

United Nations Educational, Scientific and Cultural Organization  
and  
International Atomic Energy Agency  
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

**ADDITIVE DERIVATIONS ON ALGEBRAS  
OF MEASURABLE OPERATORS**

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**Abstract**

Given a von Neumann algebra  $M$  we introduce the so-called central extension  $mix(M)$  of  $M$ . We show that  $mix(M)$  is a  $*$ -subalgebra in the algebra  $LS(M)$  of all locally measurable operators with respect to  $M$ , and this algebra coincides with  $LS(M)$  if and only if  $M$  does not admit type II direct summands. We prove that if  $M$  is a properly infinite von Neumann algebra then every additive derivation on the algebra  $mix(M)$  is inner. This implies that on the algebra  $LS(M)$ , where  $M$  is a type  $I_\infty$  or a type III von Neumann algebra, all additive derivations are inner derivations.

MIRAMARE – TRIESTE

August 2009

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## Introduction

The present paper continues the series of papers [1]-[3] devoted to the investigation of derivations on the algebra  $LS(M)$  of locally measurable operators affiliated with a von Neumann algebra  $M$  and on its various subalgebras.

Let  $\mathcal{A}$  be an algebra over the field complex number. A linear (additive) operator  $D : \mathcal{A} \rightarrow \mathcal{A}$  is called a *linear (additive) derivation* if it satisfies the identity  $D(xy) = D(x)y + xD(y)$  for all  $x, y \in \mathcal{A}$  (Leibniz rule). Further linear derivations are called simply *derivations*. Each element  $a \in \mathcal{A}$  defines a linear derivation  $D_a$  on  $\mathcal{A}$  given as  $D_a(x) = ax - xa$ ,  $x \in \mathcal{A}$ . Such derivations  $D_a$  are said to be *inner derivations*. If the element  $a$  implementing the derivation  $D_a$  on  $\mathcal{A}$ , belongs to a larger algebra  $\mathcal{B}$  containing  $\mathcal{A}$  (as a proper ideal as usual) then  $D_a$  is called a *spatial derivation*.

One of the main problems in the theory of derivations is to prove the automatic continuity, innerness or spatialness of derivations or to show the existence of non inner and discontinuous derivations on various topological algebras.

In this way A. F. Ber, F. A. Sukochev and V. I. Chilin [5] obtained necessary and sufficient conditions for the existence of non trivial derivations on commutative regular algebras. In particular, they have proved that the algebra  $L^0(0, 1)$  of all (classes of equivalence of) complex measurable functions on the interval  $(0, 1)$  admits non trivial derivations. Independently A. G. Kusraev [9], by means of Boolean-valued analysis, has also proved the existence of non trivial derivations and automorphisms on  $L^0(0, 1)$ . It is clear that these derivations are discontinuous in the measure topology, and therefore they are neither inner nor spatial. The present authors have conjectured that the existence of such pathological examples of derivations deeply depends on the commutativity of the underlying von Neumann algebra  $M$ . In this connection we have initiated the study of the above problems in the non commutative case [1]-[3], by considering derivations on the algebra  $LS(M)$  of all locally measurable operators affiliated with a von Neumann algebra  $M$  and on various subalgebras of  $LS(M)$ . In [2] we have proved that every derivation on so-called non commutative Arens algebras, affiliated with an arbitrary von Neumann algebra and a faithful normal semi-finite trace, is spatial and if the trace is finite then all derivations on this algebra are inner. In [1] and [3] we have proved the above mentioned conjecture concerning derivations on  $LS(M)$  for type I von Neumann algebras.

Recently, this conjecture was also independently confirmed for the type I case in the paper of A.F. Ber, B. de Pagter and A.F. Sukochev [6] by means of a representation of measurable operators as operator valued functions. Another approach to similar problems in the framework of type I  $AW^*$ -algebras has been outlined in the paper of A.F. Gutman, A.G.Kusraev and S.S. Kutateladze [7].

In [3] we considered derivations on the algebra  $LS(M)$  of all locally measurable operators affiliated with a type I von Neumann algebra  $M$ , and also on its subalgebras  $S(M)$  – of measurable

operators and  $S(M, \tau)$  of  $\tau$ -measurable operators, where  $\tau$  is a faithful normal semi-finite trace on  $M$ . We proved that an arbitrary derivation  $D$  on each of these algebras can be uniquely decomposed into the sum  $D = D_a + D_\delta$  where the derivation  $D_a$  is inner (for  $LS(M)$ ,  $S(M)$  and  $S(M, \tau)$ ) while the derivation  $D_\delta$  is an extension of a derivation  $\delta$  on the center of the corresponding algebra.

In the present paper we consider additive derivations on the algebra  $LS(M)$ , where  $M$  is a properly infinite von Neumann algebra.

In section 1 we introduce the so-called central extension  $mix(M)$  of a von Neumann algebra  $M$ . We show that  $mix(M)$  is a  $*$ -subalgebra in the algebra  $LS(M)$  and this algebra coincides with whole  $LS(M)$  if and only if  $M$  does not contain a direct summand of type II. The center  $Z(M)$  of  $M$  is an abelian von Neumann algebra and hence it is  $*$ -isomorphic to  $L^\infty(\Omega, \Sigma, \mu)$  for an appropriate measure space  $(\Omega, \Sigma, \mu)$ . Therefore the algebra  $LS(Z(M)) = S(Z(M))$  can be identified with the ring  $L^0(\Omega, \Sigma, \mu)$  of all measurable functions on  $(\Omega, \Sigma, \mu)$ . We also show that  $mix(M)$  is a  $C^*$ -algebra over the ring  $S(Z(M)) \cong L^0(\Omega, \Sigma, \mu)$  in the sense of [4].

In section 2 we give some necessary properties of the topology of convergence locally in measure on  $LS(M)$ .

In section 3 additive derivations on the algebra  $mix(M)$  are investigated. We prove that if  $M$  is a properly infinite von Neumann algebra then every additive derivation on the algebra  $mix(M)$  is inner. In particular, every additive derivation on the algebra  $LS(M)$  where  $M$  is of type  $I_\infty$  or of type III, is in fact an inner derivation. The latter result generalizes Theorem 2.7 from [3] to additive derivations and extends it also for type III von Neumann algebras.

## 1. Locally measurable operators affiliated with von Neumann algebras

Let  $H$  be a complex Hilbert space and let  $B(H)$  be the algebra of all bounded linear operators on  $H$ . Consider a von Neumann algebra  $M$  in  $B(H)$  with the operator norm  $\|\cdot\|_M$ . Denote by  $P(M)$  the lattice of projections in  $M$ .

A linear subspace  $\mathcal{D}$  in  $H$  is said to be *affiliated* with  $M$  (denoted as  $\mathcal{D}\eta M$ ), if  $u(\mathcal{D}) \subset \mathcal{D}$  for every unitary  $u$  from the commutant

$$M' = \{y \in B(H) : xy = yx, \forall x \in M\}$$

of the von Neumann algebra  $M$ .

