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On Chiral and Nonchiral $1D$ Supermultiplets

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On Chiral and Nonchiral 1D Supermultiplets*

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Abstract

In this talk I discuss and clarify some issues concerning chiral and nonchiral properties of the one-dimensional supermultiplets of the \mathcal{N} -Extended Supersymmetry. Quaternionic chirality can be defined for $\mathcal{N} = 4, 5, 6, 7, 8$. Octonionic chirality for $\mathcal{N} = 8$ and beyond. Inequivalent chiralities only arise when considering several copies of $\mathcal{N} = 4$ or $\mathcal{N} = 8$ supermultiplets.

Keywords: Supersymmetric Quantum Mechanics.

1 Introduction

The 1D \mathcal{N} -Extended Superalgebra, with \mathcal{N} odd generators Q_I ($I = 1, 2, \dots, \mathcal{N}$) and a single even generator H satisfying the (anti)-commutation relations

$$\begin{aligned}\{Q_I, Q_J\} &= \delta_{IJ}H, \\ [H, Q_I] &= 0,\end{aligned}\tag{1}$$

is the superalgebra underlying the Supersymmetric Quantum Mechanics [1].

In recent years the structure of its linear representations has been unveiled by a series of works [2]–[15].

The linear representations under considerations (supermultiplets) contain a finite, equal number of bosonic and fermionic fields depending on a single coordinate (the time). The operators Q_I and H act as differential operators. The linear representations are characterized by a series of properties which, for sake of consistency, are reviewed in the appendix.

The *minimal* linear representations (also called *irreducible supermultiplets*) are given by the minimal number n_{min} of bosonic (fermionic) fields for a given value of \mathcal{N} . The value n_{min} is given [2] by the formula

$$\begin{aligned}\mathcal{N} &= 8l + m, \\ n_{min} &= 2^{4l}G(m),\end{aligned}\tag{2}$$

where $l = 0, 1, 2, \dots$ and $m = 1, 2, 3, 4, 5, 6, 7, 8$.

$G(m)$ appearing in (2) is the Radon-Hurwitz function

m	1	2	3	4	5	6	7	8
$G(m)$	1	2	4	4	8	8	8	8

(3)

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Non-minimal linear representations have been discussed in [4, 12, 13, 15].

The construction of off-shell invariant actions (sigma-models) based on these representations has been given in [4, 14, 15]. In this approach the superfields techniques that can be employed to recover invariant actions for the given supermultiplets [16] are no longer necessary.

In this talk I will address and clarify a specific issue. In [17] (see also references therein) the $\mathcal{N} = 4$ minimal linear representations (given by the field content $(4, 4)$, $(3, 4, 1)$, $(2, 4, 2)$ and $(1, 4, 3)$) are given in two different forms, called “chiral” and “twisted chiral supermultiplets”. This double realization of the $\mathcal{N} = 4$ representations can also be extended to the nonlinear realizations (the $(3, 4, 1)_{nl}$ and the $(2, 4, 2)_{nl}$ which are recovered from the $(4, 4)$ root supermultiplet via a specific construction based on the supersymmetrization of the first Hopf fibration, see [18] and references therein).

For our purposes it is convenient to refer to this doubling as “chirality”. The two versions of the supermultiplets will be conveniently denoted as “chiral” and “antichiral”, respectively. Chirality seems therefore encoded in all minimal (linear and non-linear) $\mathcal{N} = 4$ representations. On the other hand, the $(4, 4)$ root supermultiplet which generates all remaining $\mathcal{N} = 4$ representations (for the linear ones such as $(3, 4, 1)$, $(2, 4, 2)$ and $(1, 4, 3)$ through the dressing, see [2] and the appendix), is in one-to-one correspondence with the (Weyl-type) realization of the $Cl(4, 0)$ Clifford algebra [2]. This realization is unique. Therefore, a natural question to be asked is whether the notion of chirality is truly there. The answer, as we will see, is rather subtle. Since chirality (if there) is inherited from the properties of the root supermultiplets, for our purpose is sufficient to address this problem for the root supermultiplets at a given \mathcal{N} . In particular, for $\mathcal{N} = 4$, its $(4, 4)$ root supermultiplet and, for $\mathcal{N} = 8$, its $(8, 8)$ root supermultiplet.

