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GENERALIZED SPIN COHERENT STATES: CONSTRUCTION AND SOME PHYSICAL PROPERTIES

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Abstract

A generalized deformation of the *su(2)* algebra and a scheme for constructing associated spin coherent states is developed. The problem of resolving the unity operator in terms of these states is addressed and solved for some particular cases. The construction is carried using a deformation of Holstein-Primakoff realization of the *su(2)* algebra. The physical properties of these states is studied through the calculation of Mandel's parameter.

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1 Introduction

Since 1926, when Shrödinger first constructed quantum states that are the closest to reproduce the classical behavior [1], the theory of coherent states has been applied in nearly all branches of physics. The notion of coherence associated with these states was first noticed by Glauber in [2, 3]. The same states were also reintroduced by Klauder in [4, 5]. The common point here was that all these coherent states were associated to the quantum harmonic oscillator. Because of their important properties these states were then generalized to other systems either from a physical or mathematical point of view. For a review of all these generalizations see Refs. [6, 7, 8].

Perelomov [9] and Gilmore [10] have independently introduced coherent states associated to any (Lie) group (not only the Heizenberg-Weyl group related to the quantum harmonic oscillator). A particular case of these states is the spin coherent state or $SU(2)$ coherent state which are associated with the $SU(2)$ group. These states describe several systems and have many applications in quantum optics, statistical mechanics and condensed matter physics [6, 7, 8].

On the other hand, the quantum groups were introduced as a mathematical description of deformed Lie algebras [11, 12] that gave the possibility to construct deformed coherent states. They were introduced as a natural extension of the notion of coherent states. Generalized deformation of Glauber states were then constructed, see [13], as related to deformed harmonic oscillators. Deformed spin coherent states were also constructed as coherent states related to the quantum algebra $su_q(2)$ [14, 15].

Recently in [16] we have studied entanglement generation using deformed spin coherent states transmitted through a beam splitter. The states used are associated to a standard deformation of the $su(2)$ algebra. In fact, deformation of the spin algebra can be achieved in many ways [17] just as the deformation of the harmonic oscillator algebra which can be carried in many different ways [13, 18, 19]. The relative importance of the results obtained in [16] with the standard deformation of spin coherent states compared with the non-deformed ones suggest that it is possible to enhance these results by using other deformations of the spin coherent states. In this paper, we will propose a general scheme for constructing coherent states associated to generalized deformations of the $su(2)$ algebra. The physical properties (classicality) of these states will then be studied through investigating Mandel's parameter.

2 The Generalized spin algebra

The generalized $su(2)$ algebra is generated by three generators $J_\pm$ and J_z obeying the following commutation relations,

$$[J_z, J_\pm] = \pm J_\pm \quad ; \quad [J_+, J_-] = [2J_z] \ , \tag{1}$$

where the "box function" in the last of the commutation relations carries the deformation of the algebra; *i.e.* by choosing a particular form for this function one gets a specific deformation of the $su(2)$ algebra. For example, for $[x] = x$ one gets the non-deformed algebra, while for the following box function [20, 21]

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad \text{for} \quad q \in R, \tag{2}$$

gives the standard deformation of the $su(2)$ algebra [11, 12]. Another widely known deformation [22, 23] can be achieved using the following box function

$$[x] = \frac{q^x - 1}{q - 1}, \quad \text{for} \quad q \in R. \tag{3}$$

While one can carry out the deformation process using a complex parameter of deformation, we are going to focus our study on the real values of q.

An important property in all these deformations (as well as in any other deformation) is that one recovers the undeformed algebra $su(2)$ for particular values of the deformation parameter(s). For instance, for both deformations given by equations (2) and (3), by taking the limit $q \longrightarrow 1$ (or $\gamma \longrightarrow 0$) one recovers the undeformed algebra.

