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ITERATED LEAVITT PATH ALGEBRAS

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Abstract

Leavitt path algebras associate to directed graphs a $\mathbb{Z}$-graded algebra and in their simplest form recover the Leavitt algebras $L(1,k)$. In this note, we introduce iterated Leavitt path algebras associated to directed weighted graphs which have natural $\oplus\mathbb{Z}$ grading and in their simplest form recover the Leavitt algebras $L(n,k)$. We also characterize Leavitt path algebras which are strongly graded.

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1. Iterated Leavitt Path Algebras

Leavitt path algebras (LPA for short), introduced by Abrams and Aranda Pino [1] and Ara, Moreno and Pardo [6], associate to a directed graph E a $\mathbb{Z}$-graded algebra $L(E)$ which is equipped with an (anti-graded) involution. In its most simplest form, when the graph E has only one vertex, and $k+1$ loops, $L(E)$ recovers the algebra constructed by Leavitt [9, p.118] which is of type $(1,k)$. The characterization of a LPA (such as simplicity, finite dimensionality, exchange, etc.) in terms of intrinsic properties of the underlying graph has been the subject of recent studies [1, 2, 3, 4, 5, 6].

In this note we introduce the *iterated* Leavitt path algebras (iLPA for short) starting from a weighted graph (i.e., a graph which each edge comes with some copies of itself). This is a $\mathbb{Z}\times\cdots\times\mathbb{Z}$-graded algebras which in the special case of a graph with weights 1 (or unweighted), it gives the Leavitt path algebras and in its most simplest form, when the graph E has only one vertex and $n+k$ loops of weights n, $L(E)$ recovers the algebra constructed by Leavitt [9, p.30] and [8, p.322] which is of type (n,k) (see Example 1.3). One could then ask for characterization of an iLPA in terms of its underlying graph and its module and algebraic structure.

The iterated Leavitt path algebras provide new classes of algebras which could not be obtained using unweighted graphs (i.e., using Leavitt path algebras); for example, note that except $L(\circlearrowleft\bullet) = R[x,x^{-1}]$ (which happens to be the only commutative LPA along with $L(\bullet)=R$, where R is an integral domain[1]), all LPA have plenty of zero divisors. However, all iLPA with one vertex and with weights greater than one and less than the number of loops are non-commutative domains (see Example 1.3).

We begin by the definition of a weighted graph. Other graph-theoretic definitions and properties are recalled in Section 2 when they are needed.

A *weighted graph* $E=(E^0,E^s,E^1,r,s,w)$ consists of three countable sets E^0 called *vertices*, E^s *structured edges* and E^1 *edges* and maps $s,r:E^s\rightarrow E^0$, $s,r:E^1\rightarrow E^0$ and a weight map $w:E^s\rightarrow\mathbb{N}$ such that if $\alpha\in E^s$ and $w(\alpha)=k$, then there are k edges $\alpha_1,\dots\alpha_k\in E^1$ with the same source and range as α, i.e., $s(\alpha_i)=s(\alpha)$ and $r(\alpha_i)=r(\alpha)$ for $1\leq i\leq k$.

We refer to the maps r_{E^s} and s_{E^s} if we need to distinguish them from those on E^1. If $s_{E^s}^{-1}(v)$ is a finite set for every $v\in E^0$, then the graph is called *row-finite*. In this note we will only consider row-finite graphs. In this setting, if the number of vertices, i.e., $|E^0|$, is finite, then the number of edges, i.e., $|E^1|$, is finite as well and we call E a *finite* graph.

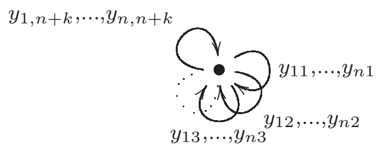
Definition 1.1. For a weighted graph E and a ring R with identity, we define the *iterated Leavitt path algebra of* E, denoted by $L_R(E)$, to be the algebra generated by the sets $\{v\mid v\in E^0\}$, $\{\alpha_1,\dots\alpha_{w(\alpha)}\mid\alpha\in E^s\}$ and $\{\alpha_1^*,\dots\alpha_{w(\alpha)}^*\mid\alpha\in E^s\}$ with the coefficients in R subjected to the relations

[1] We only consider connected graphs, as a disjoint graph would simply produce direct sum of corresponding LPAs of its connected subgraphs.

