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DETERMINATION OF NEUTRON BUILDUP FACTOR USING ANALYTICAL SOLUTION OF ONE-DIMENSIONAL NEUTRON DIFFUSION EQUATION IN CYLINDRICAL GEOMETRY

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ABSTRACT

The principal idea of this work, consist on formulate an analytical method to solved problems for diffusion of neutrons with isotropic scattering in one-dimensional cylindrical geometry. In this area were develop many works that study the same problem in different system of coordinates as well as cartesian system, nevertheless using numerical methods to solve the shielding problem. In view of good results in this works, we starting with the idea that we can represent a source in the origin of the cylindrical system by a Delta Dirac distribution, we describe the physical modeling and solved the neutron diffusion equation inside of cylinder of radius R. For the case of transport equation, the formulation of discrete ordinates S_N consists in discretize the angular variables in N directions and in using a quadrature angular set for approximate the sources of scattering, where the Diffusion equation consist on S_2 approximated transport equation in discrete ordinates. We solved the neutron diffusion equation with an analytical form by the Finite Hankel Transform. Was presented also the build-up factor for the case that we have neutron flux inside the cylinder.

1. INTRODUCTION

Nowadays the scientists research for new sources of energy, because we know that the kernel of the atoms are provided of many energy that extracted with conscious form do not be danger for the environment, this way explain our interesse in nuclear energy. Inside this context, the engineering of nuclear reactors as been a theme so much speakable, above all in physical innovative nuclear reactors. Our basic idea was develop an analytical method for determinate he buildup factor for neutrons case, the description of neutrons' migration inside of the nuclear reactor is phenomenon that have many applications, for example, in nuclear reactor, radiology protection, nuclear medicine, agronomy and more others cases. For making the mathematical modeling of the problem, taking the diffusion equation of neutrons that consist in S_2 approximate of transport equation of Boltzman. This equation represents the balance between production and lose of these particles. In the next section will be describe the mathematical modeling of the neutron diffusion equation included the propose conditions for boundary problem. In the section 3 and 4 we will show the advantages of solve this problem analytically using the Finite Hankel Transform that apply perfectly in cylindrical coordinates problem.

2 PROBLEM FORMULATION

For solve this problem for specific group of energy, we going to solve the diffusion equation applying Hankel transform in r with cylindrical geometry. Let the diffusion equation given by

$$\frac{\partial}{\partial t}\phi = D\triangle_r\phi - \Sigma_R\phi + S(r,t) \tag{1}$$

Where $\triangle_r$ is the elliptic operator given by

$$\triangle_r = r^{-1}\partial_r(r\partial_r) \tag{2}$$

Where D is the coefficient of diffusion of neutrons, ϕ is the neutron flux, Σ_R is the macroscopic cross section of remotion and $S(r,t)$ is the source of problem depends of r and t. With the follow boundary conditions,

$$\frac{\partial\phi}{\partial r}(0,t) = 0 \ , \ \phi(R,t) = 0 \tag{3}$$

3 APPLYING THE FINITE HANKEL TRANSFORM

Now, we will apply the Finite Hankel transform of order zero in (1). Remember that the Hankel transform have as definition, with order p, is given by

$$H_p[f(r); r \to \alpha_n] = \int_0^a rf(r)J_p(r\xi)dr \tag{4}$$

where α_n are the values such that $J_p(\alpha_n a) = 0$ for $n \in \mathbf{N}$, and the inversion form, can be described by

$$f(r) = \frac{2}{a^2}\sum_{n=1}^{\infty}\bar{f}(\alpha_n)\frac{J_p(\alpha_n r)}{[J'_p(\alpha_n a)]^2} \tag{5}$$

If we apply the Finite Hankel Transform, because we wont consider flux outside the cylinder limited by R. For establish the boundary conditions, will takes the idea that $\phi(R,t) = 0$ and $\phi(0,t)$ is limited. The unique initial condition is $\phi(r,0) = 0$.

