Solution to Bethe-Salpeter equation via Mellin-Barnes transform

Pedro Allendes [(a)], Bernd Kniehl [(b)], Igor Kondrashuk [(c)],
Eduardo A. Notte Cuello [(d)], Marko Rojas Medar [(c)]

(a) Departamento de Física, Universidad de Concepción, Casilla 160-C, Concepción, Chile
(b) II. Institut fur Theoretische Physik, Universitat Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany
(c) Departamento de Ciencias Básicas, Universidad del Bío-Bío, Campus Fernando May, Casilla 447, Chillán, Chile
(d) Departamento de Matemáticas, Facultad de Ciencias, Universidad de La Serena, Av. Cisternas 1200, La Serena, Chile

Abstract

We consider Mellin-Barnes transform of triangle ladder-like scalar diagram in $d = 4$ dimensions. It is shown how multi-fold MB transform of the momentum integral corresponding to any number of rungs is reduced to two-fold MB transform. For this purpose we use Belokurov-Usyukina reduction method for four-dimensional scalar integrals in the position space. The result is represented in terms of Euler ψ-function and its derivatives. We derive new formulas for MB two-fold integration in the complex planes of two complex variables. We demonstrate that these formulas solve Bethe-Salpeter equation. We comment on further applications of solution to Bethe-Salpeter equation for vertices in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. We show that the recursive property of MB transforms observed in the present work for that kind of diagrams has nothing to do with quantum field theory, theory of integral transforms, or with theory of polylogarithms in general, but has an origin in a simple recursive property for smooth functions which can be shown by using basic methods of mathematical analysis.

Keywords: Mellin-Barnes transform, UD functions, continuous functions

1 Introduction

In this paper we study Mellin-Barnes (MB) transform of three-point scalar ladder-like integrals in $d = 4$ space-time dimensions. These integrals contribute to Green's functions of four-dimensional massless scalar theories and can be represented in terms of UD functions [1, 2]. As has been proven in Refs. [4, 3, 5] at the level of graphs, the UD functions transform to themselves under Fourier transform. Also, this property can be proven by making use of MB transform [6].

As has been shown in Ref. [5], any other scalar three-point Green function even in non-integer $d = 4 - 2\epsilon$ dimensions possesses such a property of invariance with respect to Fourier transform. This happens due to the fact that any three-point function can diagrammatically be considered as a net of consequent three-point or four-point integrations in the position space since any of those integrations in the position space can be represented in terms of MB integral transforms with powers of spacetime intervals in denominators depending on the MB transform parameters. Doing consequently momentum integration via MB transform, we come to an expression in which there is no momentum integral but there are integrations in the complex planes of MB transform parameters. We can apply the arguments of Ref. [5] to the expressions of such a type in order to demonstrate their invariance with respect to Fourier transformation.

In general, the technique of MB transform is a powerful method to perform complicate multiloop calculations in the quantum field theory [7, 8, 9, 10]. For example, the three-point function in the position space in massless scalar theory is a combination of Appell's hypergeometric functions [11], which have appeared as a result of the residue calculation via MB transform. In Ref. [6] it has been shown that three-point integrations in the position space the powers of space-time intervals in the denominators equal to integer numbers shifted by some multipliers of ϵ, where ϵ is a parameter of dimensional regularization, $d = 4 - 2\epsilon$, the Appell's hypergeometric function is reduced to a combination of the UD functions. On the other side, four-point function in the position space in the scalar theory in $d = 4$ dimensions can be represented in terms of UD functions too [12]. The invariance of those four-point ladder integrals under Fourier transform can be proved by means of MB transform too [12]. All these properties make UD functions very attractive for further investigation by making use of MB technique. The main advantage of this method that we use in the present paper is that the three-point d-dimensional integration is transformed to power-like form, which can be integrated in further loops as a three-point integration with powers of space-time intervals in denominators depending on the MB transform complex variables.

The first aim of this paper is to investigate further the properties of UD functions via MB transform. As a by-product, we derive new formulas for MB two-fold integration in complex planes of two variables. The third aim is to demonstrate the compatibility of the formulas for the MB transform and Bethe-Salpeter equation for the infinite sum of triangle-like and box-like scalar ladders.

The fourth motivation for the investigation developed in this paper is to apply the experience obtained with studying the Bethe-Salpeter equation for the scalar ladders to the calculation of an auxiliary vertex or Green function, which in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory does not depend on any scale, ultraviolet or infrared, in the limit of removing dimensional regularization, $\epsilon = 0$, $d = 4$. That vertex should satisfy to the Bethe-Salpeter equation, particular for this Green function. This is Lcc vertex in which the auxiliary field L couples to two (self-adjoint) Faddeev-Popov ghost fields c. To investigate that vertex by Bethe-Salpeter equation has a sense since the ghost fields are only fields of this vertex which stand in the measure of the path integral, and therefore this equation restricts the vertex strongly.

As has been shown in Refs. [13, 14, 15], in $\mathcal{N}=4$ supersymmetric Yang-Mills theory starting with this vertex all other correlators can be found in four-dimensional limit (when $d=4$) by using Slavnov-Taylor identity [16, 17, 18, 19, 20, 21, 22, 23]. The explicit two-loop result for that vertex has been found in Refs. [24, 25, 26]. All the poles in ϵ disappear in all number of loops for this vertex, however it cannot be analysed by the methods of conformal field theory since the auxiliary field L does not propagate. Nevertheless, the three-point function of dressed mean gluon of Refs. [13, 14, 15] can be fixed by conformal symmetry in analogy to [27, 28, 29, 30, 31], however, we cannot find higher-point correlators of dressed mean field from this Green function by using ST identity.

As it can be seen from Refs. [13, 14, 15], the explicit structure of Lcc correlator includes logarithms and UD functions of ratios of spacetime distances between arguments of Green functions. As it follows from Ref.[4], the ladder-like diagrams with horizontal gluon lines, that are important subclass of all the diagrams contributing to that vertex can be represented in terms of UD functions too for any number of horizontal gluon lines. In the next paper we will demonstrate that any of the contributions to the correlator can be represented in terms of UD functions [32].