(1) $v_i v_j = \delta_{ij} v_i$ for every $v_i, v_j \in E^0$.
(2) $s(\alpha_i)\alpha_i = \alpha_i r(\alpha_i) = \alpha_i$ and $r(\alpha_i)\alpha_i^* = \alpha_i^* s(\alpha_i) = \alpha_i^*$ for all $\alpha \in E^s$ and $1 \le i \le w(\alpha)$.
(3) $\sum_{\{\alpha \in E^s, s(\alpha)=v\}} \alpha_i \alpha_j^* = \delta_{ij} s(\alpha)$ for fixed $1 \le i, j \le \max\{w(\alpha) \mid \alpha \in E^s, s(\alpha) = v\}$.
(4) $\sum_{1 \le i \le \max\{w(\alpha), w(\alpha')\}} \alpha_i^* \alpha_i' = \delta_{\alpha\alpha'} r(\alpha)$, for all $\alpha, \alpha' \in E^s$.

Example 1.2. Leavitt Path Algebras
Let the weight map $w : E^s \to \mathbb{N}$ be the constant function $w(\alpha) = 1$ for any $\alpha \in E^s$. Then $L(E)$ is the usual Leavitt path algebra (with the coefficients in the ring R) as defined in [1] and [6].

Example 1.3. Leavitt algebra of type (n, k)
Let D be a division ring. For positive integers n and k, let E^s consists of $n + k$ loops, i.e., $s(y) = r(y)$ for $y \in E^s$ and the weight function be the constant map $w(y) = n$ for all $y \in E^s$.

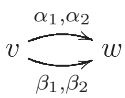


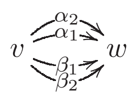
Then the iterated Leavitt path algebra associated to E, $L_D(E)$, is the algebra constructed by Leavitt in [7, p.190], for $n = 2$ and $k = 1$, where he showed that this algebra has no zero divisors, in [8, p.322], for arbitrary n and $k = 1$ and in [9, p.130] for arbitrary n and k and established that these algebras are domain and of type (n, k). Recall that a ring R is of type (n, k) if n and k are the least positive integers such that $R^n \cong R^{n+k}$ as R-modules. To recover Leavitt's algebra from Definition 1.1 (and to arrive to his notations), let $E^s = \{y_1, \dots, y_{n+s}\}$ be the structured edges and denote $(y_s)_r = y_{rs} \in E^1$, for $1 \le r \le n$ and $1 \le s \le n + k$. Denote $y_{rs}^* = x_{sr}$ and arrange the y's and x's in the matrices

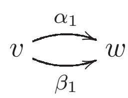
$$Y = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1,n+k} \\ y_{21} & y_{22} & \cdots & y_{2,n+k} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n,1} & y_{n,2} & \cdots & y_{n,n+k} \end{pmatrix}, \quad X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1,n} \\ x_{21} & x_{22} & \cdots & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n+k,1} & x_{n+k,2} & \cdots & x_{n+k,k} \end{pmatrix}$$

Then Condition (3) of Definition 1.1 precisely says that $Y \cdot X = I_{n,n}$ and Condition (4) is equivalent to $X \cdot Y = I_{n+k,n+k}$ which is how Leavitt defines his algebra.

Example 1.4. We compare the relations of the iterated Leavitt path algebra $L(E)$ and the usual Leavitt path algebras $L(E')$ and $L(E'')$ in the following.

$E =$ 

$E' =$ 

$E'' =$ 

$$\begin{aligned}
\alpha_1\alpha_1^* + \beta_1\beta_1^* &= v \\
\alpha_2\alpha_2^* + \beta_2\beta_2^* &= v \\
\alpha_1\alpha_2^* + \beta_1\beta_2^* &= 0 \\
\alpha_2\alpha_1^* + \beta_2\beta_1^* &= 0 \\
\alpha_1^*\alpha_1 + \alpha_2^*\alpha_2 &= w \\
\beta_1^*\beta_1 + \beta_2^*\beta_2 &= w \\
\alpha_1^*\beta_1 + \alpha_2^*\beta_2 &= 0 \\
\beta_1^*\alpha_1 + \beta_2^*\alpha_2 &= 0
\end{aligned}$$

$$\begin{aligned}
\alpha_1\alpha_1^* + \alpha_2\alpha_2^* + \beta_1\beta_1^* + \beta_2\beta_2^* &= v \\
\alpha_1^*\alpha_1 = \alpha_2^*\alpha_2 = \beta_1^*\beta_1 = \beta_2^*\beta_2 &= w \\
\alpha_i^*\alpha_j = \beta_i^*\beta_j = 0 \text{ if } i \neq j \\
\alpha_i^*\beta_j = \beta_j^*\alpha_i = 0 \text{ for all } i, j
\end{aligned}$$

$$\begin{aligned}
\alpha_1\alpha_1^* + \beta_1\beta_1^* &= v \\
\alpha_1^*\alpha_1 = \beta_1^*\beta_1 &= w \\
\alpha_1^*\beta_1 = \beta_1^*\alpha_1 &= 0
\end{aligned}$$

Remark. In the similar manner, one can define iterated graph algebras and iterated graph C^*-algebras.