If we multiplying both sides of (1) by $rJ_0(\alpha_n r)$, taken $a = R$ and integrate from 0 to R, we obtain

$$\int_0^R \frac{\partial}{\partial t}\phi r J_0(\alpha_n r)dr = D\int_0^R r\left(\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial \phi}{\partial r}\right]\right)J_0(\alpha_n r)dr + \int_0^R \Sigma_R \phi r J_0(\alpha_n r)dr + \int_0^R rS(r,t)J_0(\alpha_n r)dr \tag{6}$$

where $S(r,t)$ is the source of problem. This way, (6) can be rewrite by

$$\frac{\partial}{\partial t}\bar{\phi}(\alpha_n,t) = D\int_0^R r\left(\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial \phi}{\partial r}\right]\right)J_0(\alpha_n r)dr + \Sigma_R\bar{\phi} + \bar{S}(\alpha_n,t) \tag{7}$$

Making analysis of the first term of the right side, we can make integration by parts, thus obtain

$$\int_0^R r\left(\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial \phi}{\partial r}\right]\right)J_0(\alpha_n r)dr = rJ_0(\alpha_n r)\frac{\partial \phi}{\partial r}\Big|_0^R + \int_0^R \alpha_n r J_1(\alpha_n r)\frac{\partial \phi}{\partial r}dr \tag{8}$$

but how we choosing α_n such that $J_0(\alpha_n R) = 0$, implies that the first term of the right side in (8) vanishes, therefore

$$\int_0^R r\left(\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial \phi}{\partial r}\right]\right)J_0(\alpha_n r)dr = \int_0^R \alpha_n r J_1(\alpha_n r)\frac{\partial \phi}{\partial r}dr \tag{9}$$

and integrate by parts again, we have that

$$\int_0^R \alpha_n r J_1(\alpha_n r)\frac{\partial}{\partial r}\phi dr = \phi\alpha_n r J_0(\alpha_n r)\Big|_0^R - \alpha_n^2\int_0^R r\phi J_0(\alpha_n r)dr = -\alpha_n^2\bar{\phi} \tag{10}$$

That implies

$$\frac{d}{dt}\bar{\phi}(\alpha_n,t) + (D\alpha_n^2 + \Sigma_R)\bar{\phi}(\alpha_n,t) = \bar{S}(\alpha_n,t) \tag{11}$$

This equation is hold on to the initial condition $\bar{\phi}(\alpha_n,0) = 0$ because $\phi(r,0) = 0$, and this way have a solution expressed by

$$\bar{\phi}(\alpha_n, t) = \int_0^t \bar{S}(\alpha_n, t') e^{-(D\alpha_n^2 + \Sigma_R)(t-t')} dt' \tag{12}$$

with

$$\bar{S}(\alpha_n, t) = \int_0^R r S(r,t) J_0(\alpha_n r) dr \tag{13}$$

For make the inversion, we need apply the equation (12) in (5), therefore we obtain as result

$$\phi(r,t) = \frac{2}{R^2} \sum_{n=1}^{\infty} \bar{\phi}(\alpha_n) \frac{J_0(\alpha_n r)}{[J_0'(\alpha_n R)]^2} \tag{14}$$

expressed in terms of equation (12), thus

$$\phi(r,t) = \frac{2}{R^2} \sum_{n=1}^{\infty} \left\{ \int_0^t \bar{S}(\alpha_n, t') e^{-(D\alpha_n^2 + \Sigma_R)(t-t')} dt' \right\} \frac{J_0(\alpha_n r)}{[J_1(\alpha_n R)]^2} \tag{15}$$

that is the solution of initial problem.

3.1 Fixed Source

We can consider a fixed source, in other words, a source without time dependence, given by

$$S(r,t) = S(r) \text{ and consequently } \overline{S}(\alpha_n, t) = \overline{S}(\alpha_n) \tag{16}$$

In this case, the expression (36), can be represented by

$$\phi(r,t) = \frac{2}{R^2} \sum_{n=1}^{\infty} \left\{ \bar{S}(\alpha_n) \int_0^t e^{-(D\alpha_n^2 + \Sigma_R)(t-t')} dt' \right\} \frac{J_0(\alpha_n r)}{[J_1(\alpha_n R)]^2} \tag{17}$$

but, how we know that

$$\int_0^t e^{-(D\alpha_n^2 + \Sigma_R)(t-t')} dt' = \frac{1 - e^{-(D\alpha_n^2 + \Sigma_R)t}}{(D\alpha_n^2 + \Sigma_R)} \tag{18}$$

and therefore, the final expression for flux is

$$\phi(r,t) = \frac{2}{R^2}\sum_{n=1}^{\infty}\left\{\bar{S}(\alpha_n)\frac{1-e^{-(D\alpha_n^2+\Sigma_R)t}}{D\alpha_n^2+\Sigma_R}\right\}\frac{J_0(\alpha_n r)}{[J_1(\alpha_n R)]^2} \tag{19}$$