A linear operator  $x$  on  $H$  with the domain  $\mathcal{D}(x)$  is said to be *affiliated* with  $M$  (denoted as  $x\eta M$ ) if  $\mathcal{D}(x)\eta M$  and  $u(x(\xi)) = x(u(\xi))$  for all  $\xi \in \mathcal{D}(x)$ .

A linear subspace  $\mathcal{D}$  in  $H$  is said to be *strongly dense* in  $H$  with respect to the von Neumann algebra  $M$ , if

- 1)  $\mathcal{D}\eta M$ ;
- 2) there exists a sequence of projections  $\{p_n\}_{n=1}^\infty$  in  $P(M)$  such that  $p_n \uparrow \mathbf{1}$ ,  $p_n(H) \subset \mathcal{D}$  and  $p_n^\perp = \mathbf{1} - p_n$  is finite in  $M$  for all  $n \in \mathbb{N}$ , where  $\mathbf{1}$  is the identity in  $M$ .

A closed linear operator  $x$  acting in the Hilbert space  $H$  is said to be *measurable* with respect to the von Neumann algebra  $M$ , if  $x\eta M$  and  $\mathcal{D}(x)$  is strongly dense in  $H$  [14]. Denote by  $S(M)$  the set of all measurable operators with respect to  $M$ .

A closed linear operator  $x$  in  $H$  is said to be *locally measurable* with respect to the von Neumann algebra  $M$ , if  $x\eta M$  and there exists a sequence  $\{z_n\}_{n=1}^\infty$  of central projections in  $M$  such that  $z_n \uparrow \mathbf{1}$  and  $z_n x \in S(M)$  for all  $n \in \mathbb{N}$ .

It is well known [10] that the set  $LS(M)$  of all locally measurable operators affiliated with  $M$  is a unital  $*$ -algebra when equipped with the algebraic operations of strong addition and multiplication and taking the adjoint of an operator, and contains  $S(M)$  as a solid  $*$ -subalgebra.

Let  $\tau$  be a faithful normal semi-finite trace on  $M$ . We recall that a closed linear operator  $x$  is said to be  $\tau$ -*measurable* with respect to the von Neumann algebra  $M$ , if  $x\eta M$  and  $\mathcal{D}(x)$  is  $\tau$ -dense in  $H$ , i.e.  $\mathcal{D}(x)\eta M$  and given  $\varepsilon > 0$  there exists a projection  $p \in M$  such that  $p(H) \subset \mathcal{D}(x)$  and  $\tau(p^\perp) < \varepsilon$ . The set  $S(M, \tau)$  of all  $\tau$ -measurable operators with respect to  $M$  is a solid  $*$ -subalgebra in  $S(M)$  (see [12], [14]).

Consider the topology  $t_\tau$  of convergence in measure or *measure topology* on  $S(M, \tau)$ , which is defined by the following neighborhoods of zero:

$$V(\varepsilon, \delta) = \{x \in S(M, \tau) : \exists e \in P(M), \tau(e^\perp) \leq \delta, xe \in M, \|xe\|_M \leq \varepsilon\},$$

where  $\varepsilon, \delta$  are positive numbers, and  $\|\cdot\|_M$  denotes the operator norm on  $M$ .

It is well known [12] that  $S(M, \tau)$  equipped with the measure topology is a complete metrizable topological  $*$ -algebra.

Note that if the trace  $\tau$  is finite then

$$S(M, \tau) = S(M) = LS(M).$$

Given any family  $\{z_i\}_{i \in I}$  of mutually orthogonal central projections in  $M$  with  $\bigvee_{i \in I} z_i = \mathbf{1}$  and a family of elements  $\{x_i\}_{i \in I}$  in  $LS(M)$  there exists a unique element  $x \in LS(M)$  such that  $z_i x = z_i x_i$  for all  $i \in I$ . This element is denoted by  $x = \sum_{i \in I} z_i x_i$  and is called *the mixing* of  $\{x_i\}_{i \in I}$  with respect to  $\{z_i\}_{i \in I}$  (see Proposition 1.1 and further remarks in [3]).

By  $mix(M)$  we denote the set of all elements  $x$  from  $LS(M)$  for which there exists a sequence of mutually orthogonal central projections  $\{z_i\}_{i \in I}$  in  $M$  with  $\bigvee_{i \in I} z_i = \mathbf{1}$ , such that  $z_i x \in M$  for all  $i \in I$ , i.e.

$$mix(M) = \{x \in LS(M) : \exists z_i \in P(Z(M)), z_i z_j = 0, i \neq j, \bigvee_{i \in I} z_i = \mathbf{1}, z_i x \in M, i \in I\},$$

where  $Z(M)$  is the center of  $M$ . In other words  $mix(M)$  is the set of all mixings obtained by families  $\{x_i\}_{i \in I}$  taken from  $M$ .

**Proposition 1.1.** *Let  $M$  be a von Neumann algebras with the center  $Z(M)$ . Then*

i)  *$mix(M)$  is a  $*$ -subalgebra in  $LS(M)$  with the center  $S(Z(M))$ , where  $S(Z(M))$  is the algebra of operators measurable with respect to  $Z(M)$ ;*

ii)  $LS(M) = \text{mix}(M)$  if and only if  $M$  does not have direct summands of type II.

Proof. i) It is clear from the definition that  $\text{mix}(M)$  is a  $*$ -subalgebra in  $LS(M)$  and that its center  $Z(\text{mix}(M))$  is contained in  $S(Z(M)) = Z(LS(M))$ .

Let us show the converse inclusion. Take  $x \in S(Z(M))$  and let  $|x| = \int_0^\infty \lambda de_\lambda$  be the spectral resolution of  $|x|$ . Set

$$z_1 = e_1 \quad \text{and} \quad z_n = e_n - e_{n-1}, \quad n \geq 2.$$

Then it clear that  $\{z_n\}_{n \in \mathbb{N}}$  is a sequence of mutually orthogonal central projections in  $M$  such that  $\bigvee_{n \geq 1} z_n = \mathbf{1}$  and  $z_n x \in Z(M)$  for all  $n \in \mathbb{N}$ . Therefore  $x \in \text{mix}(M)$ . Since  $x$  commutes with each element from  $LS(M) \supset \text{mix}(M)$ , we have that  $x \in Z(\text{mix}(M))$ . Thus  $Z(\text{mix}(M)) = S(Z(M))$ .

ii) If  $M$  is of type I, then by [3, Proposition 1.6] we have  $LS(M) = \text{mix}(M)$ .

Let  $M$  have type III. Since any nonzero projection in  $M$  is infinite it follows that  $S(M) = M$ . Hence by the definitions of the algebras  $LS(M)$  and  $\text{mix}(M)$  we obtain that  $LS(M) = \text{mix}(M)$ . Thus if  $M = N \oplus K$  where  $N$  and  $K$  are respectively type I and type III von Neumann algebras, i.e. if  $M$  does not have type II direct summands, then  $LS(M) = \text{mix}(M)$ .

To prove the converse suppose that  $M$  is a type II von Neumann algebra. First assume that  $M$  is a type  $\text{II}_1$  von Neumann algebra with a faithful normal tracial state  $\tau$ . Let  $\Phi$  be the canonical center-valued trace on  $M$ .