Proposition 1.5. *Let E be a weighted graph and $L_R(E)$ be an iterated Leavitt path algebra with coefficients over a ring R. Then we have*

(1) $L_R(E)$ is a $\bigoplus_n \mathbb{Z}$-graded ring with an involution where $n = \max\{w(\alpha) \mid \alpha \in E^s\}$.
(2) $L_R(E)$ is a ring with local identities. If E is finite, then $L_R(E)$ is a ring with identity.

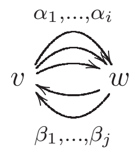
Proof. (1) For the free ring generated by $\{v \mid v \in E^0\}$, $\{\alpha_1, \dots \alpha_{w(\alpha)} \mid \alpha \in E^s\}$ and $\{\alpha_1^*, \dots \alpha_{w(\alpha)}^* \mid \alpha \in E^s\}$ with the coefficients in R, set for $v \in E^0$, $\deg(v) = 0$ and for $\alpha \in E^s$ and $1 \leq i \leq w(\alpha)$, $\deg(\alpha_i) = (0, \dots, 0, 1, 0, \dots)$ and $\deg(\alpha_i^*) = (0, \dots, 0, -1, 0, \dots) \in \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ where 1 and -1 are on i-th component, respectively. This defines a $\bigoplus_n \mathbb{Z}$-grading on this free ring where $n = \max\{w(\alpha) \mid \alpha \in E^s\}$. Note that all the relations in Definition 1.1 involve homogenous elements of the same degree, so the quotient of this algebra with the ideal generated by these relations, i.e., $L_R(E)$ is also a graded ring.

(2) Note that if a_i's are mutually orthogonal idempotents in a ring A such that $A = \sum a_i A = \sum A a_i$, then the set of $\sum_{\text{finite}} a_i$ is a set of local identities of this ring. If the number of a_i's is finite then $\sum a_i$ is an identity for this ring. Now it is easy to see that the set of vertices in E is such a system of idempotents for $L_R(E)$. $\square$

Definition 1.6. For a directed graph E, define the *opposite graph*, E^{op} as a graph with the same set of vertices and edges as E (for α in E, denote the corresponding edge in E^{op} with α^{op}), such that for an edge α^{op} in E^{op}, $s(\alpha^{\text{op}}) = r(\alpha)$ and $r(\alpha^{\text{op}}) = s(\alpha)$. This means that E^{op} is obtained from E by simply reversing the arrows.

For a directed graph E, the *weighted graph associated to* E, denoted by E^w, is obtained by considering all the edges with the same source and the same range in E as one edge with appropriate weight (i.e., the number of these edges) in E^w.

It is not clear in general how the algebras $L(E)$ and $L(E^{\text{op}})$ are related. For example for $E = \bullet \longrightarrow \bullet \rightleftarrows \bullet$ one obtains $L_K(E) = M_5(K)$, whereas, for $E^{\text{op}} = \bullet \longleftarrow \bullet \rightleftarrows \bullet$ we have $L_K(E^{\text{op}}) = M_4(K)$ [3, Prop. 3.5] (see also Example 2.5).

Example 1.7. Consider the graph E with two vertices and with no loops, $E =$ 

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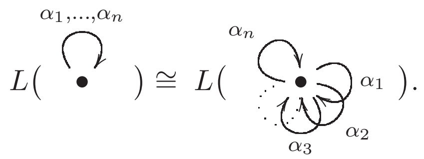
Then one can see that the map $E \longrightarrow E^{\text{op}}, (v \mapsto w, w \mapsto v, \alpha_i \mapsto \alpha_i, \beta_i \mapsto \beta_i)$ induces an isomorphism in the level of LPAs, i.e., $L(E) \cong L(E^{\text{op}})$. Now consider the associated weighted graph E^w of E (see Definition 1.6). The map $E^w \longrightarrow E^{\text{op}}, (v \mapsto v, w \mapsto w, \alpha_i \mapsto \alpha_i^*, \beta_i \mapsto \beta_i^*)$ gives the isomorphism, $L(E^w) \cong L(E^{\text{op}})$. In the same manner, one can see that for any graph E with one vertex, $L(E) \cong L(E^{\text{op}}) \cong L(E^w)$, i.e.,



