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THE ELECTROMAGNETIC DIRAC-FOCK-PODOLSKY PROBLEM AND SYMPLECTIC PROPERTIES OF THE MAXWELL AND YANG-MILLS TYPE DYNAMICAL SYSTEMS

N.N. Bogolubov (Jr.)[1]
V.A. Steklov Mathematical Institute of RAS, Moscow, Russian Federation
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy,

A.K. Prykarpatsky[2]
Department of Applied Mathematics, The AGH University of Science and Technology, Krakow 30059, Poland
and
The Ivan Franko Pedagogical State University, Drohobych, Lviv Region, Ukraine,

U. Taneri[3]
Department of Applied Mathematics and Computer Science, Eastern Mediterranean University EMU, Famagusta, North Cyprus
and
Kyrenia American University GAU, Institute of Graduate Studies, Kyrenia, North Cyprus

and

Y.A. Prykarpatsky[4]
Institute of Mathematics at NAS, Kyiv, Ukraine,
The Ivan Franko Pedagogical State University, Drohobych, Lviv Region, Ukraine
and
The Pedagogical University, Krakow, Poland.

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[1]nikolai_bogolubov@hotmail.com
[2]pryk.anat@ua.fm, prykanat@cybergal.com
[3]ufuk.taneri@gmail.com
[4]yarpry@gmail.com

Abstract

Based on analysis of reduced geometric structures on fibered manifolds, invariant under action of a certain symmetry group, we construct the symplectic structures associated with connection forms on suitable principal fiber bundles. The application to the non-standard Hamiltonian analysis of the Maxwell and Yang-Mills type dynamical systems is presented. A symplectic reduction theory of the classical Maxwell electromagnetic field equations is formulated, the important Lorentz condition, ensuring the existence of electromagnetic waves [5, 6], is naturally included into the Hamiltonian picture, thereby solving the well known Dirac, Fock and Podolsky problem [10]. The symplectically reduced Poissonian structures and the related classical minimal interaction principle, concerning the Yang-Mills type equations, are considered.

1. Introduction

When investigating different dynamical systems on canonical symplectic manifolds, invariant under action of certain symmetry groups, additional mathematical structures often appear, the analysis of which shows their importance for understanding many related problems under study. Amongst them we here mention the Cartan type connection on an associated principal fiber bundle, which enables one to study in more detail the properties of the investigated dynamical system in the case of its reduction upon the corresponding invariant submanifolds and quotient spaces, associated with them.

Problems related to the investigation of properties of reduced dynamical systems on symplectic manifolds were studied, e.g., in [1, 15, 14, 23, 22], where the relationship between a symplectic structure on the reduced space and the available connection on a principal fiber bundle was formulated in explicit form. Other aspects of dynamical systems related to properties of reduced symplectic structures were studied in [16, 17, 18], where, in particular, the reduced symplectic structure was explicitly described within the framework of the classical Dirac scheme, and several applications to nonlinear (including celestial) dynamics were given.

It is well known [5, 3, 9, 12, 13, 11] that the Hamiltonian theory of electromagnetic Maxwell equations faces a very important classical problem of introducing into the unique formalism the well-known Lorentz conditions, ensuring both the wave structure of propagating quanta and the positivity of energy. Regretfully, in spite of classical studies on this problem given by Dirac, Fock and Podolsky [10], the problem remains open, and the Lorentz condition is imposed within the modern electrodynamics as the external constraint not entering a priori the initial Hamiltonian (or Lagrangian) theory. Moreover, when trying to quantize the electromagnetic theory, as it was shown by Pauli, Dirac, Bogolubov and Shirkov and others [5, 11, 9, 6], within the existing approaches the quantum Lorentz condition could not be satisfied, except in the average sense, since it becomes not compatible with the related quantum dynamics. This problem stimulated us to study this problem from the so-called symplectic reduction theory, which allows the systematic introduction into the Hamiltonian formalism with the external charge and current conditions as well as giving rise to a solution to the Lorentz condition problem mentioned above. Some applications of the method to Yang-Mills type equations, interacting with a point charged particle, are presented. In particular, based on analysis of reduced geometric structures on fibered manifolds, invariant under the action of a symmetry group, we construct the symplectic structures associated with connection forms on suitable principal fiber bundles. We present suitable mathematical preliminaries of the related Poissonian structures on the corresponding reduced symplectic manifolds, which are often used [1, 21, 20] in various problems of dynamics in modern mathematical physics, and apply them to study the non-standard Hamiltonian properties of the Maxwell and Yang-Mills type dynamical systems. We formulate a symplectic analysis of the important Lorentz type constraints, which describe the electrodynamic vacuum properties.

We formulate a symplectic reduction theory of the classical Maxwell electromagnetic field equations and prove that the important Lorentz condition, ensuring the existence of electromagnetic waves [5, 6, 7, 8], can be naturally included into the Hamiltonian picture, thereby solving the Dirac, Fock and Podolsky problem [10] mentioned above. We also study from the symplectic reduction theory the Poissonian structures and the classical minimal interaction principle related with Yang-Mills type equations.

2. Symplectic structures and reduction on manifolds: preliminaries

2.1. **The symplectic reduction on cotangent fiber bundles with symmetry.** Consider an n-dimensional smooth manifold M and the cotangent vector fiber bundle $T^*(M)$. We equip (see [2], Chapter VII) the cotangent space $T^*(M)$ with the canonical Liouville 1-form $\lambda(\alpha^{(1)}) := pr_M^*\alpha^{(1)} \in \Lambda^1(T^*(M))$, where $pr_M : T^*(M) \to M$ is the canonical projection and, by definition,

$$\alpha^{(1)}(u) = \sum_{j=1}^{n} v_j du^j, \tag{2.1}$$

where $(u, v) \in T^*(M)$ are the corresponding canonical local coordinates on $T^*(M)$. Thus, any group of diffeomorphisms of the manifold M naturally lifted to the fiber bundle $T^*(M)$ preserves the invariance of the canonical 1-form $\lambda(\alpha^{(1)}) \in \Lambda^1(T^*(M))$. In particular, if a smooth action of a Lie group G is given on the manifold M, then every element $a \in \mathcal{G}$, where $\mathcal{G}$ is the Lie algebra of the Lie group G, generates the vector field $k_a \in T(M)$ in a natural manner. Furthermore, since the group action on M, i.e.,

$$\varphi : G \times M \to M, \tag{2.2}$$

generates a diffeomorphism $\varphi_g \in Diff\ M$ for every element $g \in G$, this diffeomorphism is naturally lifted to the corresponding diffeomorphism $\varphi_g^* \in Diff\ T^*(M)$ of the cotangent fiber bundle $T^*(M)$, which also leaves the canonical 1-form $pr_M^*\alpha^{(1)} \in \Lambda^1(T^*(M))$ invariant. Namely, the equality

$$\varphi_g^*\lambda(\alpha^{(1)}) = \lambda(\alpha^{(1)}) \tag{2.3}$$

holds [1, 2, 15] for every 1-form $\alpha^{(1)} \in \Lambda^1(M)$. Thus, we can define on $T^*(M)$ the corresponding vector field $K_a : T^*(M) \to T(T^*(M))$ for every element $a \in \mathcal{G}$. Then condition (2.3) can be rewritten in the following form for all $a \in \mathcal{G}$:

$$L_{K_a} \cdot pr_M^*\alpha^{(1)} = pr_M^* \cdot L_{k_a}\alpha^{(1)} = 0,$$

where L_{K_a} and L_{k_a} are the ordinary Lie derivatives on $\Lambda^1(T^*(M))$ and $\Lambda^1(M)$, respectively.

The canonical symplectic structure on $T^*(M)$ is defined as

$$\omega^{(2)} := d\lambda(\alpha^{(1)}) \tag{2.4}$$

and is invariant, i.e., $L_{K_a}\omega^{(2)} = 0$ for all $a \in \mathcal{G}$.

For any smooth function $H \in D(T^*(M))$, a Hamiltonian vector field $K_H : T^*(M) \to T(T^*(M))$ such that

$$i_{K_H}\omega^{(2)} = -dH \tag{2.5}$$

is defined, and vice versa, because the symplectic 2-form (2.4) is non-degenerate. Using (2.5) and (2.4), we easily establish that the Hamiltonian function $H := H_K \in D(T^*(M))$ is given by the expression $H_K = pr_M^*\alpha^{(1)}(K_H) = \alpha^{(1)}(pr_M^* K_H) = \alpha^{(1)}(k_H)$, where $k_H \in T(M)$ is the corresponding vector field on the manifold M, whose lifting to the fiber bundle $T^*(M)$ coincides with the vector field $K_H : T^*(M) \to T(T^*(M))$. For $K_a : T^*(M) \to T(T^*(M))$, where $a \in \mathcal{G}$, it is easy to establish that the corresponding Hamiltonian function $H_a = \alpha^{(1)}(k_a) = pr_M^* \ \alpha^{(1)}(K_a)$ for $a \in \mathcal{G}$ defines [1, 15, 14] a linear momentum mapping $l : T^*(M) \to \mathcal{G}^*$ according to the rule

$$H_a :=< l, a >, \tag{2.6}$$

where $< \cdot, \cdot >$ is the corresponding convolution on $\mathcal{G}^* \times \mathcal{G}$. By virtue of definition (2.6), the momentum mapping $l : T^*(M) \to \mathcal{G}^*$ is invariant under the action of any invariant Hamiltonian vector field $K_b : T^*(M) \to T(T^*(M))$ for any $b \in \mathcal{G}$. Indeed, $L_{K_b} < l, a >= L_{K_b} H_a = -L_{K_a} H_b = 0$, because, by definition, the Hamiltonian function $H_b \in D(T^*(M))$ is invariant under the action of any vector field $K_a : T^*(M) \to T(T^*(M))$, $a \in \mathcal{G}$.