2 Quaternions, octonions and the $\mathcal{N} = 4, 8$ root supermultiplets: are they chiral?

The $\mathcal{N} = 4$ root supermultiplet of field content $(4, 4)$ admits 4 bosonic fields x, x_i and 4 fermionic fields ψ, ψ_i ($i = 1, 2, 3$). Its supertransformations are expressed in terms of the quaternionic structure constants (δ_{ij} and the totally antisymmetric tensor ϵ_{ijk}). Without loss of generality we can explicitly realize them as

$$\begin{aligned}
 Q_4 x &= \psi, & Q_i x &= \psi_i, \\
 Q_4 x_j &= \psi_j, & Q_i x_j &= -\delta_{ij} \psi + s \epsilon_{ijk} \psi_k, \\
 Q_4 \psi &= \dot{x}, & Q_i \psi &= -\dot{x}_i, \\
 Q_4 \psi_j &= \dot{x}_j, & Q_i \psi_j &= \delta_{ij} \dot{x} - s \epsilon_{ijk} \dot{x}_k.
 \end{aligned}
 \tag{4}$$

A sign $s = \pm 1$ has been introduced. It corresponds to the convention of choosing the overall sign of the totally antisymmetric tensor ϵ_{ijk} . It discriminates the chiral ($s = +1$) from the antichiral ($s = -1$) $\mathcal{N} = 4$ root supermultiplet. Due to its origin, we refer to this chirality as “quaternionic chirality”.

A similar notion of chirality can be introduced for the $\mathcal{N} = 8$ $(8, 8)$ root supermultiplet which, without loss of generality, can be expressed [4] by replacing the quaternionic struc-

ture constant ϵ_{ijk} in (4) with the totally antisymmetric octonionic structure constants C_{ijk} , $i, j, k = 1, 2, \dots, 7$, such that $C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1$.

We have the following supertransformations acting on the 8 bosonic fields x, x_i and the 8 fermionic fields ψ, ψ_i :

$$\begin{aligned} Q_8 x &= \psi, & Q_i x &= \psi_i, \\ Q_8 x_j &= \psi_j, & Q_i x_j &= -\delta_{ij} \psi + s C_{ijk} \psi_k, \\ Q_8 \psi &= \dot{x}, & Q_i \psi &= -\dot{x}_i, \\ Q_8 \psi_j &= \dot{x}_j, & Q_i \psi_j &= \delta_{ij} \dot{x} - s C_{ijk} \dot{x}_k. \end{aligned} \tag{5}$$

The sign $s = \pm 1$ defines the ‘‘octonionic chirality’’ of the $\mathcal{N} = 8$ root supermultiplet.

It is easily realized that the chiral and antichiral supermultiplets are isomorphic and related by a Z_2 transformation. In the $\mathcal{N} = 4$ case this is easily achieved by exchanging $x_2 \leftrightarrow x_3$, $\psi_2 \leftrightarrow \psi_3$ and by relabeling the supersymmetry transformations $Q_2 \leftrightarrow Q_3$. It therefore looks that the chirality is an abusive notion which should be dismissed as useless. In the following we will prove that this is not quite so. It is certainly true that the notion of ‘‘quaternionic chirality’’ turns out to be useless for the $\mathcal{N} = 3$ (4, 4) root supermultiplet (it is obtained from (4) by disregarding the Q_4 supertransformation). The difference w.r.t. the $\mathcal{N} = 4$ case lies in the fact that the Z_2 isomorphism in this case can be imposed *without* relabeling the supertransformations (it is sufficient, e.g., to map $\psi \mapsto -\psi$, $\psi_i \mapsto -\psi_i$, while leaving x, x_i unchanged). The relabeling of the supertransformations, which is essential to implement the Z_2 isomorphism for the $\mathcal{N} = 4$ quaternionic chirality (and the $\mathcal{N} = 8$ octonionic chirality), makes all the difference w.r.t. the $\mathcal{N} = 3$ root supermultiplet case.