Consider the category $\mathcal{G}^w$ with objects all row-finite weighted graphs and morphisms, the complete weighted graph homomorphisms, i.e., a morphism $f : E \to F$ consists of a map $f^0 : E^0 \to F^0$ and $f^1 : E^s \to F^s$ such that $r(f^1(\alpha)) = f^0(r(\alpha))$, $s(f^1(\alpha)) = f^0(s(\alpha))$ and $w(\alpha) = w(f^1(\alpha))$ for any $\alpha \in E^s$, additionally, f^0 is injective and f^1 restricts to a bijection from $s^{-1}(v)$ to $s^{-1}(f^0(v))$ for every $v \in E^0$ which emits edges. One can check that a morphism f preserves the relations in Definition 1.1, and thus induces a graded ring homomorphism $L_R(E) \to L_R(F)$. Thus, when R is commutative, we have a functor $L : \mathcal{G}^w \to \mathcal{A}$ where $\mathcal{A}$ is the category of (non-unital) R-algebras. To any weighted graph, one can associate a directed graph by simply considering the weight as the number of edges connecting the adjacent vertices. (In Example 1.4, E' is the directed graph obtained from the weighted graph E.) One can check that this defines a (forgetful) functor from $\mathcal{G}^w$ to $\mathcal{G}$. It is not known whether there is a functor which relates the corresponding iLPA to LPA, i.e., whether there is a functor such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{G}^w & \xrightarrow{L} & \mathcal{A} \\ {\scriptstyle id}\downarrow & & \vdots\,? \\ \mathcal{G} & \xrightarrow[L]{} & \mathcal{A} \end{array}$$

In the same manner, recall that one can associate to a graph a weighted graph (see Definition 1.6), so a similar question can be raised here too:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{L} & \mathcal{A} \\ {\scriptstyle w}\downarrow & & \vdots\,? \\ \mathcal{G}^w & \xrightarrow[L]{} & \mathcal{A} \end{array}$$

For a ring R with identity, the monoid $V(R)$ is defined as the set of isomorphism classes of finitely generated projective left R-modules equipped with the direct sum as the binary operation. When R is not unital, one defines $V(R)$ as the set of equivalent classes of idempotents in $M_\infty(R)$ with $[e]+[f]=\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$ where $e\sim f$ if there are $x,y\in M_\infty(R)$ such that $e=xy$ and $f=yx$. (There is a corresponding construction based on finitely generated projective modules as well (cf. [6]).) In [6, Theorem 3.5], Ara, Moreno and Pardo show that for a directed graph E, $V(L_K(E))$ coincides with a monoid naturally constructed from the graph E. The similar construction is valid in the setting of iLPA and we sketch it here.

Theorem 1.8. *Let E be a (row-finite) weighted graph and K a field. Let M_E be the abelian monoid generated by $\{v \mid v\in E^0\}$ subjected to the relations*

$$nv=\sum_{\{\alpha\in E^s \mid s(\alpha)=v\}} r(\alpha),\ \text{for every } v\in E^0 \text{ that emits edges, where } n=\max\{w(\alpha)\mid \alpha\in E^s,\ s(\alpha)=v\}.$$

Then there is a natural monoid isomorphism $V(L_K(E))\cong M_E$.

Sketch of Proof. Define a map $\psi: E^0\to V(L(E))$ by $\psi(v)=n[v]$ where $n=\max\{w(\alpha)\mid \alpha\in E^s,\ s(\alpha)=v\}$ and extend this to the map from the free monoid on E^0 to $V(L(E))$. This induces a map $\psi_E: M_E\to V(L(E))$. To see this, we need to show that nv and $\sum_{\{\alpha\in E^s|s(\alpha)=v\}} r(\alpha)$ maps to the same element in $V(L(E))$. Let $\{\alpha_1,\cdots,\alpha_n\}$ be all the structured edges emit from v. Consider the matrices $Y=(\alpha_{ij})_{1\le j\le n, 1\le i\le w(\alpha_j)}$ where $\alpha_{ij}=(\alpha_j)_i\in E^1$ and $X=(Y^*)^t$, where ^{t} is the transpose operation. Then the conditions of Definition 1.1 guarantee that $Y.X=n[v]$ and $X.Y=\sum_{\{\alpha\in E^s|s(\alpha)=v\}}[r(\alpha)]$. So ψ_E is well-defined. Now it is easy to see that $L(E)$ is the direct limit of graph algebras corresponding to finite graphs (see [6, Lemma 3.2] for the proof in unweighted setting). Thus it is enough to prove ψ_E is an isomorphism for a finite graph E. The rest of the proof is similar to [6, Theorem 3.5] with appropriate changes. □