Since  $M$  is of type II, then there exists a projection  $p_1 \in M$  such that

$$p_1 \sim \mathbf{1} - p_1.$$

Then  $\Phi(p_1) = \Phi(p_1^\perp)$ . From  $\Phi(p_1) + \Phi(p_1^\perp) = \Phi(\mathbf{1}) = \mathbf{1}$  it follows that

$$\Phi(p_1) = \Phi(p_1^\perp) = \frac{1}{2}\mathbf{1}.$$

Suppose that there exist mutually orthogonal projections  $p_1, p_2, \dots, p_n$  in  $M$  such that

$$\Phi(p_k) = \frac{1}{2^k}\mathbf{1}, \quad k = \overline{1, n}.$$

Set  $e_n = \sum_{k=1}^n p_k$ . Then  $\Phi(e_n^\perp) = \frac{1}{2^n}\mathbf{1}$ . Take a projection  $p_{n+1} < e_n^\perp$  such that

$$p_{n+1} \sim e_n^\perp - p_{n+1}.$$

Then

$$\Phi(p_{n+1}) = \frac{1}{2^{n+1}}\mathbf{1}.$$

Hence there exists a sequence a mutually orthogonal projections  $\{p_n\}_{n \in \mathbb{N}}$  in  $M$  such that

$$\Phi(p_n) = \frac{1}{2^n}\mathbf{1}, \quad n \in \mathbb{N}.$$

Note that  $\tau(p_n) = \frac{1}{2^n}$ . Indeed

$$\tau(p_n) = \tau(\Phi(p_n)) = \tau\left(\frac{1}{2^n}\mathbf{1}\right) = \frac{1}{2^n}.$$

Since

$$\sum_{n=1}^{\infty} n\tau(p_n) = \sum_{n=1}^{\infty} \frac{n}{2^n} < +\infty$$

it follows that the series

$$\sum_{n=1}^{\infty} np_n$$

converges in measure in  $S(M, \tau)$ . Therefore there exists  $x = \sum_{n=1}^{\infty} np_n \in S(M, \tau)$ .

Let us show that  $x \in LS(M) \setminus \text{mix}(M)$ . Suppose that  $zx \in M$ , where  $z$  is a nonzero central projection. Since any  $p_n$  is a faithful projection we have that  $zp_n \neq 0$  for all  $n$ . Thus

$$\|zx\|_M = 1\|zx\|_M 1 = \|p_n\|_M \cdot \|zx\|_M \cdot \|p_n\|_M \geq \|zp_n x p_n\|_M = \|zp_n n\|_M = n,$$

i.e.

$$\|zx\|_M \geq n$$

for all  $n \in \mathbb{N}$ . From this contradiction it follows that  $x \in LS(M) \setminus \text{mix}(M)$ .

For a general type II von Neumann algebra  $M$  take a nonzero finite projection  $e \in M$  and consider the finite type II von Neumann algebra  $eMe$  which admits a separating family of normal tracial states. Now if  $f \in eMe$  is the support projection of some tracial state  $\tau$  on  $eMe$  then  $fMf$  is a type II<sub>1</sub> von Neumann algebra with a faithful normal tracial state. Hence, as above, one can construct an element  $x \in LS(M) \setminus \text{mix}(M)$ . Therefore, if  $LS(M) = \text{mix}(M)$  then  $M$  cannot have a direct summand of the type II. The proof is complete. ■

**Remark.** A similar notion (i.e. the algebra  $\text{mix}(A)$ ) for arbitrary \*-subalgebras  $A \subset LS(M)$  was independently introduced recently by M.A. Muratov and V.I. Chilin [11]. They denote this algebra by  $E(A)$  and called it the central extension of  $A$ . In particular, if  $A = M$  we have  $E(M) = \text{mix}(M)$ . Therefore following [11] we shall say that  $\text{mix}(M)$  is *the central extension of  $M$* .

An alternative proof of Proposition 1.1 follows also from Proposition 2, Theorem 1 and Theorem 3 in [11].

Let  $(\Omega, \Sigma, \mu)$  be a measure space and from now on suppose that the measure  $\mu$  has the direct sum property, i.e. there is a family  $\{\Omega_i\}_{i \in J} \subset \Sigma$ ,  $0 < \mu(\Omega_i) < \infty$ ,  $i \in J$ , such that for any  $A \in \Sigma$ ,  $\mu(A) < \infty$ , there exist a countable subset  $J_0 \subset J$  and a set  $B$  with zero measure such that  $A = \bigcup_{i \in J_0} (A \cap \Omega_i) \cup B$ .

We denote by  $L^0(\Omega, \Sigma, \mu)$  the algebra of all (equivalence classes of) complex measurable functions on  $(\Omega, \Sigma, \mu)$  equipped with the topology of convergence in measure.

Consider the algebra  $S(Z(M))$  of operators measurable with respect to the center  $Z(M)$  of the von Neumann algebra  $M$ . Since  $Z(M)$  is an abelian von Neumann algebra it is \*-isomorphic to  $L^\infty(\Omega, \Sigma, \mu)$  for an appropriate measure space  $(\Omega, \Sigma, \mu)$ . Therefore, the algebra  $S(Z(M))$  can be identified with the algebra  $L^0(\Omega, \Sigma, \mu)$  of all measurable functions on  $(\Omega, \Sigma, \mu)$ .

**Proposition 1.2.** For any  $x \in \text{mix}(M)$  there exists  $f \in S(Z(M))$  such that  $|x| \leq f$ .

Proof. Let  $x = \sum_{i \in I} z_i x \in \text{mix}(M)$ ,  $z_i x \in M$  for all  $i \in I$ . Put

$$f = \sum_{i \in I} z_i \|z_i x\|_M \in S(Z(M)).$$

Then

$$|x| = \left| \sum_{i \in I} z_i x \right| = \sum_{i \in I} z_i |z_i x| \leq \sum_{i \in I} z_i \|z_i x\|_M = f.$$

The proof is complete. ■

Proposition 1.2 implies that for any  $x \in \text{mix}(M)$  there exists the following vector-valued norm

$$\|x\| = \inf\{f \in S(Z(M)) : |x| \leq f\}. \quad (1)$$

By the definition we obtain that:

- a)  $|x| \leq \|x\|$  for all  $x \in \text{mix}(M)$ ;
- b) if  $x \in \text{mix}(M)$  then

$$\|x\| = \inf\{f \in S(Z(M)) : f \geq 0, f^{-1}x \in M, \|f^{-1}x\|_M \leq 1\};$$

- c) if  $z \in M$  is a central projection then

$$\|zx\| = z\|x\|;$$

- d) if  $x \in M$  then

$$\|x\|_M = \|\|x\|\|_M.$$

**Proposition 1.3.** Let  $x \in M$ . Then  $\|x\| = \mathbf{1}$  if and only if  $\|zx\|_M = 1$  for each nonzero central projection  $z \in M$ .