3 Irreducible $\mathcal{N} = 5, 6, 7$ supermultiplets are nonchiral, reducible supermultiplets are chiral

The $\mathcal{N} = 5, 6, 7, 8$ root supermultiplets have field content (8, 8). They all admit a decomposition into two minimal $\mathcal{N} = 4$ supermultiplets obtained by suitably picking 4 supertransformations out of the \mathcal{N} original ones. For $\mathcal{N} = 8$ we have $\binom{8}{4} = 70$ inequivalent choices of 4 supertransformations. 14 of such choices produce two minimal $\mathcal{N} = 4$ supermultiplets (in the remaining 56 cases we end up with a non-minimal, reducible but indecomposable, $\mathcal{N} = 4$ representation [15]). This number can be understood as follows: one can pick any 3 supertransformations (necessarily ending up with two $\mathcal{N} = 3$ root supermultiplets). Then one is left with 5 possible choices for the fourth supertransformation. In 1 case ($\frac{1}{5}$ of the total) we get two separate minimal $\mathcal{N} = 4$ supermultiplets. In the 4 remaining cases we end up instead with an indecomposable non-minimal $\mathcal{N} = 4$ supermultiplet.

The $\mathcal{N} = 8$ supertransformations in (5) can be associated with the octonions: Q_8 with the octonionic identity and the Q_i 's with the 7 imaginary octonions. With this identification, the 14 combinations which produce a decomposition into two separate $\mathcal{N} = 4$

supermultiplets are obtained as follows: 7 by picking up Q_8 and 3 supertransformations lying in one of the 7 lines of the Fano plane. The remaining 7 by picking up 4 supertransformations in the Fano plane which are complementary to one of the 7 lines.

Incidentally, this counting proves that for $\mathcal{N} = 5$ (and, *a fortiori*, $\mathcal{N} = 6, 7$) one can always find a decomposition into two minimal $\mathcal{N} = 4$ supermultiplets. Indeed, $\mathcal{N} = 5$ supertransformations can be obtained in two ways: either, *case a*, with 5 supertransformations lying on the Fano plane or, *case b*, Q_8 and 4 supertransformations lying on the Fano plane. In case *a*, 4 of the 5 supertransformations are necessarily complementary to a line; in case *b* we have two possibilities, either the 4 supertransformations lying on the Fano plane are complementary to a line or 3 of them belong to one of the lines. In this case to get the minimal $\mathcal{N} = 4$ decomposition we have to pick them together with Q_8 .

Well, what all this has to do with chirality and quaternionic chirality? The fact is that, in all cases, no matter which decomposition into two minimal $\mathcal{N} = 4$ supermultiplets is taken, one always ends up with two $\mathcal{N} = 4$ root supermultiplets of opposite chirality. The chirality, which is irrelevant when a single supermultiplet is concerned, suddenly turns out to be crucial when several copies of them are considered. The $\mathcal{N} = 5, 6, 7$ supermultiplets are essentially non-chiral because in their $\mathcal{N} = 4$ minimal decomposition two multiplets, a quaternionic chiral and a quaternionic antichiral one, are produced. The same is true for the $\mathcal{N} = 8$ root supermultiplet. It is quaternionic non-chiral in the sense here specified. For $\mathcal{N} = 8$ on the other hand, another notion of chirality, the octonionic chirality, can be introduced (no such notion makes sense for $\mathcal{N} = 5, 6, 7$). The $\mathcal{N} = 8$ root supermultiplet is quaternionically non-chiral and octonionically chiral.

This result admits the following reformulation: given two minimal $\mathcal{N} = 4$ supermultiplets, they can be combined into a single minimal supermultiplet for $\mathcal{N} = 5, 6, 7$ or 8 if and only if the two supermultiplets possess opposite chirality. Indeed, if one tries to link together with an extra supersymmetry transformation two supermultiplets of the same chirality, one ends up in a contradiction.