In this paper we consider a particular solution to Bethe-Salpeter equation, that represents the infinite sum of triangle scalar ladder diagrams in four space-time dimensions in MB representation. This particular solution has already been studied in the case of box-ladder diagrams by Broadhurst and Davydychev in Ref. [34]. The authors of [34] calculated a sum of those box-ladder diagrams by making use integral representation for UD functions of Ref. [2]. As has been shown in Refs.[1, 2], triangle-like and box-like ladders in four space-time dimensions are related by conformal transformation in dual space and their MB transforms coincide. The difference of the present paper with the approach of Ref. [34] is that we do the calculation via MB transformation. We derive recursive relations for MB transforms of momentum integrals corresponding to UD triangle diagrams by using Belokurov-Usyukina trick [33] for ladder diagrams in the position space, reducing multi-fold MB integrals to two-fold MB integrals.

2 Loop reduction in $d=4$ dimensions

In this section we describe a trick with a help of which a triangle ladder diagram of n loop is reduced to a diagram with $n-1$ loop. This trick has been published for the first time in paper [33]. The effect of the loop reduction has earlier been discovered in Ref. [35] for propagator-type diagram of a particular topology in a special limit for indices of diagrams. Note, that in Ref. [33] the derivation has been performed without that special limit. Later, the key intermediate point of the trick has been published in Ref. [36]. Since all of that together has never been published in one article in detail, we collect all the results and intermediate steps we need for deriving our formulas for MB transform of triangle-ladder diagram.

In the first line of fig. (1) we present the formula which we want to derive. This formula is present in paper [36], however with a bit different indices of the diagrams. Later we will see how the l.h.s. of the first line of (1) can be put in equivalence to fig. (4) of [36]. The second and third lines of fig. (1) are consequences of the first line. How exactly the second and third lines can be derived from the first line we explain further. Usually, it can be done by inserting points of integration into the lines of graphs. The third line is published in Ref. [33]. Second line has never been published. The condition for the ε-terms in the indices is

$$\varepsilon_1+\varepsilon_2+\varepsilon_3=0.$$

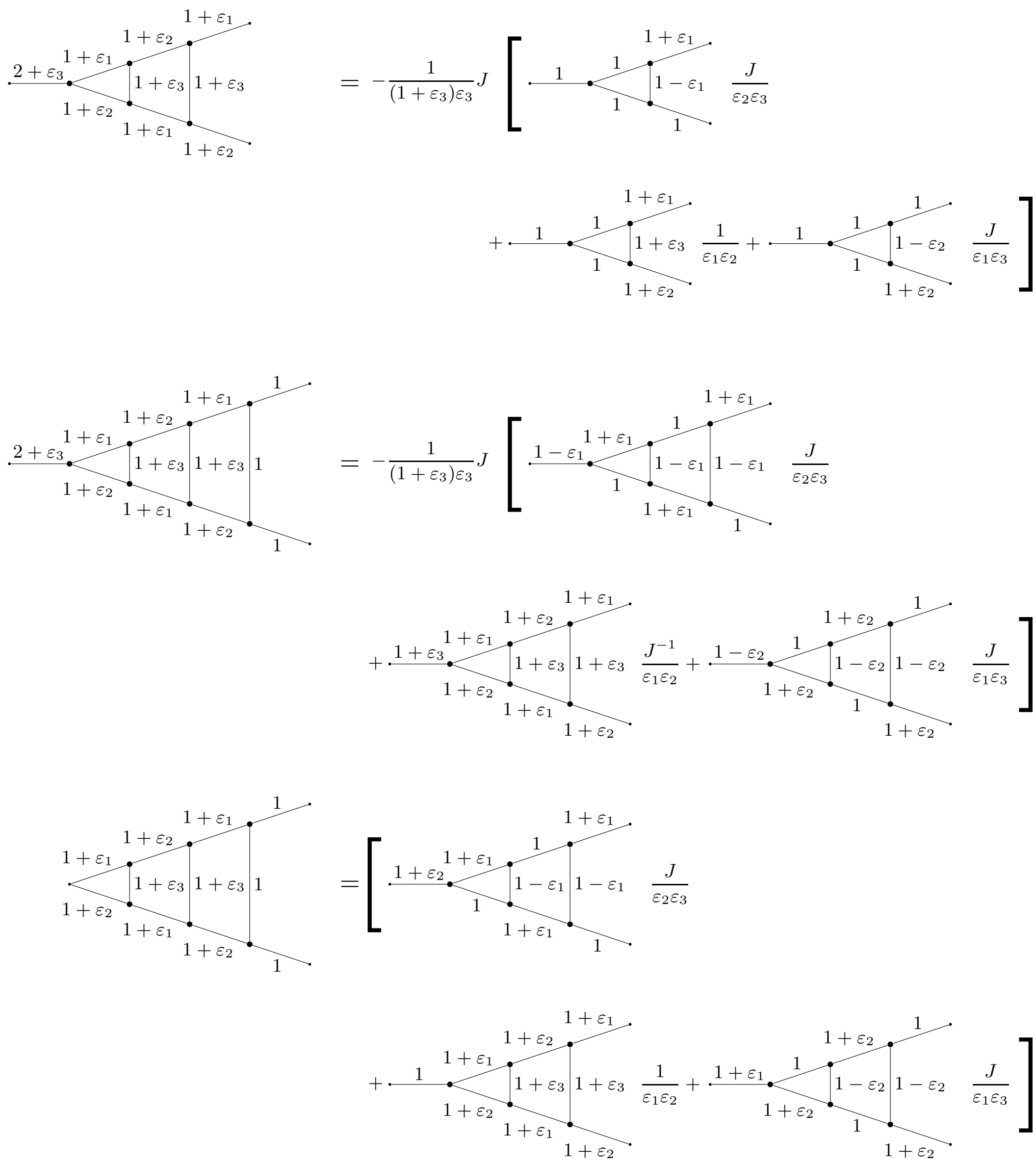

Figure 1: Basic relations

In Fig.(2) and in Fig.(3), the latter is a continuation of Fig.(2), we have shown how the first line of Fig.(1) can be derived. Actually, the only mathematics that has been used here is the uniqueness star-triangle relation and integration by part, both relations are well-known in the scalar field theories [37, 38, 39] (for a short review, Ref. [40]). In all the figures from (1) till (8) we use for