4 INFINITE LINE SOURCE DISTRIBUTION

The Delta Function, written $\delta(r)$ (in cylindrical case), is defined to be zero for all values of r except $r = 0$. The integral of $\delta(r)$ is finite, however, provided the range of integration includes the point at $r = 0$, and the value of the integral is taken to be unity. In summary,

$$\delta(r) = \begin{cases} +\infty\,, r = 0 \\ 0\,, r \neq 0 \end{cases} \tag{20}$$

And symbolic, we can write that

$$\int_{-\infty}^{\infty}\delta(r)dr = 1 \tag{21}$$

Now, let be $[a, b]$ a compact set with $a < 0 < b$. Considering $[a, b]$ a compact support for $(-\infty, \infty)$, we an assume

$$\int_{[a,b]}\delta(r)dr = 1 \tag{22}$$

By definition, $\delta(r)$ is a linear functional that evaluate a function $f(r)$ at $f(0)$, this was

$$\int_{-\infty}^{\infty} f(r)\delta(r)dr = \int_{[a,b]} f(r)\delta(r)dr = f(0) \tag{23}$$

The last equation is valid for any reange of integration which includes the point $r = 0$, then is true that

$$\int_{[0,b]} f(r)\delta(r)dr = \frac{1}{2}f(0) \tag{24}$$

How we need know our source, that needs be tranformated using the Finite Hankel transform, that is given by

$$\bar{S}(r,t) = \int_0^R rS(r,t)J_0(\alpha_n r)dr \tag{25}$$

Considering a source with dependece in time and r, like has in Lamarsh [3], let

$$S(r,t) = e^{-\lambda t}\frac{S_0\delta(r)}{\pi r} \tag{26}$$

Where S_0 is initial value for the source, λ is the time decayment constant. This way, taken $b = R$

$$\bar{S}(r,t) = \int_{[0,R]} re^{-\lambda t}\frac{S_0\delta(r)}{\pi r}J_0(\alpha_n r)dr \tag{27}$$

In our case, considering the expression (25) with $f(r) = \frac{S_0 e^{-\lambda t} J_0(\alpha_n r)}{\pi}$, thus

$$\int_{[0,R]} re^{-\lambda t}\frac{S_0\delta(r)}{\pi r}J_0(\alpha_n r)dr = \frac{1}{2}f(0) = \frac{S_0 e^{-\lambda t}}{2\pi} \tag{28}$$

Finally, we can express the final solution for the flux

$$\phi_g(r,t) = \frac{2}{R^2}\sum_{n=1}^{\infty}\left\{\int_0^t \bar{S}(\alpha_n, t')e^{-(D_g\alpha_n^2+\Sigma_{Rg})(t-t')}dt'\right\}\frac{J_0(\alpha_n r)}{[J_1(\alpha_n R)]^2} \tag{29}$$

That using (28), turns

$$\phi_g(r,t) = \frac{1}{\pi R^2}\sum_{n=1}^{\infty} S_0\left\{\int_0^t e^{-(D_g\alpha_n^2+\Sigma_{Rg})(t-(1+\lambda)t')}dt'\right\}\frac{J_0(\alpha_n r)}{[J_1(\alpha_n R)]^2} \tag{30}$$

How the result of integral is the following

$$\int_0^t e^{-(D_g\alpha_n^2+\Sigma_{Rg})(t-(1+\lambda)t')}dt' = \frac{e^{(D_g\alpha_n^2+\Sigma_{Rg})\lambda t} - e^{-(D_g\alpha_n^2+\Sigma_{Rg})t}}{(D_g\alpha_n^2+\Sigma_{Rg})(1+\lambda)} \tag{31}$$

The final solution is

$$\phi_g(r,t) = \frac{1}{\pi R^2}\sum_{n=1}^{\infty} S_0\frac{e^{(D_g\alpha_n^2+\Sigma_{Rg})\lambda t} - e^{-(D_g\alpha_n^2+\Sigma_{Rg})t}}{(D_g\alpha_n^2+\Sigma_{Rg})(1+\lambda)}\frac{J_0(\alpha_n r)}{[J_1(\alpha_n R)]^2} \tag{32}$$

Of course that when we make λ tends to zero, we have the same case with source not time dependent, that is the equation (19).

