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AN ANALYTICAL SOLUTION OF THE POINT KINETICS EQUATIONS WITH TIME VARIABLE REACTIVITY BY THE DECOMPOSITION METHOD

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ABSTRACT

In this work, we report an analytical solution for the point kinetics equations by the Decomposition method, assuming that the reactivity is an arbitrary function of time. The main idea initially consists in the determination of the point kinetics equations solution with constant reactivity by just using the well known solution results of the first order linear matrix differential equation with constant matrix entries. Applying the Decomposition method, we are able to transform the point kinetics equations with time variable reactivity into a set of recursive problems similar to the point kinetics equations with constant reactivity, which can be straightly solved by the mentioned technique. For illustration, we also report simulations for constant, linear and sinusoidal reactivity time functions as well comparisons with the results in literature.

1. INTRODUCTION

Lately, we have been noticing an increasing interest in the task of searching for analytical solutions of linear and nonlinear problems by the scientific community. Besides the mathematical elegance, the analytical solution possess a relevant aptness to generate benchmark solutions to validate computational code results. Further, the analytical solution in some sense, eliminates or at least mitigates the mathematical task of the error evaluation required by numerical methods, except for the round off error. We must mention that by analytical we mean that no approximation is done along the solution derivation. For illustration of the literature about analytical solution, we cite the works of Costa et al. [1], Buske et al. [2], Heinen et al. [3], Goulart et al. [4] and Carvalho et al. [5].

In this work, keeping us in the tracking of searching analytical solution, we apply the Decomposition method [6], to solve the point kinetics equations for a time variable reactivity. The main idea comprehends the steps: expansion of the neutron density and delay neutron concentrations in a truncated series, replacement of these ansatz in the point kinetics equations and construction of a set of recursive systems of first order differential equations similar to the point kinetics equations with constant reactivity. The solution of this recursive

system is promptly obtained recasting these equations in matrix form and using the well known solutions for the first order linear differential matrix equation with constant matrix entries. We are aware of the works in literature concerning the solution of the point kinetics equations with time variable reactivity using numerical approaches, among them we mention the works of Hansen et al. [7], Nobrega et al. [8], Kueng et al. [9], Chao et al. [10], Sanchez et al. [11], Aboander et al. [12], Kinard et al. [13], but to our knowledge the literature regarding the analytical solution is scarce, except for very particular problems. To enlighten the feasibility of the proposed methodology to solve this sort of problem, we report numerical simulations and convergence analysis as well comparisons with available results in literature for constant, liner and sinusoidal with time variable reactivity.

2. ANALYTICAL SOLUTION OF THE POINT KINETICS EQUATIONS

In order to construct the solution of the point kinetics equations for variable reactivity, let us consider the model assuming six delayed neutron families:

$$\begin{aligned} \frac{d}{dt}n\ (t) &= \frac{\rho(t)-\beta}{\Lambda}n(t)+\sum_{i=1}^{6}\lambda_i C_i(t) \quad , \\ &\vdots \\ \frac{d}{dt}C_i(t) &= \frac{\beta_i}{\Lambda}n(t)-\lambda_i C_i(t) \quad , \end{aligned} \tag{1}$$

for $i=1:6$ with the following initial condition:

$$\begin{aligned} n(0) &= n_0 \quad , \\ C_i(0) &= \frac{\beta_i}{\lambda_i \Lambda} \quad , \end{aligned} \tag{2}$$

for $i=1:6$. Here $n(t)$ denotes the neutron density, $\rho(t)$ is the time variable reactivity, β_i is the fraction of delay neutron emitted by precursor $C_i(t)$ (for i = 1 to 6), β is the summation of β_i, Λ is the average generation time, $C_i(t)$ is the delay neutron concentration, λ_i is the constant of decay of the delay neutron concentration, $n(0)$ and $C_i(0)$ are respectively the neutron density and the delayed neutron concentration at the initial time, i.e. t = 0. In order to apply the decomposition method, we expand the neutron and delayed neutron concentration in a truncated series, as:

$$
\begin{aligned}
n(t) &= \sum_{j=1}^{N} n_j(t) \quad , \\
&\vdots \\
C_i(t) &= \sum_{j=1}^{N} C_{ij}(t) \quad ,
\end{aligned}
\tag{3}
$$

for $i = 1:6$. The replacement of the expansions (3) in the point kinetics equations (1) and omitting time (t) dependence leads to:

$$
\begin{aligned}
\frac{d}{dt}(n_0 + n_1 + \ldots n_N) &= \left(\frac{\rho_0 + \rho_1(t) - \beta}{\Lambda}\right)(n_0 + n_1 + \ldots n_N) + \sum_{i=1}^{6} \lambda_i (C_{i0} + C_{i1} + \ldots C_{iN}) \quad , \\
&\vdots \\
\frac{d}{dt}(C_{i0} + C_{i1} + \ldots C_{iN}) &= \frac{\beta_i}{\Lambda}(n_0 + n_1 + \ldots n_N) - \lambda_i (C_{i0} + C_{i1} + \ldots C_{iN}) \quad ,
\end{aligned}
\tag{4}
$$

for $i = 1:6$, considering that ρ is written as $\rho = \rho_0 + \rho_1(t)$, where ρ_0 is a constant value. We must notice that this system is undetermined because 7xN unknowns exist for only seven equations. We overcome this difficulty by constructing the following recursive system:

$$
\begin{aligned}
\frac{d}{dt} n_0(t) &= \frac{\rho_0 - \beta}{\Lambda} n_0(t) + \sum_{i=1}^{6} \lambda_i C_{i0}(t) \quad , \\
\frac{d}{dt} C_{10}(t) &= \frac{\beta_1}{\Lambda} n_0(t) - \lambda_1 C_{10}(t) \quad , \\
&\vdots \\
\frac{d}{dt} C_{60}(t) &= \frac{\beta_6}{\Lambda} n_0(t) - \lambda_6 C_{60}(t) \quad ,
\end{aligned}
\tag{5}
$$

for the first term of the solution expansion and

$$
\begin{aligned}
\frac{d}{dt} n_j(t) &= \frac{\rho_0 - \beta}{\Lambda} n_j(t) + \sum_{i=1}^{6} \lambda_i C_{ij}(t) + \frac{\rho_1(t)}{\Lambda} n_{j-1}(t) \quad , \\
\frac{d}{dt} C_{1j}(t) &= \frac{\beta_1}{\Lambda} n_j(t) - \lambda_1 C_{1j}(t) \quad , \\
&\vdots \\
\frac{d}{dt} C_{6j}(t) &= \frac{\beta_6}{\Lambda} n_j(t) - \lambda_6 C_{6j}(t) \quad ,
\end{aligned}
\tag{6}
$$

for the generic term of the solution expansion, with $j = 1:N$. Here we need to remark that although not unique this decomposition is done in such manner that the fulfillment of equation (4) is guaranteed. Recasting this recursive system in matrix form, we have:

$$\begin{pmatrix} \frac{d}{dt}n_0 \\ \frac{d}{dt}C_{10} \\ \vdots \\ \frac{d}{dt}C_{60} \end{pmatrix} = \begin{pmatrix} \frac{\rho_0 - \beta}{\Lambda} & \lambda_1 & \cdots & \lambda_6 \\ \frac{\beta_1}{\Lambda} & -\lambda_1 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ \frac{\beta_6}{\Lambda} & 0 & \cdots & -\lambda_6 \end{pmatrix} \begin{pmatrix} n_0 \\ C_{10} \\ \vdots \\ C_{60} \end{pmatrix} \tag{7}$$

