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# An Improved Collision Probability Method for Thermal-Neutron-Flux Calculation in a Cylindrical Reactor Cell

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An improved collision probability method for thermal-neutron-flux calculation in a cylindrical reactor cell has been developed. Expanding the neutron flux and source into a series of even powers of the radius, one gets a convenient method for integration of the one-energy group integral transport equation.

It is shown that it is possible to perform an analytical integration in the $x$-$y$ plane in one variable and to use the effective Gaussian integration over another one. Choosing a convenient distribution of space points in fuel and moderator the transport matrix calculation and cell reaction rate integration were condensed.

On the basis of the proposed method, the computer program DISKRET for the ZUSE-Z 23 K computer has been written. The suitability of the proposed method for the calculation of the thermal-neutron-flux distribution in a reactor cell can be seen from the test results obtained. Compared with the other collision probability methods, the proposed treatment excels with a mathematical simplicity and a faster convergence.

## I. INTRODUCTION

The collision probability method for the determination of the space neutron-flux distribution in a reactor cell has become more and more attractive in recent years. Certain improvements of this method have been achieved due to the intensive application (more rigorous setting of the problem—discrete space presentation, the improvement and speeding up of the numerical procedures—an application of the Gauss numerical quadrature formula, etc.).

In this paper, by expanding the neutron flux and source into a series of even powers of the radius, a suitable method for the numerical integration of the one-group integral transport equation in a cylindrical geometry is obtained. It is shown that in the $x$-$y$ plane it is possible to perform analytical integration in one variable and to apply successfully the Gauss quadrature formula in the other one. Some difficulties in connection with the diagonal elements of the transport matrix, indicated elsewhere,[1,2,3] are completely avoided. In comparison with the similar works,[4,5] this one presents more general and mathematically simplified treatment for fast and accurate solution of the neutron transport problems in the multiregion cylindrical reactor cell.

A suitable choice of the space points reduces significantly the computing time of the transport matrix and speeds up the convergence of the

[1]K. KOBAYASHI and H. NISHIHARA, *J. Nucl. Energy*, **18**, 513 (1964).

[2]I. CARLVIK, "Integral Transport Theory in One-Dimensional Geometries," AE-227, AB Atomenergi, Stockholm (1966).

[3]I. CARLVIK, "A Method for Calculating Collision Probabilities in General Cylindrical Geometry with Applications to Flux Distributions and Dancoff Factors," *U.N. Intern. Conf. Peaceful Uses At. Energy, Geneva*, **2**, 225 (1965).

[4]H. KUSTERS and K. OTT, *Nukleonik*, **4**, 91 (1962).

[5]P. H. KIER, *Nucl. Sci. Eng.*, **26**, 230 (1966).

computation of the neutron absorption in the cell material regions.

## II. SOLUTION OF THE INTEGRAL TRANSPORT EQUATION

The integral transport equation in the space of several different material regions takes the form[6]:

$$\Sigma_{t,g}\, dV_i \phi_g(\bar{r}_i) = \sum_{k=1}^{n} \int_{V_k} |S_k(\bar{r}_j) + \Sigma_{s,k}\phi_k(\bar{r}_j)| \times P_{i,j}(\bar{r}_i, \bar{r}_j)\, dV_j \quad , \tag{1}$$

where $n$ denotes the number of material regions, and

$\Sigma_{t,g}$ = total cross section for the neutron in material region $g$

$\phi_g(\bar{r}_i)$ = neutron flux at the space point $\bar{r}_i$ situated in material region $g$

$dV_i$ = elementary volume at the point $\bar{r}_i$

$\Sigma_{s,k}$ = scattering cross section in the material region $k$

$\phi_k(\bar{r}_j)$ = neutron flux at the space point $\bar{r}_j$ situated in material region $k$

$S_k(\bar{r}_j)$ = neutron source at the point $\bar{r}_j$ situated in material region $k$

$P_{i,j}(\bar{r}_i, \bar{r}_j)$ = probability for a neutron starting at the space point $\bar{r}_j$ colliding at the space point $\bar{r}_i$

$dV_j$ = elementary volume at the point $\bar{r}_j$.