Proof. Let  $x \in M$ ,  $\|x\| = \mathbf{1}$ . Then  $\|zx\| = z\|x\| = z$  for each nonzero central projection  $z \in M$ . Thus

$$\|zx\|_M = \|\|zx\|\|_M = \|z\|_M = 1.$$

Now let  $\|zx\|_M = 1$  for each nonzero central projection  $z \in M$ , and in particular,  $\|x\|_M = 1$ . Thus  $\|x\| \leq \mathbf{1}$ . Suppose that  $\|x\| \neq \mathbf{1}$ . Then there exist a nonzero central projection  $z \in M$  and a number  $0 < \varepsilon < 1$  such that  $z\|x\| \leq \varepsilon z$ . Thus

$$\|zx\|_M \leq \varepsilon \|z\|_M = \varepsilon < 1,$$

and this contradicts to the equality  $\|zx\|_M = 1$ . Hence  $\|x\| = \mathbf{1}$ . The proof is complete. ■

A complex linear space  $E$  is said to be normed by  $L^0(\Omega, \Sigma, \mu)$  if there is a map  $\|\cdot\| : E \rightarrow L^0(\Omega, \Sigma, \mu)$  such that for any  $x, y \in E, \lambda \in \mathbb{C}$ , the following conditions are fulfilled:

- 1)  $\|x\| \geq 0$ ;  $\|x\| = 0 \iff x = 0$ ;

$$2) \|\lambda x\| = |\lambda| \|x\|;$$

$$3) \|x + y\| \leq \|x\| + \|y\|.$$

The pair  $(E, \|\cdot\|)$  is called a lattice-normed space over  $L^0(\Omega, \Sigma, \mu)$ . A lattice-normed space  $E$  is called  $d$ -decomposable, if for any  $x \in E$  with  $\|x\| = \lambda_1 + \lambda_2$ ,  $\lambda_1, \lambda_2 \in L^0(\Omega, \Sigma, \mu)$ ,  $\lambda_1 \lambda_2 = 0$ ,  $\lambda_1, \lambda_2 \geq 0$ , there exist  $x_1, x_2 \in E$  such that  $x = x_1 + x_2$  and  $\|x_i\| = \lambda_i$ ,  $i = 1, 2$ .

A net  $(x_\alpha)$  in  $E$  is said to be *(bo)*-convergent to  $x \in E$ , if the net  $\{\|x_\alpha - x\|\}$  *(o)*-converges (i.e. almost everywhere converges) to zero in  $L^0(\Omega, \Sigma, \mu)$ .

A lattice-normed space  $E$ , which is  $d$ -decomposable and complete with respect to the *(bo)*-convergence, is called a *Banach – Kantorovich space*.

It is known that every Banach – Kantorovich space  $E$  over  $L^0(\Omega, \Sigma, \mu)$  is a module over  $L^0(\Omega, \Sigma, \mu)$  and  $\|\lambda x\| = |\lambda| \|x\|$  for all  $\lambda \in L^0(\Omega, \Sigma, \mu)$ ,  $x \in E$  (see [8]).

Let  $\mathcal{A}$  be an arbitrary Banach – Kantorovich space over  $L^0(\Omega, \Sigma, \mu)$  and let  $\mathcal{A}$  be a  $*$ -algebra such that  $(\lambda x)^* = \bar{\lambda} x^*$ ,  $(\lambda x)y = \lambda(xy) = x(\lambda y)$  for all  $\lambda \in L^0(\Omega, \Sigma, \mu)$ ,  $x, y \in \mathcal{A}$ .  $\mathcal{A}$  is called a *C\*-algebra over  $L^0(\Omega, \Sigma, \mu)$*  if  $\|xy\| \leq \|x\| \|y\|$ ,  $\|xx^*\| = \|x\|^2$  for all  $x, y \in \mathcal{A}$  (see [4]).

The main result of this section is the following.

**Proposition 1.4.** *Let  $M$  be a von Neumann algebra with the center  $Z \cong L^\infty(\Omega, \Sigma, \mu)$  and let  $\|\cdot\|$  be the  $S(Z(M))$ -valued norm on  $\text{mix}(M)$  defined by (1). Then  $(\text{mix}(M), \|\cdot\|)$  is a *C\*-algebra over  $S(Z(M)) \cong L^0(\Omega, \Sigma, \mu)$* .*

*Proof.* Let  $x \in \text{mix}(M)$ ,  $x \neq 0$  and let  $|x| = \int_0^\infty \lambda de_\lambda$  be the spectral resolution of  $|x|$ . Then there exists  $\lambda_0 > 0$  such that  $e_{\lambda_0} \neq 0$ . Take an element  $f \in S(Z(M))$  such that  $|x| \leq f$ . Then

$$\lambda_0 e_{\lambda_0} \leq |x| e_{\lambda_0} \leq f e_{\lambda_0},$$

i.e.

$$\lambda_0 e_{\lambda_0} \leq f e_{\lambda_0}.$$

Thus

$$\lambda_0 z(e_{\lambda_0}) \leq f z(e_{\lambda_0}),$$

where  $z(e_{\lambda_0})$  is the central support of the projection  $e_{\lambda_0}$ . Thus

$$\lambda_0 z(e_{\lambda_0}) \leq \|x\| z(e_{\lambda_0}).$$

This means that  $\|x\| \neq 0$ .

Take  $g \in S(Z(M))$ ,  $x \in \text{mix}(M)$ . We have

$$\begin{aligned} \|gx\| &= \inf\{f \in S(Z(M)) : |gx| \leq f\} = \inf\{|g|f \in S(Z(M)) : |x| \leq f\} = \\ &= |g| \inf\{f \in S(Z(M)) : |x| \leq f\} = |g| \|x\|, \end{aligned}$$

i.e.

$$\|gx\| = |g| \|x\|.$$



Now let  $x, y \in \text{mix}(M)$ . By [10, Theorem 2.4.5] there exist partial isometries  $u, v \in M$  such that

$$|x + y| \leq u|x|u^* + v|y|v^*.$$

Thus

$$\begin{aligned} |x + y| &\leq u|x|u^* + v|y|v^* \leq u\|x\|u^* + v\|y\|v^* = \\ &= \|x\|uu^* + \|y\|vv^* \leq \|x\| + \|y\|, \end{aligned}$$

and therefore  $\|x + y\| \leq \|x\| + \|y\|$ .

Take  $x, y \in \text{mix}(M)$ . We may assume that  $\|x\| = \|y\| = \mathbf{1}$ . Then  $x, y \in M$   $\|x\|_M = \|y\|_M = 1$ , and therefore  $\|xy\|_M \leq 1$ . Hence  $|xy| \leq \mathbf{1}$ . Thus  $\|xy\| \leq \mathbf{1}$ , i.e.  $\|xy\| \leq \|x\|\|y\|$ .

Let  $x \in M$ ,  $\|x\| = \mathbf{1}$ . By Proposition 1.3 we obtain

$$\|zx\|_M = 1$$

for every nonzero central projection  $z \in M$ . Thus

$$\|zxx^*\|_M = \|(zx)(zx)^*\|_M = \|zx\|_M^2 = 1.$$

Therefore by Proposition 1.3 we obtain that  $\|xx^*\| = \mathbf{1}$ , i.e.  $\|xx^*\| = \|x\|^2$ .