The generators must also obey the following relations

$$(J_+)^\dagger = J_- \ , \qquad (J_-)^\dagger = J_+ \ , \qquad (J_z)^\dagger = J_z \ . \tag{4}$$

The unitary irreducible representations of the deformed $su(2)$ algebra are similar to those of the undeformed one; they are indexed using a single positive and half-interger parameter j. The orthonormal basis of the space of representation is denoted as, $|j, m\rangle$, with $m = j, j-1, ..., -j$. The generators act on this basis following the rules,

$$\begin{aligned} J_z |j, m\rangle &= m \, |j, m\rangle \ , \\ J_\pm |j, m\rangle &= ([j \mp m][j \pm m + 1])^{\frac{1}{2}} \, |j, m \pm 1\rangle \ . \end{aligned} \tag{5}$$

It is well known that the $su(2)$ algebra operators can be mapped to harmonic oscillator operators in different fashions, Jordan-Schwinger map [24, 25], Holstein-Primakoff realization [26]... *A priori* all these mappings and realizations can be easily generalized, allowing to realize the operators of the generalized $su(2)$ algebra by means of creation and annihilation operators of generalized harmonic oscillators. This is an important result as it allows to relate the generalization of the harmonic oscillators [13, 18, 19] to the generalization of $su(2)$ algebra [17].

For the purpose of our study we will use a generalization of Holstein-Primakoff realization:

$$J_+ = a^\dagger \sqrt{[2j - N]} \quad , \quad J_- = \sqrt{[2j - N]} \, a \quad , \quad J_z = N - j \ , \tag{6}$$

where a, $a^\dagger$ are generalized annihilation and creation operators acting on Fock states $|n\rangle$, such that

$$a|n\rangle = \sqrt{[n]} \, |n-1\rangle \quad , \quad a^\dagger |n\rangle = \sqrt{[n+1]} \, |n+1\rangle \quad , \quad N|n\rangle = n|n\rangle \ . \tag{7}$$

They obey the following generalized commutation relations

$$\begin{aligned} \left[a, a^+\right] &= aa^+ - a^+a = & \Delta' \\ [a, \Delta] &= a\Delta - \Delta a = & \Delta' a \\ \left[a^+, \Delta\right] &= a^+\Delta - \Delta a^+ = & -a^+\Delta' \\ &\vdots \end{aligned} \tag{8}$$

where $\Delta = a^+a$, and Δ' is to be interpreted as a "generalized or deformed derivative" of Δ.

This generalized harmonic oscillator (or generalized Heisenberg algebra) was already introduced and the related coherent states constructed in Refs. [13, 18, 27].

3 Generalized spin coherent states

We will see in the following that one encounters the same problems in constructing generalized spin coherent states as those when constructing generalized Glauber coherent states (especially when dealing with the resolution of the identity operator). These similarities do occur even though the starting point in both constructions is different.

Before addressing these details, let us recall that the (usual) spin coherent states are defined by applying a displacement operator on the lowest weight state of the basis [7, 10]. However, for deformed algebras, it is not trivial to construct such an operator as the structure of algebra is not preserved. An attempt to generalize Perelomov-Gilmore's group theoretical based approach for the construction of coherent states associated to Hopf-algebras was made in [28, 29].

However, for our purpose here we generalize a different approach, [14, 15], in which the deformed coherent states are constructed following a formally analogous scheme and a formally analogous operator. That is, in a given j-representation of the generalized $su(2)$ algebra the generalized spin coherent states are defined by

$$|z, j\rangle = \mathcal{N}\left(|z|^2\right) E^{zJ_+}|j, -j\rangle \ , \qquad z \in \mathcal{C} \ , \tag{9}$$

where, we introduced the deformed exponential function

$$E^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]!} \qquad \text{with} \qquad [n]! = [n][n-1]...[1] \qquad \text{and} \qquad [0]! = 1 \ .$$

In general one should pay attention to the convergence of the series defining these deformed exponential functions, however in the case where one deals with finite dimensional representations this is not an important issue.

The normalization factor in equation (9) is given by

$$\mathcal{N}\left(|z|^2\right) = \frac{1}{\sqrt{(1 + |z|^2)_Q^{2j}}} \ , \tag{10}$$

where, we have used a generalization of Newton's binomial formula:

$$(x+y)^n_Q := \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_Q x^{x-m} y^m. \tag{11}$$

Here, the generalized binomial function is,

$$\begin{bmatrix} n \\ m \end{bmatrix}_Q = \frac{[n]!}{[n]![n-m]!} \qquad \text{for} \qquad n \geq m \ . \tag{12}$$

We have used the subscript Q in the notations above to mean that these are generalized (deformed) versions (being one parametric or multi-parametric [18]) of widely known expressions. A particular form of the generalized binomial formula, corresponding to the deformation (2), was already known by mathematicians (see [30]) and was used in [15]:

$$(x+y)^n_Q := \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_Q x^{x-m} y^m = \prod_{k=1}^{n} (x + Q^{n-2k+1} y) \ . \tag{13}$$