In cylindrically symmetric geometry, as we shall show in the following, it is convenient to expand the integrand functions $S_k(\bar{r})$ and $\phi_k(\bar{r})$ in a series:

$$S_k(\bar{r}) = \sum_{p=0}^{m_k-1} S_{k,2p} r^{2p} \quad ; \quad \phi_k(\bar{r}) = \sum_{p=0}^{m_k-1} A_{k,2p} r^{2p} \quad . \tag{2}$$

By introducing the expansion, Eq. (2), into Eq. (1) one gets

$$\Sigma_{t,g}\, dV_i \phi_g(\bar{r}_i) = \sum_{k=1}^{n} \sum_{p=0}^{m_k-1} (S_{k,2p} + \Sigma_{s,k} A_{k,2p}) \int_{V_k} r^{2p} P_{i,j}(\bar{r}_i, \bar{r}_j) dV_j \quad . \tag{3}$$

Transforming the variables from the polar into the Cartesian system using

$$r^{2p} = \sum_{l=0}^{p} \binom{p}{l} y^{2(p-l)} x^{2l} \tag{4}$$

and

$$P_{i,j}(\bar{r}_i, \bar{r}_j) dV_j = \frac{\Sigma_{t,g}\, dr_i}{\sqrt{1 - \left(\frac{y}{r_i}\right)^2}} Ki_1(\tau_{i,j})\, dx dy \tag{5}$$

(where $\tau_{i,j}$ denotes the optical length between the points $\bar{r}_i$ and $\bar{r}_j$ in the $x$ direction), we get a suitable way to compute the integrals on the right-hand side of Eq. (3).

Considering only the neutrons moving parallel to the $x$-axis, by a symmetry argument one gets the total neutron flux at the circle of radius $r_i$. Then, $dV_i = 2\pi r_i\, dr_i$ and from the Eqs. (3), (4), and (5) it follows that

$$\phi_g(r_i) = \sum_{k=1}^{n} \sum_{p=0}^{m_k-1} (S_{k,2p} + \Sigma_{s,k} A_{k,2p}) \sum_{l=0}^{p} \binom{p}{l} \frac{1}{\pi} \times \int_0^{\min(r_i, r_j)} \frac{y^{2(p-l)}}{\sqrt{1 - \left(\frac{y}{r_i}\right)^2}}\, d\left(\frac{y}{r_i}\right) \times \sum_{q=1}^{2} \int_{x_k}^{x_{k+1}} Ki_1 [\Sigma_k(x_j - x_k) + \tau_{i,k}^q] x_j^{2l} dx_j \tag{6}$$

(the geometrical and optical lengths are defined in Fig. 1). Due to the symmetry with respect to the $x$-axis, the integration in the $y$ direction is performed only in the upper half-plane. A neutron starting at point $\bar{r}_j$ and moving parallel to the $x$-axis has a probability of twice crossing the circle with the radius $r_i$. Accordingly, the integral in $x$ splits in two parts.

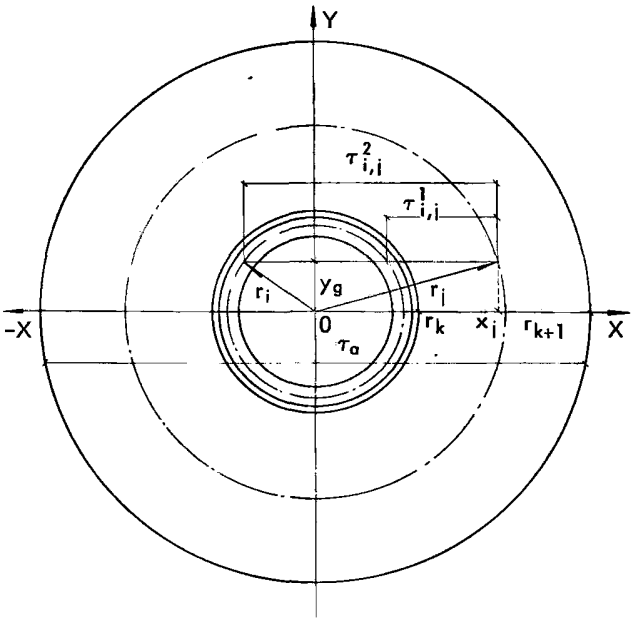


Fig. 1. Geometrical ($x$) and optical ($\tau$) lengths of the integration scheme in $X$ - $Y$ plane.

[6] B. DAVISON, *Neutron Transport Theory*, Oxford University Press, London (1958).

