Mirror Symmetry in the presence of Branes

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Zusammenfassung

Diese Arbeit behandelt Mirrorsymmetrie für $N = 1$ Kompaktifizierungen auf kompakten complex dreidimensionalen Calabi-Yau Mannigfaltigkeiten mit Branen. Das wichtigste Hilfsmittel ist ein Raum von kombinierten Deformationen der Calabi-Yau und einer Hyperfläche in der Calabi-Yau. Die Perioden dieses Deformationsraums enthalten zusätzlich zu den Daten über den geschlossenen String Informationen über Branen des B-Modells in dieser Hyperfläche. Um diese Perioden zu studieren, verallgemeinern wir Techniken aus der Mirrorsymmetrie geschlossener Strings. Wir leiten das Picard-Fuchs System her und kodieren es in erweiterten torischen Polytopen. Lösungen der Picard-Fuchs Gleichungen ergeben Superpotentiale für bestimmte Konfigurationen von Branen. Dies ist eine effiziente Methode Superpotentiale zu berechnen. Für alle Branen mit nichttrivialem Superpotential sind die studierten Deformationen massiv. Je nach Wahl der Familie von Hyperflächen hängt das Superpotential der effektiven Theorie von unterschiedlichen massiven Feldern ab. A priori gibt es keinen Grund zu der Annahme, dass diese Felder leichter sind als andere, die nicht berücksichtigt wurden. Doch wir finden Beispiele in denen das Superpotential fasst flach ist. Für diese Beispiele benutzen wir den Gauss-Manin Zusammenhang auf dem kombinierten Deformationsraum, um eine Mirror Abbildung für Deformationen der Bran zu definieren. Mit Hilfe dieser Abbildung finden wir durch Instantonen erzeugte Superpotentiale von Branen des A-Modells. Dies führt zu Vorhersagen für Ooguri-Vafa Invarianten, die holomorphe Flächen mit der Topologie einer Scheibe zählen, die auf einer Lagrange Bran auf der Quintic enden.

Eine zweite Klasse von Beispielen hat keine ausgezeichneten, fasst masselosen Deformationen und es ist möglich, das gleiche on-shell Superpotential mit Hilfe von verschiedene Familien von Hyperflächen zu berechnen. Wir berechnen Superpotentiale für Branen in Calabi-Yau Mannigfaltigkeiten mit mehreren Deformationsparametern. Wir bilden die on-shell Superpotentiale auf das A-Modell ab und erhalten Vorhersagen für Scheibeninvarianten.

Der kombinierte Deformationsraum und der durch Quanteneffekte korrigierte Kähler Deformationsraum bestimmter nichtkompakter, complex vierdimensionaler Calabi-Yau Mannigfaltigkeiten sind äquivalent. Diese vierdimensionalen Mannigfaltigkeiten sind Faserungen von dreidimensionalen Calabi-Yaus über der Ebene. Durch Studium von Monodromien der komplexen Struktur der Fasern finden wir Hinweise, dass sie mirrordual zu der Calabi-Yau mit Hyperfläche, die den kombinierten Deformationsraum definiert, sind - vorausgesetzt die Hyperfläche wird von einer NS5 Bran

gewickelt. Darauf gestützt stellen wir eine einfache Regel vor, um mirrorduale Geometrien zu Calabi-Yau Mannigfaltigkeiten mit NS5 Bran auf einer Hyperfläche zu konstruieren.

Abstract

This work deals with mirror symmetry for $N = 1$ compactifications on compact Calabi-Yau threefolds with branes. The mayor tool is a combined deformation space for the Calabi-Yau and a hypersurface within it. Periods of this deformation space contain information about B-type branes within the hypersurface in addition to the usual closed string data. To study these periods we generalize techniques used in closed string mirror symmetry. We derive the Picard-Fuchs system and encode the information in extended toric polytopes. Solutions of the Picard-Fuchs equations give superpotentials for certain brane configurations. This is an efficient way to calculate superpotentials. The deformations we consider are massive for all branes with non trivial superpotential. Depending on a choice of a family of hypersurfaces, the superpotential of the effective low energy theory depends on different massive fields. A priori there is no reason for these fields to be lighter then other fields that are not included. We find however examples where the superpotential is nearly flat. In these examples we use the Gauss-Manin connection on the combined deformation space to define an open string mirror map. We find instanton generated superpotentials of A-type branes. This gives predictions for Ooguri-Vafa invariants counting holomorphic disks that end on a Lagrangian brane on the Quintic.

A second class of examples does not have preferred nearly massless deformations and different families of hypersurfaces can be used to calculate the same on-shell superpotential. We calculate examples of superpotentials for branes in Calabi-Yau manifolds with several moduli. The on-shell superpotentials are mapped to the mirror A-model to study the instanton expansion and to obtain predictions for disk invariants.

The combined deformation spaces are equivalent to the quantum corrected Kähler deformation spaces of certain non compact Calabi-Yau fourfolds. These fourfolds are fibrations of Calabi-Yau threefolds over the plain. We study complex structure monodromies of the fibers and find evidence that they are mirror to the Calabi-Yau manifold with hypersurface that defines the combined deformation space, provided an NS5 brane is wrapped on the hypersurface. This gives a simple rule how to construct mirrors to Calabi-Yau manifolds with NS5 branes wrapped on hypersurfaces.

Motivation and Overview

After its accidental birth in nuclear physics, string theory evolved to a theory of quantum gravity with the promise to unify all of fundamental physics. Ultimately this should lead to a theory in which all properties of low energy theories like coupling constants and the field content of the standard model are in principle computable. One prediction common to all superstring theories seems however to be at odds with nature. These theories predict spacetime to be ten dimensional. There is only one way to keep this in accordance with the observation that gravity and other forces propagate only in four dimensions macroscopically. Six dimensions have to be compact and small. Low energy processes, like the ones we observe every day or in particle accelerators, do not resolve these additional dimensions. The details of the geometry of these six dimensions however determine properties of low energy physics such as the field content and couplings of an effective theory.

String theory imposes certain conditions on the compactification geometries. In two and four dimensions these turned out to be very restrictive, but unfortunately the six dimensional compactification geometry is far from unique. Contrarily there is an enormous number of possibilities.[1] While it is possible to reproduce all features of the standard model like gauge theories, chiral matter and multiple generations within string theory compactifications, it seems to be unlikely that the correct compactification could ever be chosen out of all possibilities. Unfortunately this destroys the hope of computing all properties of the standard model from first principles.

On the other hand it became clear that the different superstring theories are just descriptions of different regimes of a single unique theory and are interconnected by equivalence relations. The same physical situation can often be described in different ways within several string theories. Different constructions that describe the same physics are said to be dual to each other. Such dualities are very useful in practice. Some relate for example the strongly coupled regime of one theory to the weakly coupled regime of another one, or in general map hard calculations to easier ones. A full definition of the unique theory that is spanned by this net of dualities is still not known, but the existence of such an exceptional structure is fascinating and its study mathematically rewarding. Physically the theory can address and answer deep conceptional questions about quantum gravity like topology change and emergence of spacetime or the microscopic origin of black hole entropy in a satisfactory way.

[1] If one also considers additional possible ingredients like branes and fluxes a commonly quoted number is 10^{500}, see e.g. [1].

Because of the huge number of solutions to this unique theory, the uniqueness is however completely lost for low energy physics and it is hard to answer specific experimentally relevant questions.

While it seems impossible to extract all parameters of the standard model, the search for specific solutions of string theory that give rise to standard model like physics is still an active area of research. Apart from a "proof of principle", the main goal is to find generic properties of low energy physics. Such expected low energy properties of string theory or certain excluded combinations of parameters could be used to make contact with experiments. A major obstacle is the limited calculational control over generic string theory compactifications.

Apart from this direct application of string theory to high energy physics, string theory is often used as a new framework to study properties of conventional field theories. The rich duality net of string theory and the natural relation between physical quantities and geometry can give new handles on old problems. The most important example is the AdS/CFT duality that can be used for calculations in strongly coupled gauge theories. Amusingly, after more then 20 years of theoretical development, string theory is relevant again for nuclear physics. These valuable insights are independent from its status as a fundamental theory.

The focus of the present work is neither one of these applications, but the study of one of the key dualities of string theory. However, a better understanding of "Mirror symmetry in the presence of branes" is likely to promote the scope of research both in the direct application to high energy physics and in the use of string theory to uncover properties and unexpected relations between different physical systems.

The net of string dualities is key to the latter and mirror symmetry is one of the most important links. It is a duality between string theories compactified on two different geometries. These geometries have to fulfill certain conditions and are called Calabi-Yau manifolds. One Calabi-Yau manifold is said to be the "mirror" of the other one. The simplest example is a pair of two dimensional tori, where mirror symmetry is equivalent to a so called T-duality in one of the circles. T-duality is a property of string theories, but to hint at an explanation we start with point particles. In a space with some non compact directions and a compact circle, point particles have a quantized momentum in the compact direction. The smaller the radius R of the circle, the smaller is the maximal wavelength. So the minimal energy of a particle with momentum in the compact direction is proportional to $1/R$. In the remaining non compact dimensions the same point particle with different integer internal momenta can be interpreted as a tower of different particles with integer spaced masses proportional to $1/R$. The same momentum quantization also leads to a tower of states for a string moving in non compact dimensions times a circle. There is however a new possibility, the string could also wrap around the circle. As the string has constant finite tension, this leads to a second tower with integer spaced masses proportional to the radius of the circle R. For the physics in the non compact directions these two towers, one with masses proportional to R, the other to $1/R$, are equivalent. The theory is invariant under their exchange. This exchange is called a T-duality, it maps a circle of radius R to a circle with radius $1/R$.

This is a new feature of string theories, a theory of point particles will never show such a behavior. An extended string probes geometry in a different way than point particles. This is at the heart of some successes of string theory as a theory of quantum gravity dealing with spacetime singularities and topology change. Moreover certain geometries might be equivalent for strings even if they are not equivalent as sets of points. This is a sloppy explanation for the possibility of a phenomenon like mirror symmetry that equates string theories also on pairs of more complicated compactification geometries, not just two dimensional tori.

Mirror symmetry implies previously unexpected relations between properties of two Calabi-Yau manifolds, once studied as a complex and once as a symplectic manifold respectively. These relations led to deep mathematical insights and caused a lot of research effort among mathematicians. Some of these relations are proven rigorously by now and can in turn serve as a test of the mathematical consistency of string theory. Work by mathematicians also paved the way to an understanding of all possible brane configurations on such geometries.

Branes are non perturbative objects of string theory. Their dynamics can however be studied perturbatively by the open strings that end on them. This gave them a central role in the discovery of string dualities and turns them into a very powerful tool to test them. Moreover they are necessary ingredients for standard model like constructions in type II string theory. In the simplest case, they can be thought of as objects that fill a submanifold of spacetime. Depending on the brane this submanifold can have different numbers of dimension. Mirror symmetry maps branes within one compactification geometry to another type of brane in the mirror geometry. The explicit map between the corresponding submanifolds is complicated and not well understood. The study of explicit examples and predictions of certain properties of branes in the mirror geometry are the most interesting mathematical results of the present work. Although these results are not proven in a rigorous manner, together with the developed computational tools they could serve as valuable guides for mathematicians. A rich playground of examples is also important for physicists exploring dualities and developing new calculational methods.

Finally, to study generic properties of string compactifications to four dimensions, it is important to understand the general low energy theory for compactification geometries that are not fixed to have some specific size or shape. Mirror symmetry is an important tool to do so. One uses the comparably good computational control over properties of complex manifolds to determine properties of the mirror manifold that are related to its symplectic structure and hard to compute directly. Doing this, one can calculate important quantities of the corresponding low energy theory in four dimensions for both general shapes and general sizes of the compactification manifold. Such exact, non perturbative[2] expressions are important to decide which properties of low energy theories are generic in string theory. All compactifications of type II string theory that include standard model like properties such as gauge theories and chiral matter necessarily include branes. The study of mirror symmetry in the presence of branes is thus a first step to a non perturbative understanding of standard model like constructions in string theory.

The thesis is structured as follows. To set the stage, **chapter 1** gives an introduction to the most important ideas and tools of mirror symmetry without branes. The goal is to offer an overview over the central results and the connections between them in a short and thus readable text, but not to derive the results in a self consistent way. The focus lies on the structures that are generalized to the case with branes in the following. This are especially the construction of deformation families of Calabi-Yau manifolds and their mirrors and the study of certain functions on this deformation spaces, the period integrals. It is difficult to generalize the deformation space of a Calabi-Yau manifold to a joint deformation space for the manifold and branes. The reasons for this and how such difficulties were dealt with prior to the present work are mentioned. This leads to the main part of the thesis.

The chapters 2 to 4 contain new results that were published in [2, 3, 4]. Each chapter is a slightly modified version of one of these publications. The main technical tools that are used throughout the whole thesis are only introduced once in chapter 2, but apart from this the chapters can be read independently. At the end of each chapter we give a short summary and outlook.

The main tools that are introduced in **chapter 2** are the deformation space of a holomorphic hypersurface in a Calabi-Yau manifold, the corresponding period integrals and efficient ways to compute them. The period integrals encode information about certain branes inside the Calabi-Yau manifold, in particular they determine the superpotential, an important quantity in the low energy effective action for a compactification with branes. The structure of the deformation space determines preferred coordinates that have a natural interpretation in the dual mirror geometry. In these coordinates the superpotential is "nearly flat" in a sense made precise below. This leads to a prediction for the numbers of holomorphic disks ending on a submanifold in the mirror geometry.

Chapter 3 also deals with the calculation of brane superpotentials and predictions for numbers of holomorphic disks, albeit of a slightly different nature. The deformation space introduced in chapter 2 is only used as a calculational tool, but the deformation itself is not crucial to map the results to quantities of the mirror ge-

[2]Sizes are defined with respect to the string size, so exact expressions for arbitrary sizes can as well be read as exact in the string size if we take the latter as a free parameter. Non perturbative expressions in the string tension, the inverse of the string size, are difficult to obtain by other methods.

ometry. Several examples are calculated in detail and transition between different geometries are studied.

In **chapter 4** certain non compact Calabi-Yau geometries, whose deformations are governed by the same data as the deformations of a hypersurfaces inside another Calabi-Yau, are studied in detail. These non compact Calabi-Yau manifolds are dual to the other Calabi-Yau with a brane wrapped on the hypersurface. This duality can be understood as mirror symmetry if the brane wrapping the hypersurface is a so called NS5 brane. NS5 branes are known to turn into a purely geometrical background under T-duality. To check the correspondence under the more general mirror symmetry certain monodromies are shown to match on both sides. Finally a stack of parallel branes inside a Calabi-Yau is studied to compare with the known situation of parallel NS5 branes in flat space.

Publications

M. Alim, M. Hecht, P. Mayr and A. Mertens, *Mirror Symmetry for Toric Branes on Compact Hypersurfaces*, *JHEP* **09** (2009) 126, [arXiv:0901.2937]

M. Alim, M. Hecht, H. Jockers, P. Mayr, A. Mertens and M. Soroush, *Hints for Off-Shell Mirror Symmetry in type II/F-theory Compactifications*, *Nucl. Phys.* **B841** (2010) 303-338, [arXiv:0909.1842]

M. Alim, M. Hecht, H. Jockers, P. Mayr, A. Mertens and M. Soroush, *Type II/F-theory Superpotentials with Several Deformations and N=1 Mirror Symmetry*, *JHEP* **1106** (2011) 103, [arXiv:1010.0977]

A. Mertens, *Reflections of NS5 branes*, arXiv:1107:1457

1 Introduction: Mirror symmetry

The goal of this short introduction is to collect the main ideas and techniques that are used in this thesis. To keep the text short and thus readable as an introduction, we omit derivations and only present equations that will reappear in some way in the following chapters. There exist a lot of reviews that cover this standard material in greater detail. A general reference for the whole introduction is [5], in the different sections we refer to further texts and some original work that is often quite accessible.

1.1 The topological A- and B-model

We will treat mirror symmetry mainly as a duality between the topological A-model on a Calabi-Yau (CY) manifold Z and the topological B-model on a mirror CY Z^*. The relation between the mathematical structures on Z and Z^* that are captured by the A- and the B-model is usually referred to as mirror symmetry by mathematicians. These relations in turn are at the hart of the physical mirror symmetry between the full type II A and type II B string theories on two mirror CY threefolds Z and Z^*. The naming reflects the exchange under mirror symmetry between both topological A- and B-model and type II A and type II B, there is no preferred connection between A-model and type II A or B-model and type II B however.

Both the topological A- and B-model calculate important data of full superstring compactifications on CY manifolds. Compactifications on a CY threefold lead to theories with N=2 supersymmetry in four dimensions. The structure of such theories is very restricted and certain terms in the action have to be holomorphic functions of the fields. Such terms can be calculated, for Type II these are the Kähler potential and the holomorphic prepotential, as well as certain higher loop amplitudes. This data is extremely interesting as it determines most of the massless sector of the low energy theory.

Mathematically the topological string captures the moduli space of Ricci flat metrics for a family of CY manifolds and certain corrections to it due to the extended nature of a string. Deformations of the metric correspond to massless fields of the low energy theory. These deformations come in two forms, the metric can be changed either by varying the Kähler class or by varying the complex structure. The moduli space of a tree dimensional CY splits into a product of a Kähler moduli space and a complex structure moduli space.[1] As we will discuss, the A model probes the corrected Kähler moduli space and the B-model the complex structure moduli space.

The topological A- and B-model were defined in [6, 7]. One can understand the construction starting from the two dimensional $N = 2$ superconformal sigma model with a CY manifold Z as target space. For a review of such theories see [5]. Operators creating the supersymmetric ground states of these theories form a finite dimensional subring that is graded by $U(1)$ R-charges. They come in two forms. The so called chiral operators form the chiral ring. They are annihilated by a certain combination Q_B of the supersymmetry generators. The twisted chiral operators form the twisted chiral ring and are annihilated by Q_A, a different combination of supersymmetry generators. On the level of the supersymmetry algebra the difference is only a choice of sign and there is a Z_2 automorphism that exchanges vector with axial R-charge and Q_B with Q_A and thus chiral and twisted chiral fields. The different interpretation of chiral and twisted chiral fields in terms of a target space geometry leads to the mirror symmetry between two CY manifolds. The topological A- and B-model are theories whose only physical operators are the chiral and the twisted chiral operators respectively. They are constructed by promoting Q_A or Q_B respectively to a nilpotent BRST charge. This can be done by a "twist" [8] of the rotation generator of the two dimensional worldsheet by the vector R-charge for Q_A or by the axial R-charge for Q_B. This changes the spinorial operator $Q_{A/B}$ of the $N = 2$ superconformal theory to a scalar operator. Moreover the central charge of the superconformal algebra vanishes after the twist[2] and $Q^2_{A/B} = 0$. As the name suggests both the A- and the B-model are topological theories, so correlators do only depend on the topology of the worldsheet and the nature of the inserted operators, but not on their positions. The twisted chiral ring elements are the elements of the Q_A cohomology and thus the physical operators of the A-model. Analogously the Q_B cohomology gives the chiral ring elements of the B-model. In the presence of a fermionic symmetry generator, the quantum mechanical path integral localizes to the fixed points of this symmetry [7], due to the vanishing (Grassmann) volume of free orbits. In the case at hand the BRST operator $Q_{A/B}$ generates such a symmetry, so the whole path integral reduces to an integral over solutions to the classical equation of motion.

For the B-model the classical solutions are only the constant maps from the worldsheet to a single point in the target space. The path integral over all field configuration thus reduces to an classical integral over points in the target space! In fact, most of the technical work of this thesis is to determine such integrals. The observables are built out of the worldsheet scalar and fermion fields. As in the $N = 2$ sigma model the scalars are interpreted as maps to target space coordinates and the

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[1] In general this is only true locally.

[2] Without central charge there is a priori no condition on the dimension of the target manifold. The common restriction to complex three dimensions only comes in with the Beltrami differentials that are used to define higher genus topological string amplitudes.

fermions as sections of pullbacks of the tangent bundle. For the B-model we have after a coordinate change products of fermions that correspond to the holomorphic tangent bundle and the anti-holomorphic cotangent bundle. On these fields the BRST operator Q_B acts as the pullback of the Dolbeaut operator $\bar{\partial}$ and after a contraction with the single holomorphic $(d,0)$ form Ω of the d dimensional CY Z, physical operators are interpreted as elements of Dolbeaut cohomology. Taking into account a selection rule[3], we are particularly interested in the operators

$$\phi^{(p)} \in H^{(d-p,p)}(Z) \; .$$

Among these, the operators $\phi^{(1)}$ with $U(1)$ charge $p=1$ are particularly important as elements in $H^{(d-1,1)}$ correspond to first order deformations of the complex structure.

For correlators on the sphere, a selection rule for the remaining $U(1)$ requires $p=d$. They can be calculated as an integral over Z after an additional wedge product with Ω. For a product of three operators of total charge d we get e.g.

$$\langle \phi_a^{(p_a)} \phi_b^{(p_b)} \phi_c^{(d-p_a-p_b)} \rangle_{g=0} = C_{abc} \int_Z \phi^{(d)} \wedge \Omega = C_{abc} \; ,$$

where we used that the only charge d operator is proportional to $\bar{\Omega}$ and implicitly normalized it in the second step. C_{abc} is the coefficient of the corresponding ring product. To calculate it one starts with a multiplication of a charge one operator with a given other operator. Up to normalization this amounts to a first order deformation of the complex structure. This changes the Hodge type of the other operator and determines the product. All other products can be built up from these, so solving the topological B-model on the sphere reduces to a study of the deformation of the complex structure and the associated variation of Hodge structure. This defines a commutative operator product as the Gauss-Manin connection that governs the Hodge variation over the moduli space is flat.

The complex structure is determined by the choice of a holomorphic $(d,0)$ form Ω, forms of other Hodge type can be constructed as derivatives of Ω. To study the change of complex structure one needs a fixed reference frame to compare Ω with. A convenient frame is given by the d dimensional integral cycles $\gamma_k \in H_d(Z,\mathbb{Z})$. So studying the variation of complex structure boils down to a calculation of the period integrals $\Pi_k = \int_{\gamma_k} \Omega$.

In the A-model the classical configurations are maps to holomorphic curves in the target space Z. These curves can have different homology class and include also the constant map to single points. So the path integral reduces to a sum over different sectors. The first is the path integral over all constant maps, as above this reduces to a classical integral in Z. Its contribution to the full path integral is called the

[3]For these operators the vector $U(1)$ charge vanishes, as the corresponding symmetry is not anomalous. This is necessary for non vanishing correlators.

classical term. In addition there are path integrals over all field configurations that map to holomorphic curves of any given homology class $\beta \in H_2(Z,\mathbb{Z})$. The action is constant and proportional to the complexified Kähler volume of the curve for all such instanton configurations, $S = i\int_\beta \omega$. The path integrals for each β then give a contribution $\sim V\exp[-\int_\beta \omega]$. For the common case that the holomorphic curves are isolated, the constant volume factor V just counts the number of curves in the class β and possibly multiple wrappings of curves whose classes add up to β as well. Multiple wrappings can give rise to fractional numbers, but their contribution can be systematically removed and the remaining numbers are integer. This is even true if there is a continuous family of curves. The path integral calculates the Euler number of the corresponding moduli space in this case. In this sense the path integral "counts" the number of holomorphic curves in Z.

Similarly as in the B-model, the physical operators can be interpreted in terms of cohomology classes of the target space. Q_A acts as the de Rham differential d and after imposing a selection rule one is interested in the operators

$$\psi^{(p)} \in H^{(p,p)}(Z)\ .$$

Among these the operators $\psi^{(1)}$ with charge $p = 1$ are again of special interest as they span the Kähler moduli space of Z.

Due to the flatness of the Kähler moduli space, the classical part of a correlator on the sphere is even easier than in the B-model. It is given by an integral over the wedge product of the inserted operators,

$$\langle \psi_1\psi_2...\psi_n\rangle^{cl}_{g=0} = \int_Z \psi_1 \wedge \psi_2 \wedge ... \wedge \psi_n\ . \tag{1.1}$$

Again the total $U(1)$ charge has to be equal to d for a non vanishing result. One can chose the Poincaré duals ψ_a to integer cycles $D_a \in H_{2d-2p}(Z,\mathbb{Z})$ as basis for the operators. In this case (1.1) has a particularly easy interpretation, it is the intersection number of the dual integer cycles. This simple picture gets corrected by the instanton sectors of the path integral. They give a contribution whenever the holomorphic curve in the target space hits all of the dual integer cycles.

$$\langle \psi_1\psi_2...\psi_n\rangle_{g=0} = \int_Z \psi_1 \wedge \psi_2 \wedge ... \wedge \psi_n\ + \sum_{\beta\in H_2(Z,\mathbb{Z})} n_{\beta,D_1,D_2,..,D_n} \mathrm{e}^{-\int_\beta \omega}\ ,$$

where $n_{\beta,D_1,D_2,..,D_n}$ is the number of maps from the sphere to holomorphic curves in the class β that meet the cycles D_1, D_2, ..., D_n.

The contribution of the instanton sectors to the full path integral are exponentially suppressed by the instanton action, $\exp[-\int_\beta \omega]$, while the action for the constant maps is zero. For large real values of the complexified Kähler class these contributions are negligible. This corresponds to a CY Z whose volume is large when measured in the string scale. In this case the extended nature of strings is negligible and the classical Kähler moduli space is a good approximation to the space of vacua of the A-model. For small volumes the instanton contributions become dominant and completely change the moduli space. As we will discuss in the next section this can even glue the Kähler moduli spaces of different CY manifolds. Mirror symmetry requires these corrections to the flat Kähler moduli space as the dual moduli space of complex structures is curved.

As we stated above, in the $N = 2$ sigma model the chiral ring and the twisted chiral ring can be exchanged by a Z_2 automorphism. In terms of the target space interpretation such a change is however dramatic. If we start with a sigma model with target space Z and exchange the twisted chiral with the chiral ring, the physical observables of the A-model on Z have to be interpreted as observables of some B-model and vice versa. New B- and A-models equivalent to the old A- and B-model can indeed be found and they describe the complex structure and Kähler moduli space of a different CY, the mirror Z^* to Z.

Before we come back to the correct map from the B-model on Z^* to the A-model on Z in 1.5 we discuss some technical tools needed to construct mirror families and to solve the B-model.

1.2 The Gauged Linear Sigma Model

The gauged linear sigma model (GLSM) was introduced in [9] to study different phases of Kähler moduli spaces. It is a two dimensional $N = 2$ supersymmetric gauge theory, but it is not conformal. It flows however to a large class of $N = 2$ superconformal sigma models under renormalization. It is the properties and especially the corrected Kähler moduli space of these conformal theories that one is ultimately interested in. This includes in particular Landau Ginzburg models and non linear sigma models with a complete intersection CY as target.

The data to define a GLSM give also a very intuitive way to define a CY manifold in a toric ambient space or a non compact toric CY. These are the number s of $U(1)$ gauge groups, the number n of chiral multiplets, their charges under the gauge groups and Fayet-Iliopoulos parameters. In addition one can add a superpotential. The charges for chiral multiplets with scalar component x_i under the gauge group $U(1)_a$ are summarized in "charge vectors" l_i^a, the corresponding Fayet-Iliopoulos terms are denoted by t^a. The action together with a detailed discussion of the following points can be found e.g. in [5].

Even though we only consider GLSMs with superpotential in this thesis, we first comment on the case without. The potential for the scalars in this case reads

$$V = \sum_{a=1}^{s} \frac{e_a^2}{2} \left(\sum_{i=1}^{n} l_i^a |x_i|^2 - t^a \right)^2 , \tag{1.2}$$

where e_a are $U(1)$ coupling constants. The zeros of this potential modulo the action of the gauge groups gives the space of classical ground states

$$M = \{x \in \mathbb{C}^n | \sum_{i=1}^{n} l_i^a |x_i|^2 = t^a\} / \prod_a U(1)_a \,, \tag{1.3}$$

where the gauge groups act as $x_i \to \mathrm{e}^{i l_i^a \phi_a} x_i$. If the t^a are chosen such that this space has the maximal possible complex dimension $n - s$, this is a toric variety. If the charges fulfill the conditions $\sum_{i=1}^{n} l_i^a = 0$ for all a, these toric varieties are moreover non compact CY manifolds. In this case the parameters t^a, that determine the size of M, do not run under the renormalization group. They are thus good coordinates on the classical Kähler moduli space for the non linear sigma model as well.[4]

For certain values of t^a, e.g. when passing from a positive to a negative value, M might cease to be a toric variety or its topology can change. The latter is the case for non compact CY spaces. The GLSM however continues to make sense. The classical Kähler moduli spaces for two non compact CY spaces that are related by a change of the parameters t^a are connected by quantum effects or sometimes even generated by them.

The situation is similar if one adds an additional chiral multiplet p and a gauge invariant superpotential $W = pP(x_i)$. For a certain choice of t^a, the space of classical solutions is

$$Z = \{x \in \mathbb{C}^n | \sum_{i=1}^{n} l_i^a |x_i|^2 = t^a \ \wedge \ P(x_i) = 0\} / \prod_a U(1)_a \,. \tag{1.4}$$

This is a hypersurface in a toric ambient space. The charge of p fixes the charge of the hypersurface equation $P(x_i)$.[5] In the following the charge of p is always given by the first entry of the charge vector, l_0^a. The CY condition is fulfilled, if P is a section of the anti-canonical bundle over the ambient space. This translates again to a condition on the charges, $\sum_{i=0}^{n} l_i^a = 0$ for all a, this time including the charge of p and thus $P(x_i)$. The simplest example for a three dimensional CY hypersurface is the quintic

$$\begin{array}{c c|c c c c c} & P & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline l & \text{-5} & 1 & 1 & 1 & 1 & 1 \end{array} \,. \tag{1.5}$$

[4]One can show that charge vectors l^a correspond to two-cycles with volume t^a in the toric variety. These volumes are controlled by the Kähler form, the parameters t^a can thus serve as coordinates on the Kähler moduli space. For the complexified Kähler moduli space that includes the B-field, these parameters are complexified. We use the same letter for both the Fayet-Iliopoulos parameter and its complexification.

[5]In general there exists a whole family of $P(x_i)$ for given charges of p, the different members will be related by a change of the complex structure.

Here we used in a slight abuse of notation P instead of p to label the first entry of the charge vector, but did not switch the sign of l_0. For $t > 0$ eq. (1.4) defines the quintic hypersurface in $\mathbb{P}^4$. The Kähler modulus t controls the size of the ambient space and descends to the Kähler modulus of the CY hypersurface.[6] For $t = 0$ the classical geometry shrinks to a single point and finally for $t < 0$ the geometric interpretation of a target space breaks down completely. As in the non compact case the GLSM however continues to be well defined and still flows to a conformal theory, in this case to a Landau Ginzburg orbifold with superpotential P. The quantum corrected Kähler moduli space interpolates smoothly between these two different "phases". It can be studied as the complex structure moduli space of the mirror CY. The GLSM allows for a first classical analysis of the Kähler moduli space and its various geometric and non geometric phases and limiting points. This is especially useful in more complicated examples with more then one Kähler moduli. In this case it is helpful to visualize the possible different regions of values for the t^a by the "secondary fan" [10]. Apart from sigma model and Landau Ginzburg phases also mixed phases of gauged Landau Ginzburg models are possible.

In sect. 1.4 we will see that the charge vector also offers a fast way to determine the complex structure moduli space of the mirror. To define families of CY hypersurfaces in the main text, we will thus give the charge vectors. Usually we chose the basis of charge vectors such that $t^a > 0$, for all a, corresponds to a geometric phase.

1.3 Polytopes and the construction of Batyrev

The first construction of mirror pairs is due to Greene and Plesser [11]. Their orbifold construction was generalized to a construction applicable to any complete intersection CY by Batyrev[12]. It builds on two different ways to encode the data of normal, projective toric varieties and the relations of these data to Kähler and complex structure moduli of a hypersurface in these toric varieties. One of these we introduced implicitly already in the last section.

All the operations in toric geometry are linear and thus in principle "easy", especially for higher dimensional cases with a lot of moduli it is however difficult to picture all objects and the notation can be unintuitive. Here we will only try to highlight some facts that are important to understand Batyrev's construction and not try to introduce toric geometry in a self consistent way. Toric geometry and physical applications are introduced with a lot of examples in [5], an other introduction with the focus of the construction of mirror pairs is [13]. An efficient tool to construct pairs of dual polytopes is the software package PALP [14].

[6] In more complicated examples not all Kähler moduli of the ambient space descend necessarily to the hypersurface. This happens if a generic hypersurface does not "hit" the two cycle whose volume is determined by the parameter t^a.

To each family of complex d-dimensional CY hypersurfaces Z in a toric ambient space one can associate two different $d+1$ dimensional lattice polytopes, the reflexive polytope Δ and its dual reflexive polytope Δ^*. We denote the integral points by $\mu_j \in \Delta$ and $\nu_i \in \Delta^*$, both Δ and Δ^* have an unique integral interior point that we denote by μ_0 and ν_0. There is a natural pairing $\langle \mu, \nu \rangle \to \mathbb{Z}$. We choose $\langle \mu_j, \nu_0 \rangle = \langle \mu_0, \nu_i \rangle = 1$. To do this we embed Δ and Δ^* in $\mathbb{R}^{d+2}$ instead of $\mathbb{R}^{d+1}$ and take them to lie in a hypersurface of distance one to the origin. Each of these two polytopes in turn define the CY Z. Exchanging the role of Δ and Δ^* defines the mirror CY Z^*.

The data of the first polytope Δ^* defines the fan[7] of a normal toric variety. For a given toric ambient space there is a unique family of hypersurfaces that fulfills the CY condition, this determines the CY Z. The vectors $\nu_j - \nu_0$ span the one dimensional cones of the fan. These correspond to torus invariant divisors $x_i = 0$ of the ambient space and thus to coordinates. There are relations between the points of Δ^* of the form $\sum_{i=0}^{*} l_i \nu_i = 0$[8]. For each such relation there is a $U(1)$ in the quotient group of the toric variety. The whole construction amounts to (1.3), or if we include the hypersurface to (1.4). In giving the GLSM charges we basically define the CY hypersurface in the same way. Note however that the existence of a reflexive polytope and thus a geometrical interpretation is not guaranteed for any GLSM charges that fulfill the CY condition $\sum_i^a l_i = 0$.

The second polytope, Δ is directly interpreted as the polytope of a projective toric variety. Its $m+1$ points μ_j are interpreted as holomorphic maps from a torus that will later act on the toric variety to homogeneous coordinates of $\mathbb{P}^m$. We call these maps y_j. Linear relations between the points, $\sum_j \tilde{l}_j \mu_j = 0$ give rise to the relations

$$\prod_{\tilde{l}_j > 0} y_j^{\tilde{l}_j} = \prod_{\tilde{l}_j < 0} y_j^{-\tilde{l}_j} . \tag{1.6}$$

The set of solutions to these relations is the toric ambient space. To connect this with the coordinates x_i, we look for expressions in x_i that fulfill the same relations. These are given by $y_j = x^{\mu_j} := \prod_i x_i^{\langle \mu_j, \nu_i \rangle}$. These monomials define sections in the anticanonical bundle, so the hypersurface equation is a sum $P(x_i) = \sum_{\mu_j \in \Delta} a_j x^{\mu_j}$.

We see that the points $\nu_i \in \Delta^*$ are associated to divisors and thus to Kähler classes while the points $\mu_j \in \Delta$ are associated to monomials in the hypersurface equation and thus to choices of the complex structure. If we exchange the role of Δ and Δ^* in the definition of a family of CY hypersurfaces, we get the mirror CY Z^*. To identify individual members of the two deformation families, we need an explicit map between coordinates on the deformation spaces, the so called mirror map. We will

[7] A fan is a collection of cones of different dimensions subject to some rules. Each cone corresponds to a submanifold that is invariant under the toric actions. One dimensional cones correspond to devisors, two dimensional cones to the intersection of two devisors etc.

[8] As we took all points ν_i to lie at finite distance from the origin, this includes the condition $\sum_i l_i = 0$

come back to this in sect. 1.5. A distinguished point in the Kähler moduli space is the large volume limit, where all quantum corrections are exponentially suppressed and the classical geometry is a good approximation. This point is mapped to a point of maximal unipotent monodromy in the complex structure moduli space, where the period integrals take a certain form with logarithmic singularities, the leading logarithmic singularities in the B-model can be mapped to classical volumes in the A-model. For each fan with one dimensional cones that correspond to points of Δ^* there is one CY with a large volume limit. It is possible that the same Δ^* can be subdivided in different ways to define a fan, in this case there are different CY manifolds Z, one for each fan. Their Kähler moduli spaces are "glued" together to one moduli space by quantum corrections and mirror to the complex structure moduli space of Z^*. The complex structure moduli space of such a Z^* has several point of maximal unipotent monodromy.

In the appendix we give the polytopes Δ and Δ^* for the CY manifolds studied in the main part of the thesis.

1.4 Picard Fuchs equations and GKZ system

As we explained in sect. 1.1, the first step to solve the B-model on Z^* is the calculation of period integrals $\Pi_k = \int_{\gamma_k} \Omega$, where Ω is the holomorphic $(d,0)$-form and γ_k is a basis of integral d-cycles, $\gamma \in H_d(Z^*, \mathbb{Z})$. Together with the topological intersection metric this data is enough to determine all correlators on the sphere as we will explain in the next section.