Two chiral supermultiplets are isomorphic (via the Z_2 isomorphism) with two antichiral supermultiplets, while they are not isomorphic with a chiral and an antichiral supermultiplet, due to the fact that one cannot flip the chirality of the first multiplet without flipping the chirality of the second multiplet. The reason is that the Z_2 isomorphism requires a relabeling of the supertransformations, as discussed in the previous section, not just a transformation of the fields entering the supermultiplets (as it is the case for $\mathcal{N} = 3$).

Given a certain number n of $\mathcal{N} = 4$ (4, 4) root supermultiplets, inequivalent chiralities are discriminated by the modulus (to make it Z_2 -invariant) m given by

$$m = \left| \sum_i s_i \right| = |n_+ - n_-|, \quad (6)$$

where $n = n_+ + n_-$, with n_{\pm} denoting the number of chiral (antichiral) supermultiplets.

For $n = 1$ we have that $m = 1$. The non-chiral case ($m = 0$) is only possible for n even.

It is certainly possible and perhaps even convenient to think of m as an energy. In this interpretation the non-chiral state corresponds to the vacuum. For odd n , the vacuum energy is positive. The vacuum is doubly degenerated and spontaneously broken. For the

single ($n = 1$) supermultiplet the two (equivalent) choices of the chirality (± 1) can be interpreted as a spontaneous breaking of the Z_2 symmetry.

For a collection of n root supermultiplets of $\mathcal{N} = 8$, the previous steps are repeated in terms of the notion of ‘‘octonionic chirality’’ (each $\mathcal{N} = 8$ supermultiplet is non-chiral for what concerns the quaternionic chirality).

4 Weyl on the Leibniz-Clarke debate on the nature of space

The problem of understanding the nature of chirality for quaternionic (anti)chiral $\mathcal{N} = 4$ root supermultiplets and octonionic (anti)chiral $\mathcal{N} = 8$ root supermultiplets is similar to the problem of understanding the nature of parity (mirror symmetry) of the Euclidean space. A nice framework was provided by Weyl in his popular book on Symmetry [19].

The famous Clarke-Leibniz debate concerning the nature of the space (either absolute, thesis defended by the Newtonians, Clarke was one of them) or relative (thesis defended by Leibniz who anticipated some of the arguments later used by Mach) is well-known. The Clarke-Leibniz debate was expressed in the metaphysical and theological language of the time. Weyl, in his book, reformulates the position of Leibniz (and also Kant) by expressing the relative nature of the space for the specific Z_2 parity transformation associated with the mirror symmetry. Weyl, in his argument, mimicks the theological framework used by Clarke and Leibniz.

The Weyl argument goes as follows. Let us suppose that God at the beginning creates out of nothing, in the empty space, a hand. We have no way to say whether this hand is left or right (in the empty space the hand is so good as its mirror image). Now, let us suppose that after creating the first hand, God creates a second hand. It is only after this second hand has been created that the notion of right or left has been introduced. The second hand can be aligned with the first one or be of opposite type. Right-handedness or left-handedness is a relative notion based on the referential provided by the first hand.

The argument of Weyl clearly applies to the notion of quaternionic (or octonionic) chirality for $\mathcal{N} = 4$ ($\mathcal{N} = 8$) supermultiplets. In the case of supermultiplets on the other hand we can still go further. Let us focus on the quaternionic chirality of the $\mathcal{N} = 4$ supermultiplets. We can go on with theological speculation. Let us suppose now that, after the second hand has been created (so that we have now two hands floating in the empty space) God performs a third act of creation, creating a handless body. This handless body, floating in the empty space, try to grasp and attach the two floating hands to its handless arms. He/she can only do that if the two hands are of opposite chirality. He/she is unable to do that if they come with the same chirality. Needless to say, the handless body can stay for the full $\mathcal{N} = 8$ supersymmetry which can be obtained by linking together two minimal $\mathcal{N} = 4$ root supermultiplets. With this baroque image of handless bodies desperately seeking floating hands in the empty space, we can leave the Clarke-Leibniz-Weyl theological speculations.