2. Strongly graded Leavitt path algebras

In this section we specialize to unweighted Leavitt path algebras (i.e., the weight function is the constant function 1). These algebras have a $\mathbb{Z}$-grading and a natural question is when these algebras are strongly graded. Recall that for a graded ring $R=\oplus_{g\in G}R_g$, graded by a group G, the support of R is all $g\in G$ such that $R_g\neq 0$. Then R is called *strongly graded* if $R_gR_h=R_{gh}$ for any g,h in the support of R. Note that, with this definition, path algebras are strongly graded as any path of length $i+j$ consists of a path of length i followed by a path of length j. However, the case of LPA's are more involved. Note that if the support of R is G, then the above definition of strongly graded ring coincides with the standard definition. We will see in (iv) below that if E^1 is not empty (i.e., the graph is not one isolated vertex) then the support is the whole group.

We briefly recall some graph-theoretic definitions and properties. For a graph $E=(E^0,E^1,r,s)$, a vertex v for which $s^{-1}(v)$ is empty is called a *sink*, while a vertex w for which $r^{-1}(w)$ is empty is called a *source*. A path μ in a graph E is a sequence of edges $\mu=\mu_1\dots\mu_k$, such that

$r(\mu_i) = s(\mu_{i+1})$. In this case, $s(\mu) := s(\mu_1)$ is the *source* of μ, $r(\mu) := r(\mu_k)$ is the *range* of μ, and k is the *length* of μ which is denoted by $|\mu|$. If μ is a path in E, and if $v = s(\mu) = r(\mu)$, then μ is called a *closed path based at* v. If $\mu = \mu_1 \dots \mu_k$ is a closed path based at $v = s(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for every $i \neq j$, then μ is called a *cycle*. If $\mu = \mu_1 \dots \mu_k$ is a path, then we denote by μ^* the element $\mu_k^* \dots \mu_1^* \in L(E)$.

One can write $L_R(E) = \bigoplus_{k\in\mathbb{Z}} L(E)_k$ where

$$L(E)_k = \{\sum_i r_i\alpha_i\beta_i^* \mid \alpha_i, \beta_i \text{ paths with finite lengths}, r_i \in R, \text{ and } |\alpha_i| - |\beta_i| = k \text{ for all } i\}.$$

The elements of the form $\alpha\beta^*$ are called *monomials*. We define an involution on $L(E)$ by $\overline{\alpha\beta^*} = \beta\alpha^*$ for the monomials and extend it to the whole $L(E)$ in the obvious manner. Note that if $x \in L(E)_n$, then $\overline{x} \in L(E)_{-n}$. For vertices v and w, by $v \geq w$ we mean there is a path which connects v to w. By $v \geq_n w$, we mean there is a path of length n connecting these vertices. In this note, by $v \geq w' \geq w$, it is understood that there is a path connecting v to w and going through w' (i.e., w' is on the path connecting v to w). We say a vertex v is on an *infinite path* if there are real paths and ghost paths of arbitrary length starting from v, i.e., for any $n \in \mathbb{N}$ there are vertices w and w' such that $v \geq_n w$ and $v \leq_n w'$, respectively. A vertex v is *connected to an infinite path* if there is a vertex w such that $v \geq w$ and w is on an infinite path.

Let $\mathcal{P} \subseteq E^0$ be a subset of vertices with a certain property. We say that $w \in \mathcal{P}$ is *immediate* to v, if $v \geq w$ but there is no $w' \in \mathcal{P}$ such that $v \geq w' > w$ (with our convection this means that w' is on the same path connecting v to w). Clearly if $v \in \mathcal{P}$, then v itself is the only vertex which is immediate to v.