In principle these period integrals can be calculated directly, see e.g. [15], but the explicit construction of a basis of d-cycles can be difficult and the evaluation of the integral can usually be only done approximately in a power series. More often one uses the so called "Picard Fuchs equations". These are differential equations whose set of solutions is given by linear combinations of period integrals. Solutions that correspond to integrals over integer cycles can then by chosen e.g. by comparison with the classical A-model expectations in certain limits.

One uses that the cohomology groups $H^{(d-p,p)}(Z)$ are finite dimensional and that derivatives of the holomorphic top form Ω with respect to parameters z_i that determine the complex structure span all of the Dolbeaut cohomology $\bigoplus_p H^{(d-p,p)}(Z)$. More precisely the unique top form Ω spans $F^d := H^{(d,0)}$, the first derivatives $\partial_{z_i}\Omega$ with respect to all z_i together with Ω span $F^{d-1} := H^{(d,0)} \oplus H^{(d-1,1)}$ and in general Ω and all its possible derivatives up to maximal order q span $F^{d-q} := \bigoplus_{p \leq q} H^{(d-p,p)}$. This property is called griffith transversality. We immediately see that the dimension of F^{d-q} does not grow as fast with q as the number of possible derivatives, so there have to be non trivial relations between the derivatives of Ω. Especially derivatives of order $p > d$ will never generate new forms. These relations give rise to differential operators that annihilate Ω and thus all period integrals Π_k.

These relations can be found via "Griffith Dwork reduction". One realizes the cohomology elements explicitly as residues and constructs equivalence relations [16]. In practice one uses however a shortcut offered by the generalized hypergeometric system of Gel'fand, Kapranov and Zelevinsky (GKZ)[17]. The GKZ system of differential operators annihilates periods of Ω and is determined directly by the charge vectors l^a for the GLSM of the mirror A-model. The same charge vectors, that correspond to Fayet-Iliopoulos parameters and thus Kähler moduli, determine differential equations for the periods of the mirror B-model [18].

The GKZ system consists of operators of two types. The first set is due to the invariance of the period integrals Π_k under the torus action of the ambient space and under a rescaling of the hypersurface equation. This can be best seen in inhomogeneous coordinates X_k, $k = 1, ..., d+1$, in which the monomials read $y_i = \prod_j x_j^{\langle \nu_i, \mu_j \rangle} = \prod_k X_k^{\nu_{i,k}}$, where $\nu_{i,k}$ is the k-th component of the embedding $\nu_i \in \Delta^* \subset \mathbb{R}^{d+1}$. Here we use Δ^* to define the monomials as we are interested in the complex structure of the mirror Z^*. In this coordinates, the hypersurface equation reads

$$P(a_i) = \sum_a a_i y_i = \sum_i a_i \prod_k X_k^{\nu_{i,k}} \,. \tag{1.7}$$

The variables a_i determine P and thus the period integrals, the GKZ operators are differential operators in these variables. The fundamental period can be defined as

$$\Pi_0 = \frac{1}{(2\pi i)^{d+1}} \int_{|X_k|=1} \frac{1}{P(a_i)} \prod_{k=1}^{d+1} \frac{dX_k}{X_k} \,. \tag{1.8}$$

Here the holomorphic $(d,0)$ form is represented as a residue $\mathrm{Res}_{P=0} \frac{a_0}{P(a_i)} \prod_{k=1}^{d+1} \frac{dX_k}{X_k}$. Eq. (1.8) is invariant under the torus action $X_k \to \lambda X_k$ with $\lambda \in \mathbb{C}^*$. Such a rescaling can be absorbed into a rescaling of a_i, so the period is annihilated by operators $\mathcal{Z}_k$, $k = 1, ..., d+1$. Similarly the transformation under an overall rescaling of P is given by $\mathcal{Z}_0$.

$$\mathcal{Z}_k = \sum_i \nu_{i,k} \, \vartheta_i \ , k = 1, ..., d+1; \qquad \mathcal{Z}_0 = \sum_i \vartheta_i + 1 \,, \tag{1.9}$$

where $\vartheta_i = a_i \partial_{a_i}$. The other period integrals can be defined similarly, they are all invariant under (1.9). These imply that the periods depend, up to normalization, only on a special combination of the parameters a_i, $\Pi(a_i) = a_0 \Pi(z_a)$, where

$$z_a = (-)^{l_0^a} \prod_j a_i^{l_i^a} \,. \tag{1.10}$$

The combinations z_a give a set of coordinates on the complex structure moduli space.

The second set of differential equations follows from relations between the monomials of P, see (1.6). Because of $\prod_{l_i^a>0} y_i^{l_i^a} = \prod_{l_i^a<0} y_i^{-l_i^a}$ for all charge vectors l^a, the operators

$$\mathcal{L}(l^a) = \prod_{l_i>0} \left(\frac{\partial}{\partial a_i}\right)^{l_i} - \prod_{l_i<0} \left(\frac{\partial}{\partial a_i}\right)^{-l_i} \tag{1.11}$$

annihilate the holomorphic $(d,0)$ form $\Omega = \mathrm{Res}_{P=0}\frac{a_0}{P(a_i)}\prod_{k=1}^{d+1}\frac{dX_k}{X_k}$ in cohomology. The operators (1.11) are closely related to the Picard-Fuchs equations one would get by Griffith Dwork reduction. In general however the order of the operators is too high and some solutions to the GKZ system are not solutions of the Picard-Fuchs equations. But in most examples it is easy to choose the correct solutions and to factorize the operators (1.11) to obtain the Picard-Fuchs equations [19]. Both the complex structure moduli z_a and the operators $\mathcal{L}(l^a)$ are determined by l^a. The information about the charge vectors for the GLSM of Z is enough to set up the GKZ system for the mirror Z^*.

Solutions to the operators (1.9) and (1.11) are generalized hypergeometric functions in terms of the variables z_a defined in (1.10). For an appropriate choice of basis vectors l^a these can be expressed in terms of the generating functions

$$B_{\{l^a\}}(z_a;\rho_a) = \sum_{n_1,\ldots,n_N\in\mathbb{Z}_0^+} \frac{\Gamma\left(1-\sum_a l_0^a(n_a+\rho_a)\right)}{\prod_{i>0}\Gamma\left(1+\sum_a l_i^a(n_a+\rho_a)\right)}\prod_a z_a^{n_a+\rho_a}\,. \qquad (1.12)$$

At a point of maximal unipotent monodromy all solutions are e.g. spanned by $B_{\{l^a\}}(z_a;0)$ and the derivatives $\partial_\rho^n B_{\{l^a\}}(z_a;\rho_a)|_{\rho=0}$ where $n=1,..,d$ with d the complex dimension of the manifold.

1.5 The mirror map and correlators

The coordinates z_a that we introduced in the last section parameterize the complex structure moduli space, but they are not natural coordinates from a worldsheet point of view. One would like to use coordinates t_a such that derivatives ∂_{t_a} correspond to the insertion of charge one operators $\phi^{(1)}$, so that the infinitesimal deformations of the complex structure are also canonically normalized deformations of the B-model. It is these coordinates that map to the flat Kähler moduli of the A-model, the change of coordinates $t^a(z_i)$ is thus usually called the mirror map. For a basis $\alpha^{(q)}$ of forms that correspond to operators $\phi^{(q)}$ the derivative ∂_{t_a} thus acts as

$$\partial_{t^a}\begin{pmatrix}\alpha^{(0)}\\ \alpha_b^{(1)}\\ \alpha_b^{(2)}\\ \alpha_b^{(3)}\\ \vdots\end{pmatrix} = \begin{pmatrix} 0 & \delta_{ac} & 0 & 0 & \\ 0 & 0 & C_{abc}^{(2)} & 0 & \\ 0 & 0 & 0 & C_{abc}^{(3)} & \cdots \\ 0 & 0 & 0 & 0 & \\ & & \vdots & & \end{pmatrix}\begin{pmatrix}\alpha^{(0)}\\ \alpha_c^{(1)}\\ \alpha_c^{(2)}\\ \alpha_c^{(3)}\\ \vdots\end{pmatrix}, \qquad (1.13)$$

where $C_{abc}^{(p)}$ are operator product coefficients.[9] They depend on the moduli t_a. It is these quantities that we want to calculate as they completly determine all correlation

[9] Note that the subspaces of charge q can have different dimensions, so the labels b and c can run over different sets for different lines of the equation.

functions. For a CY threefold there is only a unique operator of the highest charge three, so in a canonical normalization we have $C^{(3)}_{abc} = \delta_{ab}$ and there are no further entries in (1.13).

As we explained in sect. 1.1, the charge zero operator is identified with the holomorphic top form Ω up to a normalization S_0, so we have $\alpha^{(0)} = \Omega/S_0$ and for the charge one operators $\alpha^{(1)}_a = \partial_{t_a}(\Omega/S_0)$. In the last section we explained how to calculate the period integrals Π_k that express Ω with respect to a fixed reference basis. The forms α can be expressed similarly in this basis by taking the respective ∂_{t_a} derivatives of the period vector. If (1.13) holds, this period matrix necessarily has an upper diagonal form with respect to some basis of cycles $\gamma \in H_d(Z^*, \mathbb{C})$

$$\underline{\Pi} = \begin{pmatrix} 1 & * & * & * & \\ 0 & 1\!\!1_{d_1\times d_1} & * & * & \\ 0 & 0 & 1\!\!1_{d_2\times d_2} & * & \cdots \\ 0 & 0 & 0 & 1\!\!1_{d_3\times d_3} & \\ & & \vdots & & \end{pmatrix}, \qquad (1.14)$$

where the q-th line corresponds to the forms $\alpha^{(q)}$ of charge q, d_q is the dimension of this subspace and $1\!\!1$ the unit matrix. At the point of maximal unipotent monodromy of a CY threefold with only one modulus z, the period matrix with respect to an integer basis of cycles for example reads

$$\underline{\Pi} = \begin{pmatrix} \alpha^{(0)} \\ \alpha^{(1)} \\ \alpha^{(2)} \\ \alpha^{(3)} \end{pmatrix} = \begin{pmatrix} 1 & t & \partial_t F & 2F - t\partial_t F \\ 0 & 1 & \partial_t\partial_t F & \partial_t F - t\partial_t\partial_t F \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (1.15)$$

where $t = log(z) + S_1(z)$, $\partial_t F = log^2(z) + S_2(z)$ etc. are functions of the modulus z with logarithmic singularities. The first line is the renormalized period vector of Ω, so the functions are given by solutions of the Picard-Fuchs equations. The existence of a function F called prepotential is non trivial and follows from the special geometry of the $N = 2$ moduli space. Once the period matrix is known, correlators can be calculated easily by paring the periods according to the intersection matrix η of $H_d(Z^*)$,

$$\langle \phi^{(p)}_a \phi^{(q)}_b \rangle = \alpha^{(p)}_a \eta \alpha^{(q)}_b \, . \qquad (1.16)$$

For a CY threefold with one modulus we have

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

so with eq. (1.15) and (1.16) the only non vanishing correlators are $\langle \phi^{(3)} \rangle = \langle \phi^{(1)}\phi^{(2)} \rangle = 1$. To calculate $\langle \phi^{(1)}\phi^{(1)}\phi^{(1)} \rangle$ we use the operator product expansion $\phi^{(1)}\phi^{(1)} = C\phi^{(2)}$

and from eqs. (1.13) and (1.15) we see that $C = \partial_t\partial_t\partial_t F$, so $\langle\phi^{(1)}\phi^{(1)}\phi^{(1)}\rangle = \partial_t\partial_t\partial_t F$. More generally for CY threefolds with more moduli t^a one finds

$$\langle\phi_a^{(1)}\phi_b^{(1)}\phi_c^{(1)}\rangle = \partial_{t^a}\partial_{t^b}\partial_{t^c}F\,.$$

The structure of eqs. (1.13) and (1.14) can be used to determine the correct flat CFT coordinates t^a and correlation functions everywhere one the modulus space constructively. One starts with any basis for $H^3(Z^*)$ given by Ω and derivatives $\partial^*_{z_a}\Omega$ to form a period matrix $\underline{\Pi}$. As $\underline{\Pi}$ already forms a basis, derivatives of the components of $\underline{\Pi}$ can again be expressed in terms of these components. This is nothing else then the Picard Fuchs equations,

$$\partial_{z_a}\underline{\Pi} = M\,\underline{\Pi}\,. \tag{1.17}$$

By coordinate redefinitions $t^a(z_a)$ and rescalings $\underline{\Pi}$ can be brought into the block diagonal form (1.14) and (1.17) in the form (1.13). If this is achieved, the coordinates t^a are the searched for flat coordinates and correlators can be read of once the intersection matrix is known.

1.6 The inclusion of branes

A similar understanding of mirror symmetry in the presence of branes would be interesting for several reasons. Type II compactifications on CY manifolds with branes lead to low energy theories with $N = 1$ supersymmetry. Such theories are phenomenologically interesting. They allow to include necessary features of realistic models like chiral matter, that are not possible in their $N = 2$ "parents" that arise from type II compactifications on CY manifolds without branes. Mirror symmetry in the presence of branes can be a tool to calculate nonperturbative effects for some theories with $N = 1$ supersymmetry and thus address important questions like moduli stabilization. The additional freedom of $N = 1$ theories also leads to a rich net of dualities and relations between different formulations of equivalent physical situations. As one among these mirror symmetry is important to check the mutual consistency of different conjectured relations and thus to obtain a deeper understanding of $N = 1$ theories in general. From the mathematical side a strong motivation is the homological mirror symmetry conjecture by Kontsevich [20], that states the equivalence between the category of branes of the topological A- and B-model. The former[10] is very difficult to treat explicitly, while the later is under reasonable control. Using mirror symmetry to map results for the category of B-branes to the category of A-branes gives thus a method to reduce a hard problem to

[10] The category of A-branes is the so called Fukaya category, whose objects are generalizations of special Lagrangian submanifolds. The category of B-branes is the derived category of coherent sheaves whose objects are coherent sheaves, a generalization of unions of holomorphic submanifolds of different codimension. Morphisms are in both cases the open strings that can be stretched between the branes.

a simpler one, just as in closed string mirror symmetry. This can be used to make statements about the stability of A-branes and the structure of the Fukaya category as a whole or to map properties of a single brane, such as the superpotential. The superpotential of an A-brane should have an interpretation in terms of worldsheet instantons wrapping disks in the target space when expanded around points in the moduli space that allow for a geometric interpretation. This is one focus of this thesis.

Following a similar route as in closed string mirror symmetry, one could try to map deformation families of branes on Calabi-Yau manifolds onto each other. However, one immediately runs into a problem: Branes that have a true deformation space have a constant superpotential along this deformation. So the superpotential can not be studied by analyzing this deformation space. If we however deform the brane in such a way that a varying superpotential is generated, we have no canonical way to choose a particular deformation out of an a priori infinite dimensional deformation space. If we take for example a B-brane wrapping a holomorphic submanifold, we will never generate a superpotential by a holomorphic deformation of this submanifold. The space of inequivalent non holomorphic deformations is however infinite.

Nevertheless a lot of interesting results were obtained in the past. For non compact CY a preferred deformation family of non holomorphic curves was introduced in [21], such that the mirror is a family of special Lagrangian cycles on a toric CY. The superpotential for branes wrapped on this Lagrangian cycles is only generated by instanton effects. The size of the disk instantons controls the position of the Lagrangian cycle and varies as an open string modulus within the deformation family. This "almost flat" superpotential can be calculated as an integral in the B-model using the mirror family of non holomorphic curves. Following this work, Ooguri-Vafa invariants [22] that count holomorphic disks wrapped by instantons, were calculated for many non compact CY manifolds [23, 24, 25, 26].

For compact CY manifolds, superpotentials with instanton expansion were first obtained in [27, 28], albeit of a slightly different nature. The instanton generated superpotential for a certain A-brane is again calculated in the B-model. There is however no deformation family of branes defined. The superpotential is calculated for a B-brane wrapping a single rigid holomorphic cycle. This superpotential does therefore only depend on closed string moduli. Expanded around a large volume point in the A-model, this superpotential has an instanton expansion in the closed string moduli. The volumes of disks ending on the A-brane is a fraction of the volume of holomorphic spheres in this CY.

In this thesis we deal with the afore mentioned problem in the following way: we study the unobstructed deformation space of an holomorphic divisor in a compact CY. Wrapping a brane on such a divisor does not give rise to a superpotential. But we can switch on an additional gauge flux on such a brane. This adds the charge of a brane that wraps a lower dimensional curve embedded in the divisor. The superpotential is the same as for a brane only wrapping the curve [29]. Such curves can cease to be holomorphic even if we vary the divisor holomorphically. Using the moduli space of the divisor we thus define a deformation family of curves that are holomorphic only for certain values of the open modulus but non holomorphic for generic values.

We use this strategy both for families of branes with "almost flat" superpotential that have an instanton expansion with respect to an open modulus, and for isolated branes with superpotential that admits an instanton expansion in the closed string modulus. In chapter 2 we deal with the first situation. For non compact CY this was already done in [25, 26]. We analyze the deformation family of a holomorphic divisor in a compact CY in detail and find evidence that some lower dimensional brane charges generate superpotentials that admit an instanton expansion in the open string modulus of the deformation family. In chapter 3 we use the period integrals associated to the deformation family of a holomorphic divisor as a computation tool to calculate superpotentials for B-branes wrapping rigid holomorphic cycles. After a map to the A-side these superpotentials allow an instanton expansion in closed string moduli.

The period integrals associated to the deformation space of a divisor in a CY 3-fold are equivalent to period integrals of certain CY 4-folds. For non compact CY 3-fold this was first realized in [25]. This relation and equivalences between different $N = 1$ compactifications on these spaces was explained by a chain of string dualities in [30]. In chapter 4 we give an alternative explanation for this structure. We propose that the appearing non compact CY 4-folds can be interpreted as mirrors to CY 3-folds with NS5 branes wrapped on divisors. To test this proposal we compare the shift of B-fields that signals the presence of an NS5 brane with a complex structure monodromy within a certain fibration structure of the 4-fold. We find perfect agreement with the expectations.

2 Branes with nearly flat superpotential

We study the unobstructed deformation space of a holomorphic hypersurface inside a CY manifold. We derive Picard-Fuchs equations and solve them to obtain period integrals. Some of these we interpret as superpotentials for branes wrapping cycles within the hypersurface. We explain this in detail and use the structure of the deformation space to define open string mirror maps. For the mirror to the quintic hypersurface in $\mathbb{P}^4$ we find a superpotential that allows for an instanton expansion in these coordinates. We conjecture that this corresponds to a Lagrangian A-brane with instanton generated superpotential on the quintic and compare the predictions for the numbers of holomorphic disks with calculations of holomorphic spheres in a degenerate case. This is the result of joined work with Murad Alim, Michael Hecht, Hans Jockers, Peter Mayr and Masoud Soroush that was published in [2]. This chapter is a shortened and in small parts slightly modified version of this publication.

2.1 Geometry and deformation space of the B-model

We start with the definition of the geometrical structure that will be taken as a model for the open-closed deformation space $\mathcal{M}$, following refs. [26, 31, 32]. Let (Z, Z^*) be a mirror pair of CY three-folds and (L, E) a mirror pair of A/B-type branes on it. On-shell, the classical A-type brane geometry is perturbatively defined by a special Lagrangian submanifold $L \in H_3(Z)$ together with a flat bundle on it [33]. At the quantum level non-perturbative open worldsheet instantons may couple to the special Lagrangian submanifold L. Then an on-shell quantum A-type brane arises if the classical geometry is not destabilized by such instanton corrections [34, 35]. The mirror B-type geometry consists of a holomorphic sheaf E on Z^* describing a D-brane with holomorphic gauge bundle wrapped on an even-dimensional cycle. The concrete realization and application of open string mirror symmetry to this brane geometry, which will be central to all of the following, has been formulated in the pivotal work [21]. More details on the action of mirror symmetry on brane geometries can be found in refs. [36, 5].

The moduli space of the closed string B-model on Z^* is the space $\mathcal{M}_{CS}$ of complex structures, parametrizing the family $\mathcal{Z}^* \to \mathcal{M}_{CS}$ of 3-folds with fiber $Z^*(z)$ at

$z \in \mathcal{M}_{CS}$. Here $z = \{z_a\}$, $a = 1, ..., h^{2,1}(Z^*)$ denote some local coordinates on $\mathcal{M}_{CS}$. An important concept in the Hodge theoretic approach to open string mirror symmetry of refs. [26, 31, 32] is the definition of an off-shell deformation space $\mathcal{M}$, which includes open string deformations. To study the obstruction superpotential on $\mathcal{M}$, one first defines $\mathcal{M}$ as an *unobstructed* deformation space for a relative homology problem and studies the functions $\underline{\Pi}^\Sigma : \mathcal{M} \to \mathbb{C}$ defined by integration over the dual cohomology space. In a second step, one adds an obstruction, which can be shown to induce a superpotential on $\mathcal{M}$ proportional to a linear combination of these 'relative periods' $\underline{\Pi}^\Sigma$.

The unobstructed moduli space for the family of relative cohomology groups can be defined as the moduli space of a *holomorphic* family of hypersurfaces embedded into the family $\mathcal{Z}^*$ of CY 3-folds [26, 31]

$$i: \qquad \mathcal{D} \hookrightarrow \mathcal{Z}^* \tag{2.1}$$
$$\mathcal{D}(z,\hat{z}) \hookrightarrow Z^*(z) \tag{2.2}$$

where[1] $\hat{z} = \{\hat{z}_\alpha\}$ are local coordinates on the moduli space of the embeddings $i : D(z,\hat{z}) \hookrightarrow Z^*(z)$ for fixed complex structure z. The total moduli space $\mathcal{M}$ of this family is the fibration

$$\begin{array}{ccc} \hat{\mathcal{M}}(\hat{z}) & \longrightarrow & \mathcal{M} \\ & & \downarrow \pi \\ & & \mathcal{M}_{CS}(z) \end{array} \tag{2.3}$$

where the point $z \in \mathcal{M}_{CS}$ on the base specifies the complex structure on the CY 3-fold $Z^*(z)$ and the point $\hat{z} \in \hat{\mathcal{M}}$ on the fiber the embedding. In the context of string theory, the moduli z and $\hat{z}$ arise from states in the closed and open string sector, respectively. Note that the fields associated with the fiber and the base of $\mathcal{M}$ couple at a different order in string perturbation theory. This will be relevant when defining a metric on $T\mathcal{M}$ in sect. 7.

Following [26, 37, 31, 32], we consider functions on the unobstructed deformation space $\mathcal{M}$ given by 'period integrals' on the relative cohomology group defined by the brane geometry. The embedding $i : \mathcal{D} \hookrightarrow Z^*$ defines the space $\Omega^*(Z^*, \mathcal{D})$ of relative p-forms via the exact sequence

$$0 \leftrightarrow \Omega^*(Z^*, \mathcal{D}) \leftrightarrow \Omega^*(Z^*) \overset{i^*}{\leftrightarrow} \Omega^*(\mathcal{D}) \leftrightarrow 0 \ .$$

The associated long exact sequence defines the relative three-form cohomology group

$$H^3(Z^*, \mathcal{D}) \ \simeq \ \ker \left(H^3(Z^*) \to H^3(\mathcal{D})\right) \oplus \text{coker} \left(H^2(Z^*) \to H^2(\mathcal{D})\right), \tag{2.4}$$

[1] Here and in the following we often use a hat to distinguish data associated with the open string sector.

which provides the geometric model for the space of groundstates of the open-closed topological B-model. In a generic[2] situation, the first summand equals $H^3(Z^*)$ and represents the closed string sector capturing the deformations of the complex structure of Z^*. The relation of the above sheaf cohomology groups considered in [26, 31] and the Ext groups studied in ref. [38] will be discussed in sect. 2.5.

By eq. (2.4), a closed relative three-form $\underline{\Phi} \in \Omega^3(Z^*, \mathcal{D})$, representing an element of $H^3(Z^*, \mathcal{D})$, can be described by a pair (Φ, ϕ), where Φ is a 3-form on Z^* and ϕ a 2-form on $\mathcal{D}$. The differential is $d\underline{\Phi} = (d\Phi, i^*\Phi - d\phi)$ and the equivalence relation $(\Phi, \phi) \sim (\Phi, \phi) + (d\alpha, i^*\alpha - d\beta)$ for $\alpha \in \Omega^2(Z^*)$, $\beta \in \Omega^1(\mathcal{D})$. The duality pairing between a 3-chain class $\gamma_\Sigma \in H_3(Z^*, \mathcal{D})$ and a relative p-form class $[\underline{\Phi}]$ is given by the integral

$$\int_{\gamma_\Sigma} \underline{\Phi} = \int_{int(\gamma_\Sigma)} \Phi - \int_{\partial\gamma_\Sigma} \phi \,. \tag{2.5}$$

The fundamental holomorphic objects of the open-closed topological B-model are particular examples of eq. (2.5), namely the relative period integrals of the holomorphic $(3,0)$ form $\underline{\Omega}$ on Z^*, viewed as the element $(\Omega, 0) \in H^3(Z^*, \mathcal{D})$, over a basis $\{\gamma_\Sigma\}$ of topological 3-chains:

$$\underline{\Pi}^\Sigma(z, \hat{z}) = \int_{\gamma_\Sigma} \underline{\Omega}, \qquad \gamma_\Sigma \in H_3(Z^*, \mathcal{D})\,. \tag{2.6}$$

The cohomology group $H^3(Z^*, D)$ is constant over $\mathcal{M}$, but the Hodge decomposition $F^pH^3(Z^*, D)$ and the direction of the (3,0) form $\underline{\Omega}$ varies with the moduli. The period integrals $\underline{\Pi}^\Sigma(z, \hat{z})$ thus define a set of moduli dependent local functions on $\mathcal{M}$. Despite the fact, that there is not yet a superpotential on $\mathcal{M}$, these functions should have an important physical meaning in the unobstructed theory as well. In sect. 7 of [2] it is argued that they define a Kähler metric on $\mathcal{M}$ and thus determine the kinetic terms of the bulk and brane moduli in the effective action.

Further details on the relation between relative cohomology and open-closed deformation spaces can be found in refs. [26, 39, 37, 31, 40, 41]. For the mathematical background, see e.g. refs. [42, 43] and for a formal treatment of the deformation problem and the associated differential equations see [44].

Obstructed deformation problem

The physical meaning of the period integrals is altered after adding an additional lower-dimensional brane charge on a 2-cycle, which induces an obstruction on $\mathcal{M}$.

[2] That is $H^1(Z^*) \simeq 0$ and we made the simplifying assumption that D is ample, which is a reasonable condition on the divisor wrapped by a B-type brane. The Lefschetz hyperplane theorem then implies $H^1(D) \simeq 0$ and, by Poincaré duality, $H^3(D) \simeq 0$.

From a physics point of view this perturbation may be realized by either adding an additional brane on a 2-cycle in $\mathcal{D}$ or by switching on a 2-form gauge flux on the original brane on $\mathcal{D}$. A world-sheet derivation of the obstruction from the relevant Ext groups in the open string CFT will be given in sect. 2.5.

In the Hodge theoretic approach of refs. [26, 31, 32], the superpotential on $\mathcal{M}$ in the obstructed theory is given by a certain linear combination of the relative periods (2.6) of the unobstructed theory, as reviewed below. This is similar to the case of closed string flux compactifications, where the flux superpotential on the space $\mathcal{M}_{CS}$ of complex structures can be computed in the unobstructed theory with $\mathcal{M}_{CS}$ as a true moduli space [45, 46, 47].

Let C_i denote the irreducible components of the 2-cycle carrying the additional brane charge and $C = \sum_i C_i$ their sum. If $[C] = 0$ as a class in $H_2(Z^*)$, there exists a 3-chain Γ in the sheaf cohomology group (2.4), with $\partial\Gamma = C$. In particular, the choice of the brane cycle C restricts the relevant co-homology to the subspace

$$H_3(Z^*, D) \ \longrightarrow\ H_3(Z^*, \sum_i C_i)\ . \tag{2.7}$$

The open-closed string superpotential $W(z,\hat{z})$ on $\mathcal{M}$ for this brane configuration is computed by a relative period integral $\underline{\Pi}(z,\hat{z})$ on this subspace [26, 31, 32].

It was argued in [48], that a superpotential, that has the correct critical points to describe a supersymmetric brane on C, is given by the chain integral

$$\mathcal{T} = \int_{\gamma(C)} \Omega \qquad \partial\gamma(C) \neq 0\ . \tag{2.8}$$

This expression was later derived from a dimensional reduction of the holomorphic Chern-Simons functional of ref. [33] in refs. [35, 21].[3]

As it stands, eq. (2.8) can be viewed either as a definition in absolute cohomology, or in relative cohomology, replacing $\Omega \to (\Omega, 0)$ and including the explicit boundary term in eq. (2.5). The difference is important only off-shell and in this way the relative cohomology ansatz of refs.[26, 31, 32], building on the results of [21], can be viewed as a particular proposal for an off-shell definition of the superpotential.

In absolute cohomology, the integral (2.8) is a priori ill-defined because of non-vanishing boundary contributions from exact forms, which do not respect the equivalence relation $[\Omega] = [\Omega + d\omega]$. To obtain a well-defined pairing one may restrict homology to chains with boundary $\partial\gamma$ a holomorphic curve and cohomology to sections of the Hodge subspace $F^2H^3 = H^{3,0} \oplus H^{2,1}$ [43].[4] This is the normal function

[3] More precisely, the chain integral gives the tension $\mathcal{T}$ of a domain wall realized by a brane wrapped on the 3-chain $\gamma(C)$.

[4] The potentially ambiguous boundary terms then vanish as $\int_\gamma \Omega + d\omega = \int_\gamma \Omega + \int_{\partial\gamma} \omega = \int_\gamma \Omega$ for ω a (2,0) form and $\partial\gamma$ a 2-cycle of type $(1,1)$.

point of view taken in refs. [27, 28]. Since the curve $C = \partial\gamma$ being holomorphic corresponds to a critical point $d\mathcal{W} = 0$ of the superpotential with respect to the open string moduli [48], continuous open-string deformations are excluded from the beginning and one obtains the critical value $W_{\rm crit}(z)$ of the superpotential as a function of the closed-string deformations z, only. The dependence of the critical superpotential $W_{\rm crit}(z)$ on the closed string moduli z is still a highly interesting quantity and at the center of the works [27, 28] on open string mirror symmetry, which gave the first computation of disc instantons in compact CY 3-fold from mirror symmetry. The dependence of the superpotential on open string deformations $\hat{z}$ is not captured by this definition.

In the relative cohomology ansatz of refs. [26, 31, 32], the pairing (2.8) is well-defined in cohomology also away from the critical points as a consequence of enlarging the co-homology spaces as in (2.4). The extra contribution to $H^3(Z^*, \mathcal{D})$ from the second factor in (2.4) describe additional degrees of freedom in the brane sector. According to this proposal, the relative periods $\underline{\Pi}(z, \hat{z})$ on the subspace $H^3(Z^*, C)$ describe the 'off-shell' superpotential $W(z, \hat{z})$ depending on brane deformations $\hat{z}$. For consistency, $\mathcal{W}(z, \hat{z})$ should reduce to the critical superpotential $W_{\rm crit}(z)$ at the critical points. This has been verified for particular examples in refs.[31, 32].

Although we eventually end up with studying the periods on the restricted subspace $H^3(Z^*, C)$ in (2.7) for a fixed brane charge C, the introduction of the larger relative cohomology space $H_3(Z^*, D)$ was not redundant, even for fixed choice of obstruction brane C, as it was crucial for the definition of the finite-dimensional off-shell deformation space $\mathcal{M}$, on which the obstruction superpotential can be defined. The off-shell deformation space for a brane on C is generically infinite-dimensional, with most of the deformations representing heavy fields in space-time that should be integrated out. To define an effective superpotential we have to pick an appropriate set of 'light' fields and integrate out infinitely many others.

The ansatz of refs. [26, 31, 32] to define $\mathcal{M}$ by perturbing the unobstructed moduli space of a family $\mathcal{D}$ of hypersurfaces is thus not a circuitry, but rather a systematic way to define a finite-dimensional deformation space with parametric small obstruction, together with a local coordinate patch, on which a meaningful off-shell superpotential can be defined. As C can be embedded in different families of hyperplanes, the parametrization of the deformation space depends on the choice of the family $\mathcal{D}$ and this corresponds to a different choice of light fields for the effective superpotential.[5] Each choice covers only a certain patch of the off-shell deformation space and there will be many choices to parametrize the same physics and mathematics near a critical locus by slightly different relative cohomology groups. This choice of a set of light fields is inherent to the use of effective actions and should not be confused with an ambiguity in the definition.[6]

[5] One could always combine these 'different' families into a single larger family at the cost of increasing the dimension of the deformation space $\mathcal{M}$.

[6] An attempt to reformulate the relative cohomology approach of refs. [26, 31, 32] by using the exci-

In the context of open string mirror symmetry, the most interesting aspect of the deformation spaces $\mathcal{M}$ constructed in this way is the presence of 'almost flat' directions in the open string sector, which lead to the characteristic A-model instanton expansion of the superpotential, as will be shown in in sect. 2.4. The result passes some non-trivial consistency checks which provides some evidence in favor of this definition of off-shell string mirror symmetry.[7] On the other hand, for general massive deformations, one would not expect the simple notions of flatness and an integral instanton expansion observed in this paper.

The most general superpotential captured by the relative cohomology group $H^3(Z^*, C)$ includes also a non-trivial closed string flux on $H^3(Z^*)$ and the two contributions can be combined in the general linear combination of relative period integrals [26, 31]

$$\mathcal{W}_{\mathcal{N}=1}(z,\hat{z}) = \sum_{\gamma_\Sigma \in H^3(Z^*,\mathcal{D})} \underline{N}_\Sigma \, \underline{\Pi}^\Sigma(z,\hat{z}) \; = \mathcal{W}_{closed}(z) + \mathcal{W}_{open}(z,\hat{z}), \tag{2.9}$$

where the contributions from the closed and open string sector are, with $\underline{N}_\Sigma := (N_\Sigma, \hat{N}_\Sigma)$,

$$\mathcal{W}_{closed}(z) = \sum_{\gamma_\Sigma, \partial\gamma_\Sigma = 0} N_\Sigma \, \underline{\Pi}^\Sigma(z), \qquad \mathcal{W}_{open}(z,\hat{z}) = \sum_{\gamma_\Sigma, \partial\gamma_\Sigma \neq 0} \hat{N}_\Sigma \, \underline{\Pi}^\Sigma(z) \ .$$

This is the superpotential for a four-dimensional $\mathcal{N} = 1$ supersymmetric string compactification with N_Σ and $\hat{N}_\Sigma$ quanta of background "fluxes" in the closed and open string sector, respectively. The first term $\mathcal{W}_{closed}(z)$ is proportional to the periods over *cycles* $\gamma_\Sigma \in H^3(Z^*)$ and represents the closed string "flux" superpotential for N_Σ "flux" quanta [50, 46, 47]. The second term captures the chain integrals (2.8). Note that there are contributions to the superpotential from different orders in the string coupling and the instanton expansion of the mirror A-model will involve *sphere and disc instantons* at the same time.[8]

There are two important points missing in the above discussion, which will be further studied in the following. One is the selection of the proper homology element $\gamma(C)$ that computes the superpotential, given a 2-cycle C representing the lower-dimensional brane charge. The other one is the mirror map, which allows to extract a prediction for the disc and sphere instanton expansion for the A-model, starting from the result obtained from the relative periods of the B-model. The additional information needed to answer these questions comes from the variation of mixed Hodge structure on the Hodge bundle with fiber the relative cohomology group

sion theorem, as contemplated on in ref. [41], is thus likely to produce just another parametrization corresponding to a slightly different choice of light fields, rather then a distinct description.

[7] See also refs. [32, 49] for additional examples and arguments.

[8] This links to the open-closed string duality to Calabi-Yau 4-folds of refs. [25, 32]; see also [51] for a recent discussion.

$H^3(Z^*, D)$. The Hodge filtration defines a grading by Hodge degree p of the cohomology space at each point $(z, \hat{z})$. In closed string mirror symmetry, restricting to $H^3(Z^*) \subset H^3(Z^*, D)$, this grading is identified with the $U(1)$ charge of the chiral ring elements in the conformal field theory on the string world-sheet. A similar interpretation in terms of an open-closed chiral ring has been proposed in refs. [26, 31]. The upshot of this extra structure is, that there are *two* relevant relative period integrals associated with the brane charge C, distinguished by the grading, such that one gives the mirror map to the A-model, and the other one the superpotential [26, 31, 32].

In the following sections we thus turn to a detailed study of the variation of the mixed Hodge structure on the relative cohomology group $H^3(Z^*, \mathcal{D})$, which we take as a geometric model for the variation of the states of the open-closed B-model over the deformation space $\mathcal{M}$. In sect. 2.2 we derive a set of differential equations, whose solutions determine a basis for the periods $\underline{\Pi}^\Sigma(z, \hat{z})$ on $\mathcal{M}$ in terms of generalized hypergeometric functions. In sect. 2.3 we study the mixed Hodge variation on the relative cohomology bundle, which leads to the selection of the proper functions for the mirror map and the superpotential.

2.2 Generalized hypergeometric systems for relative periods

In the first step we derive a generalized hypergeometric system of differential operators for the deformation problem defined above, in the concrete framework of toric branes on toric CY hypersurfaces defined in ref. [21] and further scrutinized in [32, 49]. The result is a system of differential equations acting on the relative cohomology space and its periods, whose associated Gauss-Manin system and solutions will be studied in the next section. The result is summarized in eq. (2.22); the reader who is not interested in the derivation may safely skip this section.