for the homogeneous equation and

$$\begin{pmatrix} \frac{d}{dt}n_j \\ \frac{d}{dt}C_{1j} \\ \vdots \\ \frac{d}{dt}C_{6j} \end{pmatrix} = \begin{pmatrix} \frac{\rho_0 - \beta}{\Lambda} & \lambda_1 & \cdots & \lambda_6 \\ \frac{\beta_1}{\Lambda} & -\lambda_1 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ \frac{\beta_6}{\Lambda} & 0 & \cdots & -\lambda_6 \end{pmatrix} \begin{pmatrix} n_j \\ C_{1j} \\ \vdots \\ C_{6j} \end{pmatrix} + \begin{pmatrix} \frac{\rho_1(t)}{\Lambda} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} n_{j-1} \\ C_{1j-1} \\ \vdots \\ C_{6j-1} \end{pmatrix} \tag{8}$$

for the nonhomogeneous equations, with $j = 1:N$. To this point, we must underline that the solution of equations (7) and (8) are straightly determined by the well known results of first order linear differential matrix equations. In fact, let us consider the first order linear differential matrix equation of order N:

$$\frac{dY}{dt} - AY = H(t) \quad , \tag{9}$$

where Y is the unknown column vector, H is the source vector and A is a matrix with constant entries. The well known solution for matrix equation (9) reads as:

$$Y(t) = \exp(At)\,Y(0) + \int_0^t \exp(At)\,H(t-\tau)\,d\tau \quad . \tag{10}$$

On the other hand, for the special problems in which the eigenvalues of the matrix A are distincts, the exponential of matrix A is expressed by:

$$\exp(At) = X \exp(D) X^{-1} X(0) \quad , \tag{11}$$

where X is the matrix of the eigenvectors of A and X^{-1} is its inverse. The matrix D is the diagonal matrix whose elements are the eigenvalues of the matrix A. Bearing in mind that the eigenvalues of the matrix appearing in equations (7) and (8) are distinct, the solution of problem (1) is well determined by equation (3), recalling that the solution of the recursive systems (7) and (8) are determined by equations (10) and (11). To complete the solution analysis we claim that the homogeneous problem (7) satisfies the initial condition (2), meanwhile for the nonhomogeneos problems we impose null initial condition. Finally, for the sake of completeness, we must recall the existence of several approaches to derive the solution of equation (9). For illustration we refer to the one based upon the application of the combined Laplace Transform technique and matrix decomposition, due to its aptness to handle matrix problem with large N (N up to 1500). We expect that we can handle the kinetic problem with even larger matrix orders using proper numerical schemes to invert the Laplace Transform solution. Further, this methodology also allows us to solve matrix problems with multiple eigenvalues, see Moler et al. [16].

3. NUMERICAL RESULTS

We show the feasibility of the Decomposition method to work out the point kinetics problem with time variable reactivities. We begin assuming constant reactivity. For such example let us assume the following parameters: $n(0)=1$, $\Lambda = 0.00002$, $\beta = 0.007$, $\beta_i = (0.000266,\ 0.001491,\ 0.001316,\ 0.002849,\ 0.000896,\ 0.000182)$ and $\lambda_i = (0.0127,\ 0.0317,\ 0.155,\ 0.311,\ 1.4,\ 3.87)$.

Table 1: Comparisons of the numerical results for the neutron density with constant reactivity $\rho(t) = \rho_0$.

ρ_0	Time (s)	Decomposition Solution	Stiffness Confinement Method	Piecewise Constant Approximation
0.03	t =1	2.2098	2.2254	2.2098
	t = 10	8.0192	8.0324	8.0192
	t = 20	2.8295	2.8351	2.8297
0.07	t = 0.01	4.5088	4.5001	4.5088
	t = 0.5	5.3459e+3	5.3530 e+3	5.3459 e +3
	t = 20	2.0592e+11	2.0627e+11	2.0591e+11
0.08	t = 0.01	6.2029	6.2046	6.0229
	t = 0.5	1.4104 e+3	1.4089 e+3	1.4104 e+3
	t = 20	6.1633e+23	6.1574 e+23	6.1634 e+23