We now fix a regular value of the momentum mapping $l(u,v) = \xi \in \mathcal{G}^*$ and consider the corresponding submanifold $\mathcal{M}_\xi := \{(u,v) \in T^*(M) : l(u,v) = \xi \in \mathcal{G}^*\}$. On the basis of definition (2.1) and the invariance of the 1-form $pr_M^* \ \alpha^{(1)} \in \Lambda^1(T^*(M))$ under the action of the Lie group G on $T^*(M)$, we can write the equalities

$$\begin{aligned} &< \ l(g \circ (u,v)), a >= pr_M^*\alpha^{(1)}(K_a)(g \circ (u,v)) = \\ &= \ pr_M^*\alpha^{(1)}(K_{Ad_{g^{-1}}a})(u,v) := \\ &= \ < l(u,v), Ad_{g^{-1}}a >=< Ad_{g^{-1}}^* l(u,v), a > \end{aligned} \tag{2.7}$$

for any $g \in G$ and all $a \in \mathcal{G}$ and $(u,v) \in T^*(M)$. Using (2.7) we establish that, for every $g \in G$ and all $(u,v) \in T^*(M)$, the following relation is true: $l(g \circ (u,v)) = Ad_{g^{-1}}^* l(u,v)$. This means that the diagram

$$\begin{array}{ccc} T^*(M) & \stackrel{l}{\to} & \mathcal{G}^* \\ g \downarrow & & \downarrow Ad_{g^{-1}}^* \\ T^*(M) & \stackrel{l}{\to} & \mathcal{G}^* \end{array}$$

is commutative for all elements $g \in G$; the corresponding action $g : T^*(M) \to T^*(M)$ is called equivariant [1, 15].

Let $G_\xi \subset G$ denote the stabilizer of a regular element $\xi \in \mathcal{G}^*$ with respect to the related co-adjoint action. It is obvious that in this case the action of the Lie subgroup G_ξ on the submanifold $\mathcal{M}_\xi \subset T^*(M)$ is naturally defined; we assume that it is free and proper. According to this action on $\mathcal{M}_\xi$, we can define [1, 17, 18, 19, 20] a so-called reduced space $\bar{\mathcal{M}}_\xi$ by taking the factor with respect to the action of the subgroup G_ξ on $\mathcal{M}_\xi$, i.e.,

$$\bar{\mathcal{M}}_\xi := \mathcal{M}_\xi / G_\xi. \tag{2.8}$$

The quotient space (2.8) induces a symplectic structure $\bar{\omega}_\xi^{(2)} \in \Lambda^2(\bar{\mathcal{M}}_\xi)$ on itself, which is defined as follows:

$$\bar{\omega}_\xi^{(2)}(\bar{\eta}_1, \bar{\eta}_2) = \omega_\xi^{(2)}(\eta_1, \eta_2), \tag{2.9}$$

where $\bar{\eta}_1, \bar{\eta}_2 \in T(\bar{\mathcal{M}}_\xi)$ are arbitrary vectors onto which vectors $\eta_1, \eta_2 \in T(\mathcal{M}_\xi)$ are projected, taken at any point $(u_\xi, v_\xi) \in \mathcal{M}_\xi$, being uniquely projected onto the point $\bar{\mu}_\xi \in \bar{\mathcal{M}}_\xi$, according to (2.8).

Let $\pi_\xi : \mathcal{M}_\xi \to T^*(M)$ denote the corresponding imbedding mapping into $T^*(M)$ and let $r_\xi : \mathcal{M}_\xi \to \bar{\mathcal{M}}_\xi$ denote the corresponding reduction to the space $\bar{\mathcal{M}}_\xi$. Then relation (2.9) can be rewritten equivalently in the form of the equality

$$r_\xi^* \bar{\omega}_\xi^{(2)} = \pi_\xi^* \omega^{(2)}, \tag{2.10}$$

defined on vectors on the cotangent space $T^*(\mathcal{M}_\xi)$. To establish the symplecticity of the 2-form $\omega_\xi^{(2)} \in \Lambda^2(\bar{\mathcal{M}}_\xi)$, we use the corresponding non-degeneracy of the Poisson bracket $\{\cdot,\cdot\}_\xi^r$ on $\bar{\mathcal{M}}_\xi$. To calculate it, we use a Dirac type construction, defining functions on $\bar{\mathcal{M}}_\xi$ as certain G_ξ-invariant functions on the submanifold $\mathcal{M}_\xi$. Then one can calculate the Poisson bracket $\{\cdot,\cdot\}_\xi$ of such functions that corresponds to symplectic structure (2.4) as an ordinary Poisson bracket on $T^*(M)$, arbitrarily extending these functions from the submanifold $\mathcal{M}_\xi \subset T^*(M)$ to a certain neighborhood $U(\mathcal{M}_\xi) \subset T^*(M)$. It is obvious that two extensions of a given function to the neighborhood $U(\mathcal{M}_\xi)$ of this type differ by a function that vanishes on the submanifold $\mathcal{M}_\xi \subset T^*(M)$. The difference between the corresponding Hamiltonian fields of these two different extensions to $U\mathcal{M}_\xi)$ is completely controlled by the conditions of the following lemma (see also [1, 15, 18, 17, 23]).

Lemma 2.1. *Suppose that a function $f : U(\mathcal{M}_\xi) \to \mathbb{R}$ is smooth and vanishes on $\mathcal{M}_\xi \subset T^*(M)$, i.e., $f|_{\mathcal{M}_\xi} = 0$. Then, at every point $(u_\xi, v_\xi) \in \mathcal{M}_\xi$ the corresponding Hamiltonian vector field $K_f \in T(U(\mathcal{M}_\xi))$ is tangent to the orbit $Or(G; (u_\xi, v_\xi))$.*

Proof. It is obvious that the submanifold $M_\xi \subset T^*(M)$ is defined by a certain collection of relations of the type

$$H_{a_s} = \xi_s, \qquad \xi_s := <\xi, a_s>, \tag{2.11}$$

where $a_s \in \mathcal{G}, s = \overline{1, dimG}$, is a certain basis of the Lie algebra $\mathcal{G}$, which follows from definition (2.6). Since a function $f : U(\mathcal{M}_\xi) \to \mathbb{R}$ vanishes on $\mathcal{M}_\xi$, we can write the following equality:

$$f = \sum_{s=1}^{\dim \mathcal{G}} (H_{a_s} - \xi_s) f_s,$$

where $f_s : U(\mathcal{M}_\xi) \to \mathbb{R}$, $s = \overline{1, dimG}$, is a certain collection of functions in the neighborhood $U(\mathcal{M}_\xi)$. We take an arbitrary tangent vector $\eta \in T(U(\mathcal{M}_\xi))$ at the point $(u_\xi, v_\xi) \in \mathcal{M}_\xi$ and calculate the expression

$$
\begin{aligned}
< \ df(u_\xi, v_\xi), \eta(u_\xi, v_\xi) >= & \sum_{s=1}^{\dim \mathcal{G}} < dH_{a_s}(u_\xi, v_\xi), \eta(u_\xi, v_\xi) > f_s(u_\xi, v_\xi) = \\
= & \ -\sum_{s=1}^{\dim \mathcal{G}} \omega^{(2)}(K_{a_i}(u_\xi, v_\xi), \eta(u_\xi, v_\xi)) f_s(u_\xi, v_\xi) = \\
= & \ -\omega^{(2)}(\sum_{s=1}^{\dim \mathcal{G}} K_{a_s}(u_\xi, v_\xi) f_s(u_\xi, v_\xi), \eta(u_\xi, v_\xi)) = \\
= & \ - < i_{(\sum_{s=1}^{\dim \mathcal{G}} K_{a_s}(u_\xi, v_\xi) f_s(u_\xi, v_\xi))} \omega^{(2)}, \eta(u_\xi, v_\xi) > .
\end{aligned}
\tag{2.12}
$$