To avoid lengthy repetitions, we refer to refs. [21, 32] for the definitions of the family of toric branes in compact toric hypersurfaces, to refs. [52, 5] and the introduction for background material on mirror symmetry and toric geometry and to refs. [18, 19, 12] for generalized hypergeometric systems for the closed-string case. The notation is as follows : Δ is a reflexive polyhedron in $\mathbb{R}^5$ defined as the convex hull of p integral vertices $\nu_i \subset \mathbb{Z}^5 \subset \mathbb{R}^5$ lying in a hyperplane of distance one to the origin.[9] $W = P_{\Sigma(\Delta)}$ is the toric variety with fan $\Sigma(\Delta)$ defined by the set of cones over the faces of Δ. $\Delta^\star$ is the dual polyhedron and W^* the toric variety obtained from $\Sigma(\Delta^\star)$. The mirror pair of toric hypersurfaces in (W, W^*) is denoted by (Z, Z^*).

[9]We use the standard convention, identify the interior point ν_0 of Δ with the origin, and specify the vertices by four components $\nu_{i,k}$, $k = 1, ..., 4$, i.e. $\nu_0 = (0, 0, 0, 0)$; see refs. [12, 19] for more details.

The derivation of a GKZ hypergeometric system which annihilates the relative period integrals (2.6) on the relative cohomology group proceeds similarly as in the closed string case scratched in sect. 1.4. The definition of the (union of) hypersurfaces $\mathcal{D}$ cannot preserve all torus symmetries. Instead (some of) the torus actions move the position of the branes.[10] As a consequence, the relative periods are no longer annihilated by all the operators $\mathcal{Z}_k$ and depend on additional parameters specifying the geometry of $\mathcal{D}$. The differential eqs. (1.9) and (1.11) for the period integrals imply on the level of forms

$$\mathcal{L}(l)\ \Omega = d\omega(l)\ ,$$
$$\mathcal{Z}_k\ \Omega = d\omega_k\ .$$

The exact terms on the r.h.s. contribute only to integrals over 3-chains $\hat{\gamma} \in H_3(Z^*, \mathcal{D})$ with non-trivial boundaries $\partial\hat{\gamma}$. The modification of the differential equations for the relative periods can be computed from these boundary terms. To keep the discussion simple we derive the differential operators for the relative periods on the mirror of the quintic and present general formulae at the end of this section.

The holomorphic (3,0) form of the mirror quintic can be explicitly represented as the residuum

$$\Omega = \text{Res}\ \frac{a_0}{P}\prod_k \frac{dX_k}{X_k}\ , \tag{2.10}$$

at $P = 0$. The GLSM for the quintic is specified by one charge vector $l^1 = (-5; 1, 1, 1, 1, 1)$ which defines one differential operator $\mathcal{L}_1 = \mathcal{L}(l^1)$ annihilating the periods on the mirror. For this operator we do not get an exact term, and we find

$$\left(\prod_{i=1}^{5}\vartheta_i + z_1\prod_{i=1}^{5}(\vartheta_0 - i)\right)\ \Omega = 0\ . \tag{2.11}$$

Furthermore the operators $\mathcal{Z}_k$ give rise to the relations

$$\sum_{i=0}^{5}\vartheta_i\ \Omega = 0, \qquad (\vartheta_i - \vartheta_5)\ \Omega = d\omega_i\ , \quad i = 1, ..., 4, \tag{2.12}$$

with

$$\omega_i = (-)^{i+1}\text{Res}\ \frac{a_0}{P}\prod_{\substack{j=1\\ j\neq i}}^{4}\frac{dX_j}{X_j}\ . \tag{2.13}$$

Eq. (2.12) expresses the torus invariance of the period integrals in absolute cohomology and implies that the integrals depend only on the single invariant complex modulus z_1 defined as in eq. (1.10). In relative cohomology, the exact terms on the r.h.s descend to non-trivial 2-forms on $\mathcal{H}$ by the equivalence relation

$$H^3(Z^*, \mathcal{H})\ \ni\ (\Xi, \xi) \sim (\Xi + d\alpha, \xi + i^*\alpha - d\beta)\ , \tag{2.14}$$

[10] This is what one would expect intuitively from the formulation of mirror symmetry as T duality [53].

where $i : \mathcal{H} \hookrightarrow Z^*$ is the embedding. The exact pieces in (2.12) may give rise to boundary terms that break the torus symmetry and introduce an additional dependence on moduli $\hat{z}_\alpha(a_i)$ associated with the geometry of the embedding of $\mathcal{H}$.

To proceed we need to specify the family of hypersurfaces $\mathcal{H}$. As in refs. [31, 32] we consider a 1-parameter family $\mathcal{H}_1$ of hypersurfaces defined by the equation

$$\mathcal{H}_1 : \ Q = b_0 + b_1 X_1 = 0 \ . \tag{2.15}$$

The relative period integrals depend only on special combinations of the parameters a_i and b_j. First we will discuss the symmetries that fix these combinations. In order to determine the preserved torus symmetries we examine the boundary contributions (2.12) with respect to the hypersurfaces $\mathcal{H}_1$ by evaluating the pullbacks of the two forms (2.13) .

$$i^*\omega_1 = \mathrm{Res}\frac{a_0}{P_D}\prod_{i=2}^{4}\frac{dX_i}{X_i} \ , \qquad i^*\omega_k = 0 \ , k = 2,3,4 \ . \tag{2.16}$$

Here $a_0^{-1}P_D = a_0^{-1}P(Z^*)|_{Q=0} = (1+z_2) + X_2 + X_3 + X_4 - \frac{z_1}{z_2}(X_2X_3X_4)^{-1}$. As noted in ref. [32] the hypersurface $P_D = 0$ defines a fourfold covering of the mirror of the quartic K3 surface.

Thus in the presence of the hypersurface $\mathcal{H}_1$ the torus action $X_1 \to \lambda X_1$ with $\lambda \in \mathbb{C}^*$ generated by the operator $\mathcal{Z}_1$ is broken. However, as we have argued, the broken torus symmetry moves the position of the brane. On the other hand the position of the brane is captured in the 1-parameter family $\mathcal{H}_1$ of hypersurfaces parametrized by the variables b_j. Therefore, the generator of the torus symmetry $\mathcal{Z}_1$ is replaced by a modified generator $\hat{\mathcal{Z}}_1 = \mathcal{Z}_1 + \delta\mathcal{Z}_1$, where the operator $\delta\mathcal{Z}_1$ compensates the (infinitesimal) replacement by (infinitesimally) adjusting the parameters b_j. As a consequence the operator $\delta\mathcal{Z}_1$ should only depend on the parameters of the hypersurface $\mathcal{H}_1$.

A change in the position of the brane shifts the boundary of the 3-chains $\Gamma(b_0, b_1)$ and thus the period integrals $\int_{\Gamma(b_0,b_1)}\Omega$ change. Instead of changing the chains Γ we can also work with a fixed topological base of chains and transport Ω along the vector field $-v_{b_{0/1}} = -\frac{\partial Q}{\partial b_{0/1}}\partial_Q$. This changes the holomorphic (3,0) form by an exact piece

$$\vartheta_{b_1}\ \Omega = -b_1 \mathcal{L}_{v_{b_1}}\ \Omega = -d\left(\mathrm{Res}\ \frac{a_0}{P}\prod_{i=2}^{4}\frac{dX_i}{X_i}\right) \ . \tag{2.17}$$

Here $\mathcal{L}_{v_{b_1}}$ is the Lie derivative with respect to the vector field v_{b_1}. This exact piece compensates (2.16) , so $\delta\mathcal{Z}_1 = \vartheta_{b_1}$. We find the following generators for symmetries:

$$\begin{aligned} &\hat{\mathcal{Z}}_1 = \mathcal{Z}_1 + \delta\mathcal{Z}_1 = \vartheta_{a_1} - \vartheta_{a_5} + \vartheta_{b_1} \ , \qquad &&\hat{\mathcal{Z}}_i = \mathcal{Z}_i = \vartheta_{a_i} - \vartheta_{a_5} \ , \ i = 2,3,4 \ , \\ &\hat{\mathcal{Z}}_5 = \vartheta_{b_0} + \vartheta_{b_1} \ , &&\hat{\mathcal{Z}}_0 = \mathcal{Z}_0 = \sum_{i=0}^{5}\vartheta_{a_i} \ . \end{aligned} \tag{2.18}$$

The additional operator $\hat{\mathcal{Z}}_5$ generates the rescaling of the defining equation for the hypersurface $Q \to \lambda Q$ with $\lambda \in \mathbb{C}^*$.

From the six differential operators $\mathcal{Z}_k$ for the eight parameters a_i it follows that the family $\mathcal{H}_1 \subset Z^*$ depends only on the two moduli

$$z_1 = -\frac{a_1a_2a_3a_4a_5}{a_0^5}, \qquad \hat{z}_2 = -\frac{b_0a_1}{b_1a_0} \ . \tag{2.19}$$

Here z_1 is the complex structure modulus of Z^* and $\hat{z}_2$ is the open-string position modulus of the brane.

Note that it is also possible to start with a rigid hypersurface $Q = 1 + X_1$ in the ambient space. This gives a family of moving hypersurfaces inside the Calabi-Yau once one changes the equation $P(a_i),\ i = 0,..5$. In this case the combinations of $\{a_i\}$ are only restricted by the unbroken torus symmetries of the Calabi-Yau $\mathcal{Z}_i,\ i = 2,3,4$ while the torus symmetry $\mathcal{Z}_1$ is broken and gives rise to a second modulus $z_2 = -\frac{a_1}{a_0}$ in addition to the closed string modulus $z_1 = -\frac{a_1a_2a_3a_4a_5}{a_0^5}$. In the following we will again work with the hypersurface as defined in (2.15) as it simplifies the next steps.

The next task is to determine the differential operators $\hat{\mathcal{L}}_a$ of the analyzed extended GKZ hypergeometric system. Due to eq. (2.11) the operator $\mathcal{L}_1$ annihilates the three form Ω even on the level of relative forms (and not just on the level of the absolute three-form cohomology). As a consequence the operator $\mathcal{L}_1$ annihilates the relative form $\underline{\Omega}$. Therefore we identify $\mathcal{L}_1$ as one of the differential operators of the extended GKZ hypergeometric system:

$$\hat{\mathcal{L}}_1 \equiv \mathcal{L}_1 \ .$$

Other differential operators can be derived as usually by taking consecutive derivatives with respect to the parameters a_i, now also including b_0 and b_1. As discussed above (2.17) we have

$$\partial_{b_{0/1}}\Omega = -\mathcal{L}i_{v_{b_{0/1}}}\Omega = d\left(\mathrm{Res}\frac{a_0}{b_1}\frac{1}{P}\prod_{i=2}^{4}\frac{dX_i}{X_i}\frac{1}{X_1}\frac{\partial Q}{\partial b_{0/1}}\right) \ .$$

Differentiating with respect to $a_{0/1}$ and using $\partial_{a_0}P\partial_{b_1}Q = \partial_{a_1}P\partial_{b_0}Q$ we find the operator

$$\hat{\mathcal{L}}_2 = \vartheta_{a_1}\vartheta_{b_0} + \hat{z}_2(\vartheta_{a_0} - 1)\vartheta_{b_1} \ .$$

With the differential relations (2.18) it is straightforward to rewrite the logarithmic derivatives ϑ_{a_i} and ϑ_{b_j} in the extended GKZ operators $\hat{\mathcal{L}}_a$ in terms of the logarithmic derivates $\theta_1 = z_1\partial_{z_1}$ and $\theta_2 = \hat{z}_2\partial_{\hat{z}_2}$. Then the extended GKZ operators $\hat{\mathcal{L}}_a$ in terms of the moduli $z_1, \hat{z}_2$ become

$$\begin{aligned}\hat{\mathcal{L}}_1 &= (\theta_1+\theta_2)\theta_1^4 + z_1\prod_{i=1}^{5}(-5\theta_1-\theta_2-i) =: \mathcal{L}_1^{bulk} + \mathcal{L}_1^{bound}\theta_2 \ , \\ \hat{\mathcal{L}}_2 &= \big((\theta_1+\theta_2) - \hat{z}_2(-5\theta_1-\theta_2-1)\big)\theta_2 =: \mathcal{L}_2^{bound}\theta_2 \ .\end{aligned} \tag{2.20}$$

Here $\mathcal{L}_1^{bulk} = \theta_1^5 + z_1 \prod_{i=1}^5(-5\theta_1 - i)$ is the $\hat{z}_2$ independent Picard-Fuchs operator of the quintic and the operators $\mathcal{L}_a^{bound}$ are always accompanied by at least one derivative θ_2 and thus are only sensitive to boundary contributions

$$0 = \hat{\mathcal{L}}_a \int_{\gamma_\Sigma} \underline{\Omega} = \mathcal{L}_a^{bulk} \int_{int(\gamma_\Sigma)} \Omega - \mathcal{L}_a^{bound} \int_{\partial\gamma_\Sigma} \theta_2 \xi \ , \quad a = 1, 2 \ ,$$

with $\mathcal{L}_2^{bulk} \equiv 0$.

In summary we have obtained the two differential operators $\hat{\mathcal{L}}_a,\ a = 1, 2$, together with the six differential operators $\hat{\mathcal{Z}}_k,\ k = 0, \ldots, 5$, annihilating the relative periods. These operators can be rewritten in a concise form by realizing that they represent the differential operators $\hat{\mathcal{L}}(l)$ and $\hat{\mathcal{Z}}_k$ for a different GKZ system specified by the two charge vectors

	a_0	a_1	a_2	a_3	a_4	a_5	b_0	b_1
l	-5	1	1	1	1	1	0	0
$\hat{l}$	-1	1	0	0	0	0	1	-1

. (2.21)

The five-dimensional enhanced toric polyhedron $\hat{\Delta}$ can be found in the appendix.

Note that the moduli of the relative cohomology problem defined in (2.19) coincide with the local coordinates $\hat{z}_a(\hat{l}^a)$ defined by the general relation (1.10). Similarly the operators $\hat{\mathcal{Z}}_k(\hat{\Delta})$ given by the formula (1.9) agree with the six operators (2.18), and finally the GKZ operators $\hat{\mathcal{L}}_a = \mathcal{L}(\hat{l}^a)$, obtained from the general expression

$$\mathcal{L}(l) = \prod_{k=1}^{l_0}(\vartheta_0 - k) \prod_{l_i>0} \prod_{k=0}^{l_i-1}(\vartheta_i - k) - (-1)^{l_0} z_a \prod_{k=1}^{-l_0}(\vartheta_0 - k) \prod_{l_i<0} \prod_{k=0}^{-l_i-1}(\vartheta_i - k) \ , \quad (2.22)$$

coincide with the two operators (2.20) of the relative cohomology problem.

In fact one can show that all differential operators $\mathcal{L}(\tilde{l})$ for $\tilde{l}$ a linear combination of $l, \hat{l}$ also annihilate the relative periods.[11] The coincidence of the system of differential operators for the periods on the relative cohomology group $H^3(Z^*, \mathcal{D})$ and the GKZ system for the dual four-folds constructed by the method of ref. [32] holds more generally for relative cohomology groups associated with the class of toric branes on toric hypersurfaces defined in ref. [21]. A mathematically rigorous derivation of such Picard-Fuchs systems appeared in [44].

[11] For the quintic the additional operators are of the form $\mathcal{L}^{bdry}\theta_2$, where $\mathcal{L}^{bdry} i^* \omega_1 = 0$ modulo exact 2-forms on $\mathcal{D}_1$.

2.3 Gauss-Manin connection and integrability conditions

The Picard-Fuchs system for the relative periods derived in the previous section captures the variation of Hodge structure on the relative cohomology group $H^3(Z^*, \mathcal{D})$. In this section we work out some relations and predictions for mirror symmetry for a family of A- and B-branes from the associated Gauss-Manin connection.

2.3.1 Gauss-Manin connection on the open-closed deformation space

Geometrically we can view $H^3(Z^*, \mathcal{D})$ as the fiber of a complex vector bundle over the open-closed deformation space $\mathcal{M}$. As the relative cohomology group[12] H^3 depends only on the topological data, the fiber is up to monodromy constant over $\mathcal{M}$, and there is a trivially flat connection, ∇, called the Gauss-Manin connection. The Hodge decomposition $H^3 = \oplus_{p=0}^3 H^{3-p,p}$ varies over $\mathcal{M}$, as the definition of the Hodge degree depends on the complex structure. The Hodge filtrations F^p

$$H^3 = F^0 \supset F^1 \supset F^2 \supset F^3 \supset F^4 = 0 \ , \qquad F^p = \oplus_{q \geq p} H^{q,3-q} \subset H^3 \ ,$$

define holomorphic subbundles $\mathcal{F}^p$ whose fibers are the subspaces $F^p \subset H^3$. The action of the Gauss-Manin connection ∇ on these subbundles has the property $\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes T^*_{\mathcal{M}}$, known as Griffiths transversality.

Concretely, the mixed Hodge structure on the relative cohomology space $H^3(Z^*, \mathcal{D})$ looks as follows. The Hodge filtrations are

$$\begin{aligned}
F^3 &= H^{3,0}(Z^*, \mathcal{D}) &&= H^{3,0}(Z^*) \ , \\
F^2 &= F^3 \oplus H^{2,1}(Z^*, \mathcal{D}) &&= F^3 \oplus H^{2,1}(Z^*) \oplus H^{2,0}_{var}(\mathcal{D}) \ , \\
F^1 &= F^2 \oplus H^{1,2}(Z^*, \mathcal{D}) &&= F^2 \oplus H^{1,2}(Z^*) \oplus H^{1,1}_{var}(\mathcal{D}) \ , \\
F^0 &= F^1 \oplus H^{0,3}(Z^*, \mathcal{D}) &&= F^1 \oplus H^{0,3}(Z^*) \oplus H^{0,2}_{var}(\mathcal{D}) \ ,
\end{aligned} \qquad (2.23)$$

where the equations to the right display the split $H^3(Z^*, \mathcal{D}) \simeq \ker \big(H^3(Z^*) \to H^3(\mathcal{D})\big) \oplus \mathrm{coker}\, \big(H^2(Z^*) \to H^2(\mathcal{D})\big)$. The weight filtration is defined as

$$W_2 = 0 \ , \qquad W_3 = H^3(Z^*) \ , \qquad W_4 = H^3(Z^*, \mathcal{D}) \ ,$$

such that the quotient spaces $W_3/W_2 \simeq H^3(Z^*)$ and $W_4/W_3 \simeq H^2(\mathcal{D})$ define pure Hodge structures. Variations in the closed (δ_z) and open ($\delta_{\hat{z}}$) string sector act schematically as

$$\begin{array}{ccccccc}
F^3 \cap W_3 & \xrightarrow{\delta_z} & F^2 \cap W_3 & \xrightarrow{\delta_z} & F^1 \cap W_3 & \xrightarrow{\delta_z} & F^0 \cap W_3 \\
& \searrow^{\delta_{\hat{z}}} & & \searrow^{\delta_{\hat{z}}} & & \searrow^{\delta_{\hat{z}}} & \\
& & F^2 \cap (W_4/W_3) & \xrightarrow{\delta_z, \delta_{\hat{z}}} & F^1 \cap (W_4/W_3) & \xrightarrow{\delta_z, \delta_{\hat{z}}} & F^0 \cap (W_4/W_3)
\end{array} \qquad (2.24)$$

[12]Letters without arguments refer to relative cohomology over $\mathbb{C}$, e.g. $H^3 = H^3(Z^*, \mathcal{D}; \mathbb{C})$.

The variation of the Hodge structure over $\mathcal{M}$ can be measured by the period matrix

$$\underline{\Pi}_A^\Sigma = \int_{\gamma_\Sigma} \alpha_A\,, \qquad \alpha_A \in H^3\ ,$$

where γ_Σ is a fixed topological basis for $H_3(Z^*, \mathcal{D})$ and $\{\alpha_A\}$ with $A = 1, ..., \dim(H^3)$ denotes a basis of relative 3-forms. One may choose an ordered basis $\{\alpha_A^{(q)}\}$ adapted to the Hodge filtration, such that the subsets $\{\alpha_A^{(q')}\}$, $q' \leq q$ span the spaces F^{3-q} for $q = 0, ..., 3$.

To make contact between the Hodge variation and the B-model defined at a point $m \in \mathcal{M}$, the Gauss-Manin connection has to be put into a form compatible with the chiral ring properties of a SCFT. Chiral operators of definite $U(1)$ charge are identified with forms of definite Hodge degree, which requires a projection onto the quotient spaces F^p/F^{p+1} at the point m. Moreover, the canonical CFT coordinates t_a, centered at $m \in \mathcal{M}$, should flatten the connection ∇ and we require

$$\nabla_a \alpha_a^{(q)}(m) = \partial_{t_a} \alpha_a^{(q)}(m) \stackrel{!}{=} (C_a(t) \cdot \alpha_a^{(q)})(m)\ \in F^{3-q-1}/F^{3-q}|_m\ . \qquad (2.25)$$

The second equation is an important input as it expresses the non-trivial fact, that in the CFT, an infinitesimal deformation in the direction t_a is generated by an insertion of (the descendant) of a chiral operator $\phi_a^{(1)}$ in the path integral, which in turn can be described by a naive multiplication by the operator $\phi_a^{(1)}$ represented by the connection matrix $C_a(t)$. The above condition assumes that such a simple relation holds on the full open-closed deformation space for all deformations in F^2/F^3. Thus $\phi_a^{(1)}$ can be either a bulk field of left-right $U(1)$ charge $(1,1)$ or a boundary operator of total $U(1)$ charge 1. The consistency of the results obtained below with this ansatz and the correct matching with the CFT deformation space discussed in sect. 2.5 provides evidence in favor of a proper CFT realization of this structure.

Phrased differently, we consider the $\alpha_a^{(q)}$ as flat sections of an "improved" flat connection D_a in the sense of refs. [54, 55]

$$D_a \alpha_a^{(q)} = 0, \qquad D_a = \partial_{z_a} - \Gamma_a(z) - C_a(z), \qquad [D_a, D_b]\ =\ 0\ ,$$

where z_a are local coordinates on $\mathcal{M}$, the connection terms $\Gamma_a(z)$ and $C_a(z)$ are maps from $\mathcal{F}^{3-q}$ to $\mathcal{F}^{3-q}$ and $\mathcal{F}^{3-q-1}$, respectively, and $\Gamma_a(z)$ vanishes in the canonical CFT coordinates t_a.[13]

Instead of working in generality, we study the Gauss-Manin connection for the relative cohomology group on the two parameter family of branes on the quintic defined by (2.15). We consider a large volume phase with moduli (1.10) defined by the following linear combination of charge vectors (2.21):

[13]See sect. 2.6 of ref. [55] for the definition of canonical coordinates from the (closed-string) CFT point of view.

	a_0	a_1	a_2	a_3	a_4	a_5	b_0	b_1
l	-4	0	1	1	1	1	-1	1
$\hat{l}$	-1	1	0	0	0	0	1	-1

.

A complete set of differential operators derived from eq. (2.22) is given by

$$\begin{aligned}\mathcal{L}_1 &= \theta_1^4 + (4z_1(\theta_1-\theta_2) - 5z_1z_2(4\theta_1+\theta_2+4))\prod_{i=1}^3(4\theta_1+\theta_2+i)\ ,\\ \mathcal{L}_2 &= \theta_2(\theta_1-\theta_2) + z_2(\theta_1-\theta_2)(4\theta_1+\theta_2+1)\ ,\\ \mathcal{L}_3 &= \theta_1^3(\theta_1-\theta_2) + (4z_1\prod_{i=1}^3(4\theta_1+\theta_2+i) + z_2\theta_1^3)(\theta_1-\theta_2)\ .\end{aligned} \tag{2.26}$$

Computing the ideal generated by the $\mathcal{L}_k$ acting on $\underline{\Omega}$ shows that H^3 is a seven-dimensional space spanned by the multiderivatives $(1,\theta_1,\theta_2,\theta_1^2,\theta_1\theta_2,\theta_1^3,\theta_1^2\theta_2)$ of $\underline{\Omega}$. The dimensions $d_q = \dim(F^{3-q}/F^{3-q+1})$ are $1,2,2,2$ for $q=0,1,2,3$, respectively. The $d_1 = 2$ directions tangent to $\mathcal{M}$ represent the single complex structure deformation $z = z_1z_2$ of the mirror quintic Z^* and the parameter $\hat{z} = z_2$ parametrizing the family of hypersurfaces.

To implement a CFT like structure at a point $m \in \mathcal{M}$, one may take linear combinations of the multiderivatives acting on $\underline{\Omega}$ to obtain ordered bases $\{\alpha_a^{(q)}\}$ and $\{\gamma_\Sigma\}$ which bring the period matrix into a block upper triangular form[14]

$$\underline{\Pi}_\Sigma^A = \begin{pmatrix} 1 & * & * & * \\ 0 & 1\!\!1_{d_1\times d_1} & * & * \\ 0 & 0 & 1\!\!1_{d_2\times d_2} & * \\ 0 & 0 & 0 & 1\!\!1_{d_3\times d_3}\end{pmatrix}\ . \tag{2.27}$$

Griffiths transversality then implies that in the local coordinates at a point of maximal unipotent monodromy

$$\begin{pmatrix}\nabla\alpha_1^{(0)}\\ \nabla\alpha_2^{(1)}\\ \nabla\alpha_3^{(1)}\\ \nabla\alpha_4^{(2)}\\ \nabla\alpha_5^{(2)}\\ \nabla\alpha_6^{(3)}\\ \nabla\alpha_7^{(3)}\end{pmatrix} = \begin{pmatrix} 0 & \sum_{a=1}^2 \frac{dz_a}{z_a}M_a^{(1)} & 0 & 0 \\ 0 & 0 & \sum_{a=1}^2 \frac{dz_a}{z_a}M_a^{(2)} & 0 \\ 0 & 0 & 0 & \sum_{a=1}^2 \frac{dz_a}{z_a}M_a^{(3)} \\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\alpha_1^{(0)}\\ \alpha_2^{(1)}\\ \alpha_3^{(1)}\\ \alpha_4^{(2)}\\ \alpha_5^{(2)}\\ \alpha_6^{(3)}\\ \alpha_7^{(3)}\end{pmatrix}, \tag{2.28}$$

where the moduli-dependent matrices $M_a^{(q)}$ of dimension $d_{q-1}\times d_q$ are derivatives of the entries of $\underline{\Pi}$ in eq. (2.27). The above expression is written in logarithmic variables $\ln(z_a)$, anticipating the logarithmic behavior of the periods at the point of maximal unipotent monodromy centered at $z_a = 0$. In local coordinates x_a centered

[14] It is understood, that all entries in the following matrices are block matrices operating on the respective subspaces of definite $U(1)$ charge, with dimensions determined by the numbers d_q.

at a generic point $m \in \mathcal{M}$, the periods are analytic in x_a and dz_a/z_a should be replaced with dx_a.

The left upper block can be brought into the form

$$\sum_{a=1}^{2} \frac{dz_a}{z_a} M_a^{(1)} = (\frac{dq_1}{2\pi i\, q_1}, \frac{dq_2}{2\pi i\, q_2})$$

by the variable transformation

$$q_a(z) = \exp(2\pi i\, \underline{\Pi}_1^{a+1}(z)) \ . \tag{2.29}$$

It has been proposed in ref. [26], that eq. (2.29) represents the mirror map between the A-model Kähler coordinates $t_a = \frac{1}{2\pi i}\ln(q_a)$ on the open-closed deformation space of an A-type compactification (Z, L) and the coordinates z_a on the complex structure moduli space of an B-type compactification (Z^*, E) near a large complex structure point. We propose that the above flatness conditions defines more generally the mirror map between the open-closed deformation spaces for any point $m \in \mathcal{M}$.[15] It is worth stressing that the mirror map defined by the above flatness argument coincides with the mirror map obtained earlier in refs. [21, 23] for non-compact examples by a physical argument, using domain wall tensions and the Ooguri-Vafa expansion at a large complex structure point. This coincidence can be viewed as experimental evidence for the existence of a more fundamental explanation of the observed flat structure from the underlying topological string theory, as advocated for in this note.

Identifying $\alpha_1^{(0)}$ with the unique operator $\phi^{(0)} = 1$ and the $\alpha_{a+1}^{(1)}$ with the charge one operators $\phi_a^{(1)}$ associated with the flows parametrized by $\ln(q_a)$, eq. (2.28) implements the CFT relation

$$\nabla_{q_a}\phi^{(0)} = \phi_a^{(1)} = \phi_a^{(1)} \cdot \phi^{(0)} \ ,$$

discussed below (2.25). The above series of arguments and manipulations is standard material in closed string mirror symmetry and led to the deep connection between the geometric Hodge variations of CY three-folds in the B-model and A-model quantum cohomology on the mirror.[16] After the variable transformation (2.29) and restricting to the subspace $H^3(Z^*)$ describing the complex moduli space $\mathcal{M}_{CS}$ of the mirror quintic, eq. (2.28) becomes

$$\begin{pmatrix} \nabla\alpha_{cl}^{(0)} \\ \nabla\alpha_{cl}^{(1)} \\ \nabla\alpha_{cl}^{(2)} \\ \nabla\alpha_{cl}^{(3)} \end{pmatrix} = \begin{pmatrix} 0 & \frac{dq}{2\pi i\, q} & 0 & 0 \\ 0 & 0 & C(q)\frac{dq}{2\pi i\, q} & 0 \\ 0 & 0 & 0 & -\frac{dq}{2\pi i\, q} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{cl}^{(0)} \\ \alpha_{cl}^{(1)} \\ \alpha_{cl}^{(2)} \\ \alpha_{cl}^{(3)} \end{pmatrix} , \tag{2.30}$$

---

[15] A non-trivial example will be described in ref. [56].

[16] See refs. [5, 52] for background material and a comprehensive list of references.

with $\alpha_{cl}^{(p)} \in H^{3-p,p}(Z^*, \mathbb{C})$ and $q = q_1 q_2 = e^{2\pi i t}$. Under mirror symmetry these data get mapped to the Kähler volume t and the so-called Yukawa coupling $C(q) = 5+\mathcal{O}(q)$, which describes the classical intersection and the Gromov-Witten invariants on the quintic. In the CFT, the quantities $C(q)$ represent the moduli-dependent structure constants of the ring of chiral primaries defined in ref. [57].

The point which we are stressing here is that at least part of these concepts continue to make sense for the Hodge variation (2.28) on the full relative cohomology space $H^3(Z^*, \mathcal{D})$ over the open-closed deformation space $\mathcal{M}$ fibered over $\mathcal{M}_{CS}$. More importantly, the Hodge theoretic definition of mirror symmetry described above gives correct results for the open string analogues of the Gromov-Witten invariants in those cases, where results have been obtained by different methods, such as space-time arguments involving domain walls [21, 23].[17]

In this sense, the existence of a flat structure observed above, and the agreement of the Hodge theoretic results with other methods, if available, urges for a proper CFT description of the deformation families defined over $\mathcal{M}$ and an appropriate open-string extension of A-model quantum cohomology. In the following we collect further evidence in favor of an interesting integrable structure on the open-closed deformation space, working in the B-model.

2.3.2 Integrability conditions

The correlation functions of the topological family of closed-string CFTs satisfy the famous WDVV integrability condition [59, 60]. In the context of the B-model on a CY three-fold, this condition becomes part of $\mathcal{N} = 2$ special Kähler geometry of the complex structure moduli space, which implies, amongst others, the existence of a single holomorphic prepotential $\mathcal{F}$ that determines all entries of the period matrix in the canonical CFT coordinates t_a.

There exists no prepotential for the period matrix (2.27) on the relative cohomology group $H^3(Z^*, \mathcal{D})$, but certain aspects of the $\mathcal{N} = 2$ special geometry of the closed-string sector $H^3(Z^*) \subset H^3(Z^*, \mathcal{D})$ generalize to the larger cohomology space, justifying the term $\mathcal{N} = 1$ special geometry [25, 26].[18] Some aspects of this $\mathcal{N} = 1$ special geometry have been worked out for non-compact Z^* in [26, 40] and we add here some missing pieces for the compact case. In the following we work at a "large complex structure point" $m_0 \in \mathcal{M}$ of maximal unipotent monodromy. The existence of such points m_0 follows from the general property of the GKZ systems described in sect. 2.2. We start from the following general ansatz for the 7-dimensional period

[17]See [58, 26, 40, 31, 32, 49] for various examples.

[18]$\mathcal{N} = 1, 2$ denotes the number of 4d space-time supersymmetries of the CY compactification of the physical type II string to four dimensions with and without branes.

vector of the holomorphic 3-form

$$\underline{\Pi}_1^\Sigma = \big(1,\ t,\ \hat{t},\ F_t(t),\ W(t,\hat{t}),\ F_0(t),\ T(t,\hat{t})\ \big)\ , \tag{2.31}$$

where t is the closed- and $\hat{t}$ the open-string deformation, related to the flat normal crossing divisor coordinates $(t_1(z_1,z_2), t_2(z_1,z_2))$ of eq. (2.29) by the linear transformation $t = t_1 + t_2$, $\hat{t} = t_2$. The subset of periods in the closed-string sector is determined by the prepotential $\mathcal{F}$ as $(1, t, F_t = \partial_t\mathcal{F}, F_0(t) = 2\mathcal{F}(t) - t\partial_t\mathcal{F})$ and depends only on t. The additional periods $(\hat{t}, W(t,\hat{t}), T(t,\hat{t}))$ are so far arbitrary functions, except that the leading behavior at m_0 at $z_a = 0$ is, schematically,

$$t,\hat{t} \sim \ln(z_\cdot), \qquad F_t, W \sim \ln^2(z_\cdot), \qquad F_0, T \sim \ln^3(z_\cdot)\ .$$

The function W is in some sense the closest analogue of the closed string prepotential and indeed has been conjectured to be a generating function for the open-string disc invariants in [26, 31].

For an appropriate choice of basis $\{\alpha_a^{(q)}\}$, the period matrix takes the upper triangular form (2.27) with entries

$$(\underline{\Pi}) = \begin{pmatrix} 1 & t & \hat{t} & F_t & W & F_0 & T \\ 0 & 1 & 0 & F_{t,t} & W_{,t} & F_{0,t} & T_{,t} \\ 0 & 0 & 1 & 0 & W_{,\hat{t}} & 0 & T_{,\hat{t}} \\ 0 & 0 & 0 & 1 & 0 & -t & \mu \\ 0 & 0 & 0 & 0 & 1 & 0 & \rho \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.32}$$

where the derivatives w.r.t. t and $\hat{t}$ are denoted by subscripts and the functions μ and ρ are defined by

$$\mu = \frac{W_{,t\hat{t}}T_{,tt} - W_{,tt}T_{,t\hat{t}}}{CW_{,t\hat{t}}}, \qquad \rho = \frac{T_{,t\hat{t}}}{W_{,t\hat{t}}}, \qquad C = \mathcal{F}_{,ttt} = F_{t,tt}\ .$$

The connection matrices read

$$M_t = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & W_{,tt} & 0 & 0 \\ 0 & 0 & 0 & 0 & W_{,t\hat{t}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & \mu_t \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_{,t} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ M_{\hat{t}} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & W_{,t\hat{t}} & 0 & 0 \\ 0 & 0 & 0 & 0 & W_{,\hat{t}\hat{t}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_{,\hat{t}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_{,\hat{t}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.33}$$

The integrability condition $\partial_t M_{\hat{t}} - \partial_{\hat{t}} M_t + [M_{\hat{t}}, M_t] = 0$ implies that

$$\rho(t,\hat{t}) = aW_{,\hat{t}} + b\ , \qquad \mu(t,\hat{t}) = aC^{-1}(t)\left(\int (W_{,t\hat{t}}^2 - W_{,tt}W_{,\hat{t}\hat{t}})\, d\hat{t} + g(t)\right)\ , \tag{2.34}$$

with a, b some complex constants and $g(t)$ an undetermined function. The relation (2.25) then implies that the period T is of the form

$$T(t,\hat{t}) = \int \left(\frac{a}{2}W_{,\hat{t}}^2 + bW_{,\hat{t}}\right) d\hat{t} + f(t) \ , \tag{2.35}$$

with $\partial_t^2 f(t) = g(t)$. The integrability condition (2.35) determines the top period in the open-string sector in terms of the other periods, up to the function $f(t)$. In this sense it is similar to the relation in the closed-string sector, that determines the top period F_0 in terms of the other periods. The integration constants can be fixed by determining the leading behavior of the periods in the large volume limit.

The above argument and the integrability relation (2.35) applies to any two parameter family with one closed- and one open-string modulus and can be straightforwardly generalized to more parameter cases. For a given geometry, such as the quintic family described by the operators $\mathcal{L}_k$ in (2.26), one can of course reach the same conclusion by studying the explicit solutions and also determine the function $f(t)$. The relative forms in the open-string sector of this family can be associated with the Hodge variation on a quartic K3 surface. The period vector π of the K3 surface is spanned by the solutions $\partial_{\hat{t}}(\hat{t}, W, T) = (1, W_{,\hat{t}}, T_{,\hat{t}})$ and the integrability condition (2.35) represents an algebraic relation $\pi^T\,\hat{\eta}\,\pi = 0$ amongst the K3 periods, where $\hat{\eta}$ is the intersection matrix.

The above discussion was essentially independent of the choice of the large complex structure point m_0 and a similar argument for other m shows that the integrability condition (2.35) holds for any $m \in \mathcal{M}$ in the local coordinates defined by (2.29).