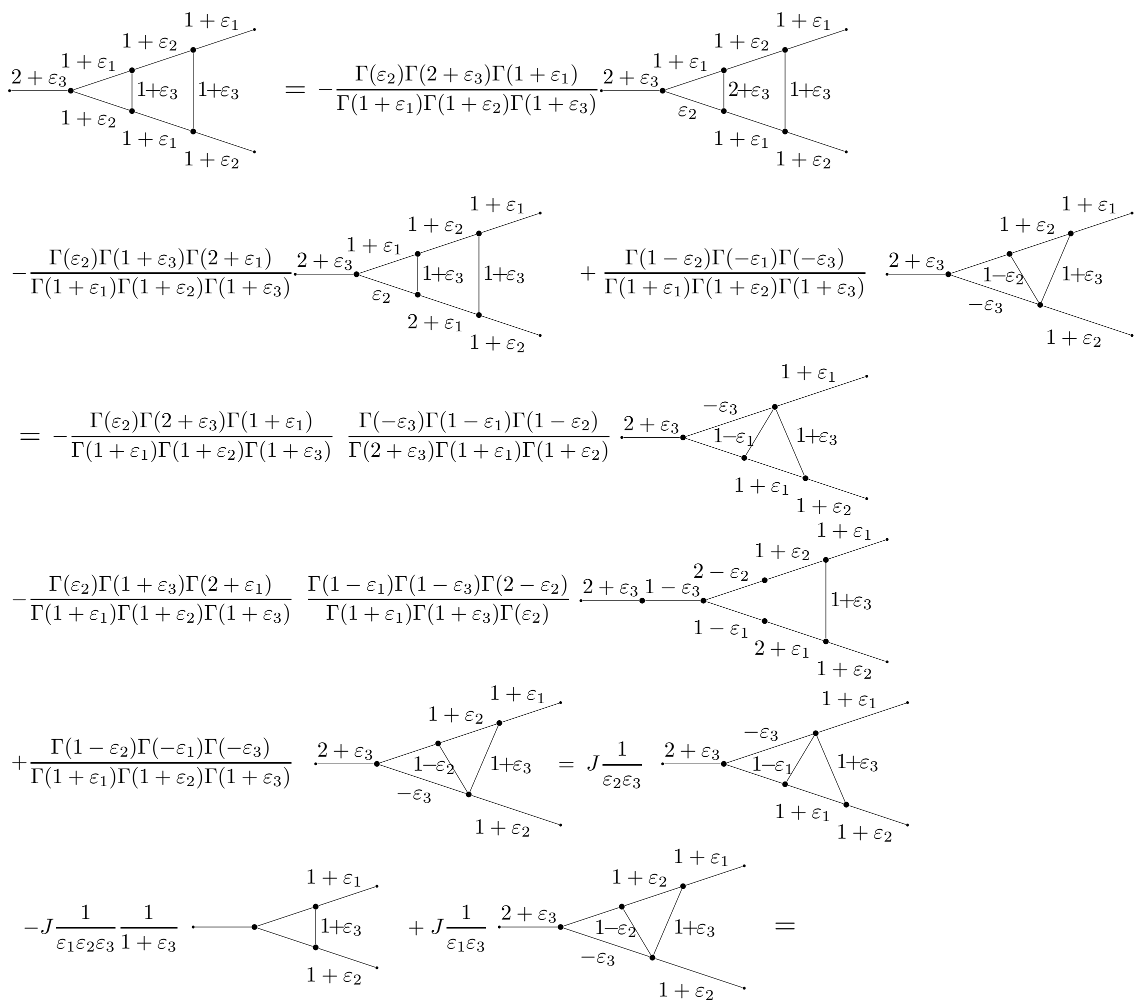

Figure 2: Derivation of the basic relation. Part I. Lines without any index are the lines with index 1

factor J taken from Ref. [33] the definition

$$J = \frac{\Gamma(1-\varepsilon_1)\Gamma(1-\varepsilon_2)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3)}.$$

In fig. (4) we show how the first diagram on the r.h.s. of the first line of fig. (1) transforms to the first diagram on the r.h.s. of the second line of figure (1). Again, we converted non-unique vertices to unique vertices by inserting the point of integration in one of the propagators attached