It follows from the arbitrariness of the vector $\eta \in T(\mathcal{M}_\xi)$ at the point $(u_\xi, v_\xi) \in \mathcal{M}_\xi$ and relation (2.12) that

$$
K_f = \sum_{s=1}^{\dim \mathcal{G}} K_{a_s} f_s,
$$

i.e., $K_f : \mathcal{M}_\xi \to T(Or(G))$, which was to be proved. □

As a corollary of Lemma 2.1, we obtain an algorithm for the determination of the reduced Poisson bracket $\{\cdot, \cdot\}_\xi^r$ on the space $\bar{\mathcal{M}}_\xi$ according to definition (2.10). Namely, we choose two functions defined on $\mathcal{M}_\xi$ and invariant under the action of the subgroup G_ξ and arbitrarily smoothly extend them to a certain open domain $U(\mathcal{M}_\xi) \subset T^*(M)$. Then we determine the corresponding Hamiltonian vector fields on $T^*(M)$ and project them onto the space tangent to $\mathcal{M}_\xi$, adding, if necessary, the corresponding vectors tangent to the orbit $Or(G)$. It is obvious that the projections obtained depend on the chosen extensions to the domain $U(\mathcal{M}_\xi) \subset T^*(M)$. As a result, we establish that the reduced Poisson bracket $\{\cdot, \cdot\}_\xi^r$ is uniquely defined via the restriction of the initial Poisson bracket upon $\mathcal{M}_\xi \subset T^*(M)$. By virtue of the non-degeneracy of the latter and the functional independence of the basis functions (2.11) on the submanifold $U(\mathcal{M}_\xi) \subset T^*(M)$, the reduced Poisson bracket $\{\cdot, \cdot\}_\xi^r$ appears to be [1, 15, 17] non-degenerate on $\bar{\mathcal{M}}_\xi$. As a consequence of the non-degeneracy, we establish that the dimension of the reduced space $\bar{\mathcal{M}}_\xi$ is even. Taking into account that the element $\xi \in \mathcal{G}^*$ is regular and the dimension of the Lie algebra of the stabilizer $\mathcal{G}_\xi$ is equal to $dim\ G_\xi$, we easily establish that $dim\ \bar{\mathcal{M}}_\xi = dim\ T^*(M) - 2dim\ \mathcal{G}_\xi$. Since, by construction, $dim\ T^*(M) = 2n$, we conclude that the dimension of the reduced space $\bar{\mathcal{M}}_\xi$ is necessarily even.

For the correctness of the algorithm, it is necessary to establish the existence of the corresponding projections of Hamiltonian vector fields onto the tangent space $T(\mathcal{M}_\xi)$. The following statement is true.

Theorem 2.2. *At every point $(u_\xi, v_\xi) \in \mathcal{M}_\xi$, one can choose a vector $V_f \in T(Or(G))$ such that $K_f(u_\xi, v_\xi) + V_f(u_\xi, v_\xi) \in T_{(u_\xi, v_\xi)}(\mathcal{M}_\xi)$. Furthermore, the vector $V_f \in T(Or(G))$ is determined uniquely up to a vector tangent to the orbit $Or(G_\xi)$.*

Proof. Note that the orbit $Or(G;(u_\xi,v_\xi))$ passing through the point $(u_\xi,v_\xi)\in\mathcal{M}_\xi$ is always symplectically orthogonal to the tangent space $T_{(u_\xi,v_\xi)}(\mathcal{M}_\xi)$. Indeed, for any vector $\eta\in T(\mathcal{M}_\xi)$ and $a\in\mathcal{G}$, we have $\omega^{(2)}(\eta,K_a)=-i_{K_a}\omega^{(2)}(\eta)=dH_a(\eta)=0$, because the submanifold $\mathcal{M}_\xi$ $\subset T^*(M)$ is defined by the equality $<\xi,a>=H_a$ for all $a\in\mathcal{G}$, i.e., $dH_a=0$ on $\mathcal{M}_\xi$. Thus, $T(\mathcal{M}_\xi)\cap T(Or(G))=T(Or(G))$ because $H_a\circ g_\xi=H_a$ for all $g_\xi\in G_\xi$, which follows from the invariance of the element $\xi\in\mathcal{G}^*$ under the action of the Lie group $G\xi$. We now solve the imbedding condition $K_f+V_f\in T(\mathcal{M}_\xi)$, or the equation

$$\omega^{(2)}(K_f+V_f,K_a)=0 \tag{2.13}$$

on the manifold $\mathcal{M}_\xi\subset T^*(M)$ for all $a\in\mathcal{G}$. We rewrite equality (2.13) in the form

$$K_af=\omega^{(2)}(V_f,K_a) \tag{2.14}$$

on $\mathcal{M}_\xi$ for all $a\in\mathcal{G}$; it is obvious that the 2-form on the right-hand side of (2.14) depends only on the element $\xi\in G^*$. Taking into account the equivariance of the group action on $T^*(M)$ and the obvious equality

$$\omega^{(2)}(K_a,K_b)=pr_M^*\alpha^{(1)}([K_a,K_b])=-pr_M^*\alpha^{(1)}(K_{[a,b]})$$

for all $a,b\in\mathcal{G}$, we establish that there exists an element $a_f\in\mathcal{G}$ such that $V_f=K_{a_f}\in T(Or(G))$ and

$$\begin{aligned}\omega^{(2)}(V_f,K_a)&=\omega^{(2)}(K_{a_f},K_a)=pr_M^*\alpha^{(1)}([K_a,K_{a_f}])=\\&=pr_M^*\alpha^{(1)}(K_{[a_f,a]})=H_{[a_f,a]}=<l,[a_f,a]>=\\&=<\xi,[a_f,a]>=<ad^*_{a_f}\xi,a>\end{aligned} \tag{2.15}$$

on $\mathcal{M}_\xi$ for all $a\in\mathcal{G}$. Since $ad^*_{a_f}\xi=0$ for any $a_f\in\mathcal{G}_\xi$, we conclude that, on the quotient space $\mathcal{G}/\mathcal{G}_\xi$ the right-hand side of (2.15) defines a non-degenerate skew-symmetric form associated with the canonical isomorphism $\hat{\xi}:\mathcal{G}/\mathcal{G}_\xi\to(\mathcal{G}/\mathcal{G}_\xi)^*$, where, by definition,

$$<\hat{\xi}(\tilde{a}),\tilde{b}>:=<\xi,[a,b]> \tag{2.16}$$

for any $\tilde{a}$ and $\tilde{b}\in\mathcal{G}/\mathcal{G}_\xi$ with the corresponding representatives a and $b\in\mathcal{G}$. Further, since the function $f:\mathcal{M}_\xi\to\mathbb{R}$ is G_ξ-invariant on $\mathcal{M}_\xi\subset T^*(M)$, the right-hand side of (2.14) defines an element $\mu_f\in(\mathcal{G}/\mathcal{G}_\xi)^*$ by the equality

$$\mu_f:\tilde{a}:=-K_af$$

for all $a\in G$. Using relations (2.15)) and (2.16), we establish that there exists the element

$$\tilde{a}_f=\hat{\xi}^{-1}\circ\mu_f\in\mathcal{G}/\mathcal{G}_\xi.$$

Since the element $\tilde{a}_f\in\mathcal{G}/\mathcal{G}_\xi$ is associated with the element a_f $(mod\ \mathcal{G}_\xi)\in\mathcal{G}$, which uniquely generates a locally defined vector field $K_{a_f}:Or(G)\to T(Or(G))$, using the fact that $V_f=K_{a_f}$ on $\mathcal{M}_\xi$, we complete the proof of the theorem. □

Now assume that two functions $f_1, f_2 \in D(\mathcal{M}_\xi)$ are G_ξ-invariant. Then their reduced Poisson bracket $\{f_1, f_2\}^r_\xi$ on $\bar{\mathcal{M}}_\xi$ is defined according to the rule:

$$\{f_1, f_2\}^r_\xi := -\omega^{(2)}(K_{f_1} + V_{f_1}, K_{f_2} + V_{f_2}) = \{f_1, f_2\} + \omega^{(2)}(V_{f_1}, V_{f_2}), \tag{2.17}$$

where we have used the following identities on $\mathcal{M}_\xi \subset T^*(M)$:

$$\omega^{(2)}(K_{f_1} + V_{f_1}, V_{f_2}) = 0 = \omega^{(2)}(K_{f_2} + V_{f_2}, V_{f_1}),$$

being simple consequences of equality (2.13) on $\mathcal{M}_\xi$. Regarding (2.15), relation (2.17) takes the form

$$\{f_1, f_2\}^r_\xi = \{f_1, f_2\} + \frac{1}{2}(V_{f_1} f_2 - V_{f_2} f_1), \tag{2.18}$$

where $f_1, f_2 \in D(\mathcal{M}_\xi)$ are arbitrary smooth extensions of the G_ξ-invariant functions defined earlier on the domain $U(\mathcal{M}_\xi)$. Thus, the following theorem holds.

Theorem 2.3. *The reduced Poisson bracket of two functions on the quotient space $\bar{\mathcal{M}}_\xi = \mathcal{M}_\xi / G_\xi$ is determined with the use of their arbitrary smooth extensions to functions on an open neighborhood $U(\mathcal{M}_\xi)$ according to the Dirac-type formula (2.18).*

2.2. **The symplectic reduction on principal fiber bundles with connection.** We begin by reviewing the backgrounds of the reduction theory subject to Hamiltonian systems with symmetry on principle fiber bundles. The material is partly available in [4, 16], so here it will be only sketched in notations suitable for us.