A cautious remark

The similarities of the above arguments with those in the case of closed string mirror symmetry may have obscured the fact that one crucial datum is still incomplete: the topological metric η on the open-closed state space. In the closed-string sector, the topological metric η_{cl} is given by the classical intersection matrix on $H_3(Z^*)$ and its knowledge permits, amongst others, the determination of the true geometric periods as a particular linear combinations of the solutions to the Picard-Fuchs equations. More importantly, the topological metric is needed to complete the argument that identifies $C(q)$ in eq. (2.30) with the structure constants of the chiral ring of ref. [57], as well as to access the non-holomorphic and higher genus sector of the theory using the tt^* equations [54] and the holomorphic anomaly equation [61]. Our present lack of a precise understanding of the topological metric in the open-string sector renders the following sections somewhat fragmentary.

In the next section we derive predictions for disc invariants by fixing the geometric periods in a way that avoids the knowledge of the topological metric. Some observations on the other issues mentioned above, will be discussed in sect. 2.5, for a proposal for the full topological metric see [2].

2.4 Large radius invariants for the A-model

The geometry of A-branes is notoriously more difficult to study than that of the B-branes. For the type of B-branes studied above, the A-model mirror geometry can be in principle obtained from the toric framework of ref. [21]. The conjectural family of A-branes mirror to the B-brane family studied above, is defined on the quintic hypersurface Z in the toric ambient space $W = \mathcal{O}(-5)_{\mathbb{P}^4}$ with homogeneous coordinates

	P	x_1	x_2	x_3	x_4	x_5
l	-5	1	1	1	1	1

.

The Kähler moduli (t_1, t_2) mirror to the complex structure coordinates (z_1, z_2) are defined by the equations

$$\sum_{i=1}^{5} |x_i|^2 - 5|P|^2 = \Im t = \Im t_1 + \Im t_2, \qquad |x_1|^2 - |P|^2 = \Im t_2 \ , \tag{2.36}$$

with $\Im t_1, \Im t_2 > 0$. The first constraint holds on all of Z and the Kähler modulus t describes the closed string deformation, the overall Kähler volume of Z. The second constraints holds only on the Lagrangian submanifold L and describes the open-string deformation.

The toric framework of ref. [21] gives an explicit description of the geometry of Lagrangian subspaces in the ambient space W, which has been used to study an interesting class of non-compact branes for CY ambient spaces, see e.g. [21, 23, 26]. The clear geometric picture of the toric description is lost for hypersurfaces and the searched for subspace $L \subset Z$ carrying the mirror A-brane has no simple description, at least at general points in the (full) moduli space and to our knowledge. However, by the homological mirror symmetry conjecture [20], we expect that a corresponding A-brane, which is mirror to the B-brane given in terms of the discussed divisor together with its curvature 2-form, should be present in the A-model geometry. Clearly, in order to complete our picture a constructive recipe of mapping B-branes to the corresponding A-branes for compact CY geometries is desirable.

Since the A-model geometry is naively independent of the complex structure moduli, one is tempted to choose a very special form of the hypersurface constraint to simplify the geometry. In ref. [62] it is shown how the number 2875 of lines in the generic quintic can be determined from the number of lines in a highly degenerate quintic, defined by the hypersurface constraint

$$P(Z_\alpha) = p_1.p_4 + \alpha\, p_5, \qquad \alpha \in \Delta \ . \tag{2.37}$$

Here α is a parameter on the complex disc Δ and the p_k are degree k polynomials in the homogeneous coordinates of $\mathbb{P}^4$. At the point $\alpha = 0$ the quintic splits into two components of degree one and four. Katz shows, that there are 1600+1275=2875 holomorphic maps to lines in the two components of the central fiber that deform to the fiber at $\alpha \neq 0$.

The $N_1 = 2875$ curves of degree one contribute to the tension $\mathcal{T}$ of a D4-brane wrapping the 4-cycle $\Gamma = H \cap Z$ as

$$\mathcal{T} = -\frac{5}{2}t^2 + \frac{1}{4\pi^2}\left(2875\sum_k \frac{q^k}{k^2} + 2\cdot 609250\sum_k \frac{q^{2k}}{k^2} + ...\right)$$

where $q = \exp(2\pi i t)$, H is the hyperplane class and the dots denote linear and constant terms in t as well as instanton corrections from maps of degree $d > 2$. In the singular CY the generic 4-cycle splits into two components and one expects two separate contributions

$$\mathcal{T}^{(1)} = c^{(1)}\, t^2 + \frac{N_d^{(1)}}{4\pi^2}\sum_k \frac{q^{dk}}{k^2} + ..., \qquad \mathcal{T}^{(2)} = \mathcal{T} - \mathcal{T}_1\ ,$$

with $N_1^{(1)} = 1600$ and $N_1^{(2)} = 1275$.

As explained in ref. [62], there are other genus zero maps to the two components, that develop nodes at the intersection locus $p_1 = p_4 = 0$ upon deformation, and they do not continue to exist as maps from S^2 to S^2. The idea is that in the presence of the Lagrangian A-brane on the degenerate quintic, the nodes of the spheres can open up to become the boundary of holomorphic disc instantons ending on L. Indeed the two independent double logarithmic solutions of the Picard-Fuchs system (2.26) can be written in the flat coordinates (2.29) as

$$\begin{aligned} \mathcal{T}^{(1)} &= -2t^2 + \tfrac{1}{4\pi^2}\textstyle\sum_k\ \tfrac{1}{k^2}\left(1600q^k + 2\cdot 339800q^{2k} + ...\right) + \mathcal{T}^{(o)}(t_1,t_2) \\ \mathcal{T}^{(2)} &= -\tfrac{1}{2}t^2 + \tfrac{1}{4\pi^2}\textstyle\sum_k\ \tfrac{1}{k^2}\left(1275q^k + 2\cdot 269450q^{2k} + ...\right) - \mathcal{T}^{(o)}(t_1,t_2) \end{aligned} \quad (2.38)$$

showing the expected behavior and adding up to the closed-string period $\mathcal{T}$. The split of the degree two curves, $N_2 = 609250 = 339800 + 269450 = (258200 + \frac{1}{2}163200) + (187850 + \frac{1}{2}163200)$ is compatible with the results of ref. [63].

The extra contribution $\mathcal{T}^{(o)}(t_1,t_2)$ can be written as

$$\mathcal{T}^{(o)}(t_1,t_2) = 4tt_2 - 2t_2^2 + \frac{1}{4\pi^2}\sum_{\substack{k,n_1,n_2\\ n_1\neq n_2}} \frac{1}{k^2}\ N_{n_1,n_2}(q_1^{n_1}q_2^{n_2})^k\ .$$

The first few coefficients N_{n_1,n_2} for small n_i, including the contributions from $n_1 = n_2$, are listed in table 1 below.

$n_2 \backslash n_1$	0	1	2	3	4	5
0	0	−320	13280	−1088960	119783040	−15440622400
1	20	1600	−116560	12805120	−1766329640	274446919680
2	0	2040	679600	−85115360	13829775520	−2525156504560
3	0	−1460	1064180	530848000	−83363259240	16655092486480
4	0	520	−1497840	887761280	541074408000	−95968626498800
5	0	−80	1561100	−1582620980	931836819440	639660032468000
6	0	0	−1152600	2396807000	−1864913831600	1118938442641400
7	0	0	580500	−2923203580	3412016521660	−2393966418927980
8	0	0	−190760	2799233200	−5381605498560	4899971282565360
9	0	0	37180	−2078012020	7127102031000	−9026682030832180
10	0	0	−3280	1179935280	−7837064629760	14557931269209000
11	0	0	0	−502743680	7104809591780	−20307910970428360
12	0	0	0	155860160	−5277064316000	24340277955510560
13	0	0	0	−33298600	3187587322380	−24957649473175420
14	0	0	0	4400680	−1549998228000	21814546476229120
15	0	0	0	−272240	597782974040	−16191876966658500
16	0	0	0	0	−178806134240	10157784412551120
17	0	0	0	0	40049955420	−5351974901676280

Table 1: Predictions for Ooguri–Vafa invariants.

According to the general philosophy of the Hodge theoretic mirror map described in the previous sections, the double logarithmic solutions represent the generating function of holomorphic discs ending on the A-brane L. In the basis of sect. 2.3 we find

$$F_t = \mathcal{T}^{(1)} + \mathcal{T}^{(2)} = \mathcal{T}, \qquad W = \mathcal{T}^{(1)} .$$

Assuming that the normalization argument leading to (2.38) is correct, the numbers N_{n_1,n_2} of table 1 are genuine Ooguri-Vafa invariants for the A-brane geometry predicted by mirror symmetry. It would be interesting to justify the above arguments and the prediction for the disc invariants in table 1 by an independent computation. Further evidence for the above results is given in [2], by deriving the same result from the afore mentioned duality to Calabi-Yau four-folds.

2.5 Relation to CFT correlators

The relevant closed-string observables in the BRST cohomology of the topological B-model of a Calabi-Yau manifold are locally given by [7]

$$\phi^{(p)} = \phi^{(p)j_1\cdots j_p}_{\bar{\imath}_1\cdots\bar{\imath}_p}\eta^{\bar{\imath}_1}\cdots\eta^{\bar{\imath}_p}\theta_{j_1}\cdots\theta_{j_p} , \qquad (2.39)$$

where the world-sheet fermions, $\eta^{\bar\imath} = \psi_+^{\bar\imath} + \psi_-^{\bar\imath}$ and $\theta_i = g_{i\bar\jmath}\left(\psi_+^{\bar\jmath} - \psi_-^{\bar\jmath}\right)$, are sections of the pullbacks of the holomorphic tangent bundle and the anti-holomorphic cotangent bundle of target-space Calabi-Yau manifold. For the Calabi-Yau three-fold Z^* these observables $\phi^{(p)}$ are identified geometrically with representatives in the sheaf cohomology groups

$$\phi^{(p)} \in H^p(Z^*, \Lambda^p TZ^*) \simeq H^{(3-p,p)}(Z^*) , \qquad p = 0, 1, 2, 3 . \tag{2.40}$$

The last identification is due to the contraction with the unique holomorphic (3,0) form of the Calabi-Yau three-fold Z^*. The integer p represents the left and right $U(1)$ charge of the bulk observable $\phi^{(p)}$.

The local open-string observables for a worldsheet with B-type boundary are analogously given by

$$\hat\phi^{(p+q)} \;=\; \hat\phi^{(p+q)}{}_{\bar\imath_1\cdots\bar\imath_p}^{j_1\cdots j_q}\eta^{\bar\imath_1}\cdots\eta^{\bar\imath_p}\theta_{j_1}\cdots\theta_{j_q} . \tag{2.41}$$

In the absence of a background gauge field on the worldvolume of the brane the fermionic modes θ_j vanish along Neumann directions whereas the fermionic modes $\eta^{\bar\imath}$ vanish along Dirichlet directions on the boundary of the worldsheet [64]. Hence, locally we view the fermionic modes θ_j as sections of the normal bundle and the fermionic modes $\eta^{\bar\imath}$ as sections of the anti-holomorphic cotangent bundle of the brane. With background fluxes on the brane worldvolume the boundary conditions become twisted and obey [33]

$$\theta_i = F_{i\bar\jmath}\eta^{\bar\jmath} . \tag{2.42}$$

In ref. [38] it is explicitly demonstrated that the observables (2.41) in the BRST cohomology of the open-string sector for a brane E arise geometrically as elements of the extension groups

$$\hat\phi^{(p+q)} \in \mathrm{Ext}^{p+q}(E, E) , \qquad p + q = 0, 1, 2, 3 . \tag{2.43}$$

In the present context, the integer $p + q$ is equal to the total $U(1)$ charge of the open-string observable $\hat\phi^{(p+q)}$.

Deformations of the topological B-model are generated by the marginal operators, which correspond to BRST observables with $U(1)$ charge one, and hence they appear in the cohomology groups $H^{(2,1)}(Z^*)$ and $\mathrm{Ext}^1(E, E)$ for the closed and open deformations, respectively.

In order to make contact with the Hodge filtration of $H^3(Z^*, \mathcal{D})$ we interpret the divisor $\mathcal{D}$ of the Calabi-Yau three-fold Z^* as the internal worldvolume of a B-type brane. For a divisor the extension groups (2.43) simplify [38], and in particular $\mathrm{Ext}^1(\mathcal{D}, \mathcal{D})$ reduces to $H^0(\mathcal{D}, N\mathcal{D}) \simeq H^{(2,0)}(\mathcal{D})$, where the last identification results again from the contraction with the holomorphic (3,0) form. For our particular example the cohomology groups $H^{(2,1)}(Z^*)$ and $H^{(2,0)}(\mathcal{D}_1)$ are both one-dimensional and therefore are generated by the closed- and open-string marginal operators ϕ and $\hat\phi$

$$\phi^{(1)} \in H^{(2,1)}(Z^*) \subset F^2/F^3 , \qquad \hat\phi^{(1)} \in H^{(2,0)}(\mathcal{D}_1) \subset F^2/F^3 .$$

Due to the identification $F^2/F^3 = H^{(2,1)}(Z^*,\mathcal{D}_1) \simeq H^{(2,1)}(Z^*) \oplus H^{(2,0)}(\mathcal{D}_1)$ we observe that the infinitesimal deformations $\nabla_t \alpha_1^{(0)} \sim \phi^{(1)}$ and $\nabla_{\hat{t}} \alpha_1^{(0)} \sim \hat{\phi}^{(1)}$ in eq. (2.25) precisely agree with the closed and open marginal operators $\phi^{(1)}$ and $\hat{\phi}^{(1)}$. As a consequence the discussed Picard-Fuchs equations, governing the Hodge filtration F^p, describe indeed the deformation space associated to the closed and open marginal operators $\phi^{(1)}$ and $\hat{\phi}^{(1)}$.

In the presence of B-type boundaries infinitesimal deformations are generically obstructed at higher order. These obstructions are encoded in the moduli-dependent superpotential generated by disc correlators with insertions of bulk and boundary marginal operators [65, 66, 67, 68]. The relevant disc correlators arise from non-trivial ring relations involving marginal operators and the (unique) boundary top element $\hat{\phi}^{(3)} \in \mathrm{Ext}^3(\mathcal{D},\mathcal{D})$. Hence the superpotential is extracted by identifying the element $\hat{\phi}^{(3)}$ in the relative cohomology group $H^3(Z^*,\mathcal{D})$. For the family of hypersurfaces $\mathcal{D}$ the extension group $\mathrm{Ext}^3(\mathcal{D},\mathcal{D})$ becomes [38]

$$\hat{\phi}^{(3)} \in \mathrm{Ext}^3(\mathcal{D},\mathcal{D}) \simeq H^2(\mathcal{D},N\mathcal{D}) \simeq H^{(2,2)}(\mathcal{D}) \ ,$$

where locally $\hat{\phi}^{(3)} = \hat{\phi}^{(3)}{}^{k}_{\bar{\imath}\bar{\jmath}}\eta^{\bar{\imath}}\eta^{\bar{\jmath}}\theta_k$. It is obvious that the cohomology group $H^{(2,2)}(\mathcal{D})$ does not appear in the filtration F^p of the relative cohomology group $H^3(Z^*,\mathcal{D})$. On the other hand the variation of mixed Hodge structure encodes by construction the ring relations of the observables generated by the marginal operators $\phi^{(1)}$ and $\hat{\phi}^{(1)}$. Therefore we conclude that these marginal operators do not generate the boundary-boundary top element $\hat{\phi}^{(3)}$. Thus the analyzed deformation problem is unobstructed and does not give rise to a non-vanishing superpotential.

From a physics point of view the family of divisors $\mathcal{D}$ describes a family of holomorphic hypersurfaces, which all give rise to supersymmetric B-brane configurations, and hence we should not expect any obstructions resulting in a superpotential.

However, the result of the above analysis drastically changes as we add a $D5$-brane charge on 2-cycles in $\mathcal{D}$, e.g. by adding non-trivial background fluxes on the world-volume of the B-type brane. From a space-time perspective [69, 70], we expect the appearance of F-terms precisely for those two-form background fluxes, whose field strength takes values in the variable cohomology of the hypersurface $\mathcal{D}$

$$F \in \mathrm{coker}\,\big(H^2(Z^*,\mathbb{Z}) \rightarrow H^2(\mathcal{D},\mathbb{Z})\big) \ . \tag{2.44}$$

These fluxes induce a macroscopic superpotential [69, 70, 71, 29]

$$W = \int_D F \wedge \omega = \int_\Gamma F \wedge \Omega, \tag{2.45}$$

where $\omega \in H^{2,0}(D)$ is obtained by contracting the bulk (3,0) form Ω with a section of the normal bundle to D. The second expression, derived in a more general context in ref. [29], is equivalent to the first one for an appropriate choice of 5-chain with boundary D.

In the microscopic worldsheet description the worldvolume flux $F_{i\bar{\jmath}}$ yields twisted boundary conditions (2.42), and the fermionic modes θ_i of the open-string observables (2.41) are in general no longer sections of the normal bundle $N\mathcal{D}$. Instead they should be viewed as appropriate section in the restricted tangent bundle, $TZ^*|_{\mathcal{D}}$ [38]. As a consequence we can trade (without changing the $U(1)$ charge) fermionic modes $\eta^{\bar{\jmath}}$ with appropriate fermionic modes θ_i. As a result the boundary top element $\hat{\phi}^{(3)}$ can now be associated with an element in the variable two-form cohomology

$$\mathrm{Ext}^3(\mathcal{D},\mathcal{D}) \ni \hat{\phi}^{(3)k}_{\bar{\imath}\bar{\jmath}}\eta^{\bar{\imath}}\eta^{\bar{\jmath}}\theta_k \overset{F_{i\bar{\jmath}}}{\longleftrightarrow} \hat{\phi}^{(3)jk}_{\bar{\imath}}\eta^{\bar{\imath}}\theta_j\theta_k \overset{\Omega_{ijk}}{\longleftrightarrow} \hat{\phi}^{(3)}_{i\bar{\jmath}}\, dx^i \wedge dx^{\bar{\jmath}} \in \mathrm{coker}\, \left(H^2(Z^*) \to H^2(\mathcal{D})\right) . \tag{2.46}$$

Thus in the presence of worldvolume background fluxes the boundary top element $\hat{\phi}^{(3)}$ *does* correspond to an element in the Hodge structure filtration F^p, and the superpotential is described by a solution of the Picard-Fuchs equations. In this way the a priori unobstructed deformation problem of divisors $\mathcal{D}$ in the Calabi-Yau three-fold is capable to describe superpotentials associated to $D5$-brane charges in $H_2(\mathcal{D})$ [26, 31, 32].

On the other hand, since the discussed F-term fluxes (2.44) are elements of the variable cohomology of the hypersurface $\mathcal{D}$, *i.e.* the field strength of the fluxes can be extended to exact two forms in the ambient Calabi-Yau space, they do not modify the $D5$-brane K-theory charges. Therefore if a suitable $D5$-brane interpretation is applicable the flux-induced superpotentials describe domain-wall tensions between pairs of $D5$-branes, which wrap homologically equivalent two cycles.

When written in the flat coordinates $t_a = \frac{1}{2\pi i}\ln(q_a)$ in (2.29), the Gauss-Manin connection on the total cohomology space takes the form:

$$\begin{pmatrix} \nabla\alpha_1^{(0)} \\ \nabla\alpha_2^{(1)} \\ \nabla\alpha_3^{(1)} \\ \nabla\alpha_4^{(2)} \\ \nabla\alpha_5^{(2)} \\ \nabla\alpha_6^{(3)} \\ \nabla\alpha_7^{(3)} \end{pmatrix} c = \sum_b \begin{pmatrix} 0 & C_b^{(0)}(q_a)\frac{dq_b}{q_b} & 0 & 0 \\ 0 & 0 & C_b^{(1)}(q_a)\frac{dq_b}{q_b} & 0 \\ 0 & 0 & 0 & C_b^{(2)}(q_a)\frac{dq_b}{q_b} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1^{(0)} \\ \alpha_2^{(1)} \\ \alpha_3^{(1)} \\ \alpha_4^{(2)} \\ \alpha_5^{(2)} \\ \alpha_6^{(3)} \\ \alpha_7^{(3)} \end{pmatrix} . \tag{2.47}$$

The most notable difference to the closed-string case (cf. eq. (2.30)) is that, whereas the matrix $C_b^{(0)}$ still is of the canonical form $(C_b^{(0)})_1{}^l = \delta_{bl}$, the matrices $C_b^{(q)}$ are now both moduli dependent for $q = 1, 2$:

$$(C_t^{(1)}) = \begin{pmatrix} C & W_{,tt} \\ 0 & W_{,t\hat{t}} \end{pmatrix}, \; (C_{\hat{t}}^{(1)}) = \begin{pmatrix} 0 & W_{,t\hat{t}} \\ 0 & W_{,\hat{t}\hat{t}} \end{pmatrix},$$
$$(C_t^{(2)}) = \begin{pmatrix} -1 & \mu_{,t} \\ 0 & \rho_{,t} \end{pmatrix}, \; (C_{\hat{t}}^{(2)}) = \begin{pmatrix} 0 & \mu_{,\hat{t}} \\ 0 & \rho_{,\hat{t}} \end{pmatrix}.$$

In correspondence with the closed-string sector it is tempting to interprete the $d_{q-1} \times d_q$ matrices $C_b^{(q)}$ as the structure constants of a ring of open and closed chiral operators

$$\phi_b^{(1)} \cdot \phi_k^{(q)} \stackrel{?}{=} (C_b^{(q)})_k{}^l \phi_l^{(q+1)} ,$$

as described in [26, 39]. A rigorous CFT derivation of such a relation is non-trivial, as the Hodge variation mixes bulk and boundary operators and describes the bulk-boundary ring in the sense of ref. [38], about which little is known in the context of topological strings (see however refs. [72, 73, 74]). A related complication is the need of a topological metric on the space of closed and open BRST states that mixes contributions at different order of the string coupling. The most direct way to connect the closed-string periods with a CFT quantity is the interpretation as overlap functions between boundary states and chiral operators [36], and it is likely that a similar idea can be applied to the entries of the relative period matrix. It would be interesting to make this precise. It would also be interesting to understand more generally the relation of the above concepts to the CFT results obtained from matrix factorizations in refs. [75, 76, 77, 78].

2.6 Summary and outlook

In this chapter we analyzed the deformation problem of certain families of toric D-branes in compact Calabi-Yau three-folds, defined along the lines of ref.[21]. This is achieved by studying the variation of Hodge structure as described by the periods of the holomorphic three-form of the Calabi-Yau manifold while keeping track of the boundary contributions relative to a family of four-cycles describing the B-brane geometry. We demonstrate our techniques with a specific B-brane configuration in the mirror quintic. Although this geometry serves as a guiding example throughout the paper, we present also a general toric description of the generalized hypergeometric systems governing the toric brane configurations, for which our discussion applies.

We find that, similarly to the well-studied deformation problem in the pure closed-string sector, the notions of flatness and integrability of the Gauss-Manin connection continue to make sense on the open-closed deformation space $\mathcal{M}$ of the family and lead to sensible results for open string enumerative invariants. Amongst others, the Gauss-Manin connection in flat coordinates displays an interesting ring structure on the infinitesimal deformations in F^2, which is compatible with CFT expectations and gives evidence for the existence of an A-model quantum product defined by the Ooguri-Vafa invariants. Other hints in this direction are the integrability condition and the meaningful definition of the mirror map (2.29) via a flatness condition.

For geometries with a single open string modulus the integrability conditions imply that the relative period matrices and the Gauss-Manin connection matrices can all be expressed in terms of functional relations involving only the holomorphic prepotential $\mathcal{F}$ and one additional holomorphic function W.[19]

[19]This can be generalized to cases with several open-string moduli studied in [56].

The analyzed open-closed deformation problem can also be related to CFT correlators. We explained how an a priori unobstructed deformation problem of B-branes wrapping a holomorphic family of four-cycles describes an obstructed deformation problem after turning on $D5$-brane charges. In particular this effect can be described in the CFT by the change of boundary conditions induced by non-trivial fluxes on the worldvolume of the B-brane. The afore mentioned holomorphic function W in the Gauss-Manin connection then turns into a superpotential encoding these obstructions.

By now this approach has successfully passed other non-trivial checks [51, 79]. In leading order the computed off-shell superpotentials are compatible with derivations of effective superpotentials using open-string worldsheet and matrix factorization techniques [80, 81, 82, 79, 68, 77]. Beyond leading order, however, the discussed off-shell superpotentials predict in the context of type II theories higher order open-closed CFT correlators, which (at present) are difficult to compute by other means.

By mirror symmetry our analysis carries over to the quantum integrable structure of the obstructed deformation space of the mirror A-branes in the mirror three-fold. For our explicit example we obtain predictions for the Ooguri-Vafa invariants of the open-closed deformation space that satisfy the expected integrality constraints and further consistency conditions.

However, the B-model results of this chapter, obtained predominantly from a Hodge theoretic approach, raise also a number of unanswered questions. The first is about the meaning of mirror symmetry between open-closed deformation spaces in the presence of a non-trivial superpotential, which requires some sort of off-shell concept of mirror symmetry. As discussed above, a heuristic ansatz might be to define $\mathcal{M}$ first as the deformation space of an unobstructed family and then add in obstructions as a sort of perturbation, here represented by $D5$-brane charges. However, we feel that there should be a more fundamental answer to this important issue.

Another set of urgent questions concerns the A-model interpretation, such as a proper formulation of an A-model quantum ring that matches the ring structure observed on the B-model side and should include the Ooguri-Vafa invariants and Floer (co-)homology as essential ingredients. Similarly one would like to have a more explicit description of the target space geometry of the A-branes.

3 Rigid branes

We calculate superpotentials for B-type D-branes wrapping isolated holomorphic 2-cycles C of a Calabi-Yau threefold Z and, by open-string mirror symmetry, the superpotential of the A-brane geometry related to it. This was first done in [27]. Here we use the techniques introduced in the last chapter. In this setting the deformation space of the holomorphic divisor is only a computational tool. It simplifies calculations in comparison to the methods developed in [27, 28, 83] and allows for calculations in models with two or even three closed string moduli. We study the behavior of the superpotentials under geometric transitions to manifolds with a lower number of complex structure moduli. We built on the introduction and the last chapter and only clarify the relation between the superpotential, the relative periods and the periods on the divisor before we start with a study of some examples. These results are joint work with Murad Alim, Michael Hecht, Hans Jockers, Peter Mayr and Masoud Soroush that were published in [3]. This chapter is a shorted version of this publication where more examples were analyzed.

3.1 Superpotentials, domainwall tensions and periods

The superpotential W is determined by the tension of a domainwall T interpolating between a brane on the holomorphic cycle C^+ and a brane on the holomorphic cycle C^- in the same homology class, $[C^+] = [C^-]$.

$$T = W(C^+) - W(C^-) = \int_{\Gamma^\pm} \Omega\,, \tag{3.1}$$

where $\Gamma^\pm$ is a 3-chain between the 2-cycles C^+ and C^-. Even though we integrate the holomorphic 3-form Ω against a chain and not a cycle this integral is well defined for holomorphic cycles C^+ and C^-. The relevant mathematical notion is that of a normal function. The domain wall tension fulfills an inhomogeneous version of the Picard-Fuchs equations of closed string mirror symmetry [27],

$$\mathcal{L}_a^{bulk} T(z) = f_a(z)\,, \tag{3.2}$$

where a labels the different Picard-Fuchs equations of Z. Starting from the curve C one can calculate the inhomogeneous piece $f_a(z)$ by a careful evaluation of residue integrals and determine the superpotential as a solution to this equation. This was the route taken in [27, 28, 83]. A direct calculation of $f_a(z)$ is challenging for cases with more moduli.

To avoid it we use the techniques of chapter 2. The relative periods of a family of divisors $\mathcal{D}(z,\hat{z})$ are domain wall tensions $\mathcal{T}(z,\hat{z})$,

$$\underline{\Pi}(\underline{\Gamma};z,\hat{z}) = \mathcal{T}(z,\hat{z}) = \int_{\underline{\Gamma}_{(z,\hat{z})}} \underline{\Omega}(z,\hat{z})\,, \tag{3.3}$$

that coincide with the integral (3.1) upon restriction of the open modulus $\hat{z}$ in certain cases. If two holomorphic curves $C_z^{\pm}$ of the same homology class in $H_2(Z)$ are embedded into a member $\mathcal{D}(z,\hat{z}_{\text{crit}})$ of the divisor family $\mathcal{D}(z,\hat{z})$ and $[C_z^+]_{\mathcal{D}_{\hat{z}_{\text{crit}}}} - [C_z^-]_{\mathcal{D}_{\hat{z}_{\text{crit}}}} \neq 0$ in $H_2(\mathcal{D})$, the 3-chain spanned between C_z^+ and C_z^- is in the relative homology $H_3(Z,\mathcal{D})_{z,\hat{z}_{\text{crit}}}$. As the boundaries of the chain $\underline{\Gamma}_{(z,\hat{z}_{\text{crit}})}$ are holomorphic, the domain wall tension should obey the on- shell condition $\frac{d}{d\hat{z}}\mathcal{T}(z,\hat{z})|_{\hat{z}=\hat{z}_{\text{crit}}} = 0$.

Mathematically speaking, the vacuum configurations hence lie within the so-called Noether-Lefshetz locus, defined as [84]

$$\mathcal{N} \;=\; \left\{ (z,\hat{z})\in\Delta \;\middle|\; 0 \;\equiv\; \frac{d\underline{\Pi}(z,\hat{z})}{d\hat{z}} \right\}\;. \tag{3.4}$$

Equivalently the locus $\mathcal{N}$ can be specified by the vanishing condition

$$\mathcal{N} \;=\; \left\{ (z,\hat{z})\in\Delta \;\middle|\; 0 \;\equiv\; \pi(z,\hat{z};\partial\underline{\Gamma}_{(z,\hat{z})}) \right\}\;, \tag{3.5}$$

for the period vector of the divisor $D_{(z,\hat{z})}$

$$\pi(z,\hat{z};\partial\underline{\Gamma}_{(z,\hat{z})}) \;=\; \left(\int_{\partial\underline{\Gamma}_{(z,\hat{z})}} \omega_{\hat{a}}^{(2,0)}(z,\hat{z}) \right)\;,\quad \hat{a} = 1,\ldots,\dim H^{2,0}(D_{(z,\hat{z})})\;. \tag{3.6}$$

Here $\omega_{\hat{a}}^{(2,0)}(z,\hat{z})$ is a basis of two forms for $H^{2,0}(D_{(z,\hat{z})})$. Hence the critical locus of D-brane vacua is mapped to the subslice of complex structures on the surface $D_{(z,\hat{z})}$, where certain linear combinations of period vectors on the surface vanish. At such points in the complex structure the Picard lattice of the surface $D_{(z,\hat{z})}$ is enhanced due to the appearance of an additional integral $(1,1)$-form.

For concreteness, we assume that the holomorphic curves $C_z^{\pm}$ are contained in the intersection of the hypersurface $X:\ P=0$ with two hyperplanes $D_{1,2}$ defined in a certain ambient space. Choose coordinates such that the equation for D_1 does not depend on the closed-string moduli z, typically of the form[1]

$$D_1:\ x_1^a + \eta\, x_2^b = 0\,,$$

[1] Note that the equation for D_1 is a priori defined in the ambient space. However, by restriction to the hypersurface X we also identify D_1 with a divisor on the hypersurface X. For ease of notation we denote both the divisor of the ambient space and of the hypersurface with the same symbol D_1.

where x_i are some homogeneous coordinates on the ambient space, a, b some constants that depend on the details and η a fixed constant, which is a phase factor in appropriate coordinates. This hyperplane can be deformed into a family $\mathcal{D}_1 : \; x_1^a + \hat{z}\, x_2^b = 0$ by replacing the constant η by a complex parameter $\hat{z}$. The relative 3-form $\underline{\Omega}$ and the relative period integrals on the family of cohomology groups $H^3(X, D_1)$, satisfy a set of Picard-Fuchs equations [85, 26, 31, 44]

$$\mathcal{L}_a(\theta, \hat{\theta})\, \underline{\Omega} = \underline{d\omega}^{(2,0)} \quad \Rightarrow \quad \mathcal{L}_a(\theta, \hat{\theta})\; \mathcal{T}(z, \hat{z}) = 0\,, \qquad a = 1, ..., A\,,$$

where a is some label for the operators. The differential operators can be split into two pieces

$$\mathcal{L}_a(\theta, \hat{\theta}) =: \mathcal{L}_a^{bulk} - \mathcal{L}_a^{bdry}\, \hat{\theta}\,, \tag{3.7}$$

where the bulk part $\mathcal{L}_a^{bulk}(\theta)$ acts only on the closed-string moduli z and the boundary part $\mathcal{L}_a^{bdry}(\theta, \hat{\theta})\, \hat{\theta}$ contains at least one derivative in the parameters $\hat{z}$. Since the dependence on $\hat{z}$ localizes on D_1, the derivatives $2\pi i\, \hat{\theta}\, \mathcal{T}(z, \hat{z})$ are proportional to the periods (3.6) on the surface D_1

$$2\pi i\, \hat{\theta}\, \mathcal{T}(z, \hat{z}) = \pi(z, \hat{z})\ . \tag{3.8}$$

Rearranging eq. (3.7) and restricting to the critical point $\hat{z} = \eta$ one obtains an inhomogeneous Picard-Fuchs equation

$$\mathcal{L}_a^{bulk}\, T(z) = f_a(z)\,, \tag{3.9}$$

with $T(z) = \mathcal{T}(z, \eta)$ and

$$2\pi i\, f_a(z) = \; \mathcal{L}_a^{bdry}\, \pi(z, \hat{z})\big|_{\hat{z}=\eta}\ . \tag{3.10}$$

In absolute cohomology the inhomogeneous term $f_a(z)$ is due to the fact that the bulk operators $\mathcal{L}_a^{bulk}$ satisfy

$$\mathcal{L}_a^{bulk}\, \Omega = d\beta \quad \Rightarrow \quad \mathcal{L}_a^{bulk} \int_{\Gamma \in H_3(X,\mathbb{Z})} \Omega = 0\ , \tag{3.11}$$

where d is the differential in the absolute setting. This is sufficient to annihilate the period integrals over cycles, as indicated on the right hand side of the above equation, but leads to boundary terms in the chain integral (3.1). In the absolute setting and based on Dwork-Griffiths reduction the inhomogeneous term $f_a(z)$ has been determined by a residue computation in ref. [28]. Here we see that the functions $f_a(z)$ are different derivatives of the surface period $\pi(z, \hat{z})$, restricted to the critical point. Hence, together with the bulk Picard-Fuchs operators, the surface period determine both the critical locus (3.5) and the critical tension.

In the examples we find that the inhomogeneous terms $f_a(z)$ satisfy a hypergeometric differential equation as well:

$$\mathcal{L}_a^{inh}\, f_a(z) = 0\ . \tag{3.12}$$

The hypergeometric operators $\mathcal{L}_a^{inh}$ descend from the Picard-Fuchs operators $\mathcal{L}^{\mathcal{D}}$ of the surface, which annihilate the surface periods $\mathcal{L}^{\mathcal{D}}\pi(z,\hat{z}) = 0$.[2] Specifically, if $f_a(z)$ is non-zero, the operator $\mathcal{L}_a^{inh}$ can be defined as

$$\mathcal{L}_a^{inh} = \left(\mathcal{L}^{\mathcal{D}} + [\mathcal{L}_a^{bdry}, \mathcal{L}^{\mathcal{D}}]\, \mathcal{L}_a^{bdry\,-1}\right)_{\hat{z}=\eta} , \qquad (3.13)$$

where the operators on the right hand side are restricted to the critical point as indicated.

It follows from the above that the inhomogeneous terms $f_a(z)$ can be written as an infinite hypergeometric series in the closed-string moduli. However, on general grounds the $f_a(z)$ need to be well-defined over the open-closed moduli space, which simplifies *on-shell* to a finite cover of the complex structure moduli space $\mathcal{M}_{CS}(X)$ of the threefold [83]. This implies that the hypergeometric series $f_a(z)$ can be written as rational functions in the closed string moduli and the roots of the extra equations defining the curves C.[3]

In the examples we observe that already the leading terms of the surface periods $\pi(z,\hat{z})$ become rational functions at the special symmetric points on the Noether-Lefshetz locus $\mathcal{N}$ in this sense. Hence there appears to be a connection between the enhancement of the Picard-lattice of the surface at these points, rationality of its periods and D-brane vacua. The rationality property is preserved when acting with $\mathcal{L}^{bdry}$ in eq. (3.10) to obtain the inhomogeneous term f_a. In the examples we verify, that the contribution $f_a(C_{\alpha_\ell})$ of a particular boundary curve C_{α_ℓ} to the inhomogeneous term can be written in closed form as follows.

$$f_a(C_{\alpha_\ell}) \;=\; \frac{p_a(\psi,\alpha)}{q_a(\psi,\alpha)}|_{\alpha=\alpha_\ell(\psi)} \;=\; \frac{g_a(\psi,\alpha)}{\prod_i \Delta_l(C)^{\gamma_i^a}}|_{\alpha=\alpha_\ell(\psi)} , \qquad (3.14)$$

where p_a, q_a are polynomials in the variables (ψ, α). Here $\psi = \psi(z)$ is a short-hand for the fractional power of the closed string moduli z appearing in the defining equation of the hypersurface X and $\{\alpha_\ell\}$ are the roots of the extra equations defining the curves, with the root α_ℓ corresponding to the component C_{α_ℓ}. Moreover, the zeros of the denominator appear only at the zeros of the components $\Delta_i(C)$ of the open-string discriminant, where different roots/curves coincide for special values of the moduli ψ. The exponents γ_i^a are some constants and $g_a(\psi,\alpha)$ some functions without singularities in the interior of the moduli space.