In Table 1, we display the numerical results encountered for the neutron density using equation (11) considering $\rho_0 = 0.03, 0.07$ and 0.08 and the time ranging from 0.01 to 20 seconds. When we perform the comparison of these results with the ones achieved by Stiffness Confinement Method (SCM) proposed by Chao et al. [10] and Piecewise Constant Approximation (PCA) method proposed by Kinard et al. [13], we promptly figure out the expected good agreement, bearing in mind that the results considered in the comparison were obtained by numerical methods. Now we are in position to solve problem with time variable reactivity. To this end, initially we assume that the reactivity is described by a linear function of time. Considering the same parameters of the previous problem, in Table 2 we report the results determined for the neutron density obtained by equation (12), for the time taking the values $t = 2, 4$ and 6 seconds. The comparison of these results with the ones attained by Generalized Runge-Kutta (GRK) method proposed by Sanchez et all. [11] and Piecewise Constant Approximation (PCA) method proposed by Kinard et al. [13], shows a good agreement, taking again the numerical character of the methods used in the comparisons into account. To this point, we have to mention that we control the accuracy of reported results by increasing the number of the terms in the series solution (N in Table 2). So far, to reach an accuracy of four significant digits, we notice the necessity of increasing the number of terms in the series solution as the time proceeds from 2 seconds to 6 seconds that means there exists a dependence of the convergence rate of the solution on time.

Table 2: Comparisons of the numerical results for the neutron density with linear reactivity $\rho(t) = 0.0007\,t$.

Time (s)	Decomposition Method	N	Generalized Runge-Kutta	Piecewise Constant Approximation
t = 2	1.3321	8	1.3382	1.3382
t = 4	2.2271	14	2.2286	2.2278
t = 6	5.5798	54	5.5830	5.5802

To complete the analysis, we now consider the point kinetics equations with time sinusoidal reactivity, with the parameters: $\rho_0 = 0.00073$, $n(0) = 1$, $\Lambda = 0.00003$, $\beta = 0.006473$, $\beta_i = (0.000214, 0.001423, 0.001247, 0.002568, 0.000748, 0.000273)$ and $\lambda_i = (0.0124, 0.0305, 0.111, 0.301, 1.14, 3.01)$.

Table 3: Comparisons of the numerical results for the neutron density with sinusoidal reactivity $\rho(t) = 0.00073\, sin(t)$.

Time (s)	Decomposition Method	N	Hansen
t = 1	1.12351	25	1.12396
t = 2	1.16816	30	1.10688
t = 3	1.07429	53	1.07442
t = 4	0.95527	93	0.95380
t = 5	0.90454	120	0.90737
t=10	0.98172	410	0.98464

In Table 3, we report the numerical results for the neutron density obtained by the proposed methodology for time ranging from 1 to 10 seconds. Given a closer look to this table, we readily notice a good comparison of the results obtained with the ones attained by Hansen et al., as well as once more the dependence of the rate of convergence of the solution on time, in other words, increasing the number of terms summation in the series solution (N in Table 3) with the increasing of time.

4. CONCLUSIONS

Concluding we would like to stress that we succeed in the issue of searching for an analytical solution of the point kinetics equations with an arbitrary time variable reactivity, bearing in mind that to our knowledge, this sort of solution does not exists in literature, for a general reactivity function. We realize the relevance of this work noticing that, besides the analytical feature and simplicity, we are confident to point out that this methodology is a robust approach to solve the point kinetics equations either under a mathematical and a computational point of view. The proposed solution is quite general because is valid for any arbitrary time variable function of the reactivity, once the existence of the solution is guaranteed. In addition, this solution is also suitable to generate benchmark results because we can control the error by just increasing the number of terms summation in the series solution. We expect that will be possible to mitigate the observed numerical convergence rate dependence on time using the concept of analytical continuation. This approach could also serve to solve the point kinetics equations in a quasi-static framework, between two shape recalculations, and thus be used in full system dynamic computational schemes. In addition, we do believe that it is a straightforward task to extend this solution to nonlinear problem by including the effect of the nonlinearity as special polynomials on the source term appearing in equation (12). Furthermore, from the previous discussion, we are confident that with this work can pave the road to solve the spatial diffusion kinetic equation applying the spectral method with an eigenfunction expansion leading to a point-like problem. Future investigations will be conducted in these directions.

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