Let G denote a given Lie group with the unity element $e \in G$ and the corresponding Lie algebra $\mathcal{G} \simeq T_e(G)$. Consider a principal fiber bundle $p : (M, \varphi) \to N$ with the structure group G and base manifold N, on which the Lie group G acts by means of a mapping $\varphi : M \times G \to M$. Namely, for each $g \in G$ there is a group diffeomorphism $\varphi_g : M \to M$, generating for any fixed $u \in M$ the following induced mapping: $\hat{u} : G \to M$, where

$$\hat{u}(g) = \varphi_g(u). \tag{2.19}$$

On the principal fiber bundle $p : (M, \varphi) \to N$ a connection $\Gamma(\mathcal{A})$ is assigned by means of such a morphism $\mathcal{A}$: $(T(M), \varphi_{g*}) \to (\mathcal{G}, Ad_{g^{-1}})$ that for each $u \in M$ a mapping $\mathcal{A}(u) : T_u(M) \to \mathcal{G}$ is a left inverse one to the mapping $\hat{u}_*(e) : \mathcal{G} \to T_u(M)$, that is

$$\mathcal{A}(u)\hat{u}_*(e) = 1. \tag{2.20}$$

As usual, denote by $\varphi^*_g : T^*(M) \to T^*(M)$ the corresponding lift of the mapping $\varphi_g : M \to M$ at any $g \in G$. If $\alpha^{(1)} \in \Lambda^1(M)$ is the canonical G - invariant 1-form on M, the canonical symplectic structure $\omega^{(2)} \in \Lambda^2(T^*(M))$ given by

$$\omega^{(2)} := d\, pr^* \alpha^{(1)} \tag{2.21}$$

generates the corresponding momentum mapping $l : T^*(M) \to \mathcal{G}^*$, where

$$l(\alpha^{(1)})(u) = \hat{u}^*(e)\alpha^{(1)}(u) \tag{2.22}$$

for all $u \in M$. Remark here that the principal fiber bundle structure $p:(M,\varphi)\to N$ means in part the exactness of the following sequences of mappings:

$$0 \to \mathcal{G} \overset{\hat{u}_*(e)}{\to} T_u(M) \overset{p_*(u)}{\to} T_{p(u)}(N) \to 0, \tag{2.23}$$

that is

$$p_*(u)\hat{u}_*(e) = 0 = \hat{u}^*(e)p^*(u) \tag{2.24}$$

for all $u \in M$. Combining (2.24) with (2.20) and (2.22), one obtains such an embedding:

$$[1-\mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u) \in \text{ range } p^*(u) \tag{2.25}$$

for the canonical 1-form $\alpha^{(1)} \in \Lambda^1(M)$ at $u \in M$. The expression (2.25) means of course, that

$$\hat{u}^*(e)[1-\mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u) = 0 \tag{2.26}$$

for all $u \in M$. Now taking into account that the mapping $p^*(u): T^*(N) \to T^*(M)$ is for each $u \in M$ injective, it has the unique inverse mapping $(p^*(u))^{-1}$ upon its image $p^*(u)T^*_{p(u)}(N) \subset T^*_u(M)$. Thereby for each $u \in M$ one can define a morphism $p_{\mathcal{A}}:(T^*(M),\varphi_g^*) \to T^*(N)$ as

$$p_{\mathcal{A}}(u): \alpha^{(1)}(u) \to (p^*(u))^{-1}[1-\mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u). \tag{2.27}$$

Based on the definition (2.27) one can easily check that the diagram

$$\begin{array}{ccc} T^*(M) & \overset{p_{\mathcal{A}}}{\to} & T^*(N) \\ pr_M\downarrow & & \downarrow pr_N \\ M & \overset{p}{\to} & N \end{array} \tag{2.28}$$

is commutative.

Let an element $\xi \in \mathcal{G}^*$ be G-invariant, that is $Ad^*_{g^{-1}}\xi = \xi$ for all $g \in G$. Denote also by $p^{\xi}_{\mathcal{A}}$ the restriction of the mapping (2.27) upon the subset $\mathcal{M}_\xi := l^{-1}(\xi) \in T^*(M)$, that is $p^{\xi}_{\mathcal{A}}: \mathcal{M}_\xi \to T^*(N)$, where for all $u \in M$

$$p^{\xi}_{\mathcal{A}}(u): l^{-1}(\xi) \to (p^*(u))^{-1}[1-\mathcal{A}^*(u)\hat{u}^*(e)]l^{-1}(\xi). \tag{2.29}$$

Now one can characterize the structure of the reduced phase space $\bar{\mathcal{M}}_\xi := l^{-1}(\xi)/G$ by means of the following lemma.

Lemma 2.4. *The mapping $p^{\xi}_{\mathcal{A}}(u): \mathcal{M}_\xi \to T^*(N)$, where $\mathcal{M}_\xi := l^{-1}(\xi)$, is a principal fiber G-bundle with the reduced space$\bar{\mathcal{M}}_\xi$, being diffeomorphic to $T^*(N)$.*

Denote by $<.,.>_{\mathcal{G}}$ the standard Ad-invariant non-degenerate scalar product on $\mathcal{G}\times\mathcal{G}$. Based on Lemma 2.4 one derives the following characteristic theorem.

Theorem 2.5. *Given a principal fiber G-bundle with a connection $\Gamma(\mathcal{A})$ and a G-invariant element $\xi \in \mathcal{G}^*$, then every such connection $\Gamma(\mathcal{A})$ defines a symplectomorphism $\nu_\xi: \bar{\mathcal{M}}_\xi \to T^*(N)$ between the reduced phase space $\bar{\mathcal{M}}_\xi$ and cotangent bundle $T^*(N)$, where $l: T^*(M) \to \mathcal{G}^*$ is the naturally associated momentum mapping for the group G-action on M. Moreover, the following equality*

$$(p^{\xi}_{\mathcal{A}})(d\ pr^*\beta^{(1)} + pr^*\Omega^{(2)}_\xi) = d\ pr^*\alpha^{(1)}\Big|_{l^{-1}(\xi)} \tag{2.30}$$

holds for the canonical 1-forms $\beta^{(1)} \in \Lambda^1(N)$ and $\alpha^{(1)} \in \Lambda^1(M)$, where $\Omega_\xi^{(2)} :=< \Omega^{(2)}, \xi >_{\mathcal{G}}$ is the ξ-component of the corresponding curvature form $\Omega^{(2)} \in \Lambda^{(2)}(N) \otimes \mathcal{G}$.

Proof. One has that on $l^{-1}(\xi) \subset M$ the following expression, due to (2.27), holds:

$$p^*(u)p_{\mathcal{A}}^{\xi}(\alpha^{(1)}(u)) = p^*(u)\beta^{(1)}(pr_N(u)) = \alpha^{(1)}(u) - \mathcal{A}^*(u)\hat{u}^*(e)\alpha^{(1)}(u)$$

for any $\beta^{(1)} \in T^*(N)$ and all $u \in M_\xi := p_M l^{-1}(\xi) \subset M$. Thus we easily get that

$$\alpha^{(1)}(u) = (p_{\mathcal{A}}^{\xi})^{-1}\beta^{(1)}(p_N(u)) = p^*(u)\beta^{(1)})(pr_N(u)) + < \mathcal{A}(u), \xi >$$

for all $u \in M_\xi$. Recall now that in virtute of (2.28) on the manifold M_ξ the following relationships hold:

$$p \circ pr_{M_\xi} = pr_N \circ p_{\mathcal{A}}^{\xi}, \;\; pr_{M_\xi}^* \circ p^* = (p_{\mathcal{A}}^{\xi})^* \circ pr_N^*.$$

Therefore, we can now write down that

$$\begin{aligned} pr_{M_\xi}^*\alpha^{(1)}(u) &= pr_{M_\xi}^*\beta^{(1)}(p_N(u)) + pr_{M_\xi}^* < \mathcal{A}(u), \xi > \\ &= (p_{\mathcal{A}}^{\xi})^*(pr_N^*\beta^{(1)})(u) + pr_{M_\xi}^* < \mathcal{A}(u), \xi >, \end{aligned}$$

whence taking the external differential, one arrives at the following equality:

$$\begin{aligned} d\, pr_{M_\xi}^*\alpha^{(1)}(u) &= (p_{\mathcal{A}}^{\xi})^* d(pr_N^*\beta^{(1)})(u) + pr_{M_\xi}^* < d\mathcal{A}(u), \xi > \\ &= (p_{\mathcal{A}}^{\xi})^* d(pr_N^*\beta^{(1)})(u) + pr_{M_\xi}^* < \Omega(p(u)), \xi > \\ &= (p_{\mathcal{A}}^{\xi})^* d(pr_N^*\beta^{(1)})(u) + pr_{M_\xi}^* p^* < \Omega, \xi > (u) \\ &= (p_{\mathcal{A}}^{\xi})^* d(pr_N^*\beta^{(1)})(u) + (p_{\mathcal{A}}^{\xi})^* pr_N^* < \Omega, \xi > (u) \\ &= (p_{\mathcal{A}}^{\xi})^*[d(pr_N^*\beta^{(1)})(u) + pr_N^* < \Omega, \xi > (u)]. \end{aligned}$$

When deriving the above expression we made use of the following property satisfied by the curvature 2-form $\Omega \in \Lambda^2(M) \otimes \mathcal{G}$:

$$\begin{aligned} &< \; d\mathcal{A}(u), \xi >=< d\mathcal{A}(u) + \mathcal{A}(u) \wedge \mathcal{A}(u), \xi > - < \mathcal{A}(u) \wedge \mathcal{A}(u), \xi > \\ &= \; < \Omega(p_N(u)), \xi >=< p_N^*\Omega, \xi > (u) \end{aligned}$$

at any $u \in M_\xi$, since for any $A, B \in \mathcal{G}$ there holds $< [A, B], \xi >=< B, Ad^*A)\xi >= 0$ in virtue of the invariance condition $Ad_G\xi = \xi$. Thereby the proof is finished. □

Remark 2.6. As the canonical 2-form $d\, pr^*\alpha^{(1)} \in \Lambda^{(2)}(T^*(M))$ is G-invariant on $T^*(M)$ due to construction, it is evident that its restriction upon the G-invariant submanifold $\mathcal{M}_\xi \subset T^*(M)$ will be effectively defined only on the reduced space $\bar{\mathcal{M}}_\xi$,that ensures the validity of the equality sign in (2.30).