Our strategy is the following: We embed holomorphic curves C into a member of a family of divisors $\mathcal{D}(z,\hat{z})$, look for periods π of the divisor that vanish at the critical point $\hat{z} = \hat{z}_{\text{crit}}$ and integrate them in $\hat{z}$ to obtain the full domain wall tension. This is computationally easier then a direct computation of the inhomogeneous term $f_a(z)$ of eq. (3.2). We use this to calculate superpotentials for some two and three parameter models in the following, more examples can be found in [3]

[2]For simplicity we suppress an index for distinguishing several Picard-Fuchs operators $\mathcal{L}^{\mathcal{D}}$.

[3]We are grateful to Johannes Walcher for explaining to us this property of the inhomogeneous terms and for pointing out the results of ref. [86] on this issue.

3.2 Examples

We proceed with the study of type II/F-theory superpotentials for a collection of examples of brane geometries on toric hypersurfaces with several open-closed string deformations. Combining the small Hodge variation associated with the surface periods (3.6) and the GKZ system on the relative cohomology group (2.22) provides an efficient method to compute the integral relative period integrals and the mirror map for a large number of deformations. We obtain new enumerative predictions for the A-model expansion, consistent with the expectations, and study the behavior of the branes under extremal transitions between different topological manifolds through points with enhanced non-abelian gauge symmetries.

3.2.1 Degree 12 hypersurface in $\mathbb{P}_{1,2,2,3,4}$

The charge vectors of the GLSM for the A-model manifold are given by [19]

$$\begin{array}{c|c|cccccc} & a_0 & \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 & \tilde{x}_4 & \tilde{x}_5 & \tilde{x}_6 \\ \hline l^1 & -6 & -1 & 1 & 1 & 0 & 2 & 3 \\ l^2 & 0 & 1 & 0 & 0 & 1 & 0 & -2 \end{array} \; . \qquad (3.15)$$

These vectors describe the relations between the vertices of a reflexive polyhedron described in the appendix. Written in homogeneous coordinates of $\mathbb{P}_{1,2,2,3,4}$ the hypersurface constraint for the mirror manifold reads

$$\begin{aligned} P &= a_1 x_1^{12} + a_2 x_2^6 + a_3 x_3^6 + a_4 x_4^4 + a_5 x_5^3 + a_0 x_1 x_2 x_3 x_4 x_5 + a_6 x_1^6 x_4^2 && (3.16) \\ &= x_1^{12} + x_2^6 + x_3^6 + x_4^4 + x_5^3 + \psi\, x_1 x_2 x_3 x_4 x_5 + \phi\, x_1^6 x_4^2 \; . && (3.17) \end{aligned}$$

In the second equation the variables x_i have been rescaled to display the dependence on the torus invariant parameters $\psi = z_1^{-1/6} z_2^{-1/4}$ and $\phi = z_2^{-1/2}$, with the z_a given by (1.10). On the mirror manifold, the Greene-Plesser orbifold group acts as $x_i \to \lambda_k^{g_{k,i}}\, x_i$ with weights[4]

$$\mathbb{Z}_6 : \; g_1 = (1,-1,0,0,0), \quad \mathbb{Z}_6 : \; g_2 = (1,0,-1,0,0)\,, \qquad (3.18)$$

where we denote the generators by λ_k with $\lambda_{1,2}^6 = 1$. The closed-string periods near the large complex structure point can be generated by evaluating the functions $B_{\{l^a\}}(z_a;\rho_a)$ in (1.12) and its derivatives with respect to ρ_i at $\rho_1 = \rho_2 = 0$ [19].

[4]The other factors of the Greene-Plesser group give nothing new, using a homogeneous rescaling of the projective coordinates, e.g. for the factor generated by $g_3 = (1,0,0,-1,0)$ with $\lambda_3^4 = 1$ one finds $g_3 \sim g_1^3 g_2^3$.

In this geometry we consider the set of curves defined by the equations

$$C_{\alpha,\kappa} = \{x_2 = \eta x_3\ ,\ x_4 = \alpha x_1^3\ ,\ x_5 = \kappa\sqrt{\alpha\eta\psi}x_3x_1^2\}\ ,$$
$$\eta^6 = -1\,, \qquad \kappa^2 = -1\,, \qquad \alpha^4 + \phi\alpha^2 + 1 = 0\,. \tag{3.19}$$

The labels (η, α, κ) are identified as $(\eta, \alpha, \kappa) \sim (\eta\lambda_1\lambda_2^{-1}, \alpha\lambda_1^3\lambda_2^3, \kappa)$ under the orbifold group. In the following we choose to label each orbit of curves by $(\alpha, \kappa) := (e^{i\pi/6}, \alpha, \kappa)$. Note that a rotation of η corresponds to a change of sign for α in this notation, $(e^{3i\pi/6}, \alpha, \kappa) = (-\alpha, \kappa)$. Instead of choosing a fixed η we can also fix the sign of α and keep two choices for η^3.

To calculate the domain wall tensions and the superpotentials for the vacua $C_{\alpha_1,\kappa}$ and $C_{\alpha_2,\kappa}$ we will study two families of divisors. The family $Q(\mathcal{D}_1) = x_2^6 + \hat{z}x_3^6$ interpolates between vacua related by a sign flip of η^3 or of the root α of the quartic equation. The family $Q(\mathcal{D}_2) = x_4^4 + \hat{z}x_1^6x_4^2$ interpolates between any two different roots α.

First divisor

We start with the analysis of the divisor

$$Q(\mathcal{D}_1) = x_2^6 + z_3x_3^6\,. \tag{3.20}$$

To obtain some geometrical understanding of the surface defined by the intersection $P = 0 = Q(\mathcal{D}_1)$ we explicitly solve for $x_3 = (-z_3)^{-1/6}x_2$ and rescale x_2 to find

$$P_{\mathcal{D}_1} = x_1^{12} + x_2^6 + x_4^4 + x_5^3 + \tilde{\psi}x_1x_2^2x_4x_5 + \phi x_1^6x_4^2\,. \tag{3.21}$$

Here $\tilde{\psi} = u_1^{-1/6}u_2^{-1/4}$, $\phi = u_2^{-1/2}$ are expressed in terms of the previous parameters as

$$u_1 = -\frac{z_1}{z_3}(1 - z_3)^2, \qquad u_2 = z_2\,. \tag{3.22}$$

Changing coordinates to $\tilde{x}_2 = x_2^2$ displays the family $\mathcal{D}_1$ as a double cover of a family of toric K3 surfaces associated to a GLSM with charges

$\mathcal{D}_1$: (3.23)

	$\tilde{x}_0$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$
$\tilde{l}^1$	-6	-1	2	0	2	3
$\tilde{l}^2$	0	1	0	1	0	-2

and with the two algebraic K3 moduli (3.22). The two covers are distinguished by a choice of sign for x_2.

The family of algebraic K3 manifolds obtained from (3.21) by the variable change $\tilde{x}_2 = x_2^2$ generically has four parameters with the two extra moduli multiplying the monomials $x_1^3x_4^3$ and $x_1^9x_4$. Since these terms are forbidden by the Greene-Plesser group of the Calabi-Yau threefold, the embedded surface is at a special symmetric point with the coefficients of these monomials set to zero. The periods on the K3 surface at this point can be computed from the GKZ system for the two parameter family, obtained from (2.22) with the charge vectors $\{\tilde{l}\}$ in eq. (3.23):

$$\begin{aligned}\mathcal{L}_1^{\mathcal{D}} &= \tilde{\theta}_1(2\tilde{\theta}_1-1)\prod_{k=0}^{2}(-3\tilde{\theta}_1+2\tilde{\theta}_2+k)-\frac{9}{2}u_1(\tilde{\theta}_1-\tilde{\theta}_2)\prod_{k=1,2,4,5}(6\tilde{\theta}_1+k)\,,\\ \mathcal{L}_2^{\mathcal{D}} &= \tilde{\theta}_2(\tilde{\theta}_2-\tilde{\theta}_1)-u_2(2\tilde{\theta}_2-3\tilde{\theta}_1)(2\tilde{\theta}_2-3\tilde{\theta}_1+1)\,,\end{aligned}\tag{3.24}$$

where $\tilde{\theta}_a = u_a\frac{d}{du_a}$. Apart from the regular solutions this system has two extra solutions depending on fractional powers in the u_i:

$$\begin{aligned}\pi_1(u_1,u_2)&=\frac{c_1}{2}\,B_{\{\tilde{l}\}}(u_1,u_2;\tfrac{1}{2},0)=\frac{4c_1}{\pi}\sqrt{u_1}\;{}_2F_1(-\tfrac{1}{4},-\tfrac{3}{4},\tfrac{1}{2},4u_2)+\mathcal{O}(u_1^{3/2})\,,\\ \pi_2(u_1,u_2)&=\frac{c_2}{2}\,B_{\{\tilde{l}\}}(u_1,u_2;\tfrac{1}{2},\tfrac{1}{2})=\frac{12c_2}{\pi}\sqrt{u_1u_2}\;{}_2F_1(-\tfrac{1}{4},\tfrac{1}{4},\tfrac{3}{2},4u_2)+\mathcal{O}(u_1^{3/2})\,.\end{aligned}\tag{3.25}$$

Here c_a are some normalization constants not determined by the differential operators. Later they will be fixed to one by studying the geometric periods on the surface.

As indicated, the exceptional solutions vanish at the critical point $u_1 = 0$ as $\sim \sqrt{u_1}$, with the coefficient a hypergeometric series in the modulus $u_2 = z_2$. These solutions arise as the specialization of the standard solutions of the four parameter family of K3 manifolds to the special symmetric point.[5] Since $u_1 = 0$ is not at the discriminant locus of the K3 family for general u_2, there is no geometric vanishing cycle associated with the zero of $\pi_{1,2}$. Instead the zero at $u_1 = 0$ arises from the 'accidental' cancellation between the volumes of different classes at the symmetric point.[6] The periods (3.25) have the special property that their leading terms $\sim \sqrt{u_1}$ near the critical point $u_1 = 0$ can be written in closed form as

$$\lim_{z_3\to 1}\frac{\pi_a(u_1,u_2)}{(1-z_3)} = \frac{4c_a}{\pi}\cdot\left.\frac{(i\alpha)(2\alpha^2-\phi)(\alpha^2+\phi)}{\psi^3}\right|_{\alpha=\alpha_{a,+}}\,,\tag{3.26}$$

where

$$\alpha_{1,\pm}=\pm\sqrt{\frac{-\phi+\sqrt{\phi^2-4}}{2}}\,,\qquad \alpha_{2,\pm}=\pm\sqrt{\frac{-\phi-\sqrt{\phi^2-4}}{2}}\,,\tag{3.27}$$

denote the roots of the quartic equation $\alpha^4+\phi\alpha^2+1=0$ appearing in the definition (3.19). Hence the leading part of the two K3 periods near the symmetric point is proportional to a rational function in the coefficients of the defining equations for the curve, evaluated at the critical points.

[5] An explicit illustration of this fact is given in the case of the second family of divisors below.

[6] One parameter controlling the difference of these volumes is the direction of the off-shell modulus.

We will first compute the domain wall tensions by integrating the periods $\pi_{1,2}$ of the surface D_1. Note that the K3 periods π_a depend on $\xi = \sqrt{z_3}$ via their dependence on u_1 and the sign of the square root correlates with the sign of α. To obtain the off-shell tension, we integrate $\pi_a(\xi)$ as

$$\mathcal{T}_a^{(\pm)}(z_1, z_2, z_3) = \frac{1}{2\pi i} \int_{\xi_0}^{\pm\sqrt{z_3}} \pi_a(\xi)\, \frac{d\,\xi}{\xi}\,, \tag{3.28}$$

where ξ_0 denotes a fixed reference point. For example, the period π_1 integrates to

$$\begin{aligned} \frac{4\pi i\, \mathcal{T}_1^{(\pm)}}{c_1} &= \int_{\xi_0}^{\pm\sqrt{z_3}} \sum_{n_1,n_2\geq 0} \frac{\Gamma(4+6n_1)\left(-\frac{z_1}{\xi^2}(1-\xi^2)^2\right)^{n_1+\frac{1}{2}} z_2^{n_2}}{\Gamma(2+2n_1)^2\Gamma(1+n_2)\Gamma\left(\frac{1}{2}-n_1+n_2\right)\Gamma\left(\frac{5}{2}+3n_1-2n_2\right)} \frac{d\xi}{\xi} \\ &= \sum_{n_1,n_2\geq 0} \frac{\Gamma(4+6n_1)(-z_1)^{n_1+\frac{1}{2}} z_2^{n_2}\left(\xi^2-1\right)^{2n_1+2} {}_2F_1\left(1,\frac{3}{2}+n_1,\frac{1}{2}-n_1,\xi^2\right)}{(1+2n_1)\Gamma(2+2n_1)^2\Gamma(1+n_2)\Gamma\left(\frac{1}{2}-n_1+n_2\right)\Gamma\left(\frac{5}{2}+3n_1-2n_2\right)\xi^{2n_1+1}}\Bigg|_{\xi=\xi_0}^{\xi=\pm\sqrt{z_3}} \end{aligned} \tag{3.29}$$

where the contribution from the reference point ξ_0 can be set to zero by choosing $\xi_0 = i$ as the lower bound. This will be used to split the result of the integral for the domain wall tension into two contributions of the superpotentials from the endpoints as in eq. (3.1). This split is not obvious in general, and ambiguous with respect to adding rational multiples of bulk periods. In the example we can use the $\mathbb{Z}_2$ symmetry acting on the curves to require that the superpotentials obey $\mathcal{W}_1^{(+)} = -\mathcal{W}_1^{(-)}$. With this convention and the particular choice of ξ_0 above, we obtain $\frac{1}{2\pi i}\int_{\xi_0}^{\pm\sqrt{z_3}} \pi_a(\xi)\, \frac{d\xi}{\xi} = \mathcal{W}_a^{(\pm)}$ or $\frac{1}{2\pi i}\int_{-\sqrt{z_3}}^{+\sqrt{z_3}} \pi_a(\xi)\, \frac{d\xi}{\xi} = \mathcal{W}_a^{(+)} - \mathcal{W}_a^{(-)} = 2\mathcal{W}_a^{(+)}$.

According to the discussion in sect. 3.1, the superpotentials $\mathcal{W}_a^{(\pm)}(z_1, z_2, z_3)$ restrict to the on-shell superpotentials $W_a^{(\pm)}(z_1, z_2)$ with vanishing derivative in the open-string direction z_3 at the critical point:

$$W_a^{(\pm)}(z_1, z_2) = \mathcal{W}_a^{(\pm)}\big|_{z_3=1}\,, \qquad \xi\partial_\xi \mathcal{W}_a^{(\pm)}(z_1, z_2, \xi^2)\big|_{z_3=1} = \pm\frac{1}{2\pi i}\ \pi_a\big|_{u_1=0} = 0\,. \tag{3.30}$$

For the above integrals one obtains

$$W_1^{(\pm)} = \mp\frac{c_1}{8\pi}\sum_{n_1,n_2\geq 0} \frac{(-1)^{n_1+1}\,\Gamma\left(-n_1-\frac{1}{2}\right)\Gamma\left(6n_1+4\right)\, z_1^{n_1+\frac{1}{2}} z_2^{n_2}}{\Gamma\left(n_1+\frac{3}{2}\right)\Gamma\left(2n_1+2\right)\Gamma\left(3n_1-2n_2+\frac{5}{2}\right)\Gamma\left(n_2+1\right)\Gamma\left(-n_1+n_2+\frac{1}{2}\right)}\,, \tag{3.31}$$

$$W_2^{(\pm)} = \mp\frac{c_2}{8\pi}\sum_{n_1,n_2\geq 0} \frac{(-1)^{n_1+1}\,\Gamma\left(-n_1-\frac{1}{2}\right)\Gamma\left(6n_1+4\right)\, z_1^{n_1+\frac{1}{2}} z_2^{n_2+\frac{1}{2}}}{\Gamma\left(n_1+\frac{3}{2}\right)\Gamma\left(2n_1+2\right)\Gamma\left(3n_1-2n_2+\frac{3}{2}\right)\Gamma\left(n_2+\frac{3}{2}\right)\Gamma\left(-n_1+n_2+1\right)}\,.$$

These functions can be expressed in terms of the bulk generating function as

$$W_1^{(\pm)} = \mp\frac{c_1}{8}\, B_{\{l\}}\left(z_1, z_2; \tfrac{1}{2}, 0\right)\,, \qquad W_2^{(\pm)} = \mp\frac{c_2}{8}\, B_{\{l\}}\left(z_1, z_2; \tfrac{1}{2}, \tfrac{1}{2}\right)\,. \tag{3.32}$$

Complementary, the tensions $\mathcal{T}_a^{(\pm)}(z_1, z_2, z_3)$ and their on-shell restrictions $T_a^{(\pm)}(z_1, z_2)$ can be described as solutions to the large GKZ system for the relative cohomology problem derived in refs. [32, 2, 44]. For the family (3.20) the additional charge vector is

	$\tilde{x}_0$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	$\tilde{x}_7$	$\tilde{x}_8$
l^3	0	0	1	-1	0	0	0	-1	1

.

Together with the charge vectors l^1 and l^2 for the Calabi-Yau hypersurface this defines the extended hypergeometric system of the form (2.22), which can be associated with a dual fourfold X_4 for a M/F-theory compactification [32, 51, 30]. For a description of X_4 as a toric hypersurface we refer to chapter 4. From the extended charge vectors one obtains after an appropriate factorization the system of differential operators[7]

$$\begin{aligned}
\mathcal{L}_1 &= (\theta_1+\theta_3)(\theta_1-\theta_3)(3\theta_1-2\theta_2) - 36z_1(6\theta_1+5)(6\theta_1+1)(\theta_2-\theta_1+2z_2(1+6\theta_1-2\theta_2))\,,\\
\mathcal{L}_2 &= \theta_2(\theta_2-\theta_1) - z_2(3\theta_1-2\theta_2-1)(3\theta_1-2\theta_2)\,,\\
\mathcal{L}_3 &= \theta_3(\theta_1+\theta_3) + z_3\theta_3(\theta_1-\theta_3)\,. \qquad (3.33)
\end{aligned}$$

After a simple variable transformation $y = \ln(z_3)$, with the variable y centered at the critical point, the solutions to this system describe the expansion of the periods on the relative homology $H^3(Z^*, \mathcal{D}_1)$ around the critical point. These include the off-shell tensions $\mathcal{T}_a^{(\pm)}(z_1, z_2, z_3)$ (3.28), which restrict to the functions (3.32), and in addition the closed-string periods $\Pi(z_1, z_2)$. The integration from the geometric surface periods of the subsystem fixes the z_3-dependent piece. The GKZ system restricts the afore mentioned integration constant to a linear combination of the closed-string periods $\Pi(z_1, z_2)$. The rational coefficients appearing in this combination can be determined by a monodromy argument, as in ref. [27] and as exemplified for a non-compact limit of the Calabi-Yau threefold in ref. [3]

Finally one may also characterize the critical tensions $T_a^{(\pm)}$, or, for the above reasons also the critical superpotentials $W_a^{(\pm)}$, as the solution to the inhomogeneous Picard-Fuchs equation (3.9), which makes contact to the normal function approach of [28]. Due to

$$\mathcal{L}_1 = \mathcal{L}_1^{bulk}(\theta_1, \theta_2) - (3\theta_1 - 2\theta_2)\theta_3^2, \qquad \mathcal{L}_2 = \mathcal{L}_2^{bulk}(\theta_1, \theta_2), \qquad (3.34)$$

we observe that only the first operator may acquire a non-zero inhomogeneous term at the critical point. This term is determined by the leading behavior of the surface periods π_a in the limit $u_1 \to 0$. Acting with $\mathcal{L}_1^{bdry} = (3\theta_1 - 2\theta_2)\theta_3$ on the terms on the right hand side of eqs. (3.25) one obtains the inhomogeneous Picard-Fuchs equations

$$\begin{aligned}
\mathcal{L}_1^{bulk} W_1^{(\pm)} &= \mp\frac{3c_1}{2\pi^2}\sqrt{z_1}\ {}_2F_1(\tfrac{1}{4}, -\tfrac{1}{4}, \tfrac{1}{2}, 4z_2) = f_1(\alpha_{1,\pm})\ ,\\
\mathcal{L}_1^{bulk} W_2^{(\pm)} &= \mp\frac{3c_2}{2\pi^2}\sqrt{z_1 z_2}\ {}_2F_1(\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{3}{2}, 4z_2) = f_1(\alpha_{2,\pm})\ ,
\end{aligned} \qquad (3.35)$$

[7] The first operator is obtained after a factorization similar to the one described in ref. [19] for the underlying threefold.

while $\mathcal{L}_2^{bulk} W_a^{(\pm)} = 0$. The roots (3.27) of the quartic equation are identified with the label $(a, \pm)$ of the curves in the right hand side of eq. (3.35). Indeed, as a consequence of eq. (3.26), the inhomogeneous terms can again be written in closed form as

$$\mathcal{L}_a^{bulk} W(\alpha) = f_a(z, \alpha)\,,$$

with $W(\alpha_{a,\pm}) = W_a^{(\pm)}$ and the $f_a(z, \alpha)$ rational functions in the coefficients of the defining equation:

$$f_1(z, \alpha) = \frac{3c}{2\pi^2} \cdot \frac{i\,\phi\,\alpha(\alpha^2 + \phi)}{\psi^3}\,, \qquad f_2(z, \alpha) = 0\,, \tag{3.36}$$

for $c = c_1 = c_2$. As is apparent from (3.35), this function satisfies a hypergeometric equation $\mathcal{L}^{inh} f_1 = 0$. The hypergeometric operator is related to the surface operators by eq. (3.12). In the present case, the relevant operator arises from $\mathcal{L}_2^{\mathcal{D}}$, that is $\mathcal{L}^{inh} = (\mathcal{L}_2^{\mathcal{D}} + [\mathcal{L}_1^{bdry}, \mathcal{L}_2^{\mathcal{D}}]\mathcal{L}_1^{bdry^{-1}})|_{\hat{z}_{crit}}$, while $\mathcal{L}_1^{\mathcal{D}}$ becomes irrelevant. With

$$\mathcal{L}_2^{\mathcal{D}}|_{\hat{z}_{crit}} = \theta_2(\theta_2 - \tfrac{1}{2}) - 4z_2(\theta_2 - \tfrac{1}{4})(\theta_2 - \tfrac{3}{4})\,, \qquad \mathcal{L}_1^{bdry}|_{\hat{z}_{crit}} = i(\theta_2 - \tfrac{3}{4})\,,$$

one obtains

$$\mathcal{L}^{inh} = \theta_2(\theta_2 - \tfrac{1}{2}) - 4z_2(\theta_2 - \tfrac{1}{4})(\theta_2 + \tfrac{1}{4})\,. \tag{3.37}$$

In the above we have used that the relevant surface period is the solution to the Picard-Fuchs system $\{\mathcal{L}_b^{\mathcal{D}}\}$ with index $\frac{1}{2}$ in the variable u_1 to set $\tilde{\theta}_1 = \frac{1}{2}$.

A-model expansion

By mirror symmetry, these functions should have an integral instanton expansion when expressed in terms of the appropriate coordinates and taking into appropriately the contributions from multi-covers [22]. For the critical branes at fixed $\hat{z}$, we use the modified multi-cover formulae of the type proposed in refs. [27, 87, 83]:

$$\frac{W_1^{(\pm)}(z(q))}{\omega_0(z(q))} = \frac{1}{(2\pi i)^2} \sum_{k\,odd} \sum_{\substack{d_1\,odd \\ d_2 \geq 0}} n_{d_1,d_2}^{(1,\pm)} \frac{q_1^{kd_1/2}\, q_2^{kd_2}}{k^2}\,, \tag{3.38}$$

$$\frac{W_2^{(\pm)}(z(q))}{\omega_0(z(q))} = \frac{1}{(2\pi i)^2} \sum_{k\,odd} \sum_{\substack{d_1\,odd \\ d_2\,odd}} n_{d_1,d_2}^{(2,\pm)} \frac{q_1^{kd_1/2}\, q_2^{kd_2/2}}{k^2}\,. \tag{3.39}$$

In this way one obtains the integer invariants in Tab. 3.1 for $c_a = 1$. As can be guessed from these numbers, the superpotentials for $a = 1, 2$ are in fact not independent, but related by a $\mathbb{Z}_2$ symmetry. The family of Calabi-Yau hypersurfaces (3.16) develops a singularity at the discriminant locus $\Delta = 1 - 4z_2 = 0$, which is mirror to a curve of A_1 singularities [88, 89]. On the B-model side the $\mathbb{Z}_2$ monodromy around the singular locus $\Delta = 0$ exchanges the two sets of roots $\alpha_{1,\pm}$ and $\alpha_{2,\pm}$ in

$n^{(1,+)}_{d_1,d_2}$

d_1	$d_2=0$	1	2	3	4	5
1	16	48	0	0	0	0
3	-432	-480	38688	10800	0	0
5	45440	-78192	5472	92812032	146742768	26162880
7	-7212912	25141920	-165384288	61652832	327357559584	1094178697056
9	1393829856	-6895024080	49628432160	-426927933792	261880092960	1383243224519472
11	-302514737008	1905539945472	-14487202588320	131586789107520	-1448971951799232	1383991826496480
13	70891369116256	-538859226100800	4335978084777792	-39691782337561536	440278250387930640	-5799613460160838608
15	-17542233743427360	155713098595732704	-1328641212531217728	12308540119113753936	-132576278776141577664	1710971659352271824160

$n^{(2,+)}_{d_1,d_2}$

d_1	$d_2=1$	3	5	7	9	11	13
1	48	16	0	0	0	0	0
3	0	10800	38688	-480	-432	0	0
5	0	82080	26162880	146742768	92812032	5472	-78192
7	0	-10780160	241323840	88380335472	702830702688	1094178697056	327357559584
9	0	1843890480	-36172116480	932346639840	364829042312640	3751178206812144	*
11	0	-369032481792	6979488962400	-143329914498240	4246347124847520	*	*

Table 3.1: Disc invariants for the on-shell superpotentials $W_a^{(+)}$ of the threefold $\mathbb{P}_{1,2,2,3,4}[12]$.

eq. (3.27). Accordingly, the superpotentials $W_1^{(\pm)}$ and $W_2^{(\pm)}$ are also exchanged as can be seen from the structure of the inhomogeneous terms. On the level of periods this monodromy action yields

$$t_1 \to t_1 + 3t_2, \qquad t_2 \to -t_2 \ . \tag{3.40}$$

As a result the invariants of W_2 are related to that of W_1 by the $\mathbb{Z}_2$ quantum symmetry $q_1 \to q_1 q_2^3,\ q_2 \to q_2^{-1}$ generated by (3.40).[8]

Extremal transition and a non-compact limit

The above results and the normalization obtained by integration from the subsystem can be verified by taking two different one parameter limits. At the singular locus $\Delta = 0$, there is an extremal transition to the one parameter family mirror to a degree (6,4) complete intersection hypersurface in $\mathbb{P}_{1,1,1,2,2,3}$. From eq. (3.40) it follows that the transition takes place at $q_2 = 1$, predicting the relation

$$\sum_{\ell=0}^{3k} n_{k,\ell}^{(a,+)}(\mathbb{P}_{1,2,2,3,4}[12]) = n_k(\mathbb{P}_{1,1,1,2,2,3}[6,4]) \ , \quad a = 1,2 \ , \tag{3.41}$$

where (k,ℓ) denote the degree in q_1 and q_2, respectively. The finiteness of the sum over ℓ follows from the symmetry (3.40). From the left hand side of the above equation one gets

$$n_k = 64,\ 48\,576,\ 265\,772\,480,\ 2\,212\,892\,036\,032,\ 22\,597\,412\,764\,939\,776,\ \ldots \tag{3.42}$$

[8]The $\mathbb{Z}_2$ symmetry is also realized on the closed-string invariants, see the results of ref. [19].

for the first invariants of $\mathbb{P}_{1,1,1,2,2,3}[6,4]$. This can be checked by a computation for the complete intersection manifold with the inhomogeneous Picard-Fuchs equation

$$\mathcal{L}\, W(z) = \frac{4\sqrt{z}}{(2\pi i)^2}\,, \qquad \mathcal{L} = \theta^4 - 48z(6\theta+5)(6\theta+1)(4\theta+3)(4\theta+1)\,. \tag{3.43}$$

Another interesting one modulus limit is obtained for $z_2 \to 0$, where X degenerates to the non-compact hypersurface

$$X^\flat: \qquad y_1^2 + y_2^3 + y_3^6 + y_4^6 + y_5^{-6} + \hat{\psi}\, y_1 y_2 y_3 y_4 y_5 = 0, \qquad \hat{\psi} = \frac{\psi}{\sqrt{\phi}} = z_1^{-1/6} \tag{3.44}$$

in weighted projective space $\mathbb{P}^4_{3,2,1,1,-1}$, with the new variables y_i related to the x_i by

$$y_1 = \phi^{1/2} x_4 x_1^3,\ y_2 = x_5,\ y_3 = x_2,\ y_4 = x_3,\ y_5 = x_1^{-2}\,.$$

The non-compact 3-fold $X^\flat$ is a local model for a certain type of singularity associated with the appearance of non-critical strings and has been studied in detail in ref. [90].

In this limit the curves $C_{\alpha_{2,\pm},\kappa}$ of eq. (3.19) are pushed to the boundary of the local threefold geometry $X^\flat$ and the domain wall tension between $C_{\alpha_{2,+},\kappa}$ and $C_{\alpha_{2,-},\kappa}$ becomes independent of the modulus z_1, which is reflected by the fact that all the disc invariants of W_2 vanish in the limit $z_2 \to 0$. The curves $C_{\alpha_{1,\pm},\kappa}$ become

$$C^\flat_{\varepsilon,\kappa} = \left\{ y_3 = \eta\, y_4\,,\ y_1 y_5^3 = \varepsilon\,,\ y_2 y_5 = \kappa\, y_4 \sqrt{\varepsilon\eta\hat{\psi}} \right\}\,, \quad \varepsilon = \pm i\,,\ \kappa = \pm i\,, \tag{3.45}$$

where $\varepsilon = \pm i$ distinguishes between the two roots $\alpha_{1,+}$ and $\alpha_{1,-}$. One can show that the 3-chain integral representing the domain wall tension in $X^\flat$ descends to an Abel-Jacobi map on a Riemann surface, which can be computed explicitly as an geometric integral. The invariants $n^{[6]}$ obtained for the superpotential in the non-compact geometry $X^\flat$ are reported in ref. [3] and they agree with the q_2^0 term of T_1, $n_{k,0} = n_k^{[6]}$.

A second family of divisors and symmetric K3s

The same critical points can be embedded into a different family of divisors

$$Q(\mathcal{D}_2) = x_4^4 + z_3 z_2^{-1/2} x_1^6 x_4^2\,. \tag{3.46}$$

Our motivation to consider this second family in detail is two-fold. Firstly, the Hodge problem on the surface is equivalent to that of a two parameter family of K3 surfaces at a special point in the moduli, which can be studied explicitly without too many technicalities. We will explicitly show that the relevant zero of the period vector arises at an orbifold point of the K3, which has been interpreted as a point with a half-integral B-field for the closed-string compactification on the local geometry [91]. Secondly, this family tests a different direction of the off-shell deformation space of the brane, leading to a different off-shell superpotential $\mathcal{W}$ for the deformation (3.46). However, since the family contains the curves $C_{\alpha,\kappa}$ for $z_3 = -\alpha^2 z_2^{1/2}$, the critical superpotential has to be the same as the one obtained for the family $\mathcal{D}_1$ in eq. (3.32). The agreement with the previous result and normalization gives an explicit illustration of the fact that different parametrizations of the off-shell directions, corresponding to a different choice of light fields represented by different relative cohomologies, fit together consistently near the critical locus.

As the critical point is determined by the vanishing condition (3.5), we again study the subsystem $P = Q(\mathcal{D}_2) = 0$. Solving for x_4 and changing coordinates to $\tilde{x}_1 = x_1^4$, the surface can be described as a cover[9] of a mirror family of K3 hypersurfaces

$$\tilde{x}_1^3 + x_2^6 + x_3^6 + x_5^3 + \tilde{\psi}\tilde{x}_1 x_2 x_3 x_5 + \tilde{\phi}(x_2 x_3)^3 = 0 \ .$$

Here $\tilde{\psi}^{-6} := u = -\frac{z_1 z_2}{z_3^3}(z_2 - z_3 + z_3^2)^2$ and the parameter $\tilde{\phi}$ is zero for the embedded surface. At $\tilde{\phi} = 0$, the GLSM for this family is defined by the charges

	$\tilde{x}_0$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_5$
$\tilde{l}$	-6	2	1	1	2

.

The GKZ system for this one modulus GLSM has an exceptional solution

$$\pi(u) = \frac{c}{2}\, B_{\{\tilde{l}\}}(u; \tfrac{1}{2}) = \frac{c}{2} \sum_{n=0}^{\infty} \frac{\Gamma(4+6n)}{\Gamma(2+2n)^2 \Gamma(\frac{3}{2}+n)^2} u^{n+\frac{1}{2}} \ , \qquad (3.47)$$

that vanishes at the critical point $u = 0$. To get a better understanding of this solution and of the integral periods on the surface, one may describe π as a regular solution of the two parameter family of K3 surfaces parametrized by $\tilde{\psi}$ and $\tilde{\phi}$, restricted to the symmetric point $\tilde{\phi} = 0$. The charges of the GLSM for the two parameter family of K3 manifolds are

	$\tilde{x}_0$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$
$\tilde{l}^1$	-3	1	0	0	1	1
$\tilde{l}^2$	0	0	1	1	0	-2

.

The two algebraic moduli of this family are $v_1 = -\tilde{\phi}\tilde{\psi}^{-3}$ and $v_2 = \tilde{\phi}^{-2}$ and these are related to the single modulus of the embedded surface by $u = \tilde{\psi}^{-6} = v_1^2 v_2$. The principal discriminant locus for this family has the two components

$$\Delta = \Delta_0 \cdot \Delta_1 = (1 + 54 v_1 + 729 v_1^2 - 2916 v_1^2 v_2) \cdot (1 - 4 v_2) \ .$$

[9] The change from x_1 to $\tilde{x}_1$ gives a fourfold cover acted on by a remaining $\mathbb{Z}_2$ action generated by g_1 in (3.18).

The periods near $\tilde{\phi} = 0$ can be computed in the phase of the two parameter GLSM with coordinates $u_1 = v_1 v_2^{1/2}$ and $u_2 = v_2^{-1/2}$. The hypergeometric series

$$\tilde{\pi}(u_1, u_2) = \frac{c}{2\pi^2} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{\Gamma(1+3n)\Gamma(\frac{1}{2}-n+p)^2}{\Gamma(1+n)^2\Gamma(2-n+2p)} u_1^n u_2^{1+2p-n} \tag{3.48}$$

is a solution of the Picard-Fuchs equation that restricts to $\pi(\sqrt{u})$ in the limit $u_2 = 0$. This series can be expressed with the help of a Barnes type integral as

$$\tilde{\pi}(u_1, u_2) = -\frac{c}{2\pi^2} \int_{\mathcal{C}_+} \sum_{n=0}^{\infty} \frac{\Gamma(1+3n)\Gamma(\frac{1}{2}+s)^2\Gamma(1+s)\Gamma(-s)(-1)^s}{\Gamma(1+n)^2\Gamma(2+n+2s)} (u_1 u_2)^n u_2^{1+2s} \tag{3.49}$$

$$+ \frac{c}{2\pi^2} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \frac{\Gamma(1+3n)\Gamma(\frac{1}{2}-p)^2}{\Gamma(1+n)^2\Gamma(2+n-2p)} (u_1 u_2)^n u_2^{1-2p} , \tag{3.50}$$

where the contour $\mathcal{C}_+$ encloses the poles of the Gamma functions on the positive real line including zero. To relate the special solution $\tilde{\pi}(u_1, u_2)$ to the integral periods on the K3, one may analytically continue it to large complex structure by closing the contour to the left and obtains

$$\begin{aligned} \tilde{\pi}(v_1, v_2) &= \frac{c}{2\pi i} \sum_{n,p=0}^{\infty} \frac{\Gamma(1+3n) v_1^n v_2^p \left(-i\pi + \ln(v_2) + 2(\Psi(1+n-2p) - \Psi(1+p))\right)}{\Gamma(1+n)^2\Gamma(1+n-2p)\Gamma(1+p)^2} \\ &= c\,\omega_0 \left(t_2^{K3} - \frac{1}{2} \right) . \end{aligned} \tag{3.51}$$

Here $\omega_0 = B_{\tilde{l}}(v_a; 0, 0)$ is the fundamental integral period at large volume, and $t_2^{K3} = (2\pi i \omega_0)^{-1} \partial_{\rho_2} B_{\tilde{l}}|_{\rho_a = 0}$ is the integral period associated with the volume of another 2-cycle C, which is mirror to the base of the elliptic fibration defined by the GLSM of the A-model side.