5 ANALYSIS OF BUILDUP FACTOR

The buildup factor have been calculated for different magnet reponse functions which impact the shield design. The shield composition used in this work is that used in the Mirror Advanced Reactor Study (MARS) design. The buildup factor for the response function from an infinite line source is defined as

$$B_p(r\Sigma_t, t) = \frac{R(r,t)}{e^{-r\Sigma_t}/4\pi r^2} \tag{33}$$

In our case, we will consider the response function as been the flux inside the cylinder, this way

$$B_p(r\Sigma_t, t) = \frac{\phi(r,t)}{e^{-r\Sigma_t}/4\pi r^2} \tag{34}$$

Where $\Sigma_t = \Sigma_R + \Sigma_f$, the macroscopic total cross section. Therefore, the buildup factor in this case in terms of flux is

$$B_p(r\Sigma_t, t) = \frac{4S_0}{e^{-r\Sigma_t}} \left(\frac{r}{R}\right)^2 \sum_{n=1}^{\infty} \frac{e^{(D_g\alpha_n^2+\Sigma_{Rg})\lambda t} - e^{-(D_g\alpha_n^2+\Sigma_{Rg})t}}{(D_g\alpha_n^2+\Sigma_{Rg})(1+\lambda)} \frac{J_0(\alpha_n r)}{[J_1(\alpha_n R)]^2} \tag{35}$$

As the shield thickness increases from zero to a few mean free paths, the energy spectra of neutrons and gamma photons change considerably. Different buildup factors obtained for the difference response functions depending on the relative gamma contribution and the energy dependence of cross sections for the different response functions. The results show that after a few mean free paths, the neutron and gamma spectra assume the fixed shapes. This stems from the fact that the mean free path for the $14.1Mev$ source of neutrons is larger than for lower energy neutrons and gammas. This results in he same buildup factor variation with the shield thickness regardless response function.

We show also the correlation of buildup factor with the radius of our cylinder, then we have the following profile

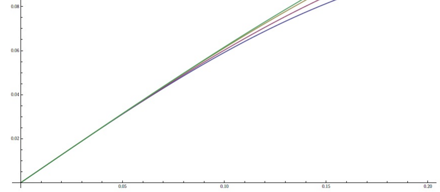

Figure 1. Profile of Buildup Factor, for different values of $t =, 2, 3, 4, 5$.

6 RESULTS

We made various computing with different parameters for calculus of expressions obtained for flux of the group "g". Considering a simple case that we have as source the expression given by (26). As expected, we obtained results similar to the literatures. The following results were obtained for different set of parameters and for different number of truncation N. The solution given (36), is represented by

$$\phi_g(r,t) = \frac{1}{\pi R^2} \sum_{n=1}^{N} S_0 \frac{e^{(D_g \alpha_n^2 + \Sigma_{Rg})\lambda t} - e^{-(D_g \alpha_n^2 + \Sigma_{Rg})t}}{(D_g \alpha_n^2 + \Sigma_{Rg})(1+\lambda)} \frac{J_0(\alpha_n r)}{[J_1(\alpha_n R)]^2} \tag{36}$$

We choice the parameteres described below for the table

$P^{(n)}$	D_1	Σ_{R1}	λ	R	S_0	N	$t=[0,T]$
$P^{(1)}$	1.43	0.39	0.99	1.0	5	10	[0,10]
$P^{(2)}$	1.43	0.39	0.58	2.0	5	10	[0,10]
$P^{(3)}$	1.43	0.39	0.58	2.0	5	10	[0,30]
$P^{(4)}$	1.43	0.39	0.58	0.5	5	10	[0,10]

Table I. Sets os parameters for make a comparison of results, the source is an infinite line.