As a consequence of Theorem 2.5 one can formulate the following useful for applications theorems.

Theorem 2.7. *Let an element $\xi \in \mathcal{G}^*$ have the isotropy group G_ξ acting on the subset $\mathcal{M}_\xi \subset T^*(M)$ freely and properly, so that the reduced phase space $(\bar{\mathcal{M}}_\xi\ , \sigma_\xi^{(2)})$ where, by definition, $\bar{\mathcal{M}}_\xi := l^{-1}(\xi)/G_\xi$, is symplectic whose symplectic structure is defined as*

$$\sigma_\xi^{(2)} := d\ pr^*\alpha^{(1)}\Big|_{l^{-1}(\xi)}. \tag{2.31}$$

If a principal fiber bundle $p : (M, \varphi) \to N$ has a structure group coinciding with G_ξ, then the reduced symplectic space $(\bar{\mathcal{M}}_\xi, \sigma_\xi^{(2)})$ is symplectomorphic to the cotangent symplectic space $(T^*(N), \omega_\xi^{(2)})$, where

$$\omega_\xi^{(2)} = d\ pr^*\beta^{(1)} + pr^*\Omega_\xi^{(2)}, \tag{2.32}$$

and the corresponding symplectomorphism is given by a relation like (2.30).

Theorem 2.8. *In order that two symplectic spaces $(\bar{\mathcal{M}}_\xi, \sigma_\xi^{(2)})$ and $(T^*(N), d\ pr^*\beta^{(1)})$ were symplectomorphic, it is necessary and sufficient that the element $\xi \in \ker\ h$, where for G-invariant element $\xi \in \mathcal{G}^*$ the mapping $h : \xi \to [\Omega_\xi^{(2)}] \in H^2(N; \mathbb{Z})$, with $H^2(N; \mathbb{Z})$ being the cohomology class of 2-forms on the manifold N.*

3. The symplectic analysis of the Maxwell and Yang-Mills type electromagnetic dynamical systems

3.1. The Hamiltonian analysis of the Maxwell electromagnetic dynamical systems.

We take the Maxwell electromagnetic equations to be

$$\begin{aligned} \partial E/\partial t &= \nabla \times B - J, \quad \partial B/\partial t = -\nabla \times E, \\ &< \nabla, E >= \rho, \qquad < \nabla, B >= 0, \end{aligned} \tag{3.1}$$

on the cotangent phase space $T^*(N)$ to $N \subset T(D; \mathbb{E}^3)$, being the smooth manifold of smooth vector fields on an open domain $D \subset \mathbb{R}^3$, all expressed in the light speed units. Here $(E, B) \in T^*(N)$ is a vector of electric and magnetic fields, $\rho : D \to \mathbb{R}$ and $J : D \to \mathbb{E}^3$ are, simultaneously, fixed charge and current densities in the domain D, satisfying the equation of continuity

$$\partial\rho/\partial t + < \nabla, J >= 0, \tag{3.2}$$

holding for all $t \in \mathbb{R}$, where we denoted by the sign "∇" the gradient operation with respect to a variable $x \in D$, by the sign "$\times$" the usual vector product in $\mathbb{E}^3 := (\mathbb{R}^3, < \cdot, \cdot >)$, being the standard three-dimensional Euclidean vector space $\mathbb{R}^3$ endowed with the usual scalar product $< \cdot, \cdot >$.

Aiming to represent equations (3.1) as those on reduced symplectic space, we define an appropriate configuration space $M \subset \mathcal{T}(D; \mathbb{E}^3)$ with a vector potential field coordinate $A \in M$. The cotangent space $T^*(M)$ may be identified with pairs $(A; Y) \in T^*(M)$, where $Y \in \mathcal{T}^*(D; \mathbb{E}^3)$ is a suitable vector field density in D. On the space $T^*(M)$ there exists the canonical symplectic form $\omega^{(2)} \in \Lambda^2(T^*(M))$, allowing, owing to the definition of the Liouville from

$$\lambda(\alpha^{(1)})(A;Y) = \int_D d^3x (< Y, dA >:= (Y, dA), \tag{3.3}$$

the canonical expression

$$\omega^{(2)} := d\lambda(\alpha^{(1)}) = (dY, \wedge dA). \tag{3.4}$$

Here we denoted by "$\wedge$" the usual external differentiation, by d^3x, $x \in D$, the Lebesgue measure in the domain D and by $pr : T^*(M) \to M$ the standard projection upon the base space M. Define now a Hamiltonian function $\tilde{H} \in \mathcal{D}(T^*(M))$ as

$$H(A, Y) = 1/2[(Y, Y) + (\nabla \times A, \nabla \times A) + (< \nabla, A >, < \nabla, A >)], \tag{3.5}$$

describing the well-known Maxwell equations in vacuum, if the densities $\rho = 0$ and $J = 0$. Really, owing to (3.4) one easily obtains from (3.5) that

$$\begin{aligned} \partial A/\partial t \ &: \ = \delta H/\delta Y = Y, \\ \partial Y/\partial t \ &: \ = -\delta H/\delta A = -\nabla \times B + \nabla < \nabla, A >, \end{aligned} \tag{3.6}$$

being true wave equations in vacuum, where we put, by definition,

$$B := \nabla \times A, \tag{3.7}$$

being the corresponding magnetic field. Now defining

$$E := -Y - \nabla W \tag{3.8}$$

for some function $W : M \to \mathbb{R}$ as the corresponding electric field, the system of equations (3.6) will become, owing to definition (3.7),

$$\partial B/\partial t = -\nabla \times E, \partial E/\partial t = \nabla \times B, \tag{3.9}$$

exactly coinciding with the Maxwell equations in vacuum, if the Lorentz condition

$$\partial W/\partial t + < \nabla, A >= 0 \tag{3.10}$$

is involved.

Since definition (3.8) was essentially imposed rather than arising naturally from the Hamiltonian approach and our equations are valid only for a vacuum, we shall try to improve upon these matters by employing the reduction approach devised in Section 2. Namely, we start with the Hamiltonian (3.5) and observe that it is invariant with respect to the following abelian symmetry group $G := \exp \mathcal{G}$, where $\mathcal{G} \simeq C^{(1)}(D; \mathbb{R})$, acting on the base manifold M naturally lifted to $T^*(M)$: for any $\psi \in \mathcal{G}$ and $(A, Y) \in T^*(M)$

$$\varphi_\psi(A) := A + \nabla\psi, \quad \varphi_\psi(Y) = Y. \tag{3.11}$$

The 1-form (3.3) under transformation (3.11) also is invariant since

$$\begin{aligned} \varphi_\psi^* \lambda(\alpha^{(1)})(A, Y) = (Y, dA + \nabla d\psi) = \\ = (Y, dA) - (< \nabla, Y >, d\psi) = \lambda(\alpha^{(1)})(A, Y), \end{aligned} \tag{3.12}$$

where we made use of the condition $d\psi \simeq 0$ in $\Lambda^1(T^*(M))$ for any $\psi \in \mathcal{G}$. Thus, the corresponding momentum mapping (2.22) is given as

$$l(A,Y) = - < \nabla, Y > \tag{3.13}$$

for all $(A,Y) \in T^*(M)$. If $\rho \in \mathcal{G}^*$ is fixed, one can define the reduced phase space $\bar{\mathcal{M}}_\rho := l^{-1}(\rho)/G$, since evidently, the isotropy group $G_\rho = G$, owing to its commutativity and the condition (3.11). Consider now a principal fiber bundle $p: M \to N$ with the abelian structure group G and a base manifold N taken as

$$N := \{B \in \mathcal{T}(D;\mathbb{E}^3) :< \nabla, B >= 0, < \nabla, E(S) >= \rho\}, \tag{3.14}$$

where, by definition,

$$p(A) = B = \nabla \times A. \tag{3.15}$$

We can construct a connection 1-form $\mathcal{A} \in \Lambda^1(M)\otimes\mathcal{G}$ on this bundle, where for all $A \in M$

$$\mathcal{A}(A) \cdot \hat{A}_*(l) = 1, d < \mathcal{A}(A), \rho >_{\mathcal{G}} = \Omega^{(2)}_\rho(A) \in H^2(M;\mathbb{Z}), \tag{3.16}$$

where $\mathcal{A}(A) \in \Lambda^1(M)$ is some differential 1-form, which we choose in the following form:

$$\mathcal{A}(A) := -(W, d < \nabla, A >), \tag{3.17}$$

where $W \in C^{(1)}(D;\mathbb{R})$ is some scalar function, still not defined. As a result, the Liouville form (3.3) transforms into

$$\lambda(\tilde{\alpha}^{(1)}_\rho) := (Y, dA) - (W, d < \nabla, A >) = (Y + \nabla W, dA) := (\tilde{Y}, dA),\ \tilde{Y} := Y + \nabla W, \tag{3.18}$$

giving rise to the corresponding canonical symplectic structure on $T^*(M)$ as

$$\tilde{\omega}^{(2)}_\rho := d\lambda(\tilde{\alpha}^{(1)}_\rho) = (d\tilde{Y}, \wedge dA). \tag{3.19}$$

