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AN ANALYTICAL SOLUTION OF THE ONE-DIMENSIONAL NEUTRON DIFFUSION KINETIC EQUATION IN CARTESIAN GEOMETRY

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ABSTRACT

In this work we report an analytical solution for the monoenergetic neutron diffusion kinetic equation in cartesian geometry. Bearing in mind that the equation for the delayed neutron precursor concentration is a first order linear differential equation in the time variable, to make possible the application of the GITT approach to the kinetic equation, we introduce a fictitious diffusion term multiplied by a positive small value ε. By this procedure, we are able to solve this set of equations. Indeed, applying the GITT technique to the modified diffusion kinetic equation, we come out with a matrix differential equation which has a well known analytical solution when ε goes to zero. We report numerical simulations as well study of numerical convergence of the results attained.

1. INTRODUCTION

The General Integral Transform Technique, dubbed as GITT approach, is a well established methodology to solve analytically linear differential equations for a broad class of problems in the area of physics and engineering. By analytical we mean that no approximation is done along the solution derivation. The main idea of this approach relies on the construction of a pair of transformation from the laplacian adjoint terms appearing in the differential equation to be solved. This fact allows us to write the solution as series expansion in terms of the orthogonal eigenfunctions attained from the solution of the auxiliary Sturm-Liouville problem constructed from the adjoint terms. The ortogonality of the eigenfunctions completes the pair of transformation. Exists a vast literature about this method and for illustration we mention the books of Cotta [1, 2]. On the other hand, the project on analytical and experimental benchmark analyses of Accelerator Driven Systems [3, 4] has given motivation to researchers to focus their attention to the task of searching analytical solution for the neutron diffusion kinetic equation, in order to determine benchmark results for computational codes validation. In this sense for instance, we are aware of the works of Maiorino et al. [5], Corno et al. [6] and Dulla et al. [7].

In this work, keeping us in the line of searching analytical solutions we present an analytical solution for the neutron diffusion kinetic equation in cartesian geometry. We specialize the application, without loosing generality, to the one-dimensional, monoenergetic diffusion kinetic equation with one delayed neutron precursor. Bearing in mind that the equation for the delayed neutron precursor concentration is a first order linear differential equation in the time variable, to make possible the application of the GITT approach to the kinetic equation, we introduce a fictitious diffusion term multiplied by a positive small value ε. By this procedure, we are able to solve this set of equations by the discussed method. In fact, applying the GITT technique to the modified diffusion kinetic equation, we come out with a matrix differential equation which has a well known solution when we make ε goes to zero. The main feature of this solution relies on its aptness to handle kinetic problems with large matrix order (up to 1500), because the eigenvalues are distinct. This fact give us certainty to affirm that this methodology is a robust technique to work out one-dimensional diffusion kinetic equation, in a straightforward manner for more realistic physical problems, we mean for the ones considering six delayed neutron precursor and also several energy groups (up to 200) as well multidimensional problems. To complete our study, we report numerical simulations as well analysis of convergence of the results attained.

2. THE ANALYTICAL SOLUTION

Let us consider, without loosing generality, the one-dimensional diffusion kinetic equation in cartesian geometry assuming monoenergetic neutrons and one delayed neutron concentration:

$$\begin{aligned} \frac{1}{v}\frac{\partial\phi(x,t)}{\partial t} &= D\frac{\partial^2\phi(x,t)}{\partial x^2} + \left(-\Sigma_a + (1-\beta)\upsilon\Sigma_f\right)\phi(x,t) + \lambda C(x,t)\,, \\ \frac{\partial C(x,t)}{\partial t} &= \beta\upsilon\Sigma_f\phi(x,t) - \lambda C(x,t)\,, \end{aligned} \qquad (1)$$

for t >0 and 0 < x < L, subjected to the vacuum boundary condition ($\phi(0,t) = \phi(L,t) = 0$) and the initial condition:

$$\begin{aligned} \phi(x,0) &= \phi_0\,, \\ C(x,0) &= \frac{\beta\nu\Sigma_f}{\lambda}\phi_0\,. \end{aligned} \qquad (2)$$

Here $\phi(x,t)$ denotes the neutron flux, ϕ_0 is the neutron flux at the initial time (*t*=0), $C(x,t)$ is the delayed neutron concentration and v, D, Σ_a, β, ν, Σ_f and λ are the standard neutron parameters.