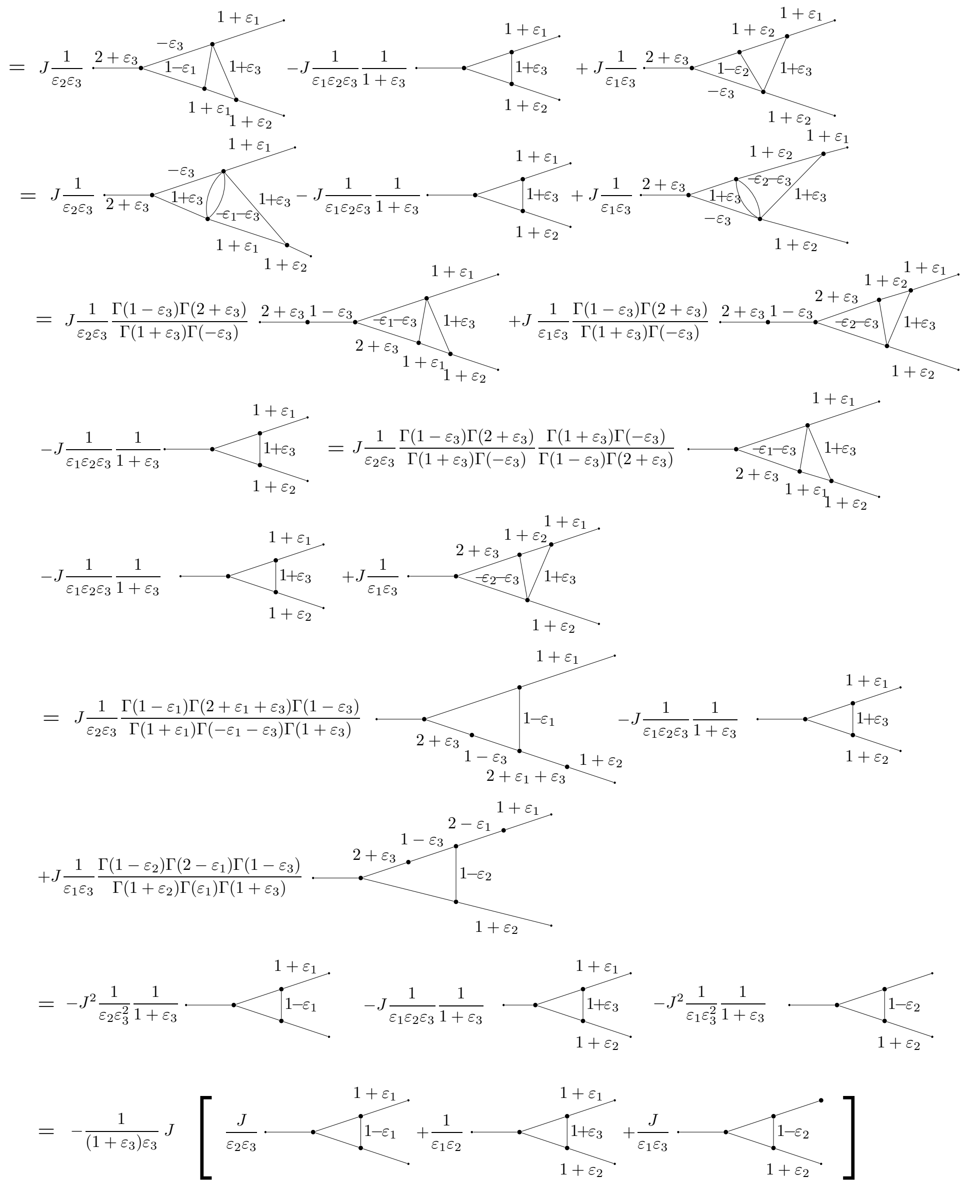

Figure 3: Derivation of the basic relation. Part II. Lines without any index are the lines with index 1

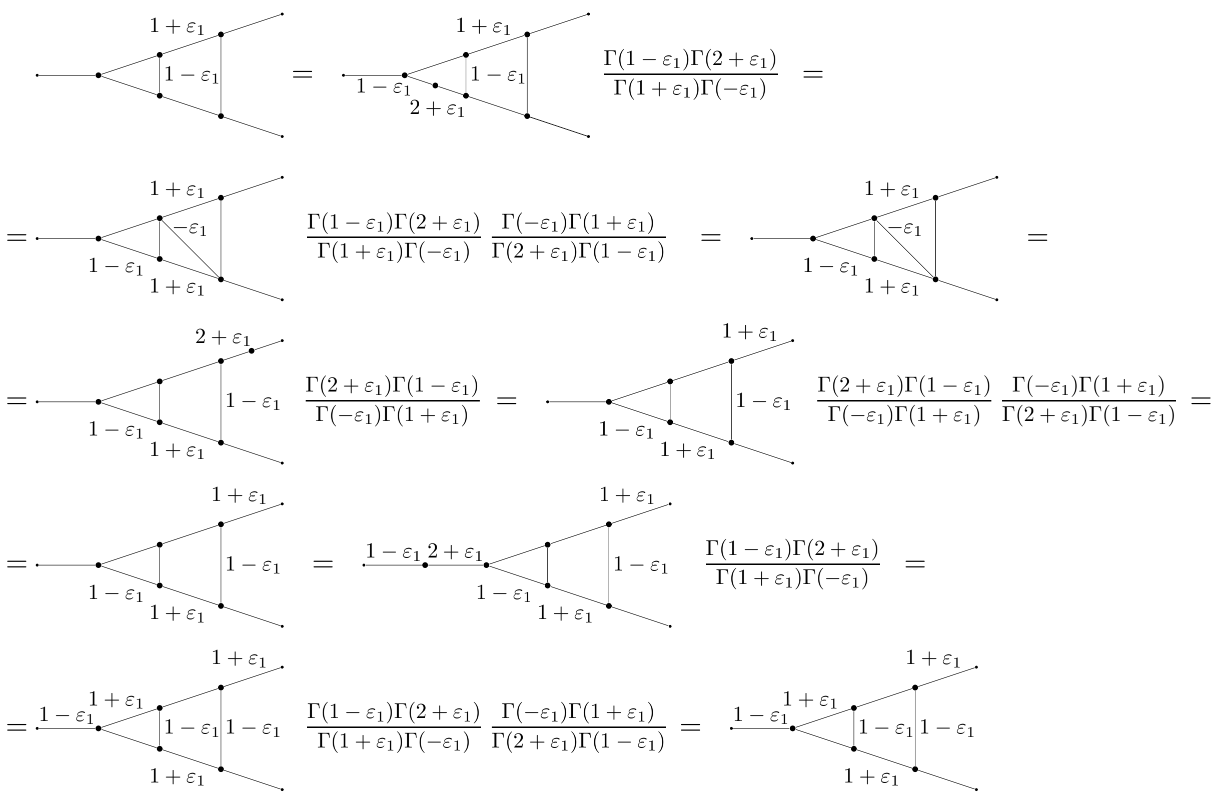

Figure 4: Transformation of the first diagram in Fig. (1). Lines without any index are the lines with index 1

to the vertex. Then, the star-triangle relation has been applied in direct way or in a triangle-star way, until the diagram is converted to the desirable values of indices. The line without any index means that the index of the line is 1. We put the exact value of this index on the corresponding propagators of fig. (1) only. In the rest of figures we omit the index 1 just not to overload the diagram with indices.