From the last expression it follows that the zero of the K3 period vector associated with the D-brane vacuum arises at the locus

$$J^{K3} = \operatorname{Im} t_2^{K3} = 0 \ , \qquad B^{K3} = \operatorname{Re} t_2^{K3} = \tfrac{1}{2} \ , \tag{3.52}$$

which, in the closed string compactification on this local K3 geometry, is interpreted as a 2-cycle of zero volume with a half-integral B-field. Indeed, in the limit $u = 0 = u_1$, eq. (3.48) becomes

$$\tilde{\pi}(u_1, u_2)|_{u_1=0} \sim \ln \left(\frac{1 - 2v_2 - \sqrt{1 - 4v_2}}{2v_2} \right) - i\pi \, ,$$

expanded around $v_2 = \infty$. The first term on the right hand side is the period for the compact cycle of the $\mathbb{C}^2/\mathbb{Z}_2$-quotient singularity studied in ref. [91], which is zero on the discriminant locus $\Delta_1 = 0$, but a constant at $v_2 = \infty$. The zero associated with the critical point hence does not appear on the principal discriminant, but at an orbifold point with non-vanishing complex quantum volume. It has been argued in refs. [27, 28], that the A-model data associated with the critical points of the present type include $\mathbb{Z}_2$-valued open-string degrees of freedom from the choice of a discrete gauge field on the A-brane. Here we see that to this discrete choice in the A-model there corresponds, at least formally, a half-integral valued B-field for the tension in the B-model geometry. It would be interesting to study this phenomenon and its $\mathbb{C}^2/\mathbb{Z}_n$ generalizations in more detail.

As in the previous parametrization, the tensions can be computed from the integrals

$$T_a = \frac{1}{2\pi i}\int_*^{\beta_a} \pi(u(\xi))\frac{d\xi}{\xi} \ ,$$

where $\beta_{1/2} = \pm i z_2^{1/4}\alpha_{1/2}$, with $\alpha_{1/2}$ defined in eq. (3.27). We again choose the reference point such that $W^{(+,\alpha)} = -W^{(-,\alpha)}$ and find

$$W^{(\pm,\alpha_1)} = \mp\frac{c}{8}\cdot B_{\{l^1,l^2\}}\left(z_1,z_2;\tfrac{1}{2},0\right) \ , \quad W^{(\pm,\alpha_2)} = \mp\frac{c}{8}\cdot B_{\{l^1,l^2\}}\left(z_1,z_2;\tfrac{1}{2},\tfrac{1}{2}\right) \ , \quad (3.53)$$

which is in agreement with (3.32) for $c = 1$.

3.2.2 Degree 14 hypersurface in $\mathbb{P}_{1,2,2,2,7}$

The charge vectors of the GLSM for the A-model manifold are given by [19]

	$\tilde{x}_0$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$
l^1	-7	-3	1	1	1	0	7
l^2	0	1	0	0	0	1	-2

.

The hypersurface constraint for the mirror manifold, written in homogeneous coordinates in $\mathbb{P}_{1,2,2,2,7}$ as well, is

$$P = x_1^{14} + x_2^7 + x_3^7 + x_4^7 + x_5^2 - \psi\, x_1x_2x_3x_4x_5 + \phi\, x_1^7x_5 \,, \quad (3.54)$$

where $\psi = z_1^{-1/7}z_2^{-1/2}$ and $\phi = z_2^{-1/2}$. The orbifold group acts as $x_i \to \lambda_k^{g_k^i}x_i$ with $\lambda_k^7 = 1$ and weights

$$\mathbb{Z}_7: \ g_1 = (1,-1,0,0,0), \quad \mathbb{Z}_7: \ g_2 = (1,0,-1,0,0), \quad \mathbb{Z}_7: \ g_3 = (1,0,0,-1,0)\,. \quad (3.55)$$

In this geometry we consider the set of curves

$$\begin{aligned} C_{\alpha,\pm} = \{x_3 = \eta x_4\,,\ x_5 = \alpha x_1^7\ ,\ x_2^3 = \pm\sqrt{\alpha\eta\psi}\, x_4x_1^4\}\,, \\ \eta^7 = -1\,, \qquad \alpha^2 + \phi\alpha + 1 = 0\,, \end{aligned} \quad (3.56)$$

with the following identification under the orbifold group: $(\eta,\alpha,\pm) \sim (\eta\lambda_2\lambda_3^{-1},\alpha,\pm)$. By choosing representatives we can fix η completely and label the orbits by $(\alpha,\pm)$.

First divisor

The family of divisors

$$Q(\mathcal{D}_1) = x_3^7 + z_3 x_4^7 \tag{3.57}$$

contains the curves $C_{\alpha,\pm}$ for the critical value $z_3 = 1$. The periods on the family of surfaces is captured by the GLSM with charges

	$\tilde{x}_0$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_5$	$\tilde{x}_6$
$\tilde{l}^1$	-7	-3	1	2	0	7
$\tilde{l}^2$	0	1	0	0	1	-2

with two algebraic moduli $u_1 = -\frac{z_1}{z_3}(1-z_3)^2$ and $u_2 = z_2$. The exceptional solutions

$$\begin{aligned}\pi_1 &= \frac{c_1}{2} B_{\{\tilde{l}\}}(u_1, u_2, \tfrac{1}{2}, 0) = -\frac{c_1}{2\pi}\sqrt{u_1}\ {}_2F_1(-\tfrac{7}{4}, -\tfrac{5}{4}, -\tfrac{1}{2}, 4u_2) + \mathcal{O}(u_1^{3/2})\,,\\ \pi_2 &= \frac{c_2}{2} B_{\{\tilde{l}\}}(u_1, u_2, \tfrac{1}{2}, \tfrac{1}{2}) = \frac{35c_2}{2\pi}\sqrt{u_1}u_2^{3/2}\ {}_2F_1(-\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{5}{2}, 4u_2) + \mathcal{O}(u_1^{3/2})\,,\end{aligned} \tag{3.58}$$

vanish at the critical point $u_1 = 0$. Note that these are series in $\sqrt{z_3}$ and the sign of the root distinguishes the two different holomorphic curves $C_{\alpha,+}$ and $C_{\alpha,-}$ in (3.56). The superpotentials obtained from integrals similar to (3.28) are

$$\begin{aligned}W_1^{(\pm)} &= \pm\frac{c_1}{8}\sum_{n_i\geq 0}\frac{\Gamma\left(7n_1+\frac{9}{2}\right)\, z_1^{n_1+\frac{1}{2}} z_2^{n_2}}{\Gamma\left(n_1+\frac{3}{2}\right)^3\Gamma\left(7n_1-2n_2+\frac{9}{2}\right)\Gamma\left(n_2+1\right)\Gamma\left(n_2-3n_1-\frac{1}{2}\right)}\,,\\ W_2^{(\pm)} &= \pm\frac{c_2}{8}\sum_{n_i\geq 0}\frac{\Gamma\left(7n_1+\frac{9}{2}\right)\, z_1^{n_1+\frac{1}{2}} z_2^{n_2+\frac{1}{2}}}{\Gamma\left(n_1+\frac{3}{2}\right)^3\Gamma\left(7n_1-2n_2+\frac{7}{2}\right)\Gamma\left(n_2+\frac{3}{2}\right)\Gamma\left(n_2-3n_1\right)}\,.\end{aligned} \tag{3.59}$$

They can be expressed in terms of the bulk generating function as

$$W_1^{(\pm)} = \pm\frac{c_1}{8}\, B_{\{l^1,l^2\}}\left(z_1, z_2; \tfrac{1}{2}, 0\right)\,, \qquad W_2^{(\pm)} = \pm\frac{c_2}{8}\, B_{\{l^1,l^2\}}\left(z_1, z_2; \tfrac{1}{2}, \tfrac{1}{2}\right)\,. \tag{3.60}$$

As in the previous example, these functions are the restrictions to the critical point $z_3 = 1$ of the off-shell tensions, which can be obtained as the solutions to the large GKZ system (2.22) of the relative cohomology problem derived in refs. [32, 2, 44]. For the family (3.57), the additional charge vector is

	$\tilde{x}_0$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	$\tilde{x}_7$	$\tilde{x}_8$
l^3	0	0	0	1	-1	0	0	-1	1

This leads to the generalized hypergeometric system

$$\begin{aligned}\tilde{\mathcal{L}}_1 &= (\theta_1+\theta_3)(\theta_1-\theta_3)\left(7\theta_1-2\theta_2\right) - 7z_1\left(z_2\left(28\theta_1-4\theta_2+18\right)-3\theta_1+\theta_2-2\right)\times\\ &\quad\times\left(z_2\left(28\theta_1-4\theta_2+10\right)-3\theta_1+\theta_2-1\right)\left(z_2\left(28\theta_1-4\theta_2+2\right)-3\theta_1+\theta_2\right)\,,\\ \tilde{\mathcal{L}}_2 &= \theta_2(\theta_2-3\theta_1) - z_2(7\theta_1-2\theta_2-1)(7\theta_1-2\theta_2)\,,\\ \tilde{\mathcal{L}}_3 &= \theta_3(\theta_1+\theta_3) + z_3\theta_3(\theta_1-\theta_3)\,,\end{aligned} \tag{3.61}$$

annihilating the relative period integrals on the relative cohomology $H^3(Z^*, D_1)$ near the critical locus $y = \ln(z_3) = 0$. Again this system has an alternative origin as the GKZ system associated to an F-theory compactification on a dual 4-fold.

Alternatively, one may characterize the normal functions as solutions to an inhomogeneous Picard-Fuchs equation. From

$$\tilde{\mathcal{L}}_1 = \mathcal{L}_1^{bulk} - (7\theta_1 - 2\theta_2)\theta_3^2, \qquad \tilde{\mathcal{L}}_2 = \mathcal{L}_2^{bulk},$$

one sees that only the first operator acquires an inhomogeneous term, which is determined by the leading part of the surface periods π_a. Acting with $(7\theta_1 - 2\theta_2)\theta_3$ on the terms in (3.58) one obtains the inhomogeneous Picard-Fuchs equations

$$\begin{aligned} \mathcal{L}_1^{bulk} W_1^{(\pm)} &= \mp\frac{7c_1}{16\pi^2}\sqrt{z_1}\, {}_2F_1\left(-\tfrac{3}{4}, -\tfrac{5}{4}, -\tfrac{1}{2}, 4z_2\right) = \pm c_1 f_1(\alpha_1)\ , \\ \mathcal{L}_1^{bulk} W_2^{(\pm)} &= \pm\frac{35c_2}{16\pi^2}\, z_1^{1/2} z_2^{3/2}\, {}_2F_1\left(\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{5}{2}, 4z_2\right) \ = \pm c_2 f_1(\alpha_2)\ . \end{aligned} \tag{3.62}$$

The inhomogeneous terms can be summarized as

$$f_1(\alpha) = -\frac{7i}{16\pi^2}\cdot\frac{\phi(\alpha+\phi)(6\alpha+\phi)}{\alpha^{1/2}\psi^{7/2}}\,, \tag{3.63}$$

where

$$\alpha_{1/2} = \frac{1}{2}\left(-\phi \pm \sqrt{\phi^2 - 4}\right)\,, \tag{3.64}$$

denote the roots of the quadratic equation in the defining equation (3.56).

A-model expansion

The superpotential $W_1^{(+)}$ is associated with the curve $C_{\alpha_1,+}$ and similarly $W_2^{(+)}$ with $C_{\alpha_2,+}$. With the normalization $c_1 = c_2 = 1$ and the multi-cover formulae (3.38) and (3.39), we obtain the integer invariants in Tab. 3.2. Similarly as in the previous example, the two superpotentials are related by a $\mathbb{Z}_2$ symmetry arising from the monodromy associated with an A_1 curve singularity [88, 89]. On the B-model side, the $\mathbb{Z}_2$ monodromy around the singular locus $\Delta = 0$ acts on the periods as $t_1 \to t_1 + 7t_2$, $t_2 \to -t_2$. The invariants of W_2 are related to that of W_1 by the $\mathbb{Z}_2$ quantum symmetry $q_1 \to q_1 q_2^7$, $q_2 \to q_2^{-1}$ induced by this monodromy.

Extremal transition and a non-compact limit

The above results and the normalization obtained by integration from the subsystem can be further verified by taking two different one parameter limits. At the singular locus $\Delta = 0$, there is an extremal transition to the one parameter family mirror to

$\frac{1}{2} \cdot n^{(1,+)}_{d_1,d_2}$

$q_1^{1/2} \backslash q_2$	0	1	2	3	4	5	6	7
1	1	-14	-35	0	0	0	0	0
3	-1	14	-56	-126	-3416	-42182	-19481	-396
5	5	-126	1351	-8358	41643	-157990	87339	-27425384
7	-42	1414	-21455	195790	-1271585	6722898	-30564891	152513340
9	429	-18200	357070	-4322640	37056327	-248175368	1390770059	-7006648980
11	-4939	252854	-6077729	91502334	-980198345	8110498760	-55066462542	322702120822
13	61555	-3691114	104989899	-1889415220	24334523486	-241697136212	1953204386721	-13402394296330

$\frac{1}{2} \cdot n^{(2,+)}_{d_1,d_2}$

$q_1^{1/2} \backslash q_2^{1/2}$	1	3	5	7	9	11
1	0	-35	-14	1	0	0
3	0	0	28	-396	-19481	-42182
5	0	0	-70	1582	-16212	179144
7	0	0	448	-13804	195552	-1907430
9	0	0	-4004	157525	-2892204	34409872

Table 3.2: Disc invariants for the on-shell superpotentials $W_a^{(+)}$ of the threefold $\mathbb{P}_{1,2,2,2,7}[14]$.

a degree eight hypersurface in $\mathbb{P}_{1,1,1,1,4}$ [92]. To study this transition, we rewrite the hypersurface constraint (3.54) as

$$P = \left(-\alpha\psi x_1^8 x_2 x_3 x_4 + x_2^7\right) + \left(x_3^7 + x_4^7\right) + \left(x_5 - \psi x_1 x_2 x_3 x_4 + (\alpha+\phi) x_1^7\right)\left(x_5 - \alpha x_1^7\right) . \quad (3.65)$$

The three summands indicated by the brackets vanish individually on the curves $C_{\alpha,\pm}$. At the singular locus $\phi = \pm 2$, the map to the hypersurface in $\mathbb{P}_{1,1,1,1,4}$ is provided by the identifications

$$x_1^8 x_2 x_3 x_4 = y_1^8, \quad x_2^7 = y_2^8, \quad x_3^7 = y_3^8, \quad x_4^7 = y_4^8, \quad x_5 \pm x_1^7 = y_5 \ ,$$

and this maps the curves $C_{\alpha,\pm}$ to the curves $C_{\zeta\mu}$ of ref. [93] in $\mathbb{P}_{1,1,1,1,4}[8]$.[10]

From the symmetry $t_2 \to -t_2$ it follows that the transition takes place at $q_2 = 1$, predicting the relation

$$\sum_{i=0}^{7k} n_{k,i}^{(a,+)}(\mathbb{P}_{1,2,2,2,7}[14]) = n_k(\mathbb{P}_{1,1,1,1,4}[8]) \,, \quad (3.66)$$

where (k, i) denote the degree in q_1 and q_2, respectively. From the left hand side of the above equation one gets from the above tables

$$-\frac{1}{2}\, n_k = \ 48, \ 65616, \ 919252560, \ ...$$

for the invariants of $\mathbb{P}_{1,1,1,1,4}[8]$. This is in agreement with the results of [93, 76], up to a sign, which is convention.

On the other hand, the the q_2^0 term of the superpotential W_1 reproduces the invariants of the superpotential in the non-compact geometry $\mathcal{O}(-3)_{\mathbb{P}^2}$ studied in ref. [94].

[10] Here μ labels the two roots of the last summand in (3.65) and ζ corresponds to a choice of the sign in (3.56).

To recover this limit geometrically from eq. (3.65) we define

$$y_0 = -\alpha\psi x_1^8 x_2 x_3 x_4, \quad y_1 = x_2^7, \quad y_2 = x_3^7, \quad y_3 = x_4^7,$$
$$x = \frac{x_5}{\phi} - \frac{\psi}{\phi} x_1 x_2 x_3 x_4 + \left(\frac{\alpha}{\phi} + 1\right) x_1^7 \,, \quad z = \phi x_5 - \alpha\phi x_1^7 \,,$$

to write the hypersurface constraint as $P = (y_0 + y_1) + (y_2 + y_3) + xz$. The two roots $\alpha_{1/2}$ behave in the limit as $\alpha_{1/2} \sim -\phi^{\mp 1}$. Choosing $\alpha = \alpha_1$ in (3.65) and rescaling $x_5 \to \frac{x_5}{\phi}$, one finds

$$x = z_1^{-1/7} x_1 x_2 x_3 x_4 + x_1^7 + \mathcal{O}(\phi^{-2}) \,.$$

Taking the root $x = 0$ imposes a constraint on the x_i, and it allows us to rewrite the terms in the first two brackets as

$$(y_0 z_1^{-1/3} + y_1) + (y_2 + y_3), \qquad y_0^3 = y_1 y_2 y_3 \;. \tag{3.67}$$

This is the equation for the Riemann surface Σ representing the mirror of $\mathcal{O}(-3)_{\mathbb{P}^2}$ [95, 23]. It can be verified that the factors in the holomorphic (3,0) form work out as well. After a final rescaling $y_0 \to z_1^{1/3} y_0$, the integral for the domain wall interpolating between the curves $C_{\alpha_1,\pm}$ becomes

$$T_1^{(+,-)}(z_1, z_2 = 0) \sim \int_{y_2 = -\sqrt{z_1}}^{y_2 = +\sqrt{z_1}} \ln(y_1) d \ln y_2 \;. \tag{3.68}$$

This is a 'half-cycle' on the Riemann surface, which reproduces the results for the local brane of ref. [83].

Second divisor

The same domain walls can be alternatively studied via the family of divisors

$$Q(\mathcal{D}_2) = x_5^2 + z_3 z_2^{-1/2} x_1^7 x_5 \,, \tag{3.69}$$

with the curves $C_{\alpha,\pm}$ contained in the divisor with $z_3 = -\alpha\, z_2^{1/2}$. Following the same steps as in the previous example, one recovers the superpotentials (3.60) as the integrals

$$W_a^{(\pm)} = \frac{1}{2\pi i} \int_*^{\beta_a} \frac{c}{2} B_{\tilde{l}}(u; \tfrac{1}{2}) \, \frac{d\xi}{\xi} \;, \quad a = 1, 2 \;,$$

where $\beta_{1/2} = \pm i (z_2^{1/2} \alpha_{1/2})^{1/2}$ with $\alpha_{1/2}$ defined in eq. (3.64). The charge vector $\tilde{l} = (-7, 4, 1, 1, 1)$ describes the subsystem defined by $\mathcal{D}_2$, and $u = -z_1 z_2^3 z_3^{-7} (z_2 - z_3 + z_3^2)^4$ is the single algebraic modulus associated with it.

3.2.3 Degree 18 hypersurface in $\mathbb{P}_{1,2,3,3,9}$

This is a three parameter CY manifold with the charge vectors of the GLSM given by [96]:

	$\tilde{x}_0$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	$\tilde{x}_7$
l^1	-6	-1	0	1	1	3	2	0
l^2	0	1	0	0	0	0	-2	1
l^3	0	0	1	0	0	0	1	-2

The hypersurface constraint is

$$P = x_1^{18} + x_2^9 + x_3^6 + x_4^6 + x_5^2 + \psi\, x_1x_2x_3x_4x_5 + \phi\, x_1^{12}x_2^3 + \chi\, x_1^6x_2^6\,, \tag{3.70}$$

where $\psi = z_1^{-1/6}z_2^{-2/9}z_3^{-1/9}$, $\phi = z_2^{-2/3}z_3^{-1/3}$ and $\chi = z_2^{-1/3}z_3^{-2/3}$. The orbifold group acts as $x_i \to \lambda_k^{g_{k,i}}\, x_i$ with the weights

$$\mathbb{Z}_9 : \, g_1 = (1,-1,0,0,0)\,, \quad \mathbb{Z}_6 : \, g_2 = (1,0,-1,0,0)\,, \quad \mathbb{Z}_6 : \, g_3 = (1,0,0,-1,0)\,, \tag{3.71}$$

with $1 = \lambda_1^9 = \lambda_{2,3}^6$. In this geometry we consider the set of curves

$$C_\pm = \{x_3^3 = \pm i x_4^3\,,\ x_5 = \pm i x_1^9\,,\ x_2 = 0\}\,, \tag{3.72}$$

with the identifications $(+,+) \sim (-,-)$ and $(+,-) \sim (-,+)$ for the possible choices of sign under the orbifold group. The divisor

$$Q(\mathcal{D}) = x_3^6 + z_4 x_4^6 \tag{3.73}$$

leads by the now familiar steps to a GLSM for a K3 manifold with charges

	$\tilde{x}_0$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	$\tilde{x}_7$
$\tilde{l}^1$	-6	-1	0	2	3	2	0
$\tilde{l}^2$	0	1	0	0	0	-2	1
$\tilde{l}^2$	0	0	1	0	0	1	-2

where the moduli of the surface are related to that of the Calabi-Yau threefold by $u_1 = -\frac{z_1}{z_4}(1-z_4)^2$, $u_2 = z_2$ and $u_3 = z_3$. The GLSM is again at a special codimension one locus in the moduli space, with the coefficient of the monomial $x_5x_1^9$ set to zero. The solution

$$\pi(u) = \frac{c}{2}\ B_{\{\tilde{l}^1,\tilde{l}^2,\tilde{l}^3\}}(u_1,u_2,u_3;\tfrac{1}{2},0,0) = \frac{4c}{\pi}\sqrt{u_1} + \mathcal{O}(u_1^{3/2})\,, \tag{3.74}$$

vanishes at the critical locus $u_1 = 0$ and integrates to the superpotential

$$W^{(\pm)} = \mp\frac{c}{8}\ B_{\{l^1,l^2,l^3\}}\left(z_1,z_2,z_3;\tfrac{1}{2},0,0\right)\,. \tag{3.75}$$

q_3^0

$q_1^{1/2} \backslash q_2$	0	1	2	3	4	5	6
1	1	1	0	0	0	0	0
3	-27	-10	-10	-27	0	0	0
5	2840	-1629	2034	2034	-1629	2840	0
7	-450807	523790	-501714	37970	37970	-501714	523790
9	87114366	-143646335	151709190	-82679940	42724232	42724232	-82679940
11	-18907171063	39698748864	-48496621950	38005868880	-25022027880	6124612608	6124612608

q_3^1

$q_1^{1/2} \backslash q_2$	0	1	2	3	4	5	6
1	0	1	0	0	0	0	0
3	0	-10	876	-10	0	0	0
5	0	-1629	-2520	595890	-2520	-1629	0
7	0	523790	-3041532	702090	393040296	702090	-3041532
9	0	-143646335	913643880	-2889725838	1131043400	248949858594	1131043400
11	0	39698748864	-261938878740	899363170080	-2195675791704	998105927940	153662218213536

q_3^2

$q_1^{1/2} \backslash q_2$	0	1	2	3	4	5	6
1	0	0	0	0	0	0	0
3	0	0	-10	-10	0	0	0
5	0	0	2034	595890	595890	2034	0
7	0	0	-501714	702090	1648025820	1648025820	702090
9	0	0	151709190	-2889725838	691571574	2721112372690	2721112372690
11	0	0	-48496621950	899363170080	-7230517669764	2911708467972	

q_3^3

$q_1^{1/2} \backslash q_2$	0	1	2	3	4	5	6
1	0	0	0	0	0	0	0
3	0	0	0	-27	0	0	0
5	0	0	0	2034	-2520	2034	0
7	0	0	0	37970	393040296	1648025820	393040296
9	0	0	0	-82679940	1131043400	2721112372690	8512061067684
11	0	0	0	38005868880	-2195675791704		

Table 3.3: Disc invariants $\frac{1}{16} \cdot n_{d_1,d_2,d_3}$ for the on-shell superpotential $W^{(+)}$ of the threefold $\mathbb{P}_{1,2,3,3,9}[18]$.

Using the multicover formula

$$\frac{W^{(\pm)}(z(q))}{\omega_0(z(q))} = \frac{1}{(2\pi i)^2} \sum_{k\, odd} \sum_{\substack{d_1\, odd \\ d_{2,3} \geq 0}} n^{(\pm)}_{d_1,d_2,d_3} \frac{q_1^{kd_1/2}\, q_2^{kd_2}\, q_3^{kd_3}}{k^2} \tag{3.76}$$

we obtain, for $c = 1$, the integer invariants in Tab. 3.3.

The closed-string type II compactification has a non-perturbative enhanced gauge symmetry with gauge group $G = SU(3)$ at the special values $t_2 = t_3 = 0$ of the closed-string moduli. The monodromy around the special locus acts as

$$m_1 : \, t_1 \to t_1{+}2t_2, \; t_2 \to -t_2, \; t_3 \to t_2{+}t_3, \qquad m_2 : \, t_1 \to t_1, \; t_2 \to t_2{+}t_3, \; t_3 \to -t_3,$$

and generates the Weyl group of $SU(3)$. The superpotential $W^{(\pm)}$ is a singlet under this group while the individual BPS states counted by the disc invariants are exchanged under the group action as $m_1 : \; n_{d_1,d_2,d_3} \to n_{d_1,2d_1-d_2+d_3,d_3}$ and $m_2 : \; n_{d_1,d_2,d_3} \to n_{d_1,d_2,d_2-d_3}$

The off-shell superpotentials are solutions of the following extended hypergeometric system

$$
\begin{aligned}
\mathcal{L}_1 &= (\theta_2-\theta_1)(\theta_2-2\theta_3) - z_2(2\theta_1-2\theta_2+\theta_3-1)(2\theta_1-2\theta_2+\theta_3)\,,\\
\mathcal{L}_2 &= \theta_3(2\theta_1-2\theta_2+\theta_3) - z_3(\theta_2-2\theta_3-1)(\theta_2-2\theta_3)\,,\\
\mathcal{L}_3 &= \theta_3(\theta_2-\theta_1) - z_2z_3(2\theta_1-2\theta_2+\theta_3)(\theta_2-2\theta_3)\,,\\
\mathcal{L}_4 &= (\theta_1+\partial_y)(\theta_1-\partial_y)(2\theta_1-2\theta_2+\theta_3)\\
&\quad - 24z_1(6\theta_1+1)(6\theta_1+5)((4z_2-1)\theta_1+(3z_2z_3-4z_2+1)\theta_2+(2z_2-6z_2z_3)\theta_3)\,,\\
\mathcal{L}_5 &= (\theta_1+\partial_y)(\theta_1-\partial_y)(\theta_2-2\theta_3) - 8z_1z_2(6\theta_1+5)(6\theta_1+3)(6\theta_1+1)\,,\\
\mathcal{L}_6 &= \partial_y(\theta_1+\partial_y) + e^y\partial_y(\theta_1-\partial_y)\,,
\end{aligned}
\tag{3.77}
$$

where $y = \ln(z_4)$.

To compute the inhomogeneous terms we note that the above differential operators are related to that of the Calabi–Yau threefold derived in [96] as

$$
\begin{aligned}
\mathcal{L}_a &= \mathcal{L}_a^{bulk}\,, \qquad a=1,2,3\,,\\
\mathcal{L}_4 &= \mathcal{L}_4^{bulk} - (2\theta_1-2\theta_2+\theta_3)\theta_4^2\,,\\
\mathcal{L}_5 &= \mathcal{L}_5^{bulk} - (\theta_2-2\theta_3)\theta_4^2\,.
\end{aligned}
\tag{3.78}
$$

to obtain from (3.74)

$$
\mathcal{L}_4^{bulk}W^{(\pm)} = \mp\frac{c}{\pi^2}\sqrt{z_1}\,, \qquad \mathcal{L}_a^{bulk}W^{(\pm)} = 0\,, a=1,2,3,5\,. \tag{3.79}
$$

Note that $\sqrt{z_1} = \psi^{-3}\phi$ is a rational function in terms of ψ and ϕ appearing in the hypersurface equation (3.70). The appearance of the square root is related to the non-trivial Greene-Plesser orbifold action on the defining equations (3.72) for the curves $C_\pm$.

As in the previous examples one may study the relation of the above brane geometry to (two and) one parameter configurations in a certain limit in the moduli. For $z_2 = z_3 = 0$ the geometry approximates a non-compact CY, the matching of the invariants is discussed in [3]. At the point $t_2 = t_3 = 0$ of $SU(3)$ gauge enhancement there is a transition to the one modulus CY $\mathbb{P}_{1,1,1,1,2,3}[3,6]$ [88], leading to the prediction

$$
\sum_{i,j} n_{k,i,j}(\mathbb{P}_{1,2,3,3,9}[18]) = n_k(\mathbb{P}_{1,1,1,1,2,3}[6,3])\,,
$$

where the first numbers are

$$
\frac{1}{16}n_k = 3,\ 735,\ 1791060,\ 6117294147,\ 25579918417320. \tag{3.80}
$$

The superpotential of the one parameter model is the solution of the inhomogeneous Picard-Fuchs equation

$$
\mathcal{L}^{bulk}W = \frac{3}{(2\pi i)^2}\sqrt{z_1}\,, \qquad \mathcal{L}^{bulk} = \theta_1^4 - 36z_1(3\theta_1+1)(3\theta_1+2)(6\theta_1+5)(6\theta_1+1)\ .
$$

3.2.4 Degree 12 hypersurface in $\mathbb{P}_{1,2,3,3,3}$

This example is very similar to the hypersurface in $\mathbb{P}_{1,2,3,3,9}$ studied above. The charge vectors of the GLSM given by [96]:

	$\tilde{x}_0$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	$\tilde{x}_7$
l^1	-4	-1	0	1	1	1	2	0
l^2	0	1	0	0	0	0	-2	1
l^3	0	0	1	0	0	0	1	-2

The hypersurface constraint is

$$P = x_1^{12} + x_2^6 + x_3^4 + x_4^4 + x_5^4 + \psi\, x_1x_2x_3x_4x_5 + \phi\, x_1^8x_2^2 + \chi\, x_1^4x_2^4\,, \tag{3.81}$$

where $\psi = z_1^{-1/4}z_2^{-1/3}z_3^{-1/6}$, $\phi = z_2^{-2/3}z_3^{-1/3}$ and $\chi = z_2^{-1/3}z_3^{-2/3}$. The orbifold group acts as $x_i \to \lambda_k^{g_{k,i}}\, x_i$ with the weights

$$\mathbb{Z}_6 :\ g_1 = (1,-1,0,0,0)\,,\quad \mathbb{Z}_4 :\ g_2 = (1,0,-1,0,0)\,,\quad \mathbb{Z}_4 :\ g_3 = (1,0,0,-1,0)\,, \tag{3.82}$$

with $1 = \lambda_1^6 = \lambda_{2,3}^4$. In this geometry we consider the set of curves

$$C_\pm = \{x_3^2 = \pm i x_4^2\ ,\ x_5^2 = \pm i x_1^6\ ,\ x_2 = 0\}\,, \tag{3.83}$$

with the identifications $(+,+) \sim (-,-)$ and $(+,-) \sim (-,+)$ for the possible choices of sign under the orbifold group. The divisor

$$Q(\mathcal{D}) = x_3^4 + z_4 x_4^4 \tag{3.84}$$

leads by the now familiar steps to a GLSM for a K3 manifold with charges

	$\tilde{x}_0$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	$\tilde{x}_7$
$\tilde{l}^1$	-4	-1	0	2	1	2	0
$\tilde{l}^2$	0	1	0	0	0	-2	1
$\tilde{l}^2$	0	0	1	0	0	1	-2

where the moduli of the surface are related to that of the CY threefold by $u_1 = -\frac{z_1}{z_4}(1-z_4)^2$, $u_2 = z_2$ and $u_3 = z_3$. The GLSM is again at a special co-dimension one locus in the moduli space. The solution

$$\pi = \frac{c}{2} B_{\{\tilde{l}^1,\tilde{l}^2,\tilde{l}^3\}}(u_1,u_2,u_3;\tfrac{1}{2},0,0) = \frac{2c}{\pi}\sqrt{u_1} + \mathcal{O}(u_1^{3/2})\,, \tag{3.85}$$

vanishes at the critical locus $u_1 = 0$ and integrates to the superpotential

$$W^{(\pm)} = \mp\frac{c}{8}\, B_{\{l^1,l^2,l^3\}}\left(z_1,z_2,z_3;\tfrac{1}{2},0,0\right)\,. \tag{3.86}$$

	$q_1^{1/2} \backslash q_2$	0	1	2	3	4	5
	1	1	0	0	0	0	0
	3	-3	0	0	0	0	0
q_3^0	5	40	0	0	0	0	0
	7	-847	0	0	0	0	0
	9	21942	0	0	0	0	0
	11	-640431	0	0	0	0	0

	$q_1^{1/2} \backslash q_2$	0	1	2	3	4	5
	1	1	1	0	0	0	0
	3	-2	-2	0	0	0	0
q_3^1	5	-45	-45	0	0	0	0
	7	1750	1750	0	0	0	0
	9	-61551	-61551	0	0	0	0
	11	2233440	2233440	0	0	0	0

	$q_1^{1/2} \backslash q_2$	0	1	2	3	4	5
	1	0	0	0	0	0	0
	3	-2	108	-2	0	0	0
q_3^2	5	50	-56	50	0	0	0
	7	-1962	-11196	-1962	0	0	0
	9	86630	439560	86630	0	0	0
	11	-3842790	-16939860	-3842790	0	0	0

	$q_1^{1/2} \backslash q_2$	0	1	2	3	4	5
	1	0	0	0	0	0	0
	3	-3	-2	-2	-3	0	0
q_3^3	5	50	11090	11090	50	0	0
	7	506	1634	1634	506	0	0
	9	-67884	-1577166	-1577166	-67884	0	0
	11	4125840	66691520	66691520	4125840	0	0

	$q_1^{1/2} \backslash q_2$	0	1	2	3	4	5
	1	0	0	0	0	0	0
	3	0	0	0	0	0	0
q_3^4	5	-45	-56	11090	-56	-45	0
	7	506	1127464	4423692	1127464	506	0
	9	28776	517288	46134	517288	28776	0
	11	-3030696	-185400024	-566257044	-185400024	-3030696	0

	$q_1^{1/2} \backslash q_2$	0	1	2	3	4	5
	1	0	0	0	0	0	0
	3	0	0	0	0	0	0
q_3^5	5	40	-45	50	50	-45	40
	7	-1962	1634	4423692	4423692	1634	-1962
	9	28776	111025794	1085027250	1085027250	111025794	28776
	11	1030368	74577268	129171092	129171092	74577268	1030368

Table 3.4: Disc invariants $\frac{1}{8} \cdot n_{d_1,d_2,d_3}$ for the on-shell superpotential $W^{(+)}$ of the threefold $\mathbb{P}_{1,2,3,3,3}[12]$.

Using the multicover formula (3.76) we obtain, for $c = 1$ the integer invariants in Tab. 3.4.

The closed-string type II compactification has a non-perturbative enhanced gauge symmetry with gauge group $G = SU(3)$ at $t_2 = t_3 = 0$. The monodromy around this special locus acts as

$$m_1: \; t_1 \to t_1, \; t_2 \to -t_2, \; t_3 \to t_2 + t_3, \qquad m_2: \; t_1 \to t_1 + t_3, \; t_2 \to t_2 + t_3, \; t_3 \to -t_3,$$

and generates the Weyl group of $SU(3)$. The superpotential $W^{(\pm)}$ is a singlet under this group while the individual BPS states counted by the disc invariants are exchanged under the group action as $m_1: \; n_{d_1,d_2,d_3} \to n_{d_1,-d_2+d_3,d_3}$ and $m_2: \; n_{d_1,d_2,d_3} \to n_{d_1,d_2,d_1+d_2-d_3}$.

The off-shell superpotentials are solutions of the following extended hypergeometric system

$$\begin{aligned}
\mathcal{L}_1 &= (\theta_2-\theta_1)(\theta_2-2\theta_3) - z_2(2\theta_1-2\theta_2+\theta_3-1)(2\theta_1-2\theta_2+\theta_3)\,,\\
\mathcal{L}_2 &= \theta_3(2\theta_1-2\theta_2+\theta_3) - z_3(\theta_2-2\theta_3-1)(\theta_2-2\theta_3)\,,\\
\mathcal{L}_3 &= \theta_3(\theta_2-\theta_1) - z_2z_3(2\theta_1-2\theta_2+\theta_3)(\theta_2-2\theta_3)\,,\\
\mathcal{L}_4 &= (\theta_1+\partial_y)(\theta_1-\partial_y)(2\theta_1-2\theta_2+\theta_3)\\
&\quad - 8z_1(4\theta_1+3)(4\theta_1+1)((4z_2-1)\theta_1+(3z_2z_3-4z_2+1)\theta_2-(6z_2z_3-2z_2)\theta_3)\,,\\
\mathcal{L}_5 &= (\theta_1+\partial_y)(\theta_1-\partial_y)(\theta_2-2\theta_3) - 4z_1z_2(4\theta_1+3)(4\theta_1+2)(4\theta_1+1)\,,\\
\mathcal{L}_6 &= \partial_y(\theta_1+\partial_y) + e^y\partial_y(\theta_1-\partial_y)\,,
\end{aligned} \tag{3.87}$$

where $y = \ln(z_4)$.

To compute the inhomogeneous terms we note that the above differential operators are related to that of the Calabi–Yau threefold as

$$\begin{aligned}
\mathcal{L}_a &= \mathcal{L}_a^{bulk}\,, \quad a=1,2,3\,,\\
\mathcal{L}_4 &= \mathcal{L}_4^{bulk} - (2\theta_1-2\theta_2+\theta_3)\theta_4^2\,,\\
\mathcal{L}_5 &= \mathcal{L}_5^{bulk} - (\theta_2-2\theta_3)\theta_4^2\,.
\end{aligned} \tag{3.88}$$

Then we obtain from eq. (3.85)

$$\mathcal{L}_4^{bulk} W^{(\pm)} = \mp\frac{c}{2\pi^2}\cdot\sqrt{z_1}\,, \qquad \mathcal{L}_a^{bulk} W^{(\pm)} = 0\,, a=1,2,3,5\,, \tag{3.89}$$

where, similarly as in the previous example, $\sqrt{z_1} = \psi^{-2}\phi$ is a rational function in ψ and ϕ, and the appearance of the square root is related to the non-trivial action of the Greene-Plesser orbifold group on the defining equations for the curves $C_\pm$.