In order to solve the problem (1) by the spectral method, known as GITT [1, 2], we introduce a fictitious diffusion term in the equation (1a) for the precursor concentration, assuming also the homogeneous boundary condition, we mean, $C(0,t) = C(L,t) = 0$. Therefore we can write the system (1) like:

$$\frac{1}{v}\frac{\partial \phi(x,t)}{\partial t} = D\frac{\partial^2 \phi(x,t)}{\partial x^2} + \left(-\Sigma_a + (1-\beta)\upsilon\Sigma_f\right)\phi(x,t) + \lambda C(x,t)\,,$$
$$\frac{\partial C(x,t)}{\partial t} = \varepsilon\frac{\partial^2 C(x,t)}{\partial x^2} + \beta\upsilon\Sigma_f\phi(x,t) - \lambda C(x,t)\,, \tag{3}$$

where ε is a positive small parameter. This assumption allows us to apply the GITT method to solve problem (3). For such, we expand the neutron flux and delay neutron concentration in the following series:

$$\phi(x,t) = \sum_{n=1}^{\infty} \varphi_n(t)\sin(\gamma_n x)\,, \tag{4}$$

$$C(x,t) = \sum_{n=1}^{\infty} \xi_n(t)\sin(\gamma_n x)\,, \tag{5}$$

with the eigenvalues γ_n taking the values $\gamma_n = n\pi/L$, for n=1,2,3,... Replacing these ansatz in the system of equations (3), taking moments and applying the orthogonality property of the eigenfunctions, we come out with the ensuing matrix equation for the coefficients of the solution expansion:

$$\frac{d}{dt}\begin{pmatrix}\varphi_n(t)\\ \xi_n(t)\end{pmatrix} = -\begin{pmatrix}a_n & b\\ c & d_n\end{pmatrix}\begin{pmatrix}\varphi_n(t)\\ \xi_n(t)\end{pmatrix}, \tag{6}$$

for n=1:N, where

$$a_n = -vD[-\gamma_n^2 + \frac{1}{D}(-\Sigma_a + (1-\beta)v\Sigma_f)],\;\; b = -v\lambda\,,\; c = -\beta v\Sigma_f \tag{6a}$$

and

$$d_n = -(\varepsilon\gamma_n^2 - \lambda)\,. \tag{6b}$$

We must recall that the first order homogeneous linear matrix equation written like

$$X'(t) + AX(t) = 0\,, \tag{7}$$

has the well known solution:

$$X(t) = \exp(-At)X(0)\,. \tag{8}$$

Further, for the matrix problems in which the eigenvalues of the A matrix are distinct, the exponential matrix can be expressed as:

$$\exp(-At) = Y\exp(-Dt)\,Y^{-1}\,, \tag{9}$$

where Y is the matrix of the eigenvectors of the matrix A, Y^{-1} is its inverse and D is the diagonal matrix of the eigenvalues of the matrix A. We must remember the existence in the literature of several approaches [8] to derive the solution appearing in equation (9). For instance, we mention the technique referring to the application of the combined Laplace transform technique and matrix decomposition. This methodology has the advantage of being general, in the sense that can be applied to solve this sort of problems with repeated eigenvalues. For more details see works of Segatto et al. [9, 10]. Concluding, we must observe that the solution of the problem (1) is well determined by the equations (4) and (5) where the expansion coefficients of the solution are evaluated by the formula (8), when ε goes to zero.

3. NUMERICAL RESULTS

To show the aptness of the proposed method to handle the diffusion kinetic equation, in the sequel we solve the following problems. We begin presenting the convergence of the results encountered for the kinetic equation to the exact solution when ε goes to zero. For such, let us solve the problem with the following parameters [4]: D = 0.96343, v = 1.103497×10^{7}, $\Sigma_a = 1.58430\times10^{-2}$, $\nu\Sigma_f = 3.33029\times10^{-2}$, L = 22.9, β = 0.0045 and λ = 0.08.

In Table 1, we display the results attained by this methodology for the neutron flux making the values of ε ranges from 10^{-3} to 10^{-7}. From a simple inspection of the displayed results, we promptly figure out a coincidence of eight significant digits between the values encountered for $\varepsilon = 10^{-6}$ and $\varepsilon = 10^{-7}$. Therefore, we may say that the fictitious problem solved converges to the exact solution of the diffusion kinetic equation.