Now, it is possible to perform an analytical integration over the $x$ variable. By changing variables

$$\frac{y}{r_i} = \sin\left[M_{i,e} \cdot z + \arcsin\left(\frac{r_e}{r_i}\right)\right] , \qquad (7)$$

where

$$M_{i,e} = \arcsin\left(\frac{r_{e+1}}{r_i}\right) - \arcsin\left(\frac{r_e}{r_i}\right)$$

[$r_e$ and $r_{e+1}$ are the inner and outer radii of the material region situated inside the circle with the radius min $(r_i, r_j)$], the integral over $y$ has a form which is convenient for the effective use of the Gauss quadrature formula:

$$\phi_g(r_i) = \sum_{k=1}^{n} \sum_{p=0}^{m_k-1} (S_{k,2p} + \Sigma_{s,k} A_{k,2p}) \sum_{l=0}^{p} \binom{p}{l} \times \sum_{e=1}^{f} \frac{M_{i,e}}{\pi} \int_0^1 f(z) y^{2(p-l)} dz \quad , \quad .... \qquad (8)$$

where $f$ is the number of material regions with the outer radius less or equal to min$(r_i, r_j)$, and

$$f(z) = \sum_{t=0}^{2l} \frac{(2l)!}{(2l-t)!} \cdot \frac{1}{\Sigma_e^{t+1}} \sum_{q=1}^{2} [Ki_{2+t}(\tau_{i,e}^q) x_e^{2l-t} - Ki_{2+t}(\tau_{i,e+1}^q) x_{e+1}^{2l-t}] \quad .$$

## III. THE BOUNDARY CONDITION

We used the white boundary condition which is defined in the following way[2]:

Consider neutrons that cross the boundary from the inside and leave the system. A certain fraction $\alpha$ (the albedo of the boundary) return again through the boundary. The returning neutrons have an isotropic distribution, that is they have an angular distribution as if they came directly from a flat source in all space outside the system.

Leakage from the system could be expressed in the form similar to Eq. (8):

$$U(r_a) = \sum_{k=1}^{n} \sum_{p=0}^{m_k-1} (S_{k,2p} + \Sigma_{s,k} \cdot A_{k,2p}) \sum_{l=0}^{p} 2\binom{p}{l} \times \sum_{e=1}^{f} M_{a,e} \cdot \int_0^1 r_a \left[1 - \left(\frac{y}{r_a}\right)^2\right]^{1/2} f^*(z) y^{2(p-l)} dz , \qquad (9)$$

where $f^*(z)$ has the same form as $f(z)$, changing formally the symbol $Ki_{2+t}$ into $Ki_{3+t}$, and $r_a$ is the radius of the system.

The part of the leakage neutrons which contributes to the flux at the circle with the radius $r_i$ can be expressed in the form[2]:

$$\Delta\phi_g(r_i) = \frac{\alpha}{1 - \alpha(1 - P_s)} V(r_i) U(r_a) \quad , \qquad (10)$$

where

$\alpha$ = the albedo of the boundary

$P_s$ = the stickling probability of the system.

$$P_s = \frac{2}{\pi r_a} \sum_{k=1}^{n} 2M_{a,k} \int_0^1 Ki_3(\tau_a) r_a \left[1 - \left(\frac{y}{r_a}\right)^2\right]^{1/2} dz , \qquad (11)$$

where $\tau_a$ is the optical chord length (Fig. 1).

$V(r_i)$ is a contribution to the flux at the circle with the radius $r_j$ from the neutrons uniformly distributed in the space outside the system:

$$V(r_i) = \frac{2}{\pi r_a} \sum_{k=1}^{n} \frac{M_{a,k}}{\pi} \int_0^1 [Ki_2(\tau_{i,a}^1) + Ki_2(\tau_{i,a}^2)] dz . \qquad (12)$$

Now, the total flux at the circle with the radius $r_i$ taking the white boundary condition is

$$\phi_g^{\text{white}}(r_i) = \phi_g(r_i) + \Delta\phi_g(r_i) . \qquad (13)$$

The coefficients $A_{k,2p}$, ($k = 1, 2, ..., n$; $p = 0, 1, ..., m_k - 1$) are the solutions of the system expressed by Eq. (13). The coefficients $S_{k,2p}$ are assumed to be known.

The fluxes at predetermined space points of the system are computed using Eq. (2).

## IV. OPTIMAL DISTRIBUTION OF SPACE POINTS IN A REACTOR CELL

Since the balance equations are set at the predetermined space points, the obtained coefficients $A_{k,2p}$ define a series expansion which is the best approximation of the flux around the given points. The determination of the integral parameters weighted by the flux can be done in two ways. Firstly, one can perform the analytic integration of the obtained series expansion and secondly, we can integrate numerically with the discrete values of the flux at the predetermined space points. Due to the fact that nothing can be said about the accuracy of the interpolated flux values, the analytical integration could give a wrong answer. The numerical integration uses only the flux values at the predetermined space points, while interpolated and/or extrapolated flux values influence the result only indirectly in the process of setting the balance equations. So we used as more accurate the numerical integration.