We show the following results:

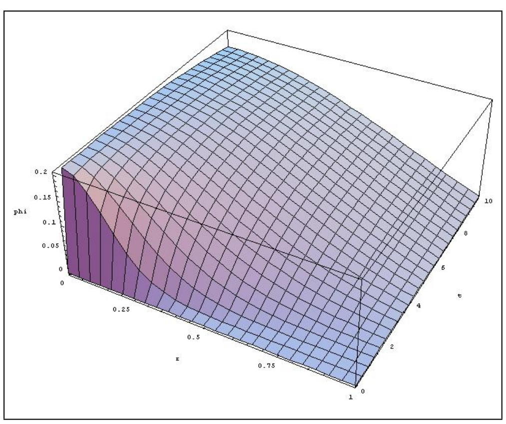

Figure 2. Profile of flux, using the parameters $P^{(1)}$.

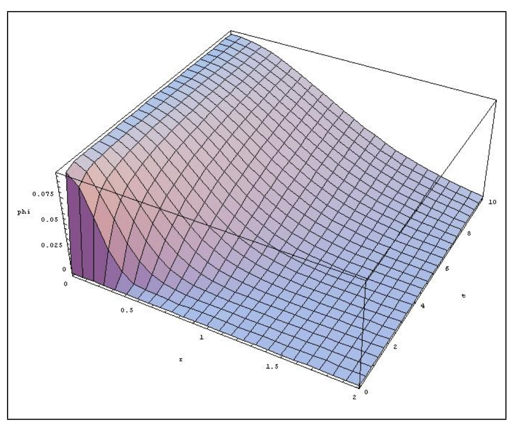

Figure 3. Profile of flux, using the parameters $P^{(2)}$.

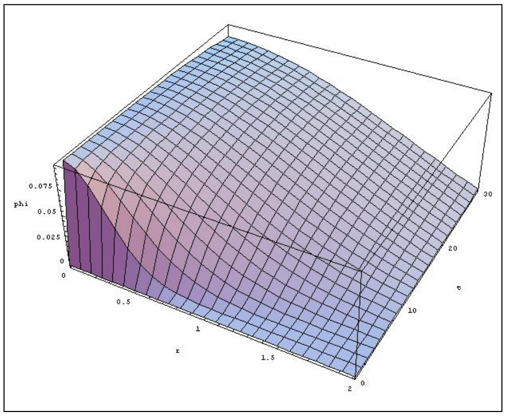

Figure 4. Profile of flux, using the parameters $P^{(3)}$.

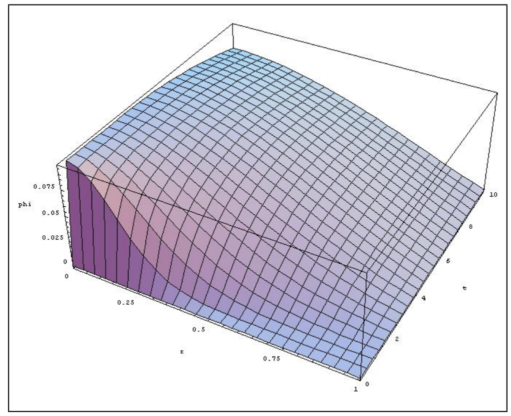

Figure 5. Profile of flux, using the parameters $P^{(4)}$.

7 CONCLUSION

In the present discussion we derived an analytical solution to the time-dependent diffusion equation in cylindrical geometry, more specifically in the apply for determination of buildup factor. We mean that no approximation is done along the solution derivation. We supposed an infinite line source distribution along to this case and did show the profile of flux in our domain and on the time. Is important to note that we consider a homogeneous cylinder. The time decay constant λ was explore for many values, in the case of λ tends to zero, we have a fixed source solution, as expected. Our motivation to choose the Finite Hankel transform for solve this problem comes from that the positive elliptic operator in diffusion equation have a simple transformation and the entire problem results in an algebraic matrix system with an eigenvalue problem, we solve with diagonalization procedure of transform matrix. In view of the good results and the mathematical elegance of the solution, we do believe that the proposed solution is a promising technique to determinate the buildup factor in this system of coordinates.

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