Finally, we shall prove the completeness of the space  $\text{mix}(M)$ . First we consider the case where the center  $S(Z(M)) \cong L^0(\Omega, \Sigma, \mu)$  satisfies the condition  $\mu(\Omega) < \infty$ .

Let  $\{x_n\}$  be a *(bo)*-fundamental sequence in  $\text{mix}(M)$ , i.e.  $\|x_n - x_m\| \rightarrow 0$   $n, m \rightarrow \infty$ . By the inequality

$$\left| \|x_n\| - \|x_m\| \right| \leq \|x_n - x_m\|$$

we obtain that the sequence  $\{\|x_n\|\}$  is *(o)*-fundamental in  $S(Z(M))$ , in particular,  $\{\|x_n\|\}$  is order bounded in  $S(Z(M))$ , i.e. there exists  $c \in S(Z(M))$  such that  $\|x_n\| \leq c$  for all  $n \in \mathbb{N}$ .

Now replacing  $x_n$  with  $(\mathbf{1} + c)^{-1}x_n$  we may assume that  $x_n \in M$ ,  $\|x_n\| \leq \mathbf{1}$  and  $\{x_n\}$  is *(bo)*-fundamental.

Since  $\mu(\Omega) < \infty$  by Egorov's theorem for any  $k \in \mathbb{N}$  there exists  $A_k \in \Sigma$  with  $\mu(\Omega \setminus A_k) \leq \frac{1}{k}$  such that  $\|\chi_{A_k}(x_n - x_m)\|_M \rightarrow 0$  as  $n, m \rightarrow \infty$ . Since  $M$  is complete, one has that  $\chi_{A_k}x_n \rightarrow a_k$  as  $n \rightarrow \infty$  for an appropriate  $a_k \in M$ .

Put

$$z_1 = \chi_{A_1}, \quad z_k = \chi_{A_k} \bigwedge_{i=1}^{k-1} (z_i)^\perp, \quad k \geq 2.$$

Then  $z_i \wedge z_j = 0$ ,  $i \neq j$ ,  $\bigvee_{k \geq 1} z_k = \mathbf{1}$ . Set

$$a = \sum_{k=1}^{\infty} z_k a_k.$$

Then  $x_n \rightarrow a$ . This means that the space  $\text{mix}(M)$  is *(bo)*-complete.

Now we consider the general case for the center  $S(Z(M)) \cong L^0(\Omega, \Sigma, \mu)$ . There exists a mutually orthogonal system  $\{\Omega_i : i \in I\}$  in  $\Sigma$  such that  $\mu(\Omega_i) < \infty$ . As above, we have that for every  $i \in I$  there exists  $a_i \in \text{mix}(M)$  such that  $\chi_{\Omega_i} x_n \rightarrow a_i$ . Set

$$a = \sum_{i \in I} \chi_{\Omega_i} a_i.$$

Then  $x_n \rightarrow a$ . This means that the space  $\text{mix}(M)$  is *(bo)*-complete. The proof is complete. ■

From Propositions 1.1 and 1.4 we obtain the following result.

**Corollary 1.5.** *Let  $M$  be a von Neumann algebra without direct summands of type II. Then  $(LS(M), \|\cdot\|)$  is a  $C^*$ -algebra over  $S(Z(M)) \cong L^0(\Omega, \Sigma, \mu)$ .*

## 2. The topology of convergence locally in measure

Let  $M$  be an arbitrary commutative von Neumann algebra. Then as we have mentioned above  $M$  is  $*$ -isomorphic to the  $*$ -algebra  $L^\infty(\Omega, \Sigma, \mu)$ , while the algebra  $LS(M) = S(M)$  is  $*$ -isomorphic to the  $*$ -algebra  $L^0(\Omega, \Sigma, \mu)$ .

The basis of neighborhoods of zero in the topology of convergence locally in measure on  $L^0(\Omega, \Sigma, \mu)$  consists of the sets

$$W(A, \varepsilon, \delta) = \{f \in L^0(\Omega, \Sigma, \mu) : \exists B \in \Sigma, B \subseteq A, \mu(A \setminus B) \leq \delta, \\ f \cdot \chi_B \in L^\infty(\Omega, \Sigma, \mu), \|f \cdot \chi_B\|_{L^\infty(\Omega, \Sigma, \mu)} \leq \varepsilon\},$$

where  $\varepsilon, \delta > 0$ ,  $A \in \Sigma$ ,  $\mu(A) < +\infty$ .

Recall the definition of the dimension functions on the lattice  $P(M)$  of projection from  $M$  (see [10]).

By  $L_+$  we denote the set of all measurable functions  $f : (\Omega, \Sigma, \mu) \rightarrow [0, \infty]$  (modulo functions equal to zero almost everywhere).

Let  $M$  be an arbitrary von Neumann algebra with the center  $Z = L^\infty(\Omega, \Sigma, \mu)$ . Then there exists a map  $D : P(M) \rightarrow L_+$  with the following properties:

- (i)  $d(e)$  is a finite function if only if the projection  $e$  is finite;
- (ii)  $d(e + q) = d(e) + d(q)$  for  $p, q \in P(M)$ ,  $eq = 0$ ;
- (iii)  $d(uu^*) = d(u^*u)$  for every partial isometry  $u \in M$ ;
- (iv)  $d(ze) = zd(e)$  for all  $z \in P(Z(M))$ ,  $e \in P(M)$ ;
- (v) if  $\{e_\alpha\}_{\alpha \in J}$ ,  $e \in P(M)$  and  $e_\alpha \uparrow e$ , then

$$d(e) = \sup_{\alpha \in J} d(e_\alpha).$$

This map  $d : P(M) \rightarrow L_+$ , is called *the dimension function* on  $P(M)$ .

The basis of neighborhoods of zero in the topology of convergence locally in measure on  $LS(M)$  consists (in the above notations) of the following sets

$$V(A, \varepsilon, \delta) = \{x \in LS(M) : \exists p \in P(M), \exists z \in P(Z(M)), xp \in M,$$

$$\{\|xp\|_M \leq \varepsilon, z^\perp \in W(A, \varepsilon, \delta), d(zp^\perp) \leq \varepsilon z\},$$

where  $\varepsilon, \delta > 0$ ,  $A \in \Sigma$ ,  $\mu(A) < +\infty$ .

We need following assertion from [10, pp. 242, 261, 265]).