In the limit $z_2 = z_3 = 0$ one can again make contact with a non-compact Calabi-Yau as discussed in [3]. At the point $t_2 = t_3 = 0$ of $SU(3)$ gauge enhancement there is again a transition to a one modulus Calabi–Yau, this time $\mathbb{P}_{1,1,1,1,1,2}[3,4]$ [88]. This leads to the prediction $\sum_{i,j} n_{k,i,j}(\mathbb{P}_{1,2,3,3,3}[12]) = n_k(\mathbb{P}_{1,1,1,1,1,2}[3,4])$, with the first invariants being

$$\frac{1}{8}n_k = 3,\ 87,\ 33252,\ 16628907,\ 10149908544,\ 6979959014559, 5196581251886028\,. \tag{3.90}$$

The superpotential is a solution of the inhomogeneous Picard-Fuchs equation of the one modulus problem

$$\mathcal{L}^{bulk} W = -\frac{3}{8\pi^2}\sqrt{z_1}\,, \qquad \mathcal{L}^{bulk} = \theta_1^4 - 12z_1(3\theta_1+1)(3\theta_1+2)(4\theta_1+1)(4\theta_1+3)\,.$$

3.3 Summary and outlook

In this chapter we studied off-shell brane superpotentials for four-dimensional type II/F-theory compactifications depending on several open-closed deformations as well as their specialization to the on-shell values in the open-string direction. Mathematically the two potentials are respectively related to the integral period integrals on the (relative) cohomology group defined by the family of branes [85, 26, 31, 32, 2, 51, 44], which depend on both open and closed deformations, and the so-called normal functions, depending only on closed-string moduli [83, 28]. Both objects can be studied Hodge theoretically by computing the variation of Hodge structure on the relevant (co-)homology fibers over the open-closed-string deformation space $\mathcal{M}$. Ultimately, this determines the superpotential as a particular solution of a system of generalized GKZ type differential equations determined by the integral (relative) homology class of the brane.

The D-brane superpotentials computed in this way are relevant in different contexts. From the phenomenological point of view, the superpotential determines the vacuum structure of four-dimensional F-theory compactifications. The complicated structure of the superpotential for this class of compactifications, described by infinite generalized hypergeometric series, should be contrasted with the simple structure of F-theory superpotentials in other classes of compactifications, as e.g. in refs. [97, 98]. These hypergeometric series have sometimes a dual interpretation as D-instanton corrections and heterotic world-sheet corrections [30], and the rich structure of non-perturbative corrections to the brane superpotential should lead to interesting hierarchies of masses and couplings in the low-energy effective theory.

As shown in ref. [30], the solutions to the generalized GKZ system representing the F-theory superpotential do not only capture the superpotentials of dual Calabi-Yau threefold compactifications, but more generally of type II and heterotic compactifications on generalized Calabi-Yau manifolds of complex dimension three.[11] This offers a powerful tool to study more generally the vacuum structure of phenomenologically interesting F-theory/type II/heterotic compactifications. It would be interesting to apply the Hodge theoretic approach described in this thesis to examples of phenomenologically motivated F-theory scenarios, as described e.g. in refs. [102, 103, 104].[12] In the search for vacua, the step of passing from relative periods depending on open and closed-string deformations to normal functions depending only on closed-string moduli provides a natural split in the minimization process, which should be helpful in a regime of small string coupling. On the other hand, this distinction between closed and open-string moduli disappears away from this decoupling limit, for finite string coupling, where the two types of fields mix in a way determined by a certain degeneration of the F-theory fourfold described in [2, 30].

A complementary aspect of the B-type superpotentials considered in this paper is the prediction of A-model disc invariants by open-closed mirror symmetry. For almost

[11] The first examples of dual compactifications of this type were given in ref. [99]. See also refs. [100, 101] for related works and examples.

[12] See also ref. [105], for a recent review on this subject, and further references therein.

flat open-string directions (characterized by a generalized large complex structure point in the open-closed deformation space as described in chapter 2), already the off-shell superpotential computed by the relative period integral has an A-model expansion in closed- and open-string parameters, leading to predictions for integral Ooguri-Vafa invariants [2, 49, 44, 30]. In the present work we instead concentrated on the critical points of the type studied in refs. [27, 28, 87, 93, 76, 106], where the A-model expansion emerges only after integrating out the open-string directions. The on-shell computations of refs. [27, 28, 87] are conceptually well understood and provided the first examples of open-string mirror symmetry in compact CY. Our main motivation to study the type of critical points accessible also in the on-shell formalism was to gain a better understanding of the minimization in the open-string direction, which relates the on-shell computation to the off-shell framework of refs. [85, 26, 31, 32, 2]. On the B-model side, the relation is provided by the connection between integral relative period integrals and normal functions described in sect. 3.1. An important datum in this correspondence is the period vector on the surface, that is the brane 4-cycle. It classifies the D-brane vacua by the vanishing condition (3.5) and determines the inhomogeneous term in the Picard-Fuchs equation for the normal function.

In the relative cohomology approach adopted in this thesis, the open-string deformations are off-shell yet one avoids working in string field theory by perturbing the unobstructed F-theory moduli space associated with the family of surfaces $\mathcal{D}$ by a probe brane representing an element in $H_2(\mathcal{D})$. This leads to well-defined *finite dimensional* off-shell deformation spaces associated with a particular parametrization by 'light' fields in the superpotential. The parametrization of off-shell deformations is adapted to the topological string and leads to a definition of off-shell mirror maps and off-shell invariants consistent with expectations. Different parametrizations are bound to fit together in an consistent way, as is explicitly demonstrated in some of the examples, where we parametrized the off-shell superpotentials by different choices of open-string deformation parameters. This means starting from a given supersymmetric configuration, we compare different off-shell deformation directions in the infinite-dimensional open-closed deformation space, and we find that the obtained on-shell tensions are independent from the chosen off-shell directions.[13] This is a gratifying result as the on-shell domain wall tensions should not depend on the details of integrating out the heavy modes.

There are many other open questions that need further exploration. For examples with a single open-string deformation a detailed analysis of the Hodge structure of the K3 surface, equivalent to the subsystem defined by the Hodge structure on the surface $\mathcal{D}$, might be rewarding. In this work we explained how the analyzed supersymmetric domain wall tensions arise at enhancement points of the Picard lattice in the K3 moduli space. The leading term of the K3 periods near these specially symmetric points is a rational function in the deformations z and the

[13] See ref. [51] for an earlier example of this kind.

roots α of the defining equation. As argued in sect. 3.2.1, the global symmetry seems to be related to the discrete symmetry in the A-type brane in the mirror A-model configuration. It would be interesting to study in detail the structure of Picard lattice enhancement loci in order to systematically explore the web of $\mathcal{N} = 1$ domain wall tensions in Calabi-Yau threefolds. Such an analysis potentially sheds light on the global structure of $\mathcal{N} = 1$ superpotentials (see e.g. refs. [107]), and should be related to the wall-crossing phenomena described in refs. [108, 109, 110].

4 Mirror Symmetry and NS5 branes

We study the fourfold geometries that appeared in the last two chapters in more detail. Generally the Picard-Fuchs systems of chapter 2 for a hypersurface in a d dimensional CY are equivalent to Picard-Fuchs systems of a $d+1$ dimensional non compact CY. In [2, 30] this was interpreted by a chain of dualities involving F-theory and heterotic strings. Here we give an alternative interpretation using the action of T-duality on NS5 branes and the relation between T-duality and mirror symmetry. In particular we study complex structure monodromies of fibers within the non compact CY geometries and find evidence that they are mirror to Calabi-Yau manifolds with NS5 brane on a divisor. This gives a simple way to construct mirrors to any Calabi-Yau hypersurface with NS5 branes wrapped on divisors. As a further application we extend the open-closed deformation space to the case of a stack of parallel hypersurfaces, compute and comment on the mirror maps and superpotentials and check that the geometries reduces locally to the known duals of parallel NS5 branes in flat space. Most of this chapter was published as [4], the discussion of parallel branes is planned to be published in [56].

4.1 Introduction

The equivalence of NS5 branes and certain Ricci flat geometries under T-duality was first shown in [111] by a study of the conformal field theory for ALE spaces with A_{N-1} singularities. These geometries are S^1 fibrations with N vanishing fibers. A T-duality along the fiber turns it into N parallel NS5 branes in flat space. The T-duality acts in a normal direction to the resulting NS5 brane, the localization of the brane in this direction is due to instanton effects [112]. The following geometric explanation based on [113] was already given in [111]: The effect of an NS5 brane localized on a point in a Torus Z^* and a point $\mathbb{C}$ is a monodromy of the B-field, $B \to B + 2\pi$ around the brane in $\mathbb{C}$. This gives a monodromy $\rho \to \rho + 1$ of the complexified Kähler class $\rho = \frac{B}{2\pi} + i\sqrt{G}$. Mirror symmetry, or T-duality in one S^1 of the torus, exchanges the Kähler class with the complex structure. After T-duality one thus expects a monodromy $\tau \to \tau + 1$ for the dual torus Z. To get such a monodromy the dual geometry has to be a fibration of Z over $\mathbb{C}$. Instead of an NS5 brane there is a singular fiber with a shrinking S^1.

Mirror symmetry should also geometrize NS5 branes on divisors in higher dimensional Calabi-Yau (CY) spaces [51]. In the Strominger Yau Zaslow picture [53] one of the T-dualities in the Lagrangian torus fiber acts in a normal direction of a generic divisor. While T-dualities in an internal direction map an NS5 brane to another NS5 brane, the single T-duality in a normal direction should turn it into a locus of a vanishing S^1. The resulting fibration has to be a consistent background preserving the same amount of supersymmetry, so it should be a non compact CY space. Based on this idea we propose a global description of a dual geometry for NS5 branes localized on a point in $\mathbb{C}$ and a divisor in a n-dimensional CY hypersurface Z^* in a toric ambient space. The dual geometry X is a fibration of the mirror CY Z over $\mathbb{C}$ and is itself a non-compact n+1 dimensional CY hypersurface. It can be constructed by toric methods. We study the complex structure monodromy of the fibers and find perfect agreement with the dual Kähler monodromies created by NS5 branes.

More concretely we propose that specific non-compact CY manifolds that already appeared in [32, 2, 3] can be interpreted in this way. These papers discuss superpotentials for branes wrapped on curves in CY 3-folds Z^*. The curves are first immersed into a divisor. Then the unobstructed deformation space of the divisor inside the CY is used to calculate volumes of chains ending on curves within the divisor. These relative period integrals were seen to be equivalent to period integrals of a non compact CY 4-fold X^*. The mirror X to this non compact 4-fold is the geometry we will mainly study in this note. The matching of relative period integrals for divisors in Z^* with quantum corrected volumes of cycles in X can be interpreted as first evidence for the present proposal. In [30] this match was explained by a different chain of dualities starting from 7-branes on the divisor and involving heterotic/F-theory duality. It would be interesting to close both proposals to a cycle of dualities. The present proposal appeared implicitly already in [51], where a similar construction involving NS5 branes is used to calculate superpotentials.

We start by repeating the construction of the non compact manifold X for a given divisor on a CY hypersurface Z^*. This construction was mainly implicit in the first chapters. Next we consider the example of an NS5 brane on a torus. We calculate the monodromy of the complex structure of the fiber in the proposed dual geometry and show that it matches with the shift of the B-field. We study the central fiber in detail on the example of a K3 fibration. In sect. 4.5 we show that complex structure monodromies for the fibers of X always map to the expected shift of a B-field. We use toric methods and the relation between monomials and divisors of mirror manifolds. After some further examples we construct the deformation system for parallel branes and study their mirror maps and superpotentials. We find the expected singularities in the dual geometry and check the monodromies. We end with a summary and some comments on a "mirror" mapping between the divisor and the degeneration locus. This could be interesting as generically the degeneration locus is not CY.

4.2 The dual geometry

We start by repeating the construction of the non compact CY fibration X, [25, 32, 2]. We use standard notation for polytopes in mirror symmetry, see e.g. [5]. Some details are summarized in sect. 4.5.

A CY hypersurface Z^* in a toric variety is given as the vanishing locus of an equation

$$\tilde{P}(Z^*) = \sum_i a_i \tilde{x}^{\nu_i} .$$

The monomials $\tilde{x}^{\nu_i} = \prod_j \tilde{x}_j^{\nu_{i,j}}$ appearing in this equation are labeled by integral points ν_i of some reflexive lattice polytope Δ^*. There are relations $\sum_i l_i^a \nu_i = 0$ between these points and therefore between the monomials, $\prod_i (\tilde{x}^{\nu_i})^{l_i^a} = 1$. These relations can be used to derive a Picard-Fuchs system for the periods of Z^* and to define the gauged linear sigma model (GLSM) [9] of the mirror CY Z.

In Z^* we study the most general divisor $\mathcal{D}$ of a given divisor class without any rigid component. If the degree of the defining equation $\tilde{Q} = 0$ for this divisor is not higher than the degree of $\tilde{P}$, it can be expressed as[1]

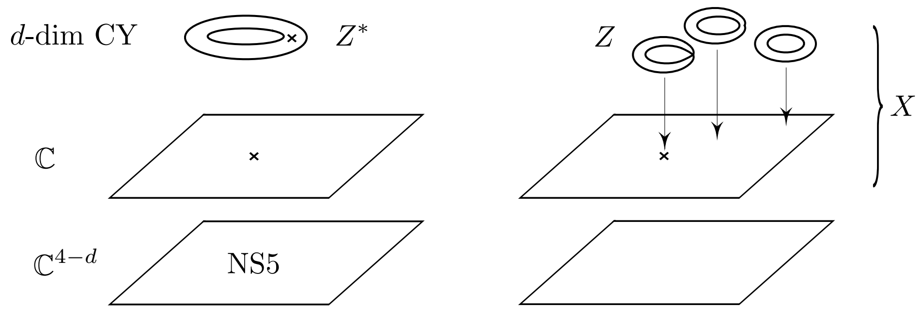
$$\tilde{Q} = (b_1 \tilde{x}^{\nu_a} + b_2 \tilde{x}^{\nu_b} + ... + b_n \tilde{x}^{\nu_*}) / \,\mathrm{gcf}\,, \tag{4.1}$$

where gcf is the greatest common factor of the monomials in $\tilde{Q}$. There are new relations between the monomials of $\tilde{Q}$ and the monomials of $\tilde{P}$. They lead to a Picard-Fuchs system governing the volume of chains ending on the divisor $\mathcal{D}$, [44, 32, 2]. For bookkeeping we express them as relations $\sum_i \hat{l}_i^a \hat{\nu}_i = 0$ between points $\hat{\nu}_i$ of an enlarged lattice polytope $\hat{\Delta}^*$. To construct it we embed Δ^* in a lattice with one additional dimension and add one point for every monomial in $\tilde{Q}$, $\hat{\Delta}^* = \{(\Delta^*, 0), (\nu_a, 1), (\nu_b, 1), ..., (\nu_*, 1)\}$. In the following we use the notation $\hat{l}$ only for the new relations that involve some of the additional points $(*, 1)$. Relations involving only the points $(\Delta^*, 0)$ are called l.

The GLSM defined by a basis for these relations gives a non compact CY X. This is the geometry we will mainly study in the following. It is always an Z fibration over $\mathbb{C}$ with a single singular fiber. In the singular fiber an S^1 shrinks over a codimension two locus. As we will show in the following, the complex structure monodromy of Z around this singular fiber matches the B-field monodromy of Z^* for an NS5 brane wrapped on $\mathcal{D}$. Moreover the relative periods of the pair $(Z^*, \mathcal{D})$ are mirror to cycles of X, including quantum corrections. In particular the moduli of the divisor $\mathcal{D}$ are mapped to Kähler moduli controlling the location of the shrinking S^1 in the singular fiber.

[1] If the degree is higher a straightforward analogous construction is still possible. In this case further new points can be added to the extended polytope. We will not consider this case to avoid cluttering the notation.

We thus conjecture that the geometry X is dual to $Z^* \times \mathbb{C}$ with an NS5 brane wrapped on the divisor $\mathcal{D}$ and localized at a point in $\mathbb{C}$. This statement appeared implicitly already in [51].[2] The internal directions of the NS5 brane fill the divisor in Z^* and the remaining unspecified directions, the dimension of Z^* does not matter. By mirror symmetry, instanton effects for X are naturally captured by the classical geometry X^* or the pair $(Z^*, \mathcal{D})$. From a supergravity point of view they cause the localization of the NS5 brane in the transverse circle [112, 114], for a recent discussion within "doubled geometry" see [115].



An NS5 brane localized on a divisor and a point in $\mathbb{C}$, and the dual geometry X.

4.3 NS5 brane on a Torus

Consider a torus Z^*, defined by a hypersurface equation $\tilde{P} = a_1\tilde{x}_1^3 + a_2\tilde{x}_2^3 + a_3\tilde{x}_3^3 + a_0\tilde{x}_1\tilde{x}_2\tilde{x}_3 = 0$ in $\mathbb{P}^2/\mathbb{Z}_3$. To this geometry we add an NS5 brane at $\tilde{Q} = b_1\tilde{x}_1^2 + b_0\tilde{x}_2\tilde{x}_3 = 0$, localized at the origin of $\mathbb{C}$ and wrapping $\mathbb{R}^6$. If we forget about the NS5 brane and apply mirror symmetry to the torus we get a dual Torus Z. From [111] we learn that we get a fibration of the dual torus Z over $\mathbb{C}$ if we take the NS5 brane into account. In the case at hand we wrap an NS5 brane around a divisor of class $2[pt]$.[3] As the class of a point, $[pt]$, is dual to the Kähler class, this gives a monodromy $\rho \to \rho + 2$. We conjecture that the dual non-compact CY 2-fold is given by the GLSM

[2] There an NS5 brane on a divisor is geometrized by a T-duality and the resulting geometry is used to calculate superpotentials. Instead of $Z^* \times \mathbb{C}$ [51] starts with a CY 3-fold Z^* times $S^1 \times \mathbb{R}$ and performs a T-duality on the S^1 to get a 4-fold Y without branes. It was noted that 3 dimensional mirror symmetry of Z^* should also geometrize the NS5 brane and that the resulting geometry could be the (4 dimensional) mirror of Y. On the level of period integrals the mirror symmetry between Y and X was checked. Y is however not identical with the mirror X^* of the non compact CY X as it appeared in [32, 2]. The pair (X, X^*) can be compactified to a mirror pair of compact CY hypersurfaces.

[3] This is the simplest example, we will explain later how to construct the dual geometry as well for a Torus with NS5 brane on a minimal divisor.

$$\begin{array}{c|c|ccccc} & P & x_1 & x_2 & x_3 & y_0 & y_1 \\ \hline l & \text{-3} & 1 & 1 & 1 & 0 & 0 \\ \hat{l} & \text{-1} & 1 & 0 & 0 & 1 & \text{-1} \end{array} . \tag{4.2}$$

The most general hypersurface equation for these charges is

$$P = x_1 p^2(x_1 y_1,\, x_2,\, x_3) + y_0\, q^3(x_1 y_1,\, x_2,\, x_3) + \mathcal{O}(y_0 y_1)\,,$$

where p^2 and q^3 are arbitrary degree two and three polynomials in x_1y_1, x_2 and x_3. This geometry is a Z torus fibration over $\mathbb{C}$, the coordinate on $\mathbb{C}$ is y_0y_1. We can see this as follows. $\{(x_1,x_2,x_3) \in \mathbb{P}^2\,|P(x_1,x_2,x_3,y_0,y_1) = 0\}$ is a torus whose complex structure depends on y_0 and y_1. By the D-term constraint for the charge vector $\hat{l}$, $|x_1|^2 + |y_0|^2 - |y_1|^2 = \hat{t}$, the two coordinates y_0 and y_1 are not independent. We can use this constraint together with the corresponding $U(1)_{\hat{l}}$ action to fix y_0 and y_1 once the product $y_0y_1 \in \mathbb{C}$ is given. We have thus a torus over each generic point y_0y_1 of the base. The only non generic point is $y_0y_1 = 0$, where a S^1 shrinks in the central fiber. For $|x_1|^2 = \hat{t}$ both y_0 and y_1 vanish and $U(1)_{\hat{l}}$ acts only on the phase of x_1. By construction this action is compatible with the hypersurface constraint for the torus at $y_0 = y_1 = 0$, $P = x_1p^2(x_2,x_3)$. So we can use a cylinder $a < |x_1| < b$ with $a < \sqrt{\hat{t}} < b$ as coordinate patch for the torus and the $U(1)_{\hat{l}}$ action cuts the cylinder into a union of two cones. As there are two solutions to $P = x_1p^2(x_2,x_3) = 0$ with $|x_1|^2 = \hat{t}$, this happens twice. The two loci are mirror to the two points $\tilde{Q} = b_1\tilde{x}_1^2 + b_0\tilde{x}_2\tilde{x}_3 = 0$. The Kähler modulus $\hat{t}$ that determines the position of the degenerating S^1 in the torus T is mirror to the modulus $\hat{z} = \frac{a_1b_0}{a_0b_1}$ that determines the position of the NS5 branes in Z^*. For details on the mirror map in slightly more complicated examples see [32, 2].

To calculate the monodromy around the origin we consider y_0, y_1 as (redundant) parameters that determine the complex structure of the fiber. The period integrals can be brought into the standard form by a rescaling $x_1 \to x_1\frac{1}{y_1^{2/3}}$, $x_2 \to x_2y_1^{1/3}$, $x_3 \to x_3y_1^{1/3}$,

$$\int \frac{\Xi}{x_1p^2(x_1y_1,\, x_2,\, x_3) + y_0\, q^3(x_1y_1,\, x_2,\, x_3)} = \int \frac{\Xi}{x_1p^2(x_1,x_2,x_3) + y_0y_1\, p^3(x_1,x_2,x_3)}\,,$$

where Ξ is the holomorphic 2-form of $\mathbb{P}^2$. After this rescaling P depends on y_0 and y_1 only in the combination y_0y_1, so we can treat it as hypersurface equation of the fiber depending on the position of the base, $P(x_1,x_2,x_3;y_0y_1)$. The geometry (4.2) is a blow-up of the fibration $\{(x_1,x_2,x_3) \in \mathbb{P}^2\,|P(x_1,x_2,x_3;y_0y_1) = 0\} \to \mathbb{C}$. We discuss this in more detail in the next example. Close to $y_0y_1 = 0$ all monomials containing only x_2 and x_3 are suppressed. After some coordinate redefinitions these are only the two monomials x_2^3 and x_3^3. Moving in a sufficiently small circle around $y_0y_1 = 0$ these are the only monomials whose prefactors in the hypersurface equation of the fiber vary and we can use standard methods [19] to determine the complex structure. We find $\tau = 2\ln(y_0y_1) + \mathcal{O}(y_0y_1)$ near $y_0y_1 = 0$. The factor of 2 comes about as both the monomials x_2^3 and x_3^3 are suppressed by y_0y_1. Alternatively, after an additional rescaling, only one of the monomials e.g. x_2^3 is suppressed by $(y_0y_1)^2$. This gives the expected monodromy $\tau \to \tau + 2$. The logarithmic singularity at $y_0y_1 = 0$ is in accordance with the expected backreaction of an NS5 brane, for a recent study of this situation in the heterotic string see [116, 117].

4.4 NS5 brane on a K3

We now want to extend this construction to more complicated geometries. The Strominger Yau Zaslow picture of mirror symmetry [53] seems to indicate that this is possible. It explains mirror symmetry as simultaneous T-dualities in all directions of a Lagrangian torus fibration. One of this directions is normal, the others transversal to a holomorphic divisor. A T-duality performed in an internal direction maps an NS5 brane to another NS5 brane, a T-duality in a normal direction should turn it into a locus where the T-dual S^1 shrinks. Mirror symmetry should thus geometrize the NS5 brane. As it exchanges Kähler and complex structure moduli, die shift of the B-field that signals the presence of an NS5 brane should be mapped to a monodromy of the complex structure.

We will study generalizations of the geometry (4.2) and show that the complex structure of the fiber Z has a monodromy around the origin of the base. This monodromy is in agreement with the interpretation as mirrors of a CY Z^* with NS5 brane. To make contact with the easier case of the torus we consider an elliptically fibered K3 Z that is fibered over $\mathbb{C}$,

$$\begin{array}{c|c|ccccccc} & P & x_1 & x_2 & x_3 & x_4 & x_5 & y_0 & y_1 \\ \hline l^1 & -3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ l^2 & 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \\ \hat{l} & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \end{array} . \tag{4.3}$$

The coordinate on $\mathbb{C}$ is y_0y_1. We call the whole fibration again X. It has a singular fiber over the origin of $\mathbb{C}$, in this singular K3 the elliptic fiber degenerates. Now we concentrate on a neighborhood of the vanishing S^1 in the degenerate elliptic fiber. Locally the geometry is a cone ($uv = 0$) over $\mathbb{P}^1 \times \mathbb{C}$. This should turn into an NS5 brane, if we can consistently implement a duality transformation that involves one T-duality in the elliptic fiber of the K3 Z. Mirror symmetry in the Z fiber over each point in $\mathbb{C}$ is such a duality, its maps the Z fibration to a Z^* fibration over $\mathbb{C}$. As the Kähler structure of Z fiber does not vary in (4.3), the complex structure of Z^* is constant in the dual fibration.

The complex structure of the fiber Z however does vary. The hypersurface equation

$$P = x_1 p^2(x_1y_1, x_2, x_3x_4^2, x_3x_5^2, x_3x_4x_5) + y_0 q^3(x_1y_1, x_2, ..) + \mathcal{O}(y_0y_1)\,, \qquad (4.4)$$

depends on parameters y_0 and y_1. With x_i we denote coordinates for the smooth blow-up of $\mathbb{P}_{1122}$. In the following this blow-up is understood whenever we write $\mathbb{P}_{1122}$ or $\mathbb{P}_{112}$.

In the period integrals we can rescale $x_1 \to x_1/y_1^{2/3}$, $x_2 \to x_2 y_1^{1/3}$, $x_3 \to x_3 y_1^{1/3}$,

$$\int \frac{\Xi}{P} = \int \frac{\Xi}{x_1 p^2(x_1, x_2, x_3x_4^2, x_3x_5^2, x_3x_4x_5) + y_0y_1 q^3(x_1, x_2, ..)}\,,$$

so the complex structure only depends on the product y_0y_1, as it should. Here we claimed that the complex structure of the Z fiber is the same as the complex structure of the hypersurface $P' = x_1p^2(x_1, x_2, x_3x_4^2, x_3x_5^2, x_3x_4x_5) + zq^3(x_1, x_2, ..) + \mathcal{O}(z^2)$ in $\mathbb{P}_{1122}$. Let us look at the two geometries more carefully. Both geometries fall apart into two components at $y_0y_1 = 0$ and $z = 0$ respectively. For the fiber $Z|_{y_0y_1=0}$ we have the components

$$|x_1|^2 \le \hat{t}\,,\, y_1 = 0 \;\text{ and}$$
$$|x_1|^2 \ge \hat{t}\,,\, y_0 = 0\,,$$

where $\hat{t}$ is the Kähler modulus for the charge vector $\hat{l}$, $|x_1|^2 + |y_0|^2 - |y_1|^2 = \hat{t}$. For the hypersurface $P' = 0$ we have

$$\{x_1 = 0\} \in \mathbb{P}_{1122} \simeq \mathbb{P}_{112} \;\text{ and}$$
$$\{p^2(x_1, x_2, x_3x_4^2, x_3x_5^2, x_3x_4x_5) = 0\} \in \mathbb{P}_{1122}\,.$$

In the first component of the fiber $Z|_{y_0y_1=0}$, $y_1 = 0$, we have the equation $x_1p^2(0, x_2, x_3x_4^2, ..) + y_0q^3(0, x_2, x_3x_4^2, ..) = 0$. This can unambiguously be solved for $\frac{x_1}{y_0}$ for any $(x_2 : x_3 : x_4 : x_5) \in \mathbb{P}_{112}$ away from $p^2(x_2, x_3x_4^2, ..) = q^3(x_2, x_3x_4^2, ..) = 0$. Once $\frac{x_1}{y_0}$ is fixed, x_1 and y_0 are determined by the Kähler parameter $\hat{t}$. However, at $p^2(x_2, x_3x_4^2, ..) = q^3(x_2, x_3x_4^2, ..) = 0$, the ratio $\frac{x_1}{y_0}$ is free and $(x_1 : y_0)$ parameterize a $\mathbb{P}^1$. So the first component is a $\mathbb{P}_{112}$, with the locus $p^2(x_2, x_3x_4^2, ..) = q^3(x_2, x_3x_4^2, ..) = 0$ blown up by a $\mathbb{P}^1$. The size of this $\mathbb{P}^1$ is the Kähler modulus $\hat{t}$. In the second component of the fiber, $y_0 = 0$, we have the equation $x_1p^2(x_1y_1, x_2, x_3x_4^2, x_3x_5^2, x_3x_4x_5) = 0$. As $x_1 \neq 0$ in this component we have $\{p^2(x_1y_1, x_2, x_3x_4^2, x_3x_5^2, x_3x_4x_5) = 0\} \in \mathbb{P}_{1122}$ as for the second component of the hypersurface $P' = 0$. The coordinates on $\mathbb{P}_{1122}$ in this case are $(y_1 : x_2 : x_3 : x_4 : x_5)$, so x_1 is exchanged for y_1. Away from the singular point we have an isomorphism between the K3 fiber and the hypersurface P' by the rescaling given above.[4]

[4]The singular fiber is a union of two Fano varieties. $Y_1 = \{p^2(x_2, x_3x_4^2, x_3x_5^2, x_3x_4x_5) = 0\} \in \mathbb{P}_{1122}$ and Y_2 is a blow-up of $\mathbb{P}_{112}$. They intersect over a Torus $D = \{p^2(x_2, x_3x_4^2, x_3x_5^2, x_3x_4x_5) = 0\} \in \mathbb{P}_{112}$, $K_D = 0$ so $D \in |-K_{Y_i}|$. The singular fiber is a normal crossing of the type described in [118], while the whole non-compact 3-fold defined by (4.3) is its smoothing. This is a generic property, one can see the toric constructions introduced in [32, 2] as a prescription how to cut a CY hypersurface into a normal crossing of Fano varieties.

The difference between the fibration (4.3) and the fibration of P' over the z-plane is the additional Kähler modulus $\hat{t}$. For $\hat{t} = 0$ the additional $\mathbb{P}^1$ shrinks and the two geometries agree, $y_0 y_1 = 0$ implies $y_0 = 0$ in this case. Especially the first component of the singular fiber $Z|_{y_0 y_1 = 0}$ is $x_1 = 0$ and the coordinates for the second one are $(x_1 : x_2 : x_3 : x_4)$ in both cases.

The complex structure of the fibration is singular at $y_0 y_1 = 0/z = 0$ and has a monodromy if we move around this point. The dual Kähler monodromy of Z^* is a shift in the B-field. This signals the presence of an NS5 brane on a divisor dual to the class of the corresponding B-field. In the next chapter we show that this is indeed the divisor (4.1) whose relative periods obey a GKZ system with charges (4.3). The modulus of this divisor is mirror to the additional Kähler modulus $\hat{t}$.

We did not discuss possible $\mathcal{O}(y_0 y_1)$ terms in the eq. (4.4). Such terms signal the additional freedom in the variation of the complex structure of the fiber over the base. Depending on the choice of these terms, the dual geometry is the trivial fibration $Z^* \times \mathbb{C}$ or an honest fibration with a varying Kähler structure.

The same construction is possible for any realization of a K3 surface or for 3 or 4 dimensional CY hypersurfaces. Above we started with an elliptic K3 to make contact with the torus. But note that locally, at the vanishing locus of the S^1, the singular fiber always looks like the product of the degeneration locus times a cone. Mirror Symmetry in the SYZ picture always involves one T-duality in the transverse geometry, so applying Mirror Symmetry fiberwise should give rise to a dual geometry involving NS5 branes. In the following we use toric methods to show that the complex structure monodromy around the central fiber always maps to the monodromy in the B-field caused by an NS5 brane.

4.5 Divisors and Monomials

First we fix the notation and repeat some facts about reflexive polytopes and associated CY hypersurfaces that we will need in the following. For more information see [12, 5]. $\nu_i \in \Delta^*$ are integral points of the lattice polytope Δ^* of the CY Z^*, $\mu_j \in \Delta$ are integral points in the dual lattice polytope Δ of Z. ν_0 and μ_0 are the unique interior points and $\langle \nu_i, \mu_j \rangle = \langle \mu_j, \nu_i \rangle \in \mathbb{Z}$ is the natural pairing. We take the whole polytope to lie in an affine plane of distance 1 to the origin, such that $\langle \nu_0, \mu_j \rangle = 1$ for all μ_j and $\langle \mu_0, \nu_i \rangle = 1$ for all ν_i.

Taking the vectors $\nu_i - \nu_0$ as generators of one dimensional cones, we can construct the fan of the ambient space of Z from Δ^* and likewise the fan of the ambient space of Z^* from Δ. One dimensional cones correspond to divisors $x_i = 0$ of the ambient space and by restriction onto the hypersurface to toric divisors of Z. So there is a correspondence $\nu_i \leftrightarrow x_i = 0$ and $\mu_j \leftrightarrow \tilde{x}_j = 0$, $i\,, j \neq 0$, between integral points and divisors and we choose to label the coordinates x and $\tilde{x}$ with the same indices as ν and μ.

Moreover all integral points μ_j of the polytope Δ correspond to a monomial x^{μ_j} in the hypersurface equation $P = 0$ of Z and likewise ν_i to monomials $\tilde{x}^{\nu_i}$ in $\tilde{P} = 0$. Here we use the notation $x^{\mu_j} := \prod_i x_i^{\langle \mu_j, \nu_i \rangle}$ and $\tilde{x}^{\nu_i} := \prod_j \tilde{x}_j^{\langle \nu_i, \mu_j \rangle}$.

The integral points μ_i, $i \neq 0$ correspond thus both to a monomial in the defining equation $P = 0$ of Z and to a toric divisor of Z^*. Mirror symmetry exchanges this data. Close to the large volume point in the Kähler moduli space of Z^* and to the point of maximal unipotent monodromy in the complex structure moduli space of Z this identification gives rise to the "monomial divisor mirror map" [119]. A change of the Kähler volume of a two cycle dual to a given toric divisor is mapped to a change of the prefactor of the corresponding monomial in the hypersurface equation $P = 0$ and thus to a change of complex structure. In particular, at the point of maximal unipotent monodromy this prefactor vanishes and moving around this point we get a monodromy $\tau \to \tau + 1$ in the complex structure moduli space of Z and $t \to t + 1$ in the Kähler moduli space of Z^*.

Kähler classes of the ambient space[5] of Z are in one to one correspondence with a certain base for the set of linear relations between points of the polytope Δ^*, $\sum_i l_i^m \nu_i = 0$. For this base, the entries of the charge vectors l^m are the intersection numbers between a curve dual to the corresponding Kähler class and the divisors $x_i = 0$. Divisors with the same entries for all l^m and thus the same intersection numbers are equivalent and dual to the same Kähler class. The relation $\sum_i l_i^m \nu_i = 0$ translates to the condition that all monomials x^{μ_j} of the hypersurface equation $P = 0$ are in the divisor class of the anticanonical bundle.

With the construction of chapter 4.2 we can choose any divisor $\mathcal{D}$ in Z^* given by $\tilde{Q} = (\tilde{x}^{\nu_1} + \tilde{x}^{\nu_2} + .. + \tilde{x}^{\nu_n}) / \operatorname{gcf}$, where gcf is the greatest common factor of the appearing monomials x^{ν_i}. In the following we explain how to identify the divisor class in terms of one dimensional cones generated by $\mu_a - \mu_0$ and thus in terms of points μ_a of the dual polytope. Next we study the proposed mirror geometry and determine which monomials of P depend on the base coordinate $y_1...y_n$ of the CY fibration. We will see that exactly the monomials x^{μ_a} get suppressed in the central fiber over $y_1...y_n = 0$, where μ_a are the points that correspond to the divisor class of $\mathcal{D}$. The monomial divisor mirror map [119] then assures a monodromy of the complex structure in agreement with the proposed picture of a geometrization of NS5 branes by mirror symmetry.

In the simplest case we have only two monomials that determine the divisor, $\tilde{Q} = (b_1 \tilde{x}^{\nu_a} + b_2 \tilde{x}^{\nu_b}) / \operatorname{gcf}$. The divisor class can be read of either the nominator or the denominator of $\frac{\tilde{x}^{\nu_a}}{\tilde{x}^{\nu_b}} = \tilde{x}^{\nu_a - \nu_b}$. Choosing the nominator we find the divisor $\tilde{x}_1^{k_1} \tilde{x}_2^{k_2} .. =$

[5]Most of these restrict to Kähler classes of the CY.