Respectively, the Hamiltonian function (3.5), as a function on $T^*(M)$, transforms into

$$\tilde{H}_\rho(A,\tilde{Y}) = 1/2[(\tilde{Y},\tilde{Y}) + (\nabla \times A, \nabla \times A) + (< \nabla, A >, < \nabla, A >)], \tag{3.20}$$

coinciding with the well-known Dirac-Fock-Podolsky [5, 10] Hamiltonian expression. The corresponding Hamiltonian equations on the cotangent space $T^*(M)$

$$\begin{aligned} \partial A/\partial t \ &: \ = \delta\tilde{H}/\delta\tilde{Y} = \tilde{Y}, \tilde{Y} := -E - \nabla W, \\ \partial\tilde{Y}/\partial t \ &: \ = -\delta\tilde{H}/\delta A = -\nabla \times (\nabla \times A) + \nabla < \nabla, A >, \end{aligned}$$

describe true wave processes related to the Maxwell equations in vacuum, which do not take into account boundary charge and current densities conditions. Really, from (3.20) we obtain that

$$\partial^2 A/\partial t^2 - \nabla^2 A = 0 \Longrightarrow \partial E/\partial t + \nabla(\partial W/\partial t + < \nabla, A >) = -\nabla \times B, \tag{3.21}$$

giving rise to the true vector potential wave equation, but the electromagnetic Farady induction law is satisfied if one to impose additionally the Lorentz condition (3.10).

To remedy this situation, we will apply to this symplectic space the reduction technique devised in Section 2. Namely, owing to Theorem 2.7, the constructed above cotangent manifold $T^*(N)$ is symplectomorphic to the corresponding reduced phase space $\bar{\mathcal{M}}_\rho$, that is

$$\bar{\mathcal{M}}_\rho \simeq \{(B;S) \in T^*(N) : <\nabla, E(S)>= \rho, <\nabla, B>= 0\} \tag{3.22}$$

with the reduced canonical symplectic 2-form

$$\omega_\rho^{(2)}(B,S) = (dB, \wedge dS = d\lambda(\alpha_\rho^{(1)})(B,S)\lambda(\alpha_\rho^{(1)})(B,S) := -(S, dB), \tag{3.23}$$

where we put, by definition,

$$\nabla \times S + F + \nabla W = -\tilde{Y} := E + \nabla W, <\nabla, F>:= \rho, \tag{3.24}$$

for some fixed vector mapping $F \in C^{(1)}(D;\mathbb{E}^3)$, depending on the imposed boundary conditions. The result (3.23) follows right away upon substituting the expression for the electric field $E = \nabla \times S + F$ into the symplectic structure (3.19), and taking into account that $dF = 0$ in $\Lambda^1(M)$. The Hamiltonian function (3.20) reduces, respectively, to the following symbolic form:

$$\begin{aligned} H_\rho(B,S) &= 1/2[(B,B) + (\nabla \times S + F + \nabla W, \nabla \times S + F + \nabla W) + \\ &+(< \nabla, (\nabla\times)^{-1}B>, <\nabla, (\nabla\times)^{-1}B>)], \end{aligned} \tag{3.25}$$

where "$(\nabla\times)^{-1}$" means, by definition, the corresponding inverse curl-operation, mapping [21] the divergence-free subspace $C^{(1)}_{\rm div}(D;\mathbb{E}^3) \subset C^{(1)}(D;\mathbb{E}^3)$ into itself. As a result from (3.25), the Maxwell equations (3.1) become a canonical Hamiltonian system upon the reduced phase space $T^*(N)$, endowed with the canonical symplectic structure (3.23) and the modified Hamiltonian function (3.25). Really, one easily obtains that

$$\begin{aligned} \partial S/\partial t &: = \delta H/\delta B = B - (\nabla\times)^{-1}\nabla <\nabla, (\nabla\times)^{-1}B>, \\ \partial B/\partial t &: = -\delta H/\delta S = -\nabla \times (\nabla \times S + F + \nabla W) := -\nabla \times E, \end{aligned} \tag{3.26}$$

where we make use of the definition $E = \nabla \times S + F$ and the elementary identity $\nabla \times \nabla = 0$. Thus, the second equation of (3.26) coincides with the second Maxwell equation of (3.1) in the classical form

$$\partial B/\partial t = -\nabla \times E.$$

Moreover, from (3.24), owing to (3.26), one obtains via the differentiation with respect to $t \in \mathbb{R}$ that

$$\begin{aligned} \partial E/\partial t &= \partial F/\partial t + \nabla \times \partial S/\partial t = \\ &= \partial F/\partial t + \nabla \times B, \end{aligned} \tag{3.27}$$

as well as, owing to (3.2),

$$<\nabla, \partial F/\partial t>= \partial\rho/\partial t = -<\nabla, J>. \tag{3.28}$$

So, we can find from (3.28) that, up to non-essential curl-terms $\nabla \times (\cdot)$, the following relationship

$$\partial F/\partial t = -J \tag{3.29}$$

holds. Really, the current density vector $J \in C^{(1)}(D;\mathbb{E}^3)$, owing to the equation of continuity (3.2), is defined up to curl-terms $\nabla \times (\cdot)$ which can be included into the right-hand side of (3.29). Having now substituted (3.29) into (3.27), we obtain exactly the first Maxwell equation of (3.1):

$$\partial E/\partial t = \nabla \times B - J, \tag{3.30}$$

being supplemented, naturally, with the external boundary constraint conditions

$$\begin{array}{c} <\nabla, B>= 0, <\nabla, E>= \rho, \\ \partial\rho/\partial t + <\nabla, J>= 0, \end{array} \tag{3.31}$$

owing to the continuity relationship (3.2) and definition (3.22).

Concerning the wave equations, related to the Hamiltonian system (3.26), we obtain the following: the electric field E is recovered from the second equation as

$$E := -\partial A/\partial t - \nabla W, \tag{3.32}$$

where $W \in C^{(1)}(D;\mathbb{R})$ is some smooth function, depending on the vector field $A \in M$. To retrieve this dependence, we substitute (3.29) into equation (3.30), having taken into account that $B = \nabla \times A$:

$$\partial^2 A/\partial t^2 - \nabla(\partial W/\partial t + <\nabla, A>) = \nabla^2 A + J. \tag{3.33}$$

With the above, if we now impose the Lorentz condition (3.10), we obtain from (3.33) the corresponding true wave equations in the space-time, taking into account the external charge and current density conditions (3.31).

Notwithstanding our progress so far, the problem of fulfilling the Lorentz constraint (3.10) naturally within the canonical Hamiltonian formalism still remains to be completely solved. To this end, we are compelled to analyze the structure of the Liouville 1-form (3.18) for Maxwell equations in vacuum on a slightly extended functional manifold $M \times L$. As a first step, we rewrite 1-form (3.18) as

$$\begin{array}{rcl} \lambda(\tilde{\alpha}^{(1)}_\rho) & : & = (\tilde{Y}, dA) = (Y + \nabla W, dA) = (Y, dA) + \\ +(W, -d & < & \nabla, A>) := (Y, dA) + (W, d\eta), \end{array} \tag{3.34}$$

where we put, by definition,

$$\eta := - <\nabla, A> . \tag{3.35}$$

Considering now the elements $(Y, A; \eta, W) \in T^*(M \times L)$ as new canonical variables on the extended cotangent phase space $T^*(M \times L)$, where $L := C^{(1)}(D;\mathbb{R})$, we can rewrite the symplectic structure (3.19) in the following canonical form

$$\tilde{\omega}^{(2)}_\rho := d\lambda(\tilde{\alpha}^{(1)}_\rho) = (dY, \wedge dA) + (dW, \wedge d\eta). \tag{3.36}$$

Subject to the Hamiltonian function (3.20) we obtain the expression

$$H(A, Y; \eta, W) = 1/2[(Y - \nabla W, Y - \nabla W) + (\nabla \times A, \nabla \times A) + (\eta, \eta)], \tag{3.37}$$

with respect to which the corresponding Hamiltonian equations take the form:

$$\begin{aligned}\partial A/\partial t &: = \delta H/\delta Y = Y - \nabla W, Y := -E,\\ \partial Y/\partial t &: = -\delta H/\delta A = -\nabla \times (\nabla \times A),\\ \partial \eta/\partial t &: = \delta H/\delta W =< \nabla, Y - \nabla W >,\\ \partial W/\partial t &: = -\delta H/\delta \eta = -\eta. \end{aligned} \tag{3.38}$$