Define the *orbit* of v with respect to $\mathcal{P}$

$$O_{\mathcal{P}}(v) = \{v \geq w \mid \; w \in \mathcal{P} \text{ is immediate to } v\}$$

We say $O_{\mathcal{P}}(v)$ is *bounded*, if there is a $n \in \mathbb{N}$, such that all the paths in $O_{\mathcal{P}}(v)$ have length at most n. The bound of $O_{\mathcal{P}}(v)$ is zero if and only if $v \in \mathcal{P}$

The followings are easy to observe and will be used in the text:

(i) If a vertex is not connected to an infinite path, then the graph has a vertex which is either a source or a sink.

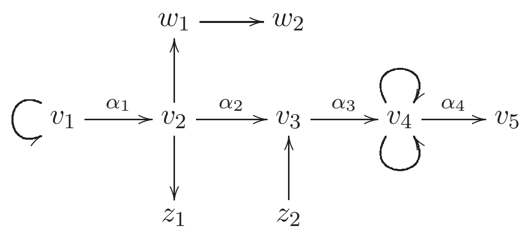
(ii) If x is a monomial, i.e., $x = \alpha\beta^*$, then $x\overline{x}x = x$.

(iii) If $R_nR_m \neq 0$, $n > 0$, then there is a vertex v and a path of length at least n emitting from v.

(iv) If there is a sink, say v, then $v \in R_0$, however $v \notin R_1R_{-1}$ as otherwise there should be an edge emitting from v by (iii). This shows that if $L(E)$ is strongly graded and E^1 is not empty, then the support of $L(E)$ is $\mathbb{Z}$.

(v) If p is a finite path, by $\mathbf{e}(p)$ we denote the set of (including the repetition) end-vertices of all exit edges from p. To be precise, if $p = \alpha_1\alpha_2 \dots \alpha_k$ and μ is an exit edge with

$s(\mu) = s(\alpha_i), 1 \le i \le k-1$, then $r(\mu) \in \mathbf{e}(p)$. By the convection, we let $r(\alpha_k) \in \mathbf{e}(p)$. Since E is a finite-row graph, $\mathbf{e}(p)$ is a finite set. For example for the path $p = \alpha_1\alpha_2\alpha_3\alpha_4$ in the graph below



we have $\mathbf{e}(p) = \{v_1, w_1, z_1, v_4, v_4, v_5\}$. Observe that if $\mu_1 \dots \mu_k$ and $\mu_t \dots \mu_l$ are two paths with $r(\mu_k) = s(\mu_t)$ then

$$\mathbf{e}(\mu_1 \dots \mu_k \mu_t \dots \mu_l) = \big(\mathbf{e}(\mu_1 \dots \mu_k) \cup \mathbf{e}(\mu_t \dots \mu_l)\big) \backslash r(\mu_k). \tag{2.1}$$

(vi) For a finite path p, if $v = s(p)$ then

$$v = \sum_{w \in \mathbf{e}(p)} (v \ge w)(v \ge w)^*. \tag{2.2}$$

Here if μ is an exit edge with $s(\mu) = s(\alpha_i)$, then $r(\mu) \in \mathbf{e}(p)$ and $v \ge r(\mu) = \alpha_1\alpha_2 \dots \alpha_{i-1}\mu$.

Proof. We prove the statement by using an induction on the length of p. Let $|p| = 1$. Then (2.2) (note the convection in the defintion of $\mathbf{e}(p)$) reduces to $v = \sum_{\{\alpha \in E^1 | s(\alpha) = v\}} \alpha\alpha^*$ which is correct. Now let $p = \mu_1 \dots \mu_n$ and suppose the statement is valid for paths of length $n-1$. Considering the path $q = \mu_2 \dots \mu_n$ with $s(q) = t$ we then have

$$v = s(\mu_1) = \sum_{\{\alpha \in E^1 | s(\alpha) = v, \alpha \ne \mu_1\}} \alpha\alpha^* + \mu_1\mu_1^*, \text{ and } \quad t = \sum_{w \in \mathbf{e}(q)} (t \ge w)(t \ge w)^*. \tag{2.3}$$

Writing $\mu_1\mu_1^*$ as $\mu_1 t \mu_1^*$ and replacing t with the above equation we have,

$$v = \sum_{\{\alpha \in E^1 | s(\alpha) = v, \alpha \ne \mu_1\}} \alpha\alpha^* + \mu_1\Big(\sum_{w \in \mathbf{e}(q)} (t \ge w)(t \ge w)^*\Big)\mu_1^*. \tag{2.4}$$