Using these definitions, we may write the generalized spin coherent states as:

$$|z,j\rangle = \left((1+|z|^2)^{2j}_Q\right)^{-\frac{1}{2}} \sum_{m=-j}^{j} \left(\begin{bmatrix} 2j \\ j+m \end{bmatrix}_Q \right)^{\frac{1}{2}} z^{(j+m)} |j,m\rangle \ . \tag{14}$$

In order to proceed with the study of physical properties of these states, we need to express the basis vectors $|j,m\rangle$ in terms of the Fock states $|n\rangle$ ($|j,m\rangle \sim |n\rangle$). One way of achieving this, is by using the generalization of Holstein-Primakoff realization given in Eq(6). Using this realization (especially the last equation) one gets the following change of variables $m = n - j$ or $n = j + m$, which when applied in Eq(14), yields the following expression of the deformed spin coherent states in terms of the Fock states

$$|z,j\rangle = \left((1+|z|^2)^{2j}_Q\right)^{-\frac{1}{2}} \sum_{n=0}^{2j} \left(\begin{bmatrix} 2j \\ n \end{bmatrix}_Q \right)^{\frac{1}{2}} z^{n} |n\rangle \ . \tag{15}$$

The first two of Klauder's three criteria are easily verified for the generalized spin coherent states (15). In fact, the states are normalized and weakly continued:

$$\Big| |z,j\rangle - |z',j\rangle \Big|^2 \longrightarrow 0 \quad \text{when} \quad |z-z'|^2 \longrightarrow 0. \tag{16}$$

The third condition, resolution of the identity operator, is less trivial to demonstrate. As we are going to demonstrate, it is not possible to resolve the identity operator in the general case however for particular cases this can be done by finding the adequate weight function. This feature was already encountered when constructing generalized Glauber coherent states [13, 18]. Resolving the identity operator there, yields a generalized moment problem, which does not admit a general solution.

For the generalized spin coherent states (15), we look for a resolution of the identity operator in the form

$$\iint |z,j\rangle\langle z,j| \mu(dz, d\bar{z}) = I \tag{17}$$

where the measure in the double integral is given by

$$\mu(dz, d\bar{z}) = w(z\bar{z})\, dz\, d\bar{z} \tag{18}$$

Thus in order to resolve the identity operator as in equation (17), one should find the adequate weight function $w(z\bar{z})$. By means of a change of the complex variables in terms of polar coordinates ($z = \mathrm{e}^{i\theta}$) and using the completeness of the states $|n\rangle$, the resolution of the identity operator (17) can be brought to solving the following integral equation

$$\int_0^{+\infty} x^n \left((1+x)_Q^{2j}\right)^{-1} w(x)dx = \frac{1}{\pi}\left(\begin{bmatrix} 2j \\ n \end{bmatrix}_Q\right)^{-1} \tag{19}$$

where $x = r^2$. Instead of solving equation (19) for $w(x)$ we investigate solutions of the following integral equation

$$\int_0^{+\infty} x^n \widetilde{w}(x)dx = \frac{1}{\pi}\left(\begin{bmatrix} 2j \\ n \end{bmatrix}_Q\right)^{-1} \tag{20}$$

for

$$\widetilde{w}(x) = \left((1+x)_Q^{2j}\right)^{-1} w(x)$$

As for the moment problem, this integral equation does not admit a general solution. In fact whether a solution exists or not the form of this solution in the positive case depends on the form of the box function. So at this stage the only thing that can be affirmed is that equation (14), or equivalently equation (15), defines generalized spin coherent states if and only if equation (20), or equivalently equation (19), admits a solution.