**Proposition 2.1.** *Let  $\varepsilon, \delta > 0$ ,  $A \in \Sigma$ ,  $\mu(A) < +\infty$ . Then:*

- a)  $\lambda V(A, \varepsilon, \delta) = V(A, |\lambda|\varepsilon, \delta)$  for all  $\lambda \in \mathbb{C}$ ,  $\lambda \neq 0$ ;
- b)  $x \in V(A, \varepsilon, \delta) \Leftrightarrow |x| \in V(A, \varepsilon, \delta)$ ;
- c)  $x \in V(A, \varepsilon, \delta) \Rightarrow x^* \in V(A, 2\varepsilon, \delta)$ ;
- d)  $x \in V(A, \varepsilon, \delta)$ ,  $y \in M \Rightarrow yx \in \|y\|_M V(A, \varepsilon, \delta)$ ;
- e) for each  $x \in LS(M)$  there exist  $\varepsilon_1, \delta_1 > 0$ ,  $B \in \Sigma$ ,  $\mu(B) < +\infty$ , such that

$$x \cdot V(B, \varepsilon_1, \delta_1) \subseteq V(A, \varepsilon, \delta).$$

In the next section we shall also use the following properties of the topology of convergence locally in measure.

**Lemma 2.2.** *Let  $\varepsilon, \delta > 0$ ,  $A \in \Sigma$ ,  $\mu(A) < +\infty$ ,  $\lambda \in \mathbb{C}$ . If  $|\lambda| \leq \varepsilon$ , then  $\lambda \mathbf{1} \in V(A, \varepsilon, \delta)$ .*

Proof. Put  $p = \mathbf{1}$ ,  $z = \mathbf{1}$ . Then  $\|\lambda \mathbf{1} p\|_M = |\lambda| \leq \varepsilon$ ,  $z^\perp = 0 \in W(A, \varepsilon, \delta)$ ,  $D(zp^\perp) = D(0) = 0 \leq \varepsilon z$ , and therefore  $\lambda \mathbf{1} \in V(A, \varepsilon, \delta)$ . The proof is complete. ■

**Lemma 2.3.** *Let  $x \in LS(M)$ ,  $\varepsilon, \delta > 0$ ,  $A \in \Sigma$ ,  $\mu(A) < +\infty$ . Then there exists  $\lambda_0 > 0$  such that  $x \in \lambda_0 V(A, \varepsilon, \delta)$ .*

Proof. By Proposition 2.1 e) there exist  $\varepsilon_1, \delta_1 > 0$ ,  $B \in \Sigma$ ,  $\mu(B) < +\infty$ , such that

$$x \cdot V(B, \varepsilon_1, \delta_1) \subseteq V(A, \varepsilon, \delta).$$

From Lemma 2.2 it follows that  $\varepsilon_1 \mathbf{1} \in V(B, \varepsilon_1, \delta_1)$ . Therefore  $x \varepsilon_1 \mathbf{1} \in V(A, \varepsilon, \delta)$ , i.e.  $x \in \lambda_0 V(A, \varepsilon, \delta)$ , where  $\lambda_0 = \varepsilon_1^{-1}$ . The proof is complete. ■

**Lemma 2.4.** *If  $x \in V(A, \varepsilon, \delta)$  and  $u, v \in M$  are partial isometries, then  $uxv \in V(A, 4\varepsilon, \delta)$ .*

Proof. The case when  $u = 0$  or  $v = 0$  is trivial. Assume that  $u, v \neq 0$ . Then  $\|u\|_M = \|v\|_M = 1$ . By Proposition 2.1 d) we obtain that  $vx \in V(A, \varepsilon, \delta)$ . From Proposition 2.1 c) it follows that  $x^* v^* = (vx)^* \in V(A, 2\varepsilon, \delta)$ . Applying Proposition 2.1 d) once more we have that  $v^* x^* u^* \in V(A, 2\varepsilon, \delta)$  and  $uxv = (v^* x^* u^*)^* \in V(A, 4\varepsilon, \delta)$ . The proof is complete. ■

**Lemma 2.5.** *If  $f_i \in S(Z(M))$ ,  $i = 1, 2$ ,  $|f_1| \leq |f_2|$  and  $f_2 \in V(A, \varepsilon, \delta)$ , then  $f_1 \in V(A, \varepsilon, \delta)$ .*

Proof. Let  $f_2 \in V(A, \varepsilon, \delta)$ . Then  $|f_2| \in V(A, \varepsilon, \delta)$ . Therefore there exist  $p_0 \in P(M)$ ,  $z_0 \in P(Z(M))$  such that

$$|f_2| p_0 \in M, \| |f_2| p_0 \|_M \leq \varepsilon, z_0^\perp \in W(A, \varepsilon, \delta), d(z_0 p_0^\perp) \leq \varepsilon z_0.$$

From  $|f_1| \leq |f_2|$  we get  $p_0 |f_1| p_0 \leq p_0 |f_2| p_0$  and  $|f_1| p_0 \leq |f_2| p_0$ . Hence  $|f_1| p_0 \in M$  and  $\| |f_1| p_0 \|_M \leq \| |f_2| p_0 \|_M \leq \varepsilon$ , i.e.  $\| |f_1| p_0 \|_M \leq \varepsilon$ . Since  $z_0^\perp \in W(A, \varepsilon, \delta)$ ,  $d(z_0 p_0^\perp) \leq \varepsilon z_0$  we see that  $|f_1| \in V(A, \varepsilon, \delta)$  or  $f_1 \in V(A, \varepsilon, \delta)$ . The proof is complete. ■

Recall [13] that a von Neumann algebra  $M$  is said to be *properly infinite*, if any nonzero central projection  $z$  in  $M$  is infinite.

**Lemma 2.6.** *Let  $M$  be a properly infinite von Neumann algebra and let  $\varepsilon, \delta > 0$ ,  $A \in \Sigma$ ,  $\delta < \mu(A) < +\infty$ ,  $0 < \varepsilon < 1$ . Then from  $\lambda \mathbf{1} \in V(A, \varepsilon, \delta)$ , where  $\lambda \in \mathbb{C}$ , it follows that  $|\lambda| \leq \varepsilon$ .*

*Proof.* Let  $\lambda \mathbf{1} \in V(A, \varepsilon, \delta)$ . Then there exist  $p \in P(M)$ ,  $z \in P(Z(M))$  such that  $zp^\perp$  is finite and  $\|\lambda p\|_M \leq \varepsilon$ ,  $z^\perp \in W(A, \varepsilon, \delta)$ . Set  $z = \chi_E$ , where  $E \in \Sigma$ . Since  $z^\perp \in W(A, \varepsilon, \delta)$  there exists  $B \in \Sigma$ ,  $B \subset A$  such that

$$\mu(A \setminus B) \leq \delta, \quad \|z^\perp \chi_B\|_M \leq \varepsilon.$$

Since  $0 < \varepsilon < 1$ , from the inequality  $\|z^\perp \chi_B\|_M \leq \varepsilon$  we have that  $(\mathbf{1} - \chi_E)\chi_B = 0$ . From  $\mu(A) > \delta$  and  $\mu(A \setminus B) \leq \delta$ , we get  $\chi_B \neq 0$ , and therefore from  $(\mathbf{1} - \chi_E)\chi_B = 0$  we obtain that  $\chi_E \neq 0$ , i.e.  $z \neq 0$ . Since  $zp^\perp$  is finite and  $M$  is properly infinite, it follows that projection  $zp$  is an infinite. Therefore  $p \neq 0$ . Thus  $|\lambda| = |\lambda| \|p\|_M = \|\lambda p\|_M \leq \varepsilon$ , i.e.  $|\lambda| \leq \varepsilon$ . The proof is complete. ■

### 3. Additive derivations on the central extensions of properly infinite von Neumann algebras

The following theorem is the main result of this paper.