In fig. (5) the second diagram of the first line of fig. (1) is transformed to the second diagram of the second line of fig. (1). Again, nothing else but the creating unique vertices and triangles by putting new points of integration in the propagators has been used to create the desired indices. Thus, we have proven the first line and the second line of fig. (1).

The third line of fig. (1) can be obtained from the second line of fig. (1) by simple integration. Indeed, we eliminate the leftmost propagator on the l.h.s. of the second line by convoluting its leftmost point with the line whose index is $2-\varepsilon_3$,

$$\int d^4x \frac{1}{\left[(x_1-x)^2\right]^{2-\varepsilon_3}\left[(x_2-x)^2\right]^{2+\varepsilon_3}} \sim \frac{\Gamma(\varepsilon_3)}{\Gamma(2-\varepsilon_3)}\frac{\Gamma(-\varepsilon_3)}{\Gamma(2+\varepsilon_3)}\delta^{(4)}(x_1-x_2). \tag{1}$$

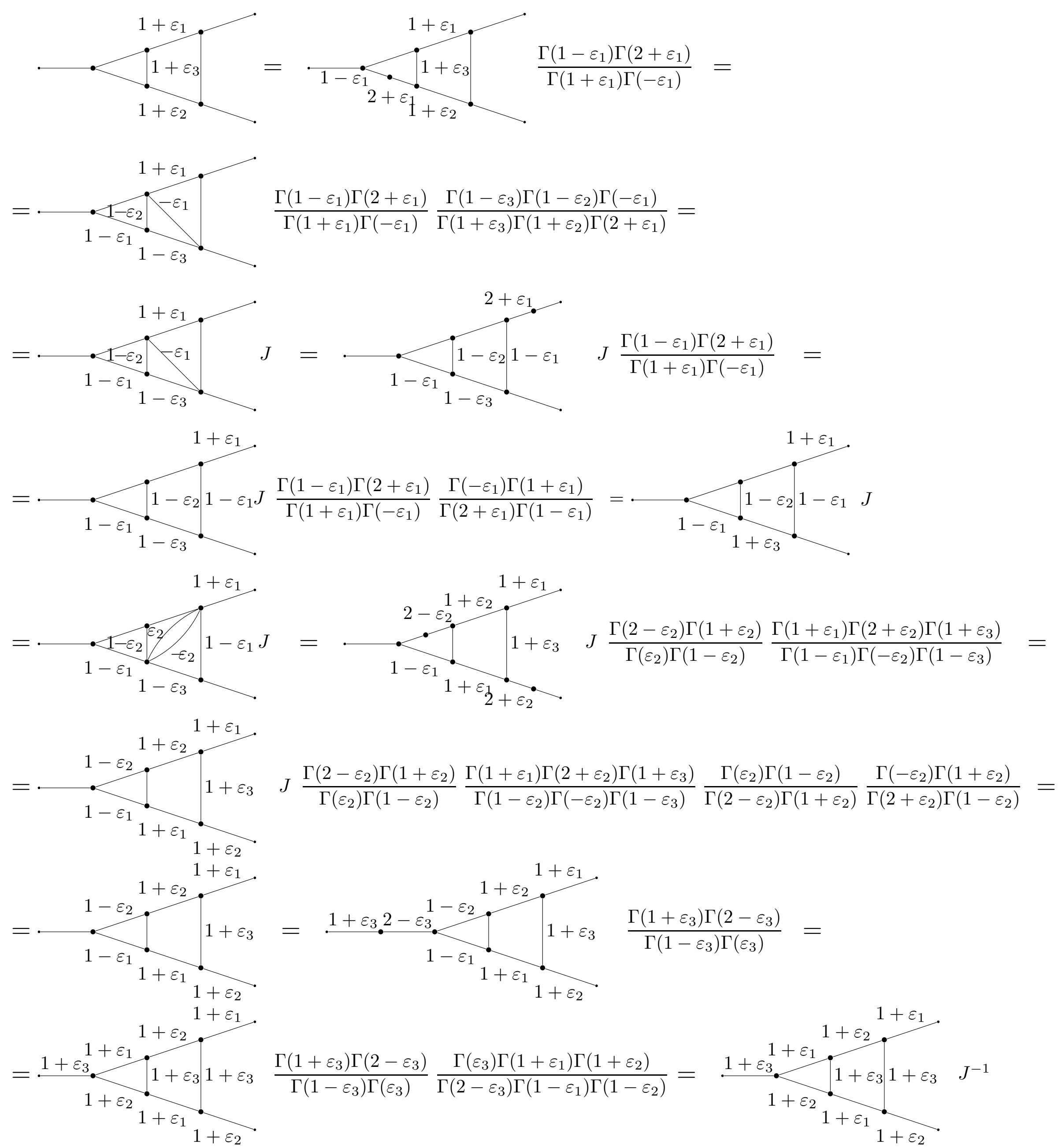

Figure 5: Transformation of the second diagram in Fig. (1)