For the usual spin coherent states, $[n] = n$, such a solution exists and is given by

$$\widetilde{w}(x) = \frac{2j+1}{\pi}\,(1+x)^{-2j-2}\,. \tag{21}$$

$$w(x) = \frac{2j+1}{\pi}(1+x)^{-2}. \tag{22}$$

For a different choice of the box function, and in the case of high values of j, we propose to solve the integral equation (20) using the Fourier transform method. Multiplying both sides of equation (19) by $\left(\frac{(iy)^n}{n!}\right)$ and summing over n yields

$$\int_0^{\infty} dx\; e^{iyx}\widetilde{w}(x) = \sum_{n\geq 0} \frac{(iy)^n}{\pi n!}\left(\begin{bmatrix} 2j \\ n \end{bmatrix}_Q\right)^{-1} := \bar{w}(y) \tag{23}$$

In the case where series defining the function $\bar{w}(x)$ converges, the inverse Fourier transform is given by

$$\widetilde{w}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx\; e^{-ixy}\bar{w}(y)dy \tag{24}$$

The weight function $w(x)$ allowing for a resolution of the identity operator, in the form given in equation (17), is given by

$$w(x) = \frac{(1+x)_Q^{2j}}{2\pi}\int_{-\infty}^{+\infty} dx\; e^{-ixy}\bar{w}(y)dy. \tag{25}$$

This completes Klauder's criteria to be imposed on the generalized spin coherent states introduced in this paper in order to be worth the nomenclature (suffix) coherent states.

It is important to notice that the resolution of the identity operator has been carried out for the particular case (3) in [23] using a different approach. In fact, this is done by using the same weight function as in the undeformed case but by altering the definition of integration and using q-integration [30].

4 Physical properties of the deformed spin coherent states

In Ref. [16], the deformed spin coherent states correspond to the box function given in equation (2). The physical properties of these states were investigated by studying their behavior through a beam splitter. It proves that these deformed states provide a richer structure than the ordinary spin coherent states. Here we investigate the physical properties of the generalized spin coherent states by computing the corresponding Mandel parameter [31].

The Mandel parameter is defined by

$$\mathbf{Q} = \frac{\langle (\Delta N)^2 \rangle - \langle N \rangle}{\langle N \rangle}. \tag{26}$$

This parameter depends, in our case, on the complex number z which labels the coherent states, on the representation parameter j and eventually on deformation parameters. In the following we are going to study the behaviour of Mandel parameter for the main known deformation of spin algebra; namely the deformation associated with the box functions (2) and the one associated with equation (3). In both cases, we study the dependence of Mandel parameter on the three parameters indicated above.

Let's start with the deformation corresponding to equation (3). Figure 1, shows the graph of Mandel's $\mathbf{Q}$ parameter as a function of q and j for $|z| = 3$ while Figure 2, shows the same function for $|z| = 1$. We see from the first graph that while being always negative, $\mathbf{Q}$ increases as q approaches the value $q = 1$ and it decreases as the value of the *spin* increases. This means that the deformed spin coherent states satisfy a sub-Poissonian statistical distribution; they are the closest to *classical* states (states satisfying Poissonian distributions) for $q = 1$ and small values of j. A completely different behavior is noticed in Figure 2 for $|z| = 1$. In fact, $\mathbf{Q}$ decreases as q approaches the value $q = 1$ and it increases with j. But the most striking phenomenon here, is that Mandel's parameter changes sign which indicates a change in the nature of the statistics described. The value at which this occurs (that is the value at which we have a Poissonian distribution) depends both on j and q, as is shown in Figure 3 and Figure 4. The difference in the behavior of Mandel's parameter noticed here between the two cases $|z| = 1$ and $|z| = 3$ is in fact genuine between $|z| \leq 1$ and $|z| > 1$.

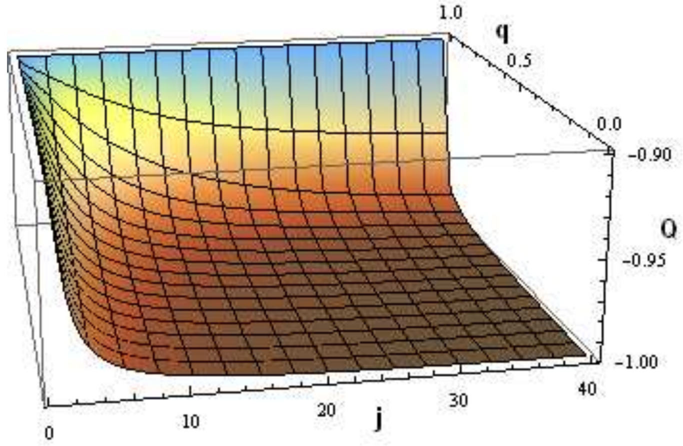

Figure 1. Mandel's parameter as a function of q and j for $|z| = 3$.