The optimal distribution of space points in a reactor cell is sought according to the minimum computing time for the determination of the reaction rates in the material regions of the reactor cell.

Due to the fact that the computation of the transport matrix is most time consuming, the boundary points $r = 0$ and $r = r_a$ can be fixed.

In the determination of the matrix elements for the point $r = 0$ the one-dimensional integration is necessary, while for all others the two-dimensional one is needed. In the computation of the matrix elements for the space point $r = r_a$ one gets simultaneously the necessary data for the boundary condition.

In the central material region of the reactor cell, the function: $F(r) = \phi(r) - \phi(0)$, behaves as $r^2$ in the vicinity of $r = 0$. This function can be effectively integrated using the Gaussian quadrature formula with the Jacobean weighting function[7]:

$$\int_0^1 x^3 f(x)\,dx = \sum_{i=1}^{n} W_i f(x_i)\,dx \quad . \tag{14}$$

The reaction rate is given now as a linear combination of the flux $\phi(0)$ and the mentioned integral.

The Gaussian quadrature formula with a fixed point[7] has been used in the computation of the reaction rates in the boundary material region.

In the other material regions the use of the

[7] V. I. KRILOV and L. T. ŠULJGINA, *Handbook for a Numerical Quadrature*, Moscow (1966) (in Russian).

TABLE I

Data for Annular Cells

| Region nr | Heavy Water Cell | | | Light Water Cell | |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 1 | 2 |
| Material | Nat. U + can + coolant ($D_2O$) | Shroud | Moderator $D_2O$ | Enriched U | Moderator $H_2O$ |
| Outer radius (cm) | 7.2 | 7.35 | 13.5631 | 0.381 | 0.644869 |
| $\Sigma_{tot}(cm^{-1})$ | 0.418940 | 0.095709 | 0.401884 | 0.780 | 1.0618 |
| $\Sigma_s(cm^{-1})$ | 0.366520 | 0.082500 | 0.401817 | 0.387 | 1.053 |
| Source strength ($cm^{-3}$ $sec^{-1}$) | 0.67695 | 0 | 1.0 | 0.0 | 1.0 |

TABLE II

Rate of Convergence of DIT, CP, and DISKRET Methods (See Ref. 9)

| Flux Form Factors in a Heavy Water Cell | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Number of Points | | | | $\bar{\phi}$ cladding/$\bar{\phi}$ fuel | | | $\bar{\phi}$ moderator/$\bar{\phi}$ fuel | | |
| Total | Fuel | Shroud | Moderator | DIT | CP | DISKRET | DIT | CP | DISKRET |
| 5 | 2 | 1 | 2 | 1.32626 | - | 1.34382 | 1.80053 | - | 1.68544 |
| 7 | 3 | 1 | 3 | 1.32936 | 1.29596 | 1.33673 | 1.75907 | 1.6435 | 1.72602 |
| 9 | 4 | 1 | 4 | 1.33345 | 1.30986 | 1.33678 | 1.74949 | 1.67765 | 1.73344 |
| 11 | 5 | 1 | 5 | 1.33556 | 1.31796 | 1.33688 | 1.74544 | 1.69691 | 1.73657 |
| 13 | 6 | 1 | 6 | 1.33656 | 1.32340 | 1.33716 | 1.74339 | 1.70786 | 1.73794 |
| 30 | 14 | 2 | 14 | 1.33768 | 1.33419 | - | 1.74042 | 1.73359 | - |

| Flux Form Factors in a Light Water Cell | | | | | |
|---|---|---|---|---|---|
| Number of Points | | | $\bar{\phi}$ moderator/$\bar{\phi}$ fuel | | |
| Total | Fuel | Moderator | DIT | CP | DISKRET |
| 4 | 2 | 2 | 1.14625 | 1.13929 | 1.13805 |
| 6 | 3 | 3 | 1.14262 | 1.14000 | 1.13953 |
| 8 | 4 | 4 | 1.14152 | 1.14028 | 1.14018 |
| 10 | 5 | 5 | 1.14110 | 1.14038 | 1.14044 |
| 12 | 6 | 6 | 1.14093 | 1.14048 | 1.14055 |
| 20 | 10 | 10 | 1.14075 | 1.14062 | - |