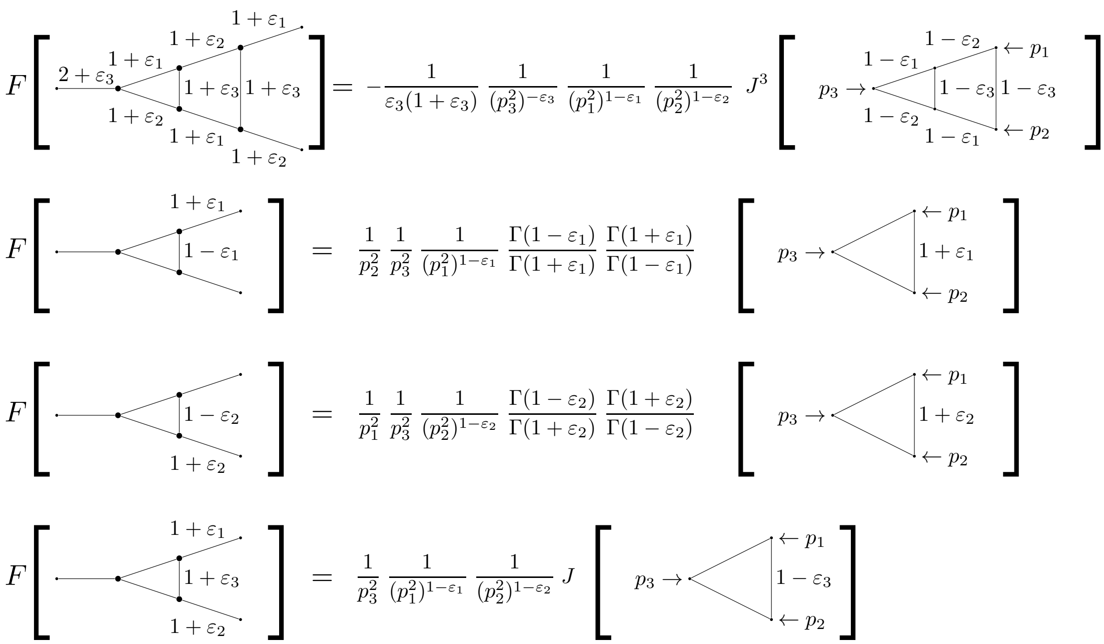

Figure 6: Fourier transform of the diagrams in Fig. (1)

This formula follows the chain integration

$$\int Dx \frac{1}{[(x_1-x)^2]^{\alpha_1}\,[(x_2-x)^2]^{\alpha_2}} = \frac{A(\alpha_1,\alpha_2,d-\alpha_1-\alpha_2)}{[(x_1-x_2)^2]^{\alpha_1+\alpha_2-d/2}},$$

$$A(\alpha_1,\alpha_2,\alpha_3) = a(\alpha_1)a(\alpha_2)a(\alpha_3), \tag{2}$$

a new d-dimensional measure $Dx \equiv \pi^{-\frac{d}{2}} d^d x$ was introduced in ref. [24]. To derive formula (1) we need to replace in Eq. (2) each factor in the integrand with its integral Fourier transform. The formulas for the Fourier transform of the factor like those can be found in Ref. [40].

2.1 Description of Fourier Transform for the diagrams

In fig. (6) the Fourier transform is made for all the diagrams that appear in the first line of fig. (1). The procedure of Fourier transform is done as follows. We replace each factor in the integrand of the position space representation with the integral Fourier transform of the corresponding factor in momentum space representation. Space-time coordinates appear in the exponentials of Fourier transforms. Integrating over the coordinates of internal vertices we crate Dirac δ-functions, corresponding to the momentum conservation in each vertex of integration in the position space (internal vertex). The corresponding momentum integrals over loop momenta will be the Fourier transforms of the integrals in the position space. For example, the described procedure in a particular case of

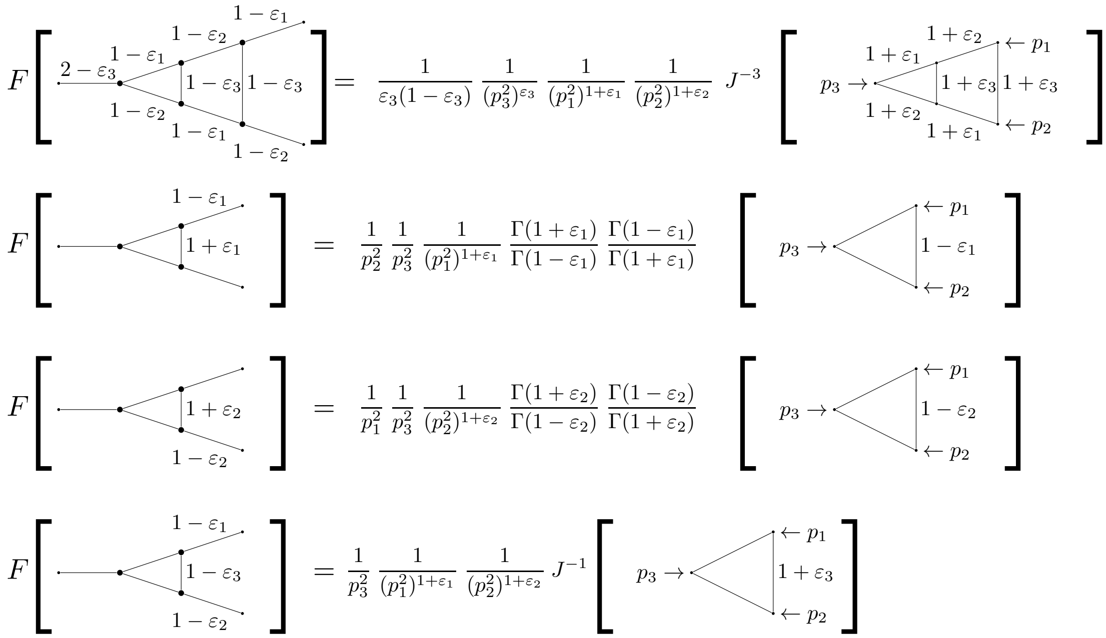

Figure 7: Fourier transform of the diagrams in Fig. (1)