0 with the multiplicities

$$\begin{array}{ll} k_j = \langle \nu_a - \nu_b, \mu_j \rangle & \text{if } \langle \nu_a - \nu_b, \mu_j \rangle > 0\,, \\ k_j = 0 & \text{if } \langle \nu_a - \nu_b, \mu_j \rangle \leq 0\,. \end{array} \tag{4.5}$$

The proposed mirror geometry is a CY fibration over $\mathbb{C}$. Enlarging the polytope Δ^* to a polytope $\hat{\Delta}^*$ with points $(\Delta^*, 0)$ and $(\nu_a, 1)$, $(\nu_b, 1)$ we find a new relation $\hat{l}$ between the points of $\hat{\Delta}^*$ and thus a condition on the possible monomials in the coordinates x_i and y_1, y_2.[6]

$$\begin{array}{c|ccccc} & (\nu_a,0) & (\nu_b,0) & \ldots & (\nu_a,1) & (\nu_b,1) \\ & x_a & x_b & \ldots & y_a & y_b \\ \hline \hat{l} & \text{-1} & 1 & \ldots & 1 & \text{-1} \end{array}\,, \tag{4.6}$$

In addition we have all the relations l^m between the points ν_i. Imposing them gives the set of x^{μ_j} as possible monomials of the hypersurface equation $P = \sum_j a_j x^{\mu_j} = 0$ of a general fiber. The additional condition forces us to multiply some of these monomials with y_a or y_b. We get a hypersurface equation $P = \sum_j a_j(y_a, y_b)\ x^{\mu_j}$. We are interested in the behavior of the coefficients $a_j(y_a, y_b) = y_a^{k_j} y_b^{l_j} (a_j^0 + \mathcal{O}(y_a y_b))$ close to $y_a y_b = 0$ so we neglect the subleading contributions $\mathcal{O}(y_a y_b)$. The monomials have to be neutral under the charges of the vector $\hat{l}$. The power of x_a in x^{μ_j} is $\langle \nu_a, \mu_j \rangle$ and similarly for x_b, so we get monomials $x^{\mu_j} y_a^{k_j} y_b^{l_j}$, where

$$\begin{array}{lll} k_j = \langle \nu_a - \nu_b, \mu_j \rangle, & l_j = 0\,, & \text{if } \langle \nu_a - \nu_b, \mu_j \rangle > 0\,, \\ k_j = 0\,, & l_j = 0\,, & \text{if } \langle \nu_a - \nu_b, \mu_j \rangle = 0\,, \\ k_j = 0\,, & l_j = -\langle \nu_a - \nu_b, \mu_j \rangle\,, & \text{if } \langle \nu_a - \nu_b, \mu_j \rangle < 0\,. \end{array} \tag{4.7}$$

By a rescaling of the x_i that leaves the holomorphic $(n, 0)$ form and thus the period integrals invariant it is always possible to combine y_a and y_b to the product $y_a y_b$. In the monomials this replaces e.g. $y_a \to y_a y_b$ and $y_b \to 1$ and we are left with $P = \sum_j (y_a y_b)^{k_j} x^{\mu_j} (a_j^0 + \mathcal{O}(y_a y_b))$. Comparing conditions (4.5) and (4.7) we see that monomials corresponding to a point μ_j are suppressed with a power k_j, if the divisor $\mathcal{D}$ contains the divisor $\tilde{x}_j = 0$ k_j times. The monomial divisor mirror map thus assures fitting monodromies.

By a different rescaling of the x_i we could as well replace $y_a \to 1$ and $y_b \to y_a y_b$. This would suppress monomials that started out with positive powers of y_b by $(y_a y_b)^{l_j}$. The corresponding points μ_j correspond to the divisors in the denominator of $\tilde{x}^{\nu_a - \nu_b}$. This reflects the equivalence of the divisor classes.

This can be generalized to divisors $\tilde{Q} = (b_1 \tilde{x}^{\nu_a} + b_2 \tilde{x}^{\nu_b} \ldots + b_n \tilde{x}^{\nu_*}) /$ gcf with more than two monomials. For each additional monomial we get a new independent relation

[6] If one of the lattice points $\nu_{a/b}$ is the interior point ν_0, the coordinate corresponding to $(\nu_{a/b}, 0)$ is P and not $x_{a/b}$. This does not change the following discussion however.

$\hat{l}_m$. Imposing one relation we multiply all monomials x^{μ_i} with new coordinates y_* that correspond to divisors $\tilde{x}_j = 0$ that differ between two of the monomials of $\tilde{Q}$. After imposing all relations, all monomials x^{μ_i} are multiplied by some power of some new coordinates y_* up to monomials that correspond to divisors in the gcf of $\tilde{Q}$. A rescaling of x_i again collects all y_* into the coordinate on the base of the fibration. In the following we give some explicit examples of divisors with several moduli.

4.6 Further examples

4.6.1 Torus, charge 3

As an example with more than two monomials in the divisor equation we again consider a torus Z^*, defined by $\tilde{P} = a_1\tilde{x}_1^3 + a_2\tilde{x}_2^3 + a_3\tilde{x}_3^3 + a_0\tilde{x}_1\tilde{x}_2\tilde{x}_3 = 0$ in $\mathbb{P}^2/\mathbb{Z}_3$. This time we add an NS5 brane on the divisor $\tilde{Q} = b_1\tilde{x}_1^3+b_2\tilde{x}_2^3+b_3\tilde{x}_3^3+b_0\tilde{x}_1\tilde{x}_2\tilde{x}_3 = 0$, again localized at the origin of $\mathbb{C}$ and wrapping $\mathbb{R}^6$.

The dual non compact 2-fold is given by the GLSM

	P	x_1	x_2	x_3	y_0	y_1	y_2	y_3
l	-3	1	1	1	0	0	0	0
$\hat{l}^1$	-1	1	0	0	1	-1	0	0
$\hat{l}^2$	-1	0	1	0	1	0	-1	0
$\hat{l}^3$	-1	0	0	1	1	0	0	-1

, (4.8)

with hypersurface

$$P = x_1x_2x_3 + y_0\, q^3(x_1y_1,\, x_2y_2,\, x_3y_3) + \mathcal{O}(y_0y_1y_2y_3)\,. \tag{4.9}$$

This is a fibration of the dual torus Z over $\mathbb{C}$, the coordinate on $\mathbb{C}$ is $y_0y_1y_2y_3$. The period integrals can be brought into the standard form by a rescaling $x_1 \to x_1\frac{(y_2y_3)^{1/3}}{y_1^{2/3}}$, $x_2 \to x_2\frac{(y_1y_3)^{1/3}}{y_2^{2/3}}$, $x_3 \to x_3\frac{(y_1y_2)^{1/3}}{y_3^{2/3}}$,

$$\int \frac{\Xi}{x_1x_2x_3 + y_0\, q^3(x_1y_1,\, x_2y_2,\, x_3y_3)} = \int \frac{\Xi}{x_1x_2x_3 + y_0y_1y_2y_3\, q^3(x_1, x_2, x_3)}\,.$$

The complex structure of the fiber behaves like $\tau = 3\ln(y_0y_1y_2y_3) + \mathcal{O}(y_0y_1y_2y_3)$ near $y_0y_1y_2y_3 = 0$. We get the factor 3 as all monomials x_i^3 are suppressed by $y_0y_1y_2y_3$. Alternatively, after a rescaling a single monomial e.g. x_3^3 is suppressed by $(y_0y_1y_2y_3)^3$. The monomials x_i^3 are related to the divisors $\tilde{x}_i = 0$ by the monomial-divisor mirror map. The class of a point is dual to the Kähler class, so we would indeed expect a monodromy $\rho \to \rho + 3$ for an NS5 brane wrapped on the divisor $\tilde{Q} = b_1\tilde{x}_1^3 + b_2\tilde{x}_2^3 + b_3\tilde{x}_3^3 + b_0\tilde{x}_1\tilde{x}_2\tilde{x}_3 = 0$.

4.6.2 Torus, charge 1

We already described NS5 branes on divisors of class $2[pt]$ and $3[pt]$ in a torus, but not the elementary situation of a single brane localized on one point. The toric realization of the dual geometry is a little bit more complicated but straightforward after the general discussion of sect. 4.5. This time we realize the torus T^* as a degree 3 hypersurface $\tilde{P} = 0$ in $\mathbb{P}^3$. There are ten possible monomials $\tilde{x}^{\nu_i}$ out of which we can choose two to define $\tilde{Q}$. Usually one restricts the number of monomials by $PGL(3,\mathbb{C})$ coordinate changes and only keeps $\tilde{x}_1^3$, $\tilde{x}_2^3$, $\tilde{x}_3^3$ and $\tilde{x}_1\tilde{x}_2\tilde{x}_3$. In the polytope the other monomials correspond to interior points of a codimension one face. On the mirror side these points correspond to divisors in the ambient space that are not hit by the generic CY hypersurface. However, if we want to express $\tilde{Q} = \tilde{x}_1 + \tilde{x}_2$ in terms of monomials $\tilde{x}^{\nu_i}$ we have to use at least one of these additional points, e.g. $\tilde{Q} = (b_1\tilde{x}_1^3 + b_2\tilde{x}_1^2\tilde{x}_2)/x_1^2$. The GLSM for of the dual geometry is given by

	P	x_1	x_2	x_3	x_4	y_1	y_2
l^1	-3	1	1	1	0	0	0
l^2	0	2	1	0	-3	0	0
				$\vdots$			
$\hat{l}$	0	-1	0	0	1	1	-1

,

where x_4 is the coordinate for one of the blow-ups of the singularities of $\mathbb{P}^2/\mathbb{Z}^3$ and for ease of notation we omitted further blow-up coordinates and relations for them. These relations however have to be included to determine the allowed monomials for P.[7] We find

$$P = x_1^3 x_4^2 y_1 + x_2^3 x_4 y_2 + x_3^3 + x_1 x_2 x_3 x_4 + \mathcal{O}(y_1 y_2)\,.$$

After a rescaling of x_i either the monomial with x_1^3 or x_2^3 is suppressed close to $y_1 y_2 = 0$ and we get the expected monodromy $\tau \to \tau + 1$.

4.6.3 Quintic

For CY 3-folds, geometries of the type discussed in chapter 4.2 were already used in [32, 2] to calculate superpotentials. The simplest example is the mirror quintic $\tilde{P} = \tilde{x}_1^5 + \tilde{x}_2^5 + \tilde{x}_3^5 + \tilde{x}_4^5 + \tilde{x}_5^5 + \tilde{x}_1\tilde{x}_2\tilde{x}_3\tilde{x}_4\tilde{x}_5$ with NS5 brane on $\tilde{Q} = \tilde{x}_1^4 + \tilde{x}_2\tilde{x}_3\tilde{x}_4\tilde{x}_5$. Here all additional coordinates needed to describe the blow-ups are scaled to one for ease of notation. The intersection of $\tilde{P}$ and $\tilde{Q}$ is a covering of a K3 surface, for more details see [2, 3]. The dual quintic fibration is

[7] In constructing the dual polytope one does so automatically, we focus here on the relations as we have to include the additional constraint by $\hat{l}$.

	P	x_1	x_2	x_3	x_4	x_5	y_0	y_1
l	-5	1	1	1	1	1	0	0
$\hat{l}$	-1	0	0	0	0	0	1	-1

,

with hypersurface

$$P = x_1 p^4(x_1y_1,\, x_2,\, x_3,\, x_4,\, x_5) + y_0\, q^5(x_1y_1,\, x_2,\, x_3,\, x_4,\, x_5) + \mathcal{O}(y_0y_1)\,.$$

After a rescaling of x_i all monomials without x_1 are suppressed by y_0y_1. As shown in sect. 4.5 this are the monomials that correspond to the divisor class of $\mathcal{D}$.

The singular locus in the central fiber is $p^4(x_2, x_3, x_4, x_5) = 0$ and $x_1 = \sqrt{\hat{t}}$ in $\mathbb{P}^4$, where $\hat{t}$ is again the Kähler parameter associated with $\hat{l}$. This is a K3 surface and it is the mirror of the K3 surface whose covering is wrapped by the NS5 brane in the mirror quintic.

4.7 Parallel branes

A set of N parallel NS5 branes turns into the blow up of an A_{N-1} singularity [111] upon a transverse T-duality. The different blow up modes are dual to the distances between the branes. This should also happen in the present construction. To check it, we study the Gauss-Manin system for relative periods of a set of parallel divisors in Z^*. After explaining the general structure we do this in detail for the conifold geometry and the mirror quintic. For the quintic we again check the complex structure monodromies in the dual fibration.

In order to describe a system of N parallel NS5 branes, we consider an equation $\tilde{Q} = 0$ of degree N. The vanishing locus $\mathcal{D}$ of this equation in Z^* gives rise to a union of N divisors for general values of the parameters. A divisor

$$\tilde{Q} = b_0(\tilde{x}^{\nu_a})^N + b_1(\tilde{x}^{\nu_a})^{N-1}\tilde{x}^{\nu_b} + ... + b_N(\tilde{x}^{\nu_b})^N\,, \tag{4.10}$$

has N open moduli $\hat{z}_i$. As usual, the differential operators for the relative cohomology group are encoded in an enlarged GKZ system. New relations after including the monomials of $\tilde{Q}$ can again be summarized by an enlarged polytope. The $N+1$ additional vertices are of the form $(n(\nu_a - \nu_b), 1)$, where n runs from 0 to N. We notice that these vertices span the Dynkin diagram of A_{N-1} [8]. In the variation of Hodge structure for this system, we find the subsystem for the closed string deformation of the CY, but the open string sector is more complicated. We have a union

[8] The open moduli parameterizing the distance of the additional branes to the first one can be understood as the blow up modes of an A_{N-1} singularity in the non compact 4-fold on the A side.

of N divisors whose embedding in the CY is parameterized by N coordinates $\hat{z}_i$, their movement however is not decoupled in these coordinates. On the other hand, volumes of chains ending on the hypersurfaces only depend on their positions and not on the coordinates used, so we expect a decoupling of the different hypersurfaces in the physical coordinates. This is exactly what happens: We can choose a basis of relative cycles such that the period vector expressed in physical coordinates takes the form

$$\underline{\Omega} : (1, t, \hat{t}_1, \hat{t}_2, \cdots, F_t(t), W^{(1)}(t, \hat{t}_1), W^{(2)}(t, \hat{t}_2), \cdots, -F_0, T^{(1)}(t, \hat{t}_1), T^{(2)}(t, \hat{t}_2), \cdots), \tag{4.11}$$

where the functional forms of $\hat{t}_i$, $W^{(i)}(t, \hat{t}_i)$ and $T^{(i)}(t, \hat{t}_i)$ are the same for different values of i and agree with the same functional form of the single brane case [2]. Because of the decoupling that occurs in the period vector, the integrability requirement for this system reduces to N copies of the flatness condition for the case of one open-string deformation, and therefore it is trivially fulfilled.

4.7.1 Conifold

First we consider parallel branes on the mirror of the conifold. In many ways this is the easiest example and it is particularly suited to study the relation between the coupled coordinates $z_i = b^{l_i}$ and the position of the different parallel components. The mirror of the conifold is given by the hypersurface (see [95] for more details)

$$y\, z = a_0\, \mathrm{e}^u + a_1\, \mathrm{e}^v + a_2\, \mathrm{e}^{u+v} + a_3 \;,$$

in $\mathbb{C}^4$. The intersection of the above hypersurface with

$$\tilde{Q} = b_0\, \mathrm{e}^{Nu} + b_1\, \mathrm{e}^{(N-1)u+v} + ... + b_N\, \mathrm{e}^{Nv} = 0 \;, \tag{4.12}$$

defines the family of divisors $\mathcal{D}$. Differential equations for the relative periods of $H^3(Z^*, \mathcal{D})$ are given by the hypergeometric system with the following charge vectors

	a_0	a_1	a_2	a_3	b_0	b_1	b_2	b_3	...	b_{N-2}	b_{N-1}	b_N
l^1	1	1	-1	-1	0	0	0	0	...	0	0	0
$\hat{l}^1$	0	0	0	0	1	-2	1	0	...	0	0	0
$\hat{l}^2$	0	0	0	0	0	1	-2	1	...	0	0	0
								.				.
									.			
										.		
$\hat{l}^N$	0	0	0	0	0	0	0	0	...	1	-2	1
$\hat{l}^{N+1}$	-1	1	0	0	0	0	0	0	...	0	1	-1

In the non-compact fourfold defined by a GLSM with these charges there are no compact 4 cycles, so apart from the regular (constant) solution there are only solutions with a simple logarithmic divergence for $z_i \to 0$. These solutions correspond to the Kähler volumes of the various $\mathbb{P}^1$ in the non-compact 4-fold, $t_1 = \ln z_1$ is the size of the $\mathbb{P}^1$ in the generic conifold fibre. $t_i = \ln z_i + ..$, $i = 2, ..N$ are the blow up modes of an A_{N-1} singularity, that is for $t_i \to 0$, $i, 2, ..N$ the geometry develops an A_{N-1} singularity in the central fibre. Finally $t_{N+1} = \ln z_{N+1} + ..$ determines the location of this A_{N-1} singularity within the central fibre. For $t_i \neq 0$ the non-compact fourfold is smooth and the central fibre is a stable degeneration of a conifold into $N+1$ components. These components intersect over parallel planes with a shrinking S^1 at $s_1 = t_{N+1}$, $s_2 = t_{N+1} + t_N$,..$s_N = \sum_{n=0}^{N-2} t_{N+1-n}$. The blow up modes of the A_{N-1} singularity parameterize the distance between these planes and for $t_i \to 0$, $2 \leq i \leq N$ two planes coincide and an A_1 singularity is formed.
These loci of vanishing S^1 are mirror to the location of NS5 branes, this can be used to get nice expressions for roots of degree N equation. If we rewrite (4.12) as $Q = (\xi_1 \mathrm{e}^u + \mathrm{e}^v)(\xi_2 \mathrm{e}^u + \mathrm{e}^v)...(\xi_N \mathrm{e}^u + \mathrm{e}^v)$ we expect a map between s_i and the roots ξ_i. For a single brane we have a linear equation $Q = \hat{z}\mathrm{e}^u + \mathrm{e}^v$ and the modulus $\hat{z} = \frac{b_0 a_1}{b_1 a_0}$ coincides with the root ξ. The corresponding map between s_i and the roots ξ_i is thus the inverse mirror map $\xi = \hat{z} = \exp[s]$ for the single brane case. This gives the following expression for the roots of a degree N equation (4.12)[9]:

$$\xi_i(z_2, ..z_N) = \exp[s_i(z_2, ..z_N)] \tag{4.13}$$

It has been known for a long time, that roots of a degree N equation can be expressed in term of hypergeometric functions in N variables [120]. Considering the action of the symmetric group S_N on both the roots and the periods of a resolved A_{N-1} singularity such a connection is natural. From this point of view the inverse mirror map for the single brane case exp[..] is necessary, as monodromies of the periods do not only exchange the different s_i but also involve the fundamental (constant) solution, for example when moving $s_1 = \ln z_{N+1} + ..$ around $z_{N+1} = 0$. By the exponential map periods that differ by such an integer shift of the fundamental period get mapped to the same point and the monodromies only generate the symmetric group S_N.
To illustrate this structure we give the solutions for two parallel branes. A quadratic hypersurface equation

$$\tilde{Q} = b_0\,\mathrm{e}^{2u} + b_1\,\mathrm{e}^{u+v} + b_2\,\mathrm{e}^{2v} = 0\ ,$$

reads after a rescaling

$$\tilde{Q} = z_3^2 z_2\,\mathrm{e}^{2u} + z_3\,\mathrm{e}^{u+v} + \,\mathrm{e}^{2v} = 0\ , \tag{4.14}$$

[9]In this simple example there is no dependence on the closed string modulus z_1 and s_i are the periods of an ALE space.

with the coordinates $z_2 = \frac{b_0 b_2}{b_1^2}$ and $z_3 = \frac{b_1 a_1}{b_2 a_0}$. Solving the differential operators we find the solutions

$$s_1 = \ln\left[\frac{z_3}{2}(1+\sqrt{1-4z_2}\right] = \ln z_3 + \mathcal{O}(z_2, z_3)\,,$$
$$s_2 = \ln\left[\frac{z_3}{2}(1-\sqrt{1-4z_2}\right] = \ln(z_3 z_2) + \mathcal{O}(z_2, z_3)\,.$$

We see that $\xi_i = \exp[s_i]$ are indeed the roots of (4.14), $\mathrm{e}^v = -\xi_i \mathrm{e}^u$. For $z_2 = \frac{1}{4}$ the blow up mode $t_2 = \ln[4z_2/(1+\sqrt{1-4z_2})^2] = \ln z_2 + \mathcal{O}(z_2)$ shrinks, on the A-side an A_1 singularity is formed, while on the B-side the two components of the hypersurface (4.14) lie on top of each other.

4.7.2 Quintic

For relative periods with at most single logarithmic divergence we find exactly the same structure for parallel branes on the quintic. These periods are mirror to the volume of two cycles of the non compact 4-fold that determine the positions of planes with a vanishing S^1 in the central fibre. Using the mirror map for the position of a single brane these positions can be translated back to positions of components of the hypersurface $\mathcal{D}$.

$$\xi_i(z_1, z_2, ..z_N) = \tau[s_i(z_1, z_2, ..z_N)]\,, \qquad (4.15)$$

where s_i are solutions with simple logarithmic divergence as above and τ is the inverse mirror map for the single brane[10], $\tau = \hat{t}^{-1}$. In this case the involved hypergeometric functions s_i depend on a closed string modulus, but this dependence drops out by the use of the inverse mirror map τ.
In addition to these relative periods we also have periods with double and triple logarithmic divergence near $z_i \to 0$. These solutions contain information about the superpotential if we turn on flux on the various parallel branes[11]. We will first discuss the case for two parallel branes in some detail and afterwards comment on the general structure for $N > 2$.

The mirror of the quintic is described by the $\mathbb{Z}_5^3$ orbifold of the zero locus of the equation

$$\tilde{P} = a_1\,\tilde{x}_1^5 + a_2\,\tilde{x}_2^5 + a_3\,\tilde{x}_3^5 + a_4\,\tilde{x}_4^5 + a_5\,\tilde{x}_5^5 + a_0\,\tilde{x}_1\tilde{x}_2\tilde{x}_3\tilde{x}_4\tilde{x}_5 = 0\ .$$

To realize the two parallel branes, we intersect the above defining equation with a quadratic equation that gives rise to two hypersurfaces

$$\tilde{Q} = b_0(x_2x_3x_4x_5)^2 + b_1\,x_1^4(x_2x_3x_4x_5) + b_2\,x_1^8\ .$$

[10]If one does not normalize the s_i with the regular fundamental solution one should use the unnormalized "mirror map".

[11]Here we think about the S-dual situation, where the NS5 brane of II B is turned into a D5 brane.

The combinations of a_i and b_i that unambiguously give rise to the good coordinates near the point of maximal unipotent monodromy can be read from the following charge vectors

$$
\begin{array}{cc|cccccccc}
 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & b_0 & b_1 & b_2 \\
 & P & x_1 & x_2 & x_3 & x_4 & x_5 & y_0 & y_1 & y_2 \\
\hline
l & -4 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
\hat{l}^1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
\hat{l}^2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{array} \,. \tag{4.16}
$$

These charge vectors also define the GKZ system of differential operators annihilating the periods. After a factorization of the differential operators analogous to the closed string case, we find ten linearly independent solutions. One regular solution which corresponds to the fundamental period and three solutions for each of simple, quadratic and cubic logarithmic divergences. For $z_2 = \frac{b_0 b_2}{b_1^2} = 0$ the quadratic equation factors trivially, one of the branes gets pushed to either $x_1^4 = 0$ or $x_2x_3x_4x_5 = 0$ while the position of the other depends only on the single modulus $\xi = z_1z_2$ or $\xi = z_1$ as in the single brane case. In the two limits $b_0 \to 0$ and $b_2 \to 0$ we expect thus to recover the solutions for a single brane. First, for z_1z_2 and z_3 finite with $z_2 \to 0$, the solutions $\hat{t}_1 = \ln z_1z_2 + ..$, $W^{(1)} = -2(\ln z_1z_2)^2 + ..$ and $T^{(1)} = -\frac{2}{3}(\ln z_1z_2)^3 + ..$ reduce to the solutions of the one parameter case. Second, for z_1 and z_2z_3 finite with $z_2 \to 0$, we find the same solutions from $\hat{t}_2 = \ln z_1 + ..$, $W^{(2)} = -2(\ln z_1)^2 + ..$ and $T^{(2)} = -\frac{2}{3}(\ln z_1)^3 + ...$ In addition, we always have the four closed string solutions that only depend on the combination $z_1z_2z_3$. The normalization of the superpotential can be fixed by comparison with [2] in these limits. Expressed in terms of the physical coordinates these superpotentials agree with the superpotential for a single brane located at $\hat{t}_1$ or $\hat{t}_2$ respectively. As the functional dependence of the superpotential on the brane position $W(\hat{t})$ is always the same we can write the combined superpotential for fluxes turned on on both branes as

$$
W = \mathrm{Tr}\, W_{single}(t)\,, \tag{4.17}
$$

where t is promoted to the matrix $\mathrm{diag}(\hat{t}_1, \hat{t}_2)$. The period vector can be written in the form (4.11) and the K3-relation $\partial_{\hat{t}_a} T^{(a)} = \frac{1}{8}(\partial_{\hat{t}_a} W^{(a)})^2$ guarantees the integrability of the Gauss-Manin connection.

The most general hypersurface equation for the GLSM (4.16) is

$$
P = x_1 p^4(x_1y_1y_2^2,\, x_2,\, x_3,\, x_4,\, x_5) + y_0^2 y_1\, q^5(x_1y_1y_2^2,\, x_2,\, x_3,\, x_4,\, x_5) + \mathcal{O}(y_0y_1)\,.
$$

This is a Quintic fibration over $\mathbb{C}$, where $\mathbb{C}$ is parameterized by $y_0y_1y_2$. In the singular fiber over the origin an S^1 shrinks at both $|x_1|^2 = \hat{t}_1$ and $|x_1|^2 = \hat{t}_2$, these loci are mirror to the locations of the two NS5 branes. After a rescaling all monomials without x_1 are suppressed by $(y_0y_1y_2)^2$. Comparing with sect. 4.6.3 we see that this is twice the monodromy of the single brane case.

The above structure holds in general for N parallel branes. We intersect the mirror of the quintic with an equation of degree N in two monomials and derive differential equations for this system. These operators can be summarized by relations between vertices given in the appendix. For each additional brane there is an additional vertex. Each additional vertex gives rise to a new independent relation and thus we get a new modulus, controlling the distance between this brane and another one. The additional vertices span the Dynkin diagram of A_{N-1} and within the dual geometry on the A-side we again find the blow up of an A_{N-1} singularity. The positions of the loci with degenerating S^1 in this geometry are given by mirror maps with the expected leading behavior. Combinations of double and triple logarithmic solutions again reduce to the functions W and T of a single brane when expressed in these coordinates. The complex structure monodromy around the base of the dual geometry X is N times the monodromy for a single brane.

4.8 Summary and outlook

We presented evidence that CY fibrations of the type discussed in chapter 4.2 can be interpreted as mirrors of CY hypersurfaces with NS5 brane on a divisor. These CY fibrations are constructed using the data of the GKZ system of the joint deformation space of divisor and CY. By construction the quantum corrected volume of cycles in these geometries thus correspond to the relative periods of the divisor in the CY. Moreover we showed in this chapter that monodromies on both sides of the proposed duality always match. This duality is a alternative to the duality chain of [30] and explains the non compact fourfolds of the last chapters in a novel way. The techniques of the last chapters can also be used to describe a set parallel branes and the appearing A_N singularities in the dual geometry point towards parallel NS5 branes.

A generalization to complete intersection CY manifolds should be straightforward and the idea should also carry over to other CY that were studied in open string mirror symmetry [121]. The construction allows to study mirror symmetry for a pair of a CY and divisor without specifying an A-type brane on the mirror. Nevertheless the geometry should encode information of an A-type brane as discussed in [32, 2]. The role of the A-type brane is played by the degeneration locus in the singular fiber. It would be interesting to investigate such a correspondence, e.g. by a lift to M-theory.

We would like to note some observations. We saw in the Quintic example 4.6.3 that the degeneration locus is the mirror of the K3 surface that determines the subset of open periods. This is true also for all examples in [3], where these K3 "subsystems" in Z^* were used to calculate numbers of disks ending on Lagrangian submanifolds in Z. It might be rewarding to study this "mirror symmetry" between degeneration locus and divisor in the light of the Strominger Yau Zaslow conjecture. The Lagrangian torus fibration has always one leg in a normal direction to the divisor. If the remaining directions restrict to a Lagrangian torus fibre of the divisor, one would expect the degeneration locus to be the mirror. Note that the construction is possible for any non rigid divisor.

For $d-1$ dimensional divisors with more then one modulus the degeneration locus in the dual geometry falls apart into different components that only meet in complex codimension one. As $dim(H^{(d-1,0)}) > 1$ for such divisors this is what one would expect for the mirror geometry, there should be more then one class of points. Such a structure appeared e.g. in [122].

Appendix: Extended polytopes

In the following table we collect the (extended) points $\hat{\nu}_i$ spanning the extended polytopes for geometries used in this thesis. All these points lie in a hypersurface of distance one to the origin, the common normal coordinate is omitted. The points ν_i for the threefold Z are given by the subset of the $\hat{\nu}_i$ with vanishing last entry, $\hat{\nu}_i = (\nu_i, 0)$.

	$\hat{\Delta}^* \supset \Delta^*$	Δ
$\mathbb{P}_{1,1,1,1,1}[5]$	$\hat{\nu}_1 = (\ 1,\ 1,\ 1,\ 1;\ 0)$	$\mu_1 = (\ 1,\ 1,\ 1,\ 1)$
	$\hat{\nu}_2 = (-1,\ 0,\ 0,\ 0;\ 0)$	$\mu_2 = (-4,\ 1,\ 1,\ 1)$
	$\hat{\nu}_3 = (\ 0,-1,\ 0,\ 0;\ 0)$	$\mu_3 = (\ 1,-4,\ 1,\ 1)$
	$\hat{\nu}_4 = (\ 0,\ 0,-1,\ 0;\ 0)$	$\mu_4 = (\ 1,\ 1,-4,\ 1)$
	$\hat{\nu}_5 = (\ 0,\ 0,\ 0,-1;\ 0)$	$\mu_5 = (\ 1,\ 1,\ 1,-4)$
$\mathcal{D}$	$\hat{\nu}_6 = (\nu_0; 1),\ \hat{\nu}_7 = (\nu_1; 1)$	
$\mathcal{D}_N$	$\hat{\nu}_6 = (\nu_0; 1), \hat{\nu}_{6+n} = (n\,\nu_1; 1)$	$0 \leq n \leq N$
$\mathbb{P}_{1,2,2,3,4}[12]$	$\hat{\nu}_1 = (\ 2,\ 2,\ 3,\ 4;\ 0)$	$\mu_1 = (\ 1,\ 1,\ 1,\ 1)$
	$\hat{\nu}_2 = (-1,\ 0,\ 0,\ 0;\ 0)$	$\mu_2 = (-5,\ 1,\ 1,\ 1)$
	$\hat{\nu}_3 = (\ 0,-1,\ 0,\ 0;\ 0)$	$\mu_3 = (\ 1,-5,\ 1,\ 1)$
	$\hat{\nu}_4 = (\ 0,\ 0,-1,\ 0;\ 0)$	$\mu_4 = (\ 1,\ 1,-3,\ 1)$
	$\hat{\nu}_5 = (\ 0,\ 0,\ 0,-1;\ 0)$	$\mu_5 = (\ 1,\ 1,\ 1,-2)$
	$\hat{\nu}_6 = (\ 1,\ 1,\ 1,\ 2;\ 0)$	
$\mathcal{D}_1$	$\hat{\nu}_7 = (\nu_2; 1),\ \hat{\nu}_8 = (\nu_3; 1)$	
$\mathcal{D}_2$	$\hat{\nu}_7 = (\nu_4; 1),\ \hat{\nu}_8 = (\nu_6; 1)$	
$\mathbb{P}_{1,2,2,2,7}[14]$	$\hat{\nu}_1 = (\ 2,\ 2,\ 2,\ 7;\ 0)$	$\mu_1 = (\ 1,\ 1,\ 1,\ 1)$
	$\hat{\nu}_2 = (-1,\ 0,\ 0,\ 0;\ 0)$	$\mu_2 = (-6,\ 1,\ 1,\ 1)$
	$\hat{\nu}_3 = (\ 0,-1,\ 0,\ 0;\ 0)$	$\mu_3 = (\ 1,-6,\ 1,\ 1)$
	$\hat{\nu}_4 = (\ 0,\ 0,-1,\ 0;\ 0)$	$\mu_4 = (\ 1,\ 1,-6,\ 1)$
	$\hat{\nu}_5 = (\ 0,\ 0,\ 0,-1;\ 0)$	$\mu_5 = (\ 1,\ 1,\ 1,-1)$
	$\hat{\nu}_6 = (\ 1,\ 1,\ 1,\ 3;\ 0)$	
$\mathcal{D}_1$	$\hat{\nu}_7 = (\nu_3; 1),\ \hat{\nu}_8 = (\nu_4; 1)$	
$\mathcal{D}_2$	$\hat{\nu}_7 = (\nu_5; 1),\ \hat{\nu}_8 = (\nu_6; 1)$	

	$\hat{\Delta}^* \supset \ \Delta^*$	Δ
$\mathbb{P}_{1,2,3,3,9}[18]$	$\hat{\nu}_1 = (\ 2,\ 3,\ 3,\ 9;\ 0)$	$\mu_1 = (\ 1,\ 1,\ 1,\ 1)$
	$\hat{\nu}_2 = (-1,\ 0,\ 0,\ 0;\ 0)$	$\mu_2 = (-8,\ 1,\ 1,\ 1)$
	$\hat{\nu}_3 = (\ 0,-1,\ 0,\ 0;\ 0)$	$\mu_3 = (\ 1,-5,\ 1,\ 1)$
	$\hat{\nu}_4 = (\ 0,\ 0,-1,\ 0;\ 0)$	$\mu_4 = (\ 1,\ 1,-5,\ 1)$
	$\hat{\nu}_5 = (\ 0,\ 0,\ 0,-1;\ 0)$	$\mu_5 = (\ 1,\ 1,\ 1,-1)$
	$\hat{\nu}_6 = (\ 1,\ 2,\ 2,\ 6;\ 0)$	
	$\hat{\nu}_7 = (\ 0,\ 1,\ 1,\ 3;\ 0)$	
$\mathcal{D}$	$\hat{\nu}_8 = (\nu_3; 1),\ \hat{\nu}_9 = (\nu_4; 1)$	
$\mathbb{P}_{1,2,3,3,3}[12]$	$\hat{\nu}_1 = (\ 2,\ 3,\ 3,\ 3;\ 0)$	$\mu_1 = (\ 1,\ 1,\ 1,\ 1)$
	$\hat{\nu}_2 = (-1,\ 0,\ 0,\ 0;\ 0)$	$\mu_2 = (-5,\ 1,\ 1,\ 1)$
	$\hat{\nu}_3 = (\ 0,-1,\ 0,\ 0;\ 0)$	$\mu_3 = (\ 1,-3,\ 1,\ 1)$
	$\hat{\nu}_4 = (\ 0,\ 0,-1,\ 0;\ 0)$	$\mu_4 = (\ 1,\ 1,-3,\ 1)$
	$\hat{\nu}_5 = (\ 0,\ 0,\ 0,-1;\ 0)$	$\mu_5 = (\ 1,\ 1,\ 1,-3)$
	$\hat{\nu}_6 = (\ 1,\ 2,\ 2,\ 2;\ 0)$	
	$\hat{\nu}_7 = (\ 0,\ 1,\ 1,\ 1;\ 0)$	
$\mathcal{D}$	$\hat{\nu}_8 = (\nu_3; 1),\ \hat{\nu}_9 = (\nu_4; 1)$	
$\mathbb{P}_{1,1,2,2}[6]$	$\hat{\nu}_1 = (\ 1,\ 2,\ 2;\ 0)$	$\mu_1 = (\ 1,\ 1,\ 1)$
	$\hat{\nu}_2 = (-1,\ 0,\ 0;\ 0)$	$\mu_2 = (-5,\ 1,\ 1)$
	$\hat{\nu}_3 = (\ 0,-1,\ 0;\ 0)$	$\mu_3 = (\ 1,-2,\ 1)$
	$\hat{\nu}_4 = (\ 0,\ 0,-1;\ 0)$	$\mu_4 = (\ 1,\ 1,-2)$
	$\hat{\nu}_5 = (\ 0,\ 1,-1;\ 0)$	
$\mathcal{D}$	$\hat{\nu}_6 = (\nu_3; 1),\ \hat{\nu}_7 = (\nu_0; 1)$	
$\mathbb{P}_{1,1,1}[3]$	$\hat{\nu}_1 = (\ 1,\ 1;\ 0)$	$\mu_1 = (\ 1,\ 1)$
	$\hat{\nu}_2 = (-1,\ 0;\ 0)$	$\mu_2 = (-2,\ 1)$
	$\hat{\nu}_3 = (\ 0,-1;\ 0)$	$\mu_3 = (\ 1,-2)$
$\mathcal{D}_{2[pt]}$	$\hat{\nu}_4 = (\nu_0; 1),\ \hat{\nu}_5 = (\nu_1; 1)$	
$\mathcal{D}_{3[pt]}$	$\hat{\nu}_4 = (\nu_0; 1),\ \hat{\nu}_5 = (\nu_1; 1)$	
	$\hat{\nu}_6 = (\nu_2; 1),\ \hat{\nu}_7 = (\nu_3; 1)$	
$\mathcal{D}_{[pt]}$		$\hat{\mu}_{11} = (\mu_1; 1),\ \hat{\mu}_{12} = (0, 1; 1)$

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