From (3.38) we obtain, owing to external boundary conditions (3.31), successively that

$$\begin{aligned}\partial B/\partial t + \nabla \times E &= 0, \partial^2 W/\partial t^2 - \nabla^2 W = \rho,\\ \partial E/\partial t - \nabla \times B &= 0, \partial^2 A/\partial t^2 - \nabla^2 A = -\nabla(\partial W/\partial t + < \nabla, A >).\end{aligned} \tag{3.39}$$

As is seen, these equations describe electromagnetic Maxwell equations in vacuum, but without the Lorentz condition (3.10). Thereby, as above, we will apply to the symplectic structure (3.36) the reduction technique devised in Section 2. We obtain that under transformations (3.24) the corresponding reduced manifold $\bar{\mathcal{M}}_\rho$ becomes endowed with the symplectic structure

$$\bar{\omega}_\rho^{(2)} := (dB, \wedge dS) + (dW, \wedge d\eta), \tag{3.40}$$

and the Hamiltonian (3.37) assumes the form

$$H(S, B; \eta, W) = 1/2[(\nabla \times S + F + \nabla W, \nabla \times S + F + \nabla W) + (B, B) + (\eta, \eta)], \tag{3.41}$$

whose Hamiltonian equations

$$\begin{aligned}\partial S/\partial t &: = \delta H/\delta B = B, \partial W/\partial t := -\delta H/\delta \eta = -\eta,\\ \partial B/\partial t &: = -\delta H/\delta S = -\nabla \times (\nabla \times S + F + \nabla W) = -\nabla \times E,\\ \partial \eta/\partial t &: = \delta H/\delta W = - < \nabla, \nabla \times S + F + \nabla W >= - < \nabla, E > -\Delta W,\end{aligned} \tag{3.42}$$

coincide completely with Maxwell equations (3.1) under conditions (3.24), describing true wave processes in vacuum, as well as the electromagnetic Maxwell equations, taking into account *a priori* both the imposed external boundary conditions (3.31) and the Lorentz condition (3.10), solving the problem mentioned in [5, 10]. Really, it is easy to obtain from (3.42) that

$$\begin{aligned}\partial^2 W/\partial t^2 - \Delta W &= \rho, \partial W/\partial t + < \nabla, A >= 0,\\ \nabla \times B &= J + \partial E/\partial t, \partial B/\partial t = -\nabla \times E,\end{aligned} \tag{3.43}$$

Based now on (3.43) and (3.31) one can easily calculate [13, 12] the magnetic wave equation

$$\partial^2 A/\partial t^2 - \Delta A = J, \tag{3.44}$$

supplementing the suitable wave equation on the scalar potential $W \in L$, finishing the calculations. Thus, we can formulate the following proposition.

Proposition 3.1. *The electromagnetic Maxwell equations (3.1) jointly with Lorentz condition (3.10) are equivalent to the Hamiltonian system (3.42) with respect to the canonical symplectic structure (3.40) and Hamiltonian function (3.41), which correspondingly reduce to electromagnetic equations (3.43) and (3.44) under external boundary conditions (3.31).*

The obtained above result can be, eventually, used for developing an alternative quantization procedure of Maxwell electromagnetic equations, being free of some quantum operator problems, discussed in detail in [5]. We hope to consider this aspect of quantization problem in a specially devoted study.

Remark 3.2. If one considers a motion of a charged point particle under a Maxwell field, it is convenient to introduce a trivial fiber bundle structure $p : M \to N$, such that $M = N \times G, N := D \subset \mathbb{R}^3$, with $G := \mathbb{R}\backslash\{0\}$ being the corresponding (abelian) structure Lie group. An analysis similar to the above gives rise to the reduced (on the space $\bar{\mathcal{M}}_\xi := l^{-1}(\xi)/G \simeq T^*(N)$, $\xi \in \mathcal{G}$) symplectic structure

$$\omega^{(2)}(q,p) = < dp, \wedge dq > + d < \mathcal{A}(q,g), \xi >_{\mathcal{G}},$$

where $\mathcal{A}(q,g) := < A(q), dq > + g^{-1}dg$ is a suitable connection 1-form on phase space M, with $(q,p) \in T^*(N)$ and $g \in G$. The corresponding canonical Poisson brackets on $T^*(N)$ are easily found to be

$$\{q^i, q^j\} = 0, \{p_j, q^i\} = \delta^i_j, \{p_i, p_j\} = F_{ji}(q) \tag{3.45}$$

for all $(q,p) \in T^*(N)$. If one introduces a new momentum variable $\tilde{p} := p + A(q)$ on $T^*(N) \ni (q,p)$, it is easy to verify that $\omega^{(2)}_\xi \to \tilde{\omega}^{(2)}_\xi := < d\tilde{p}, \wedge dq >$, giving rise to the following Poisson brackets [20, 23, 22]:

$$\{q^i, q^j\} = 0, \{\tilde{p}_j, q^i\} = \delta^i_j, \{\tilde{p}_i, \tilde{p}_j\} = 0, \tag{3.46}$$

where $i, j = \overline{1,3}$, iff the standard Maxwell field equations

$$\partial F_{ij}/\partial q_k + \partial F_{jk}/\partial q_i + \partial F_{ki}/\partial q_j = 0 \tag{3.47}$$

are satisfied on N for all $i,j,k = \overline{1,3}$ with the curvature tensor $F_{ij}(q) := \partial A_j/\partial q^i - \partial A_i/\partial q^j$, $i,j = \overline{1,3}$, $q \in N$.

Such a construction permits a natural generalization to the case of non-abelian structure Lie group yielding a description of Yang-Mills field equations within the reduction approach, to which we proceed below.

3.2. **The Hamiltonian analysis of the Yang-Mills type dynamical systems.** As above, we start with defining a phase space M of a particle under a Yang-Mills field in a region $D \subset \mathbb{R}^3$ as $M := D \times G$, where G is a (not in general semisimple) Lie group, acting on M from the right. Over the space M one can define quite naturally a connection $\Gamma(\mathcal{A})$ if we consider the following trivial principal fiber bundle $p : M \to N$, where $N := D$, with the structure group G. Namely, if $g \in G$, $q \in N$, then a connection 1-form on $M \ni (q,g)$ can be expressed [4, 15, 14, 19] as

$$\mathcal{A}(q;g) := g^{-1}(d + \sum_{i=1}^{n} a_i A^{(i)}(q))g, \tag{3.48}$$

where $\{a_i \in \mathcal{G} : i = \overline{1,n}\}$ is a basis of the Lie algebra $\mathcal{G}$ of the Lie group G, and $A_i : D \to \Lambda^1(D)$, $i = \overline{1,n}$, are the Yang-Mills fields on the physical space $D \subset \mathbb{R}^3$.

Now one defines the natural left invariant Liouville form on M as

$$\alpha^{(1)}(q;g) :=< p, dq > + < y, g^{-1}dg >_{\mathcal{G}}, \tag{3.49}$$

where $y \in T^*(G)$ and $< \cdot, \cdot >_{\mathcal{G}}$ denotes, as before, the usual Ad-invariant non-degenerate bilinear form on $\mathcal{G}^* \times \mathcal{G}$, as, evidently, $g^{-1}dg \in \Lambda^1(G) \otimes \mathcal{G}$. The main assumption we need to proceed is that the connection 1-form is compatible with the Lie group G action on M. The latter means that the condition

$$R_h^* \mathcal{A}(q;g) = Ad_{h^{-1}} \mathcal{A}(q;g) \tag{3.50}$$

is satisfied for all $(q,g) \in M$ and $h \in G$, where $R_h : G \to G$ means the right translation by an element $h \in G$ on the Lie group G.

Having stated all preliminary conditions needed for the reduction Theorem 2.7 to be applied to our model, suppose that the Lie group G canonical action on M is naturally lifted to that on the cotangent space $T^*(M)$ endowed due to (endowed owing to (3.3) with the following G-invariant canonical symplectic structure:

$$\begin{aligned} \omega^{(2)}(q,p;g,y) \quad &: = d\ pr^* \alpha^{(1)}(q,p;g,y) =< dp, \wedge dq > + \\ + \quad &< \ dy, \wedge g^{-1}dg >_{\mathcal{G}} + < ydg^{-1}, \wedge dg >_{\mathcal{G}} \end{aligned} \tag{3.51}$$

for all $(q,p;g,y) \in T^*(M)$. Take now an element $\xi \in \mathcal{G}^*$ and assume that its isotropy subgroup $G_\xi = G$, that is $Ad_h^* \xi = \xi$ for all $h \in G$. In the general case such an element $\xi \in \mathcal{G}^*$ cannot exist but trivial $\xi = 0$, as it happens, for instance, in the case of the Lie group $G = SL_2(\mathbb{R})$. Then one can construct the reduced phase space $l^{-1}(\xi)/G$ symplectomorphic to $(T^*(N), \omega_\xi^{(2)})$, where owing to (2.30) for any $(q,p) \in T^*(N)$

$$\begin{aligned} \omega_\xi^{(2)}(q,p) \quad &= \quad < dp, \wedge dq > + < \Omega^{(2)}(q), \xi >_{\mathcal{G}} = \\ &= \quad < dp, \wedge dq > + \sum_{s=1}^{n} \sum_{i,j=1}^{3} e_s F_{ij}^{(s)}(q) dq^i \wedge dq^j . \end{aligned} \tag{3.52}$$

In the above we have expanded the element $\xi = \sum_{i=1}^n e_i a^i \in \mathcal{G}^*$with respect to the bi-orthogonal basis $\{a^i \in \mathcal{G}^*, a_j \in \mathcal{G} : \ < a^i, a_j >_{\mathcal{G}} = \delta^i_j,\ i,j = \overline{1,n}\}$, with $e_i \in \mathbb{R}$, $i = \overline{1,3}$, being some constants, and we, as well, denoted by $F_{ij}^{(s)}(q)$, $i,j = \overline{1,3}$, $s = \overline{1,n}$, the corresponding curvature 2-form $\Omega^{(2)} \in \Lambda^2(N) \otimes \mathcal{G}$ components, that is

$$\Omega^{(2)}(q) := \sum_{s=1}^{n} \sum_{i,j=1}^{3} a_s F_{ij}^{(s)}(q) dq^i \wedge dq^j \tag{3.53}$$

for any point $q \in N$. Summarizing the calculations accomplished above, we can formulate the following result.