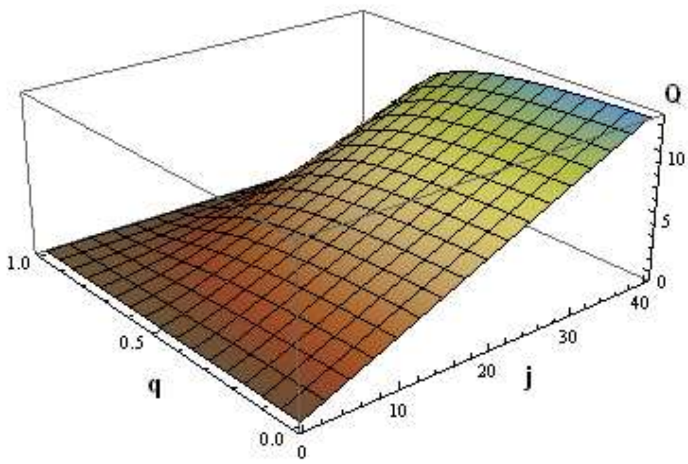

Figure 2. Mandel's parameter as a function of q and j for $|z| = 1$.

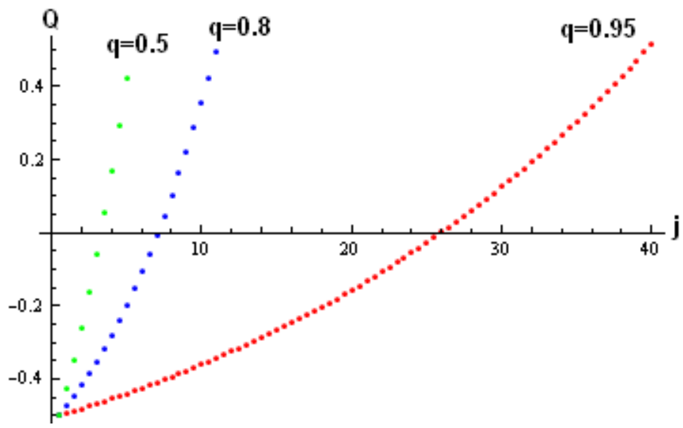

Figure 3. Mandel's parameter as a function of j for $|z| = 1$ for different values of q.

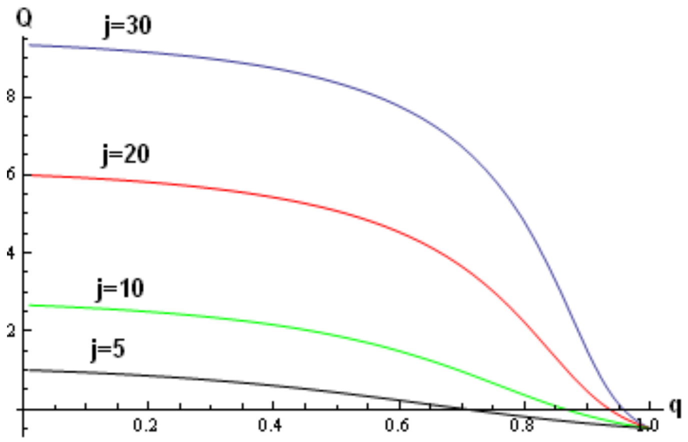

Figure 4. Mandel's parameter as a function of q for $|z| = 1$ for different values of j.

In the case of the other box function equation (2) we have a similar behavior, as in the previous case for $|z| \leq 1$ (see Figure 5). The difference occurs when $|z| > 1$, here **Q** is always negative even though, as a function of either j or q, it is not strictly monotone as in the previous cases (see Figure 6 and Figure 7).

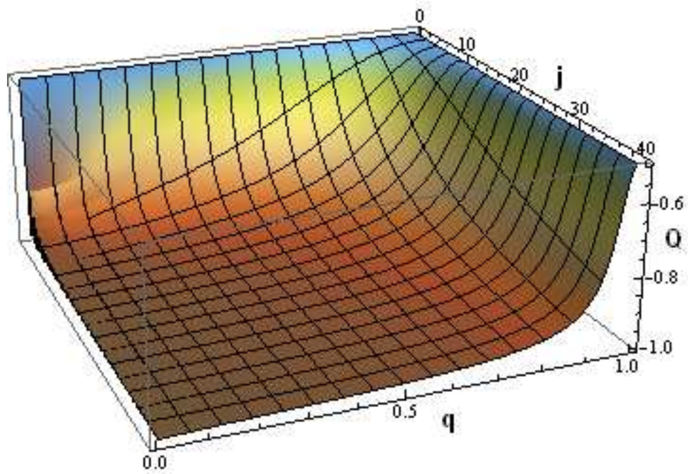

Figure 5. Mandel's parameter as a function of q and j for $|z| = 1$.