Now (2.1) guarantees that Equation (2.4) coincides with (2.2), so we are done. □

Lemma 2.1. *If the set $O_{\mathcal{P}}(v)$ is bounded then it is finite. If v' is a vertex such that $v > v' \ge w$ for $w \in \mathcal{P}$ an immediate vertex of v, then the bound of $O_{\mathcal{P}}(v')$ is strictly less than the bound of $O_{\mathcal{P}}(v)$ and $|O_{\mathcal{P}}(v')| \le |O_{\mathcal{P}}(v)|$.*

Proof. Since the graph is row-finite, there is only a finite number of paths of given length emitting from each vertex. This gives the first statement. For the second assertion, let w' be an immediate vertex for v', with the path $v' \ge w'$ connecting v' to w' with the maximum length k. Since $v > v' \ge w$, the path $v > v' \ge w'$ makes w' an immediate vertex for v with the length strictly greater than k. This argument also shows that $|O_{\mathcal{P}}(v')| \le |O_{\mathcal{P}}(v)|$. □

Let $\mathcal{P}$ be the following property: "Vertices on an infinite path". Then we have,

Theorem 2.2. *The Leavitt path algebra associated to a graph which the orbit of any vertex is nonempty and bounded is strongly graded.*

Proof. Suppose any vertex of the graph E is connected to a vertex on an infinite path, i.e., the orbits are not empty. Let $L = L(E)$. First note that $L_n \neq 0$ for any $n \in \mathbb{Z}$. We need to show that $L_{n+m} = L_n L_m$ for all $n, m \in \mathbb{Z}$. It is enough to show that any monomial $\alpha\beta^* \in L_{n+m}$ is in $L_n L_m$. Writing $\alpha\beta^* = \alpha_1 \dots \alpha_k \beta_1^* \dots \beta_l^*$, we have $k - l = n + m$.

We have the following:

(1) For any vertex v, and $n \in \mathbb{N}$, one can write $v = \sum p_i q_i^*$ with $|p_i| = |q_i| = n$. Clearly $v = v.v \in L_0 L_0$. Since any vertex is connected to an infinite path, the graph does not have a sink, it is also row-finite, so one can write

$$v = \sum_{\substack{e \in E^1 \\ s(e)=v}} ee^*. \tag{2.5}$$

So $v \in L_1 L_{-1}$. Now if $v = \sum p_i q_i^*$ with $|p_i| = |q_i| = n - 1$, write $v = \sum p_i r(p_i) q_i^*$ and use (2.5) for each $r(p_i)$ to conclude by indiction.

(2) For any vertex v on an infinite path, and $n \in \mathbb{N}$, it is easy to see that one can write $v = p^* p$ where p is a path of length n with $r(p) = v$.

We need to consider two cases.

Case $\mathbf{n \geq 0}$. If $k \geq n$ then writing $\alpha\beta^* = (\alpha_1 \dots \alpha_n)(\alpha_{n+1} \dots \alpha_k \beta_1^* \dots \beta_l^*)$, it is clear that $\alpha\beta^* \in L_n L_m$.

If $k < n$, then by (1), we can write $r(\alpha_k) = \sum p_i q_i^*$ with $|p_i| = |q_i| = n - k$. So

$$\alpha\beta^* = \alpha_1 \dots \alpha_k r(\alpha_k) \beta_1^* \dots \beta_l^* = \sum (\alpha_1 \dots \alpha_k p_i)(q_i^* \beta_1^* \dots \beta_l^*) \in L_n L_m.$$

Case $\mathbf{n < 0}$. If $m < 0$, consider $\overline{\alpha\beta^*} \in L_{-m-n}$. By the previous case now, $\overline{\alpha\beta^*} \in L_{-m} L_{-n}$. Applying the involution to this element again we have $\alpha\beta^* \in L_n L_m$.

Now for the remaining case (in fact the following argument is valid when $m < 0$ as well), let $v = r(\alpha_k)$ and consider $O_{\mathcal{P}}(v)$ which consists of finite number of paths by Lemma 2.1. We claim

$$v = \sum_{w \text{ immediate to } v} (v \geq w) w (v \geq w)^*. \tag{2.6}$$

To see this, we proceed by induction on the bound of $O_{\mathcal{P}}(v)$. If the bound is zero, Equation (2.6) trivially holds and if the bound is 1 (which prevents v to have a loop), then the (2.6) reduces to $v = \sum_{\{\alpha \in E^1 | s(\alpha) = v\}} \alpha\alpha^*$ which is also correct.