5 Conclusions

In this paper I clarified the issue of the chirality associated with $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supermultiplets. I pointed out that there are two notions of chirality which can be introduced: quaternionic chirality which applies to $\mathcal{N} = 4, 5, 6, 7, 8$ minimal supermultiplets and octonionic chirality which applies to $\mathcal{N} \geq 8$ minimal supermultiplets. For $\mathcal{N} > 4$ the minimal supermultiplets are quaternionically nonchiral (every decomposition into two minimal $\mathcal{N} = 4$ minimal supermultiplets produces two supermultiplets of opposite chirality). The same is true (concerning octonionic chirality) for the minimal supermultiplets with $\mathcal{N} > 8$. The notion of quaternionic chirality applies to the $\mathcal{N} = 4$ minimal supermultiplets, while the notion of octonionic chirality to the minimal $\mathcal{N} = 8$ supermultiplets.

In both cases a *single* chiral supermultiplet is isomorphic to its antichiral counterpart via a Z_2 transformation. Chirality cannot be detected if we deal with a single $\mathcal{N} = 4$ ($\mathcal{N} = 8$) supermultiplet.

On the other hand, the chirality issue becomes important when we are dealing with reducible $\mathcal{N} = 4$ ($\mathcal{N} = 8$) supermultiplets given by several copies of chiral and antichiral supermultiplets. Let us take the example of the reducible $\mathcal{N} = 4$ representation given by two separate root supermultiplets (the total field content is $(8, 8) = 2 \times (4, 4)$). There are two inequivalent such reducible representations ($(8, 8)_{red.,ch.}$ and $(8, 8)_{red.,nc.}$). One is chiral ($(8, 8)_{red.,ch.}$); it is given by two $\mathcal{N} = 4$ root supermultiplets of the same chirality. The other one ($(8, 8)_{red.,nc.}$) is nonchiral and given by two $\mathcal{N} = 4$ root supermultiplets of opposite chirality. Only $(8, 8)_{red.,nc.}$ can be “promoted” to a minimal irreducible representation for $\mathcal{N} = 5, 6, 7, 8$ by inserting extra supertransformations linking its two $\mathcal{N} = 4$ component supermultiplets.

The overall chirality of n $\mathcal{N} = 4$ supermultiplets is important when constructing $\mathcal{N} = 4$ off-shell invariant actions (sigma-models) for this large set of n supermultiplets.

The notion of quaternionic chirality for root supermultiplets gets extended to the remaining linear supermultiplets which are obtained by dressing (their chirality is encoded in the chirality of their associated root supermultiplets). It is also extended to the two nonlinear $\mathcal{N} = 4$ supermultiplets ($(3, 4, 1)_{nl}$ and $(2, 4, 2)_{nl}$) which are also produced, see [18], from the $\mathcal{N} = 4$ root supermultiplet. The chirality of the originating root supermultiplet has to be taken into account.

Appendix A.

For completeness we report the definitions, applied to the cases used in the text, of the properties characterizing the linear representations of the one-dimensional \mathcal{N} -Extended Superalgebra. In particular the notions of *mass-dimension*, *field content*, *dressing transformation*, *connectivity symbol*, *dual supermultiplet* and so on, as well as the association of linear supersymmetry transformations with graphs, will be reviewed following [2, 4, 3, 9, 10, 15]. The Reader can consult these papers for broader definitions and more detailed discussions.

Mass-dimension:

A grading, the mass-dimension d , can be assigned to any field entering a linear rep-

representation (the hamiltonian H , proportional to the time-derivative operator $\partial \equiv \frac{d}{dt}$, has a mass-dimension 1). Bosonic (fermionic) fields have integer (respectively, half-integer) mass-dimension.

Field content:

Each finite linear representation is characterized by its ‘‘field content’’, i.e. the set of integers (n_1, n_2, \dots, n_l) specifying the number n_i of fields of mass-dimension d_i ($d_i = d_1 + \frac{i-1}{2}$, with d_1 an arbitrary constant) entering the representation. Physically, the n_l fields of highest dimension are the auxiliary fields which transform as a time-derivative under any supersymmetry generator. The maximal value l (corresponding to the maximal dimensionality d_l) is defined to be the *length* of the representation (a root representation has length $l = 2$). Either n_1, n_3, \dots correspond to the bosonic fields (therefore n_2, n_4, \dots specify the fermionic fields) or viceversa.

