B^{-1} & l(.)B^{-1} \\ -B^{-1}(.) & B^{-1}(.)\widetilde{l} \end{pmatrix}, \quad \Phi'_{(l,\widetilde{l})} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\Phi'^{-1}_{(A,B)} = \begin{pmatrix} -(1 - l\otimes\widetilde{l}^{-1})^{-1}(.)B & (1 - l\otimes\widetilde{l}^{-1})^{-1}lB(.)\widetilde{l}^{-1} \\ -(1 - l\otimes\widetilde{l}^{-1})^{-1}(.)B & (1 - l\otimes\widetilde{l}^{-1})^{-1}B(.) \end{pmatrix},$$

$$(\Phi'^*_{(A,B)})^{-1} = \begin{pmatrix} -B(.)(1 - \widetilde{l}^{-1}\otimes l)^{-1} & -B(.)(1 - \widetilde{l}^{-1}\otimes l)^{-1} \\ \widetilde{l}^{-1}(.)Bl(1 - \widetilde{l}^{-1}\otimes l)^{-1} & (.)B(1 - \widetilde{l}^{-1}\otimes l)^{-1} \end{pmatrix} \tag{3.15}$$

jointly with the ϑ- structure (3.12), one gets from (3.14) that

$$
\begin{array}{rcl}
\eta & = & \begin{pmatrix} -(1-l\otimes\widetilde{l}^{-1})^{-1}(.)B & (1-l\otimes\widetilde{l}^{-1})^{-1}lB(.)\widetilde{l}^{-1} \\ -(1-l\otimes\widetilde{l}^{-1})^{-1}(.)B & (1-l\otimes\widetilde{l}^{-1})^{-1}B(.) \end{pmatrix} \times \\
& & \times \left(\begin{array}{c} \left[l, \left((1-l\otimes\widetilde{l}^{-1})^{-1}(.)B(1-l\otimes\widetilde{l}^{-1})^{-1}(.)\right)_+\right] - \\
\left[\left(\widetilde{l}^{-1}(.)Bl(1-\widetilde{l}^{-1}\otimes l)^{-1}\right)_+, \widetilde{l}\right] - \left[\widetilde{l}^{-1}(.)Bl(1-\widetilde{l}^{-1}\otimes l)^{-1}, \widetilde{l}\right] - \\
- \left[l, (1-l\otimes\widetilde{l}^{-1})^{-1}(.)B(1-l\otimes\widetilde{l}^{-1})^{-1}(.)\right]_+, \\
- \left[\widetilde{l}^{-1}(.)Bl(1-\widetilde{l}^{-1}\otimes l)^{-1}, \widetilde{l}\right] + \left[\left((.)B(1-\widetilde{l}^{-1}\otimes l)^{-1}\right)_+, \widetilde{l}\right], \\
- \left[\left(B(.)(1-\widetilde{l}^{-1}\otimes l)^{-1}\right)_+, l\right] + \left[\left(B(.)(1-\widetilde{l}^{-1}\otimes l)^{-1}\right), l\right]_+ \\
\left[\left((.)B(1-\widetilde{l}^{-1}\otimes l)^{-1}\right)_+, \widetilde{l}\right] - \left[(.)B(1-\widetilde{l}^{-1}\otimes l)^{-1}, \widetilde{l}\right]_+
\end{array} \right)
\end{array}
\tag{3.16}
$$

at $l = AB^{-1}$ and $\widetilde{l} = B^{-1}A \in \mathfrak{g}^*$.

Let now take any functional $\gamma \in \mathrm{I}(\mathfrak{g}^*)$ and construct the functional $\widetilde{\gamma} := \gamma_{l=AB^{-1}} \in D(\mathrm{G}_+ \times \mathrm{G}_+)$. Then one construct due to the Poissonian bracket (3.16) the following Hamiltonian flow on $\mathrm{G}_+ \times \mathrm{G}_+$:

$$\frac{d}{d\tau}(A,B)^{\intercal} = \eta \nabla \widetilde{\gamma}(A,B), \tag{3.17}$$

where $(A,B) \in \mathrm{G}_+ \times \mathrm{G}_+$ and $\tau \in \mathbb{R}$ is an evolution parameter. The flow (3.17) is characterized by the following theorem.