Suppose (2.6) is valid for any vertex with a bound less than the bound of $O_{\mathcal{P}}(v)$. Fix an immediate vertex w of v and consider the path $v \geq w$ and $\mathbf{e}(v \geq w) = \{v_1, \dots, v_h, w\}$. Then by (vi) in page 8,

$$v = (v \geq w)(v \geq w)^* + (v \geq v_1) v_1 (v \geq v_1)^* + \dots + (v \geq v_h) v_h (v \geq v_h)^*. \tag{2.7}$$

By Lemma 2.1, the bound of $O_{\mathcal{P}}(v_i)$ is smaller than the bound of $O_{\mathcal{P}}(v)$, so by induction we have

$$v_i = \sum_{w \text{ immediate to } v_i} (v_i \geq w)(v_i \geq w)^*.$$

Plugging these sums for v_i in Equation (2.7) and observing that for any v_i, if w' is an immediate vertex to v_i, then it is immediate to v and also, for any immediate exit $w'' \neq w$ of v, there is a path connecting v to w'', thus w'' is an immediate vertex for some v_i, we have obtained (2.6).

Now in Equation 2.6, since each w is on an infinite path, by (2), we can write $w = p_w^* p_w$, where $|p_w| = k + |v \geq w| + |n|$. Thus

$$\alpha\beta = \alpha_1 \dots \alpha_k v \beta_1^* \dots \beta_l^* = \sum_{w \text{ immediate}} \Big(\alpha_1 \dots \alpha_k (v \geq w) p_w^*\Big)\Big(p(v \geq w)^* \beta_1^* \dots \beta_l^*\Big).$$

A quick inspection shows that each term in the sum is in $L_n L_m$. □

Example 2.3. By Theorem 2.2, the Leavitt algebra $L(1, n)$ (and matrix algebra over $L(1, n)$) is strongly graded as the only vertex in its graph is on cycles.

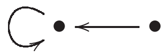
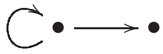
Remark. For the converse of Theorem 2.2, we have the following statement: If $L(E)$ is strongly graded, then for any vertex v and any natural number $n \in \mathbb{N}$, there exists vertices w_1, w_2, w_3, and paths $v \geq w_1$ of length n, and $v \geq w_2$ of length s, and $w_3 \geq w_2$ of length $s + n$. Although this formulation does not seem to look elegant, when the graph is finite, we have a simple characterization (see Theorem 2.4).

2.1. **Finite graphs.** When the number of vertices are finite, we can give a complete characterization of strongly graded Leavitt path algebras:

Theorem 2.4. *Let E be a finite graph. The Leavitt path algebra $L(E)$ is strongly graded if and only if any vertex is connected to a cycle.*

Proof. If any vertex is connected to a cycle then the orbit of any vertex is nonempty and bounded (as the graph is finite), thus by Theorem 2.2, $L(E)$ is strongly graded. For the converse, let R be strongly graded. Then the graph E does not have any sink (see iv in page 7). Let $|E^0| = n$. For any vertex v, consider the path $\mu_1\mu_2 \dots \mu_n$ of length n emitting from v (this is possible as there is no sink). Since the number of vertices are n, this forces $s(\mu_i) = s(\mu_j)$ for some i, j. That is v is connected to $\mu_i\mu_{i+1} \dots \mu_j$ which is a closed path. Now the following easy argument, base on an induction on the length, shows that any vertex on a closed path is connected to a cycle. If the vertex w is on a closed path of length 1, then this a loop and there is nothing to prove. Assume the statement is correct for any vertex on a closed path of length less than k. Let $\gamma_1 \dots \gamma_k$ be a closed path of length k, with $s(\gamma_1) = r(\gamma_k)$. If $s(\gamma_i) \neq s(\gamma_j)$ for all $1 \leq i \neq j \leq k$, then the path is a cycle and there is nothing to prove. Otherwise, suppose $s(\gamma_i) = s(\gamma_j)$ and consider the path $\gamma_1 \dots \gamma_{i-1}\gamma_j \dots \gamma_k$. This is clearly a closed path of smaller length and we are done by induction. □

Example 2.5. According to Theorem 2.4, the graph E below produces a strongly graded Leavitt path algebra, but the opposite graph E^{op} does not.

$E =$  $E^{\mathrm{op}} =$ 

Also combining Theorem 2.4 with [2, Theorem 11], it follows that unital purely infinite simple Leavitt path algebras are strongly graded.

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