Theorem 3.3. *Suppose the Yang-Mills field (3.48) on the fiber bundle $p : M \to N$ with $M = D \times G$ is invariant with respect to the Lie group G action $G \times M \to M$. Suppose also that an element $\xi \in G^*$ is chosen so that $Ad_G^* \xi = \xi$. Then for the naturally constructed momentum mapping*

$l: T^*(M) \to G^*$ (being equivariant) the reduced phase space $l^{-1}(\xi)/G \simeq T^*(N)$ is endowed with the symplectic structure (3.52), having the following component-wise Poisson brackets form:

$$\{p_i, q^j\}_\xi = \delta_i^j, \quad \{q^i, q^j\}_\xi = 0, \quad \{p_i, p_j\}_\xi = \sum_{s=1}^{n} e_s F_{ji}^{(s)}(q) \tag{3.54}$$

for all $i, j = \overline{1,3}$ and $(q, p) \in T^*(N)$.

The respectively extended Poisson bracket on the whole cotangent space $T^*(M)$ amounts, owing to (3.11), to the following set of Poisson relationships:

$$\begin{aligned} \{y_s, y_k\}_\xi &= \sum_{r=1}^{n} c_{sk}^r y_r, \quad \{p_i, q^j\}_\xi = \delta_i^j, \\ \{y_s, p_j\}_\xi &= 0 = \{q^i, q^j\}\{p_i, p_j\}_\xi = \sum_{s=1}^{n} y_s F_{ji}^{(s)}(q), \end{aligned} \tag{3.55}$$

where $i, j = \overline{1,n}$, $c_{sk}^r \in \mathbb{R}$, $s, k, r = \overline{1,m}$, are the structure constants of the Lie algebra $\mathcal{G}$, and we made use of the expansion $A^{(s)}(q) = \sum_{j=1}^n A_j^{(s)}(q)\ dq^j$ as well we introduced alternative fixed values $e_i := y_i$, $i = \overline{1,n}$. The result (3.55) can easily be seen if one one makes a shift within the expression (3.51) as $\sigma^{(2)} \to \sigma_{ext}^{(2)}$, where $\sigma_{ext}^{(2)} := \sigma^{(2)}\big|_{\mathcal{A}_0 \to \mathcal{A}}$, $\mathcal{A}_0(g) := g^{-1}dg$, $g \in G$. Thereby one can obtain in virtue of the invariance properties of the connection $\Gamma(\mathcal{A})$ that

$$\begin{aligned} &\sigma_{ext}^{(2)}(q, p; u, y) =< dp, \wedge dq > +d < y(g), Ad_{g^{-1}}\mathcal{A}(q; e) >_\mathcal{G}= \\ &=< dp, \wedge dq > + < dAd_{g^{-1}}^* y(g), \wedge\mathcal{A}(q; e) >_\mathcal{G}=< dp, \wedge dq > + \sum_{s=1}^{m} dy_s \wedge du^s + \\ &+ \sum_{j=1}^{n}\sum_{s=1}^{m} A_j^{(s)}(q) dy_s \wedge dq - < Ad_{g^{-1}}^* y(g), \mathcal{A}(q, e) \wedge \mathcal{A}(q, e) >_\mathcal{G} + \\ &+ \sum_{k \geq s=1}^{m}\sum_{l=1}^{m} y_l c_{sk}^l du^k \wedge du^s + \sum_{s=1}^{n}\sum_{i \geq j=1}^{3} y_s F_{ij}^{(s)}(q) dq^i \wedge dq^j, \end{aligned} \tag{3.56}$$

where coordinate points $(q, p; u, y) \in T^*(M)$ are defined as follows: $\mathcal{A}_0(e) := \sum_{s=1}^m du^i\ a_i$, $Ad_{g^{-1}}^* y(g) = y(e) := \sum_{s=1}^m y_s\ a^s$ for any element $g \in G$. Hence one gets straightaway the Poisson brackets (2.8) plus additional brackets connected with conjugated sets of variables $\{u^s \in \mathbb{R} : s = \overline{1,m}\} \in \mathcal{G}^*$ and $\{y_s \in \mathbb{R} : s = \overline{1,m}\} \in \mathcal{G}$:

$$\{y_s, u^k\}_\xi = \delta_s^k, \{u^k, q^j\}_\xi = 0, \quad \{p_j, u^s\}_\xi = A_j^{(s)}(q), \{u^s, u^k\}_\xi = 0, \tag{3.57}$$

where $j = \overline{1,n}$, $k, s = \overline{1,m}$, and $q \in N$.

Note here that the transition suggested above from the symplectic structure $\sigma^{(2)}$ on $T^*(N)$ to its extension $\sigma_{ext}^{(2)}$ on $T^*(M)$ just consists formally in adding to the symplectic structure $\sigma^{(2)}$ an exact part, which transforms it into an equivalent one. Looking now at the expressions (3.56), one can infer immediately that an element $\xi := \sum_{s=1}^m e_s a^s \in \mathcal{G}^*$ will be invariant with respect to the Ad^*-action of the Lie group G iff

$$\{y_s, y_k\}_\xi\big|_{y_s = e_s} = \sum_{r=1}^{m} c_{sk}^r e_r = 0 \tag{3.58}$$

identically for all $s,k=\overline{1,m},\ \ j=\overline{1,n}$ and $q\in N$. In this, and only this case, the reduction scheme elaborated above will go through.

Returning our attention to expression (3.57), one can easily write the following exact expression:

$$\omega^{(2)}_{ext}(q,p;u,y)=\omega^{(2)}(q,p+\sum_{s=1}^{n}y_sA^{(s)}(q);u,y), \tag{3.59}$$

on the phase space $T^*(M)\ni(q,p;u,y)$, where we abbreviated $<A^{(s)}(q),dq>$ as $\sum_{j=1}^{n}A_j^{(s)}(q)$ dq^j. The transformation like (3.59) was discussed within somewhat different contexts in articles [20, 23] containing also a good background for the infinite dimensional generalization of symplectic structure techniques. Having observed from (3.59) that the simple change of variable

$$\tilde p:=p+\sum_{s=1}^{m}y_s\ A^{(s)}(q) \tag{3.60}$$

of the cotangent space $T^*(N)$ recasts our symplectic structure (3.56) into the old canonical form (3.51), one obtains that the following new set of canonical Poisson brackets on $T^*(M)$ $\ni(q,\tilde p;u,y)$:

$$\begin{aligned}\{y_s,y_k\}_\xi &= \sum_{r=1}^{n}c^r_{sk}y_r, \{\tilde p_i,\tilde p_j\}_\xi=0, \{\tilde p_i,q^j\}=\delta^j_i,\\ \{y_s,q^j\}_\xi &= 0=\{q^i,q^j\}_\xi, \{u^s,u^k\}_\xi=0, \{y_s,\tilde p_j\}_\xi=0,\\ \{u^s,q^i\}_\xi &= 0, \{y_s,u^k\}_\xi=\delta^k_s, \{u^s,\tilde p_j\}_\xi=0,\end{aligned} \tag{3.61}$$

where $k,s=\overline{1,m}$ and $i,j=\overline{1,n}$, holds iff the nonabelian Yang-Mills type field equations

$$\begin{aligned}&\partial F^{(s)}_{ij}/\partial q^l+\partial F^{(s)}_{jl}/\partial q^i+\partial F^{(s)}_{li}/\partial q^j+\\ &+\sum_{k,r=1}^{m}c^s_{kr}(F^{(k)}_{ij}A^{(r)}_l+F^{(k)}_{jl}A^{(r)}_i+F^{(k)}_{li}A^{(r)}_j)=0\end{aligned} \tag{3.62}$$

are fulfilled for all $s=\overline{1,m}$ and $i,j,l=\overline{1,n}$ on the base manifold N. This effect of complete reduction of gauge Yang-Mills type variables from the symplectic structure (3.56) is known in literature [20] as the principle of minimal interaction and appeared to be useful enough for studying different interacting systems as in [21, 24]. We plan to continue further the study of the geometric properties of reduced symplectic structures connected with such interesting infinite-dimensional coupled dynamical systems of Yang-Mills-Vlasov, Yang-Mills-Bogolubov and Yang-Mills-Josephson types [21, 24] as well as their relationships with associated principal fiber bundles endowed with canonical connection structures.

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