Table 1. The ε-convergence of the neutron flux

ε	ϕ [$cm^{-2}s^{-1}$]
10^{-3}	0.17002198
10^{-4}	0.17002082
10^{-5}	0.17002055
10^{-6}	0.17002052
10^{-7}	0.17002052

The question left to be answered regards to the issue of the analysis of the convergence of the solution reported when we increase the number of terms in the series solution. To this end, let us consider the kinetic problem with the same parameters of the previous one. In Table 2, we show the numerical convergence of the results obtained for the neutron flux increasing the number of terms in the series solution up to seven hundred. Given a closer look to Table 2, we readily realize that we reach an accuracy of eight significant digits, when we make the summation of seven hundred terms in the series solution. This discussion give us confidence to affirm that the results appearing in Table 2 for the neutron flux has an accuracy of eight significant digits for the value $\varepsilon = 10^{-6}$ except for the round-off error.

Table 2. Neutron flux simulations for x = 11.45 cm, t = 1 sec and $\varepsilon = 10^{-6}$.

N	ϕ [$cm^{-2}s^{-1}$]	N	ϕ [$cm^{-2}s^{-1}$]
5	0.17012145	70	0.17002132
10	0.17005796	75	0.17001988
15	0.17000575	80	0.17001994
20	0.17001105	85	0.17002105
25	0.17002617	90	0.17002101
30	0.17002477	95	0.17002012
35	0.17001760	100	0.17002016
40	0.17001816	200	0.17002044
45	0.17002234	300	0.17002050
50	0.17002206	400	0.17002051
55	0.17001932	500	0.17002052
60	0.17001948	600	0.17002053
65	0.17002141	700	0.17002053

Once shown the numerical convergence, in order to visualize the flux behavior, next in Figure (1), we plot a graphic for the neutron flux as function of the spatial variable and time. Given a closer look to this figure, as expected, we promptly observe the asymptotic behavior of the neutron flux as time goes to infinity, we mean the time depending solution tends to the stationary solution with the time increasing.

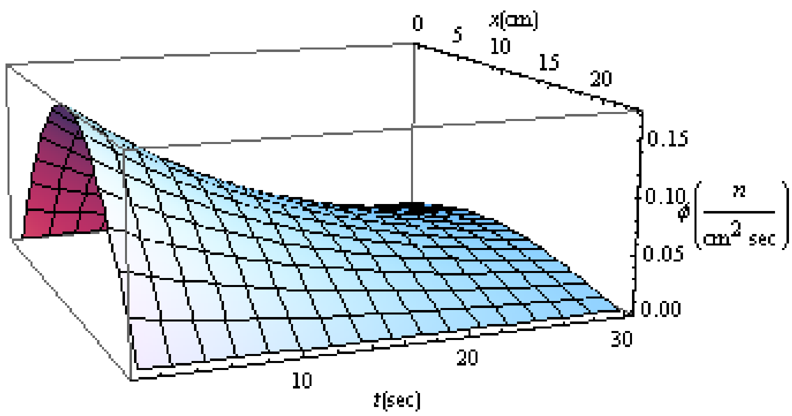

Figure 1. Neutron flux behavior as function of position and time.

4. CONCLUSIONS

Analyzing the previous results, we are certain to affirm that the proposed method is an important and promising methodology to work out the diffusion kinetic equation in analytical fashion. Besides the elegance and the analytical feature, we bolster this affirmative recalling that this approach is quite general in the sense that the matrix exponential formula solution discussed doesn't pose limitation in the calculations for a matrix of order up to 1500. That means this technique can be applied in a straightforward manner to more realistic physical problems, particularly the ones considering six delayed neutron group and also multigroup model with up two hundred groups of energy. Further, we need to say that the extension of this methodology to multilayered slab problem is an easy task by properly applying interface boundary condition. In addition, we have to underline the relevant aspect of this technique inherent to analyticity, regarding its aptness to generate benchmark solutions for computational codes validation. In fact, we claim that we can get exact results except for the round-off error with a prescribed accuracy by increasing the number of terms summation in the series solution, paying attention to the expected observed oscillatory behavior of the convergence. From the previous discussion, we obviously envisage the natural extension of this sort of solution for multidimensional diffusion kinetic equation in cartesian geometry considering properly the solution series expansion, in order to reduce the multidimensional problem to the one here solved. So far, the natural direction of our future research shall be focused in these topics.

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