**Theorem 3.1.** *Let  $M$  be a properly infinite von Neumann algebra. Then every additive derivation on the algebra  $\text{mix}(M)$  is inner.*

To prove this theorem we need several preliminary assertions.

Let  $\mathcal{A}$  be an algebra and denote by  $Z(\mathcal{A})$  its center. If  $D$  is an additive derivation on  $\mathcal{A}$  and  $\Delta = D|_{Z(\mathcal{A})}$  is its restriction onto the center of  $\mathcal{A}$ , then  $\Delta$  maps  $Z(\mathcal{A})$  into itself [3, Remark 1] (see also [6, Lemma 4.2]).

Let  $M$  be a commutative von Neumann algebra and let  $\mathcal{A}$  be an arbitrary subalgebra of  $LS(M) = S(M)$  containing  $M$ . Further we shall identify the algebra  $LS(M) = S(M)$  with an appropriate  $L^0(\Omega, \Sigma, \mu)$ .

Consider an additive derivation  $D : \mathcal{A} \rightarrow S(M)$  and let us show that  $D$  can be extended to an additive derivation  $\tilde{D}$  on the whole  $S(M)$ .

In the commutative von Neumann algebra  $M$  for an arbitrary element  $x \in S(M)$  there exists a sequence  $\{z_n\}$  of mutually orthogonal projections with  $\bigvee_{n \in \mathbb{N}} z_n = \mathbf{1}$  and  $z_n x \in M$  for all  $n \in \mathbb{N}$ . Set

$$\tilde{D}(x) = \sum_{n \geq 1} z_n D(z_n x). \quad (2)$$

Since every additive derivation  $D : \mathcal{A} \rightarrow S(M)$  is identically zero on projections of  $M$ , the equality (2) gives a well-defined additive derivation  $\tilde{D} : S(M) \rightarrow S(M)$  which coincides with  $D$  on  $\mathcal{A}$ .

Given an arbitrary additive derivation  $\Delta$  on  $S(M) = L^0(\Omega, \Sigma, \mu)$  the element

$$z_\Delta = \inf\{\pi \in \nabla : \pi \Delta = \Delta\}$$

is called the support of the additive derivation  $\Delta$ , where  $\nabla$  is the complete Boolean algebra of all idempotents from  $L^0(\Omega, \Sigma, \mu)$  (i.e. characteristic functions of sets from  $\Sigma$ ).

For any non trivial additive derivation  $\Delta : L^0(\Omega, \Sigma, \mu) \rightarrow L^0(\Omega, \Sigma, \mu)$  there exists a sequence  $\{\lambda_n\}_{n=1}^\infty$  in  $L^\infty(\Omega, \Sigma, \mu)$  with  $|\lambda_n| \leq \mathbf{1}$ ,  $n \in \mathbb{N}$ , such that

$$|\Delta(\lambda_n)| \geq nz_\Delta$$

for all  $n \in \mathbb{N}$  (see [3, Lemma 2.6]). In [3] this assertion has been proved for linear derivations, but the proof is the same for additive derivations.

**Lemma 3.2.** *Let  $M$  be a properly infinite von Neumann algebra, and let  $\mathcal{A} \subseteq LS(M)$  be a  $*$ -subalgebra such that  $M \subseteq \mathcal{A}$  and suppose that  $D : \mathcal{A} \rightarrow \mathcal{A}$  is an additive derivation. Then  $D|_{Z(\mathcal{A})} \equiv 0$ , in particular,  $D$  is  $Z(\mathcal{A})$ -linear.*

*Proof.* Let  $D$  be an additive derivation on  $\mathcal{A}$ , and let  $\Delta$  be its restriction onto  $Z(\mathcal{A})$ . Since  $M \subset \mathcal{A} \subset LS(M)$  it follows that  $Z(M) \subset Z(\mathcal{A}) \subset S(Z(M)) = L^0(\Omega, \Sigma, \mu)$ . Let us extend the additive derivation  $\Delta$  onto whole  $S(Z(M))$  as in (2) above, and denote the extension also by  $\Delta$ .

Since  $M$  is properly infinite there exists a sequence of mutually orthogonal projections  $\{p_n\}_{n=1}^\infty$  in  $M$  such that  $p_n \sim \mathbf{1}$  for all  $n \in \mathbb{N}$ , and  $\bigvee_{n=1}^\infty p_n = \mathbf{1}$ .

For any bounded sequence  $\Lambda = \{\lambda_n\}_{n \in \mathbb{N}}$  in  $Z(M)$  define an operator  $x_\Lambda$  by

$$x_\Lambda = \sum_{n=1}^\infty \lambda_n p_n.$$

Then

$$x_\Lambda p_n = p_n x_\Lambda = \lambda_n p_n \tag{3}$$

for all  $n \in \mathbb{N}$ .

Take  $\lambda \in Z(\mathcal{A})$  and  $n \in \mathbb{N}$ . From the identity

$$D(\lambda p_n) = D(\lambda)p_n + \lambda D(p_n)$$

multiplying it by  $p_n$  from the both sides we obtain

$$p_n D(\lambda p_n) p_n = p_n D(\lambda) p_n + \lambda p_n D(p_n) p_n.$$

Since  $p_n$  is a projection, one has that  $p_n D(p_n) p_n = 0$ , and since  $D(\lambda) = \Delta(\lambda) \in Z(\mathcal{A})$ , we have

$$p_n D(\lambda p_n) p_n = \Delta(\lambda) p_n. \tag{4}$$

Now from the identity

$$D(x_\Lambda p_n) = D(x_\Lambda) p_n + x_\Lambda D(p_n),$$

in view of (3) one has similarly

$$p_n D(\lambda_n p_n) p_n = p_n D(x_\Lambda) p_n + \lambda_n p_n D(p_n) p_n,$$

i.e.

$$p_n D(\lambda_n p_n) p_n = p_n D(x_\Lambda) p_n. \tag{5}$$

Now (4) and (5) imply

$$p_n D(x_\Lambda) p_n = \Delta(\lambda_n) p_n. \quad (6)$$

If we suppose that  $\Delta \neq 0$  then  $z_\Delta \neq 0$ . By [3, Lemma 2.6] there exists a bounded sequence  $\Lambda = \{\lambda_n\}_{n \in \mathbb{N}}$  in  $Z(M)$  such that

$$|\Delta(\lambda_n)| \geq n z_\Delta$$

for all  $n \in \mathbb{N}$ .

Replacing the algebra  $M$  by the algebra  $z_\Delta M$ , and the additive derivation  $D$  by  $z_\Delta D$ , we may assume that  $z_\Delta = \mathbf{1}$ , i.e.

$$|\Delta(\lambda_n)| \geq n \mathbf{1} \quad (7)$$

for all  $n \in \mathbb{N}$ .