In both cases the equality $n_1 + n_3 + \dots = n_2 + n_4 + \dots = n$ is guaranteed.

Dressing transformation:

Higher-length supermultiplets are obtained by applying a dressing transformation to the length-2 root supermultiplet. The root supermultiplet is specified by the \mathcal{N} supersymmetry operators \widehat{Q}_i ($i = 1, \dots, \mathcal{N}$), expressed in matrix form as

$$\widehat{Q}_j = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \gamma_j \\ -\gamma_j \cdot H & 0 \end{pmatrix}, \quad \widehat{Q}_{\mathcal{N}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n \cdot H & 0 \end{pmatrix}, \quad (\text{A.1})$$

where the γ_j matrices ($j = 1, \dots, \mathcal{N} - 1$) satisfy the Euclidean Clifford algebra

$$\{\gamma_i, \gamma_j\} = -2\delta_{ij}\mathbf{1}_n. \quad (\text{A.2})$$

The length-3 supermultiplets are specified by the \mathcal{N} operators Q_i , given by the dressing transformation

$$Q_i = D\widehat{Q}_iD^{-1}, \quad (\text{A.3})$$

where D is a diagonal dressing matrix such that

$$D = \begin{pmatrix} \widetilde{D} & 0 \\ 0 & \mathbf{1}_n \end{pmatrix}, \quad (\text{A.4})$$

with \widetilde{D} an $n \times n$ diagonal matrix whose diagonal entries are either 1 or the derivative operator ∂ .

Association with graphs:

The association between linear supersymmetry transformations and \mathcal{N} -colored oriented graphs goes as follows. The fields (bosonic and fermionic) entering a representation are expressed as vertices. They can be accommodated into an $X - Y$ plane. The Y coordinate can be chosen to correspond to the mass-dimension d of the fields. Conventionally, the lowest dimensional fields can be associated to vertices lying on the X axis. The higher dimensional fields have positive, integer or half-integer values of Y . A colored edge links two vertices which are connected by a supersymmetry transformation. Each one of the \mathcal{N} Q_i supersymmetry generators is associated to a given color. The edges are oriented. The orientation reflects the sign (positive or negative) of the corresponding

supersymmetry transformation connecting the two vertices. Instead of using arrows, alternatively, solid or dashed lines can be associated, respectively, to positive or negative signs. No colored line is drawn for supersymmetry transformations connecting a field with the time-derivative of a lower dimensional field. This is in particular true for the auxiliary fields (the fields of highest dimension in the representation) which are necessarily mapped, under supersymmetry transformations, in the time-derivative of lower-dimensional fields.

Each irreducible supersymmetry transformation can be presented (the identification is not unique) through an oriented \mathcal{N} -colored graph with $2n$ vertices. The graph is such that precisely \mathcal{N} edges, one for each color, are linked to any given vertex which represents either a 0-mass dimension or a $\frac{1}{2}$ -mass dimension field. An unoriented “color-blind” graph can be associated to the initial graph by disregarding the orientation of the edges and their colors (all edges are painted in black).

Connectivity symbol:

A characterization of length $l = 3$ color-blind, unoriented graphs can be expressed through the connectivity symbol ψ_g , defined as follows

$$\psi_g = (m_1)_{s_1} + (m_2)_{s_2} + \dots + (m_Z)_{s_Z}. \quad (\text{A.5})$$

The ψ_g symbol encodes the information on the partition of the n $\frac{1}{2}$ -mass dimension fields (vertices) into the sets of m_z vertices ($z = 1, \dots, Z$) with s_z edges connecting them to the $n - k$ 1-mass dimension auxiliary fields. We have

$$m_1 + m_2 + \dots + m_Z = n, \quad (\text{A.6})$$

while $s_z \neq s_{z'}$ for $z \neq z'$.

Dual supermultiplet:

A dual supermultiplet is obtained by mirror-reversing, upside-down, the graph associated to the original supermultiplet.

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