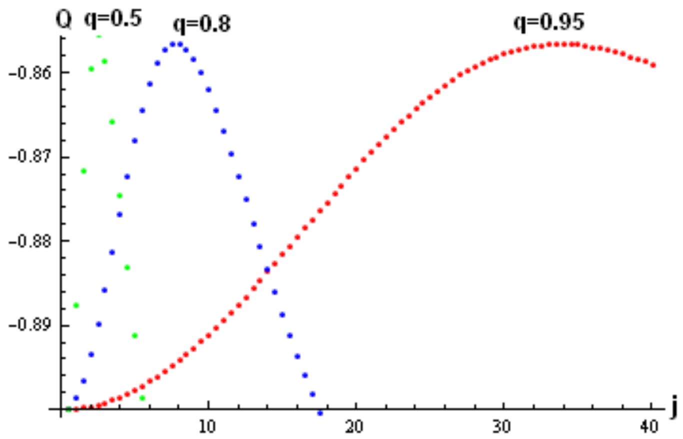

Figure 6. Mandel's parameter as a function of j for $|z| = 3$ for different values of q.

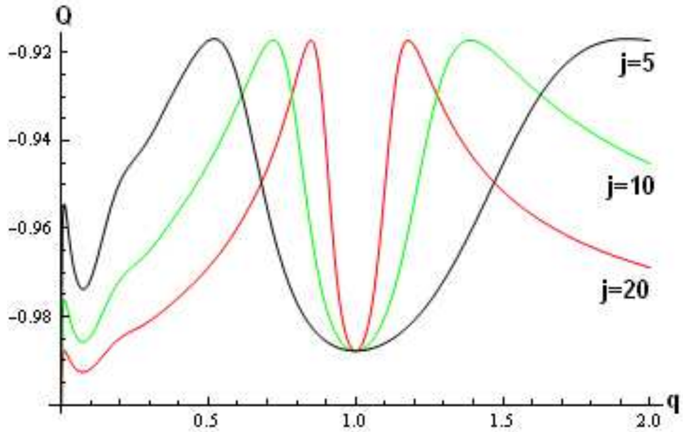

Figure 7. Mandel's parameter as a function of q for $|z| = 3$ for different values of j.

This non-monotony was also noticed in [16] where the same states (deformed spin coherent states (15) corresponding to the box function (2)). The classicality of the states, being related to the phenomenon of entanglement, was then studied by investigating the behavior of these states after being reflected on 50:50 beam-splitter. In fact, the actual results do confirm those obtained in [16] using a different approach.

5 Summary

As a summary, we presented in this paper an approach to construct generalized su(2) (spin) algebras and to build the related generalized spin coherent states. The corresponding statistical distributions obeyed by such states was studied using the Mandel parameter and this gives some hints about their *classicality*. The study was carried out for two particular deformations. In both cases it proves that the deformed spin coherent states provide a much richer structure than the undeformed ones. One can convince oneself by checking the graphs in Figures 4 and 7. In fact, the states that are the closest to the *classical* ones correspond to a value $q \neq 1$. Similarly the states that are the farthest correspond also to another value $q \neq 1$. We recall that in both cases the undeformed coherent states are obtained by taking the particular case $q = 1$. So, either one is interested in constructing non classical states (to construct maximally entangled states for example) or on the contrary classical states (to avoid quantum effects), the best choice is provided by deformed spin coherent states. What kind of deformation (box function) to be chosen is a matter of how far or how close we want our states from the classical case. For instance, the deformation (3) provides states with Poissonian distribution (i.e. $\mathbf{Q} = 0$) as shown in Figure 4. This deformation provides also states with super-Poissonian, as well as sub-Poissonain statistics, which is not the case for the other deformation considered here (2).

As mentioned at the beginning, the two particular deformations (2) and (3) are not the only cases that one can consider. Other deformations can be considered corresponding to other choices of the box function. For example, two parametric deformations have been considered in the literature [32] and it will be interesting to construct the corresponding coherent states and investigate the physical properties.

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