Now take  $\varepsilon, \delta > 0$ ,  $A \in \Sigma$ ,  $\delta < \mu(A) < +\infty$ . By Lemma 2.3 there exists a number  $\lambda_0 > 0$  such that  $D(x_\Lambda) \in \lambda_0 V(A, \varepsilon, \delta)$ . From Lemma 2.4 it follows that  $p_n D(x_\Lambda) p_n \in \lambda_0 V(A, 4\varepsilon, \delta)$  for all  $n \in \mathbb{N}$ . If we combine this with (6) we obtain

$$\Delta(\lambda_n) p_n \in \lambda_0 V(A, 4\varepsilon, \delta) \quad (8)$$

for all  $n \in \mathbb{N}$ . Since  $p_n \sim \mathbf{1}$  for each  $n \in \mathbb{N}$ , there exists a sequence of partial isometries  $\{u_n\}_{n \in \mathbb{N}}$  in  $M$  such that  $u_n u_n^* = p_n$  and  $u_n^* u_n = \mathbf{1}$  for all  $n \in \mathbb{N}$ . Using (8) and Lemma 2.4 we have

$$u_n^* \Delta(\lambda_n) p_n u_n \in \lambda_0 V(A, 16\varepsilon, \delta)$$

for all  $n \in \mathbb{N}$ . Thus from the equality  $u_n^* p_n u_n = u_n^* u_n u_n^* u_n = \mathbf{1}$ , we obtain that

$$\Delta(\lambda_n) \in \lambda_0 V(A, 16\varepsilon, \delta)$$

for all  $n \in \mathbb{N}$ . Thus by Lemma 2.5 and from the inequality (7) we have

$$n \mathbf{1} \in \lambda_0 V(A, 16\varepsilon, \delta) \quad (9)$$

for all  $n \in \mathbb{N}$ . Take a number  $n_0 \in \mathbb{N}$  such that  $n_0 > 16\lambda_0\varepsilon$ . From Proposition 2.1 a) and (9) we obtain that

$$\mathbf{1} \in V(A, 16\lambda_0\varepsilon n_0^{-1}, \delta).$$

Since  $\delta < \mu(A)$  and  $16\lambda_0\varepsilon n_0^{-1} < 1$ , from Lemma 2.6 we have the inequality  $1 \leq 16\lambda_0\varepsilon n_0^{-1}$ , which contradicts the inequality  $n_0 > 16\lambda_0\varepsilon$ . This contradiction implies that  $\Delta \equiv 0$ , i.e.  $D$  is identically zero on the center of  $\mathcal{A}$ , and therefore it is  $Z(\mathcal{A})$ -linear. The proof is complete. ■

**Remark.** A result similar to Lemma 3.2 for the case of linear derivations has been announced without proof in [6, Proposition 6.22].

In the case of linear derivations on the algebras  $\mathcal{A} = S(M)$  or  $S(M, \tau)$  a shorter proof of Lemma 3.2 can also be obtained from the following result.

**Proposition 3.3.** *Let  $M$  be a properly infinite von Neumann algebra with the center  $Z(M)$ . Then the centers of the algebras  $S(M)$  and  $S(M, \tau)$  coincide with  $Z(M)$ .*

*Proof.* Consider a central element  $z \in S(M)$ ,  $z \geq 0$ , and let  $z = \int_0^\infty \lambda de_\lambda$  be its spectral resolution. Then  $e_\lambda \in Z(M)$  for all  $\lambda > 0$ . Assume that  $e_n^\perp \neq 0$  for all  $n \in \mathbb{N}$ . Since  $M$  is properly infinite,  $Z(M)$  does not contain nonzero finite projections. Thus  $e_n^\perp$  is infinite for all  $n \in \mathbb{N}$ , which contradicts the condition  $z \in S(M)$ . Therefore there exists  $n_0 \in \mathbb{N}$  such that  $e_n^\perp = 0$  for all  $n \geq n_0$ , i.e.  $z \leq n_0 \mathbf{1}$ . This means that  $z \in Z(M)$ , i.e.  $Z(S(M)) = Z(M)$ . Similarly  $Z(S(M, \tau)) = Z(M)$ . The proof is complete. ■

Let  $M$  be a properly infinite von Neumann algebra with the center  $Z(M)$  and let  $D$  be a linear derivation on the algebra  $\mathcal{A} = S(M)$  or  $S(M, \tau)$ . Proposition 3.3 implies that  $Z(\mathcal{A}) = Z(M)$ , and therefore  $\Delta = D|_{Z(\mathcal{A})}$  is a linear derivation on the algebra  $Z(M)$ . By [13, Lemma 4.1.2] we obtain that  $\Delta = 0$  as it was asserted in Lemma 3.2.

**Proof of Theorem 3.1.** Let  $D : \text{mix}(M) \rightarrow \text{mix}(M)$  be an additive derivation. From Lemma 3.2 it follows that  $D$  is  $S(Z(M))$ -linear. By Proposition 1.4 we have that  $\text{mix}(M)$  is a  $C^*$ -algebra over  $S(Z(M)) \cong L^0(\Omega, \Sigma, \mu)$ . Since  $D$  is  $S(Z(M))$ -linear, by [4, Theorem 5] we obtain that  $D$  is a  $S(Z(M))$ -bounded, i.e. there exists  $c \in S(Z(M))$  such that  $\|D(x)\| \leq c\|x\|$  for all  $x \in \text{mix}(M)$ . Take a sequence of pairwise orthogonal central projections  $\{z_n\}_{n \in \mathbb{N}}$  in  $M$  with  $\bigvee_{n \geq 1} z_n = \mathbf{1}$  such that  $z_n c \in Z(M)$  for all  $n$ . Then for any  $x \in M$  we have

$$\|D(z_n x)\| = z_n \|D(x)\| \leq z_n c \|x\|,$$

i.e.  $\|D(z_n x)\| \in Z(M)$ . Thus

$$z_n D(x) \in z_n M.$$

Therefore, the operator  $z_n D$  maps each subalgebra  $z_n M$  into itself for all  $n \in \mathbb{N}$ . By Sakai's theorem [13, Theorem 4.1.6] there exists  $a_n \in z_n M$  such that

$$z_n D(x) = a_n x - x a_n, \quad x \in z_n M.$$

Set  $a = \sum_{n \geq 1} z_n a_n$ . Then  $a \in \text{mix}(M)$  and  $D(x) = ax - xa$  for all  $x \in \text{mix}(M)$ . This means that  $D$  is inner. The proof is complete. ■

From Theorem 3.1 and Proposition 1.1 we obtain the following result which generalizes and extends Theorem 2.7 from [3].

**Corollary 3.4.** *Let  $M$  be a direct sum of von Neumann algebras of type  $I_\infty$  and III. Then every additive derivation on the algebra  $LS(M)$  is an inner derivation.*

**Acknowledgments.** Part of this work was done within the framework of the Associateship Scheme of the Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy. The first author thank ICTP for the kind hospitality and for providing financial support and all facilities (July-August, 2009).

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