Theorem 3.2. *The Hamiltonian vector field $d/d\tau$ on $\mathrm{G}_+ \times \mathrm{G}_+$ defined by (3.17) and the vector field d/dt defined by (3.11) coincide on $\mathrm{G}_+ \times \mathrm{G}_+$.*

Proof. This theorem can be proved straightforwardly in a simple but tedious manner by calculatiing the expression (1.18). ☐

The result above completely solves a problem posed in [15] about the Hamiltonian formulation of factorized equations (3.11).

3.4 Tensor products of Poisson structures and source like factorized operator dynamical systems

Now let us assume that an operator pair $(A,B) \in G_+ \times G_+$ is transformed as $(A,B) :\to (\tilde{A}, \tilde{B})$, where $\tilde{A} := \tilde{a}^{-1}A\tilde{a}$, $\tilde{B} := \tilde{b}^{-1}B\tilde{b}$ for some $\tilde{a}, \tilde{b} \in G_+$, depending on a scalar function $J \in C^\infty(\mathbb{R};\mathbb{R})$. Take a suitable Poisson structure $\vartheta_{\tilde{J}}$ on a functional manifold $\tilde{M} \ni \tilde{J}$, and construct the following tensor Poisson structure on $\tilde{M} \times \tilde{G}_+ \times \tilde{G}_+$:

$$(\vartheta_{\tilde{J}} \otimes \vartheta_{(\tilde{A},\tilde{B})} : T^*(\tilde{M} \times \tilde{G}_+ \times \tilde{G}_+) \to T(\tilde{M} \times \tilde{G}_+ \times \tilde{G}_+). \tag{3.18}$$

Since the extended mapping

$$(\tilde{J};\tilde{A},\tilde{B}) \xrightarrow{\mathcal{B}} (J = J; A = \tilde{a}\tilde{A}\tilde{a}^{-1}, B = \tilde{b}\tilde{B}\tilde{b}^{-1} \tag{3.19}$$

is assumed to be considered as a Bäcklund transformation of the corresponding Poisson structures, we can easily calculate the resulting Poisson structure on the manifold $(M; G_+ \times G_+) \ni (J; A, B)$. Concerning the construction of Lax type integrable flows related with this mapping, it suffices to take a Casimir functional $\tilde{\gamma} := \gamma(\tilde{l})$, where $\tilde{l} := \tilde{A}\tilde{B}^{-1} \in \mathcal{G}^*$ and to find the corresponding source like flow on the manifold $(M; G_+ \times G_+)$, having taken into account that $\tilde{l} := \tilde{A}\tilde{B}^{-1} = \tilde{a}^{-1}A\tilde{a}\tilde{b}^{-1}B\tilde{b} \in \mathcal{G}^*$:

$$\frac{d}{dt}(J; A, B)^{\intercal} = -\vartheta_{(J;A,B)}\nabla\tilde{\gamma}(J; A, B), \tag{3.20}$$

where, by definition, $\mathcal{B}_*\vartheta_{(J;A,B)} := \vartheta_{(\tilde{J})} \otimes \vartheta_{(\tilde{A},\tilde{B})}$. On the other hand, the flow (3.20) is globally equivalent to the Lax type flow $d\tilde{l}/dt = [\nabla\gamma(\tilde{l}), \tilde{l}]$ on the space $\mathcal{G}^* \ni \tilde{l}$, which immediately shows that our flow possesses an infinite hierarchy of commuting conserved quantities, thereby guaranteeing that, with some additional conditions, its complete integrability by quadratures. The theoretical scheme presented above provides a complete explanation of a series of purely computational results announced in [16]. Other applications of the developed approach, being of interest in hydrodynamics and plasma physics, have recently been found and shall be presented elsewhere.

3.5 Remarks

We have presented in this work a purely Lie-algebraic solution to the problem concerning factorization of operator dynamical systems posed by Dickey [15]. Our solution employs only the standard properties of tensor-multiplied Poisson structures and some specially constructed [13, 14] Bäcklund type transformations. The approach presented in the work appears to be effective for many applications of factorized operator dynamical systems in diverse fields of mathematical physics, in particular in quantum computing mathematics [17, 2, 3], mathematical genetics and other applied fields.

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