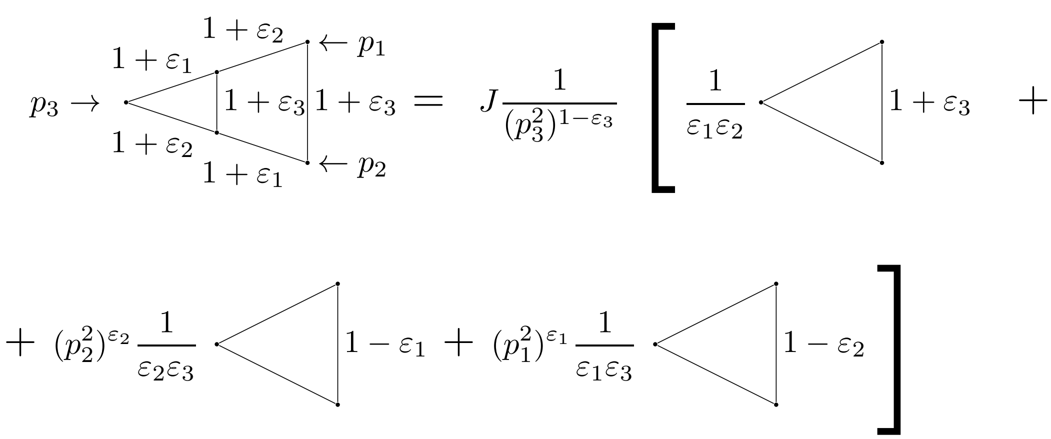

Figure 8: Formula (25) of Ref. [1]

one-rung integral in the position space would result in

$$\frac{1}{[31]}\int d^4y\; d^4z\,\frac{1}{[2y][1y][3z][yz][2z]} = \frac{1}{(2\pi)^4}\int d^4p_1 d^4p_2 d^4p_3\;\delta(p_1+p_2+p_3)\times$$
$$\times e^{ip_2x_2}e^{ip_1x_1}e^{ip_3x_3}C^{(2)}(p_1^2,p_2^2,p_3^2)$$

We assume the notation of Ref. [24], where $[Ny] = (x_N - y)^2$ and analogously for $[Nz]$ and $[yz]$, that is, $N = 1, 2, 3$ stands for $x_N = x_1, x_2, x_3$, respectively, which are external points of the triangle-like ladder diagram. In Refs. [1, 2] $C^{(n)}(p_1^2, p_2^2, p_3^2)$ is the definition for the result of momentum integrals for n-rung triangle ladder diagram in the momentum space representation. Just to make clear the definition for Fourier transform operation used in fig. (6), we provide an example of the relation

$$F\left[\frac{1}{[31]}\int\ d^4y\ d^4z \frac{1}{[2y][1y][3z][yz][2z]}\right] = C^{(2)}(p_1^2, p_2^2, p_3^2).$$

The transformation from position space to momentum space is necessary to normalize on the results for MB transform of UD functions done in [1, 2] for the momentum space integrals.

Fig. (7) is fig. (6) with signs changed for all ε-terms. After this change of signs, the Fourier transform of the first line of fig. (1) takes the form depicted in fig. (8). In that diagrammatic relation we recognize Eq. (25) of Ref. [1].

3 Recursive formula for MB transform of UD functions

We start this section with writing the definition of the momentum integral for the diagram in the r.h.s. of Fig. (8). We take a formula from paper [1] for the representation of the three-point momentum integral

$$J(\nu_1, \nu_2, \nu_3) = \int\ Dk\ \frac{1}{\left[(k+q_1)^2\right]^{\nu_1}\left[(k+q_2)^2\right]^{\nu_2}\left[(k+q_3)^2\right]^{\nu_3}}.$$

in terms of MB transform

$$\begin{aligned} J(\nu_1, \nu_2, \nu_3) = \frac{1}{\Pi_i\Gamma(\nu_i)\Gamma(d-\Sigma_i\nu_i)}\frac{1}{(p_3^2)^{\Sigma\nu_i - d/2}}\oint_C dz_2\ dz_3 x^{z_2}y^{z_3}\left\{\Gamma\left(-z_2\right)\Gamma\left(-z_3\right)\right. \\ \left.\Gamma\left(-z_2-\nu_2-\nu_3+d/2\right)\Gamma\left(-z_3-\nu_1-\nu_3+d/2\right)\Gamma\left(z_2+z_3+\nu_3\right)\Gamma\left(\Sigma\nu_i-d/2+z_3+z_2\right)\right\} \equiv \\ \equiv \frac{1}{(p_3^2)^{\Sigma\nu_i-d/2}}\oint_C dz_2\ dz_3 x^{z_2}y^{z_3}D^{(z_2,z_3)}[\nu_1,\nu_2,\nu_3], \end{aligned} \tag{3}$$

where we have introduced definition of function D,

$$\begin{aligned} D^{(z_2,z_3)}[\nu_1,\nu_2,\nu_3] = \frac{\Gamma\left(-z_2\right)\Gamma\left(-z_3\right)\Gamma\left(-z_2-\nu_2-\nu_3+d/2\right)\Gamma\left(-z_3-\nu_1-\nu_3+d/2\right)}{\Pi_i\Gamma(\nu_i)} \\ \times\frac{\Gamma\left(z_2+z_3+\nu_3\right)\Gamma\left(\Sigma\nu_i-d/2+z_3+z_2\right)}{\Gamma(d-\Sigma_i\nu_i)} \end{aligned} \tag{4}$$

3.1 Description of notation in MB representation

We absorb into the definition of the MB transform $D^{(z_2,z_3)}[\nu_1,\nu_2,\nu_3]$ of this three-point integral all the factors except for a power of the square of the external momentum p_3^2. For $d = 4$ in the denominator we have a sum of the indices minus two for the power of p_3^2.

We do not write the powers of i, supposing that we work in Euclidean space and the corresponding power of i can be recovered back after Wick rotation.