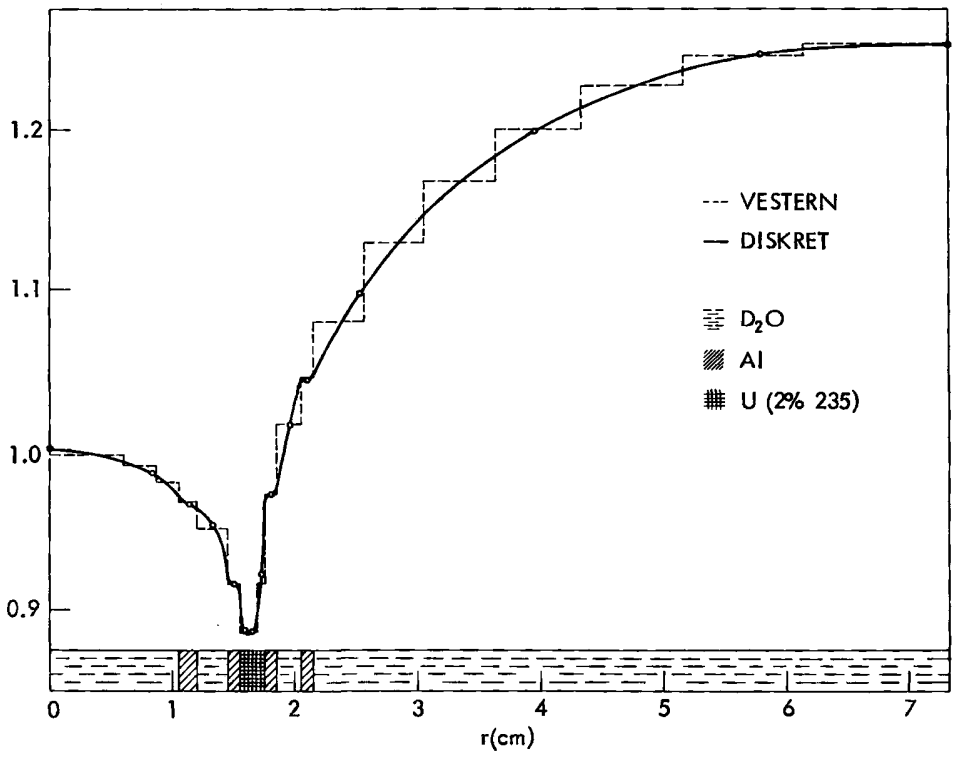


Fig. 2. Thermal-neutron-flux distribution in RA reactor cell.

standard Gaussian quadrature formula has been made.

The central and boundary material regions are usually occupied by the fuel and the moderator, respectively, so the mentioned special choice of the space points has a great influence on the convergence of the numerical procedure.

## V. PROGRAM "DISKRET" AND RESULTS OF NUMERICAL TEST

On the basis of the proposed method and making use of the rational approximations for the Bickley functions,[8] the DISKRET program for the ZUSE-Z 23 K computer has been written.

In the numerical test we treated the cases of the heavy-water and light-water cylindrical reactor cells which are specified in Table I.[9]

The speed of the convergence of the programs DIT[2,9] and DISKRET is compared in Table II. Under the heading "Number of Points" are given the numbers of predetermined space points, "total" denotes the number of predetermined space points for the system as a whole, and the rest of the symbols denote material regions. In this case the number of predetermined space points is equal to the number of the terms in the series expansion. The values in columns DIT and CP are taken from the literature.[9] The advantages of the program DISKRET in comparison with the similar program DIT are obvious.

An example of the thermal-neutron-flux distribution in the cell of the RA reactor in Vinča-Belgrade is given in Fig. 2. The solid line represents the results obtained using the program DISKRET, while the dashed line represents the results obtained using the program VESTERN.[10] The computer program VESTERN, developed for the ZUSE-Z 23K computer, is a standard routine for the neutron flux determination in a cylindrical reactor cell based on the collision probability method and a numerical procedure published earlier.[3] From Fig. 2 the convergence of the VESTERN results to those obtained using the program DISKRET is obvious.

The main advantage of the program is that the integration in one variable is performed analytically. The fast and accurate computation of Bickley's functions of higher order is of the utmost importance. Numerical integration in the other variable is also very efficient using the relatively small number of the integration points.

[8]K. MAKINO, *Nukleonik*, **7**, 351 (1967).

[9]I. CARLVIK, "Calculations of Neutron Flux Distributions by Means of Integral Transport Methods," AE-179, AB Atomenergi, Stockholm (1967).

[10]T. BOŠEVSKI, "First-Flight Collision Probability Program 'VESTERN'," IBK-635 Beograd (1968), Internal Publication.