We follow the notation of Refs.[24, 6] and absorb a power of π in the measure of integration. In the rest of the paper we use the notation

$$x \equiv \frac{p_1^2}{p_3^2}, \quad y \equiv \frac{p_2^2}{p_3^2},$$

where the d-dimensional momentums p_1, p_2, p_3 satisfy the conservation law $p_1 + p_2 + p_3 = 0$ and are related to d-dimensional momenta q_1, q_2, q_3 by a parametrization

$$\begin{aligned} p_1 &= q_3 - q_2, \\ p_2 &= q_1 - q_3, \\ p_3 &= q_2 - q_1. \end{aligned}$$

As a consequence of this definition, p_1 appears to be a momentum that enters the one-loop triangle diagram in the vertex of the triangle which is opposite to the line with index ν_1.

The contour of integration C passes a bit on the left of the imaginary axis, separates left and right poles and should be closed to the left infinity or to the right infinity. We choose to close the contour of integration in the complex plane to the right infinity. It could be closed to the left infinity too but it makes more complicate to take the residues into account since the residues in variables z_2 and z_3 are mixed in that case. Whether we have to close the contour to the right infinity or to the left infinity, the result should be the same function. We omit the factor $1/2\pi i$ that accompanies each integration over MB transform parameter. The inverse factor is generated in front of the residues.

It is known that this representation of the three-point integral can be derived by applying two-fold MB transform to the integral over Feynman parameters, producing the Euler beta functions [7] after integrating these parameters. There is a difference with the representation used in Ref.[6]. The form of Ref.[6] can be recovered from (3) by a cyclic redefinition. In the rest of the paper, we use representation (3).

3.2 Multi-fold MB transforms of UD functions

To develop the recursive equations for the MB transforms of the UD functions, we recover the definition of MB transforms for the UD functions and for the ladder triangle diagrams in $d = 4$ spacetime dimensions

$$C^{(n)}(p_1^2, p_2^2, p_3^2) = \frac{1}{(p_3^2)^n}\Phi^{(n)}\left(x,y\right) = \frac{1}{(p_3^2)^n}\oint_C dz_2\; dz_3 x^{z_2} y^{z_3} \mathcal{M}^{(n)}(z_2,z_3),$$

where we used the notation $\mathcal{M}^{(n)}(z_2,z_3)$ for the MB transform of UD function $\Phi^{(n)}\left(x,y\right)$ of two variables. The definition of the incoming momenta for the triangle ladder diagrams is like in Ref.[1, 2, 5]. We include in the definition of $\mathcal{M}^{(n)}(z_2,z_3)$ all the factors that can appear in front of MB transformation except for the power of the momentum p_3^2.

The two-fold MB transform is known for the first UD function only. For the higher number of UD function, the result is given in terms of multi-fold MB transform [1, 2]. In the next section we reduce the multi-fold MB transform to a two-fold MB transform by making use of the loop reduction trick described in the previous section.

The iterative integral relation for the triangle ladder diagram given in Refs. [1, 2, 5],

$$C^{(n)}(p_1^2,p_2^2,p_3^2) = \int \; d^4r_n \; \frac{C^{(n-1)}((p_1+r_n)^2,(p_2-r_n)^2,p_3^2)}{(p_1+r_n)^2(p_2-r_n)^2r_n^2},$$

results in an integral recursive relation for the MB transforms of UD functions

$$\Phi^{(n+1)}\left(\frac{p_1^2}{p_3^2},\frac{p_2^2}{p_3^2}\right) = p_3^2\int \; d^4r \; \Phi^{(n)}\left(\frac{(p_1+r)^2}{p_3^2},\frac{(p_2-r)^2}{p_3^2}\right)\frac{1}{(p_1+r)^2(p_2-r)^2r^2} =$$
$$p_3^2\int \; d^4r \; \oint_C dz_2\; dz_3\mathcal{M}^{(n)}(z_2,z_3)\left(\frac{(p_1+r)^2}{p_3^2}\right)^{z_2}\left(\frac{(p_2-r)^2}{p_3^2}\right)^{z_3}\frac{1}{(p_1+r)^2(p_2-r)^2r^2} =$$
$$\int \; d^4r \; \oint_C dz_2\; dz_3\mathcal{M}^{(n)}(z_2,z_3)\frac{1}{((p_1+r)^2)^{1-z_2}((p_2-r)^2)^{1-z_3}r^2}\frac{1}{(p_3^2)^{z_2+z_3-1}} =$$
$$\pi^2\oint_C dz_2\; dz_3\mathcal{M}^{(n)}(z_2,z_3)J(1-z_3,1-z_2,1)\frac{1}{(p_3^2)^{z_2+z_3-1}} =$$
$$\pi^2\oint_C dz_2\; dz_3\; du\; dv\; x^uy^v\mathcal{M}^{(n)}(z_2,z_3)D^{(u,v)}[1-z_3,1-z_2,1],$$

where $D^{(u,v)}[1-z_3,1-z_2,1]$ is defined above as MB transform of the corresponding three-point integral, according to the description done in the previous subsection.

Thus, one can derive the first recursive formula for MB transform, which relates the transforms of two neighbour UD functions

$$\mathcal{M}^{(n+1)}(u,v) = \pi^2\oint_C dz_2\; dz_3\mathcal{M}^{(n)}(z_2,z_3)\; D^{(u,v)}[1-z_3,1-z_2,1].$$

The function $D^{(u,v)}[1-z_3,1-z_2,1]$ is a nontrivial combination of the Euler gamma functions in numerator and denominator,

$$D^{(u,v)}[1-z_3,1-z_2,1] = \frac{\Gamma(z_2-u)\,\Gamma(z_3-v)\,\Gamma(1-z_2-z_3+u+v)}{\Gamma(1+z_2+z_3)\,\Gamma(1-z_2)\,\Gamma(1-z_3)}\Gamma(-u)\,\Gamma(-v)\,\Gamma(1+u+v)\,.$$

This formula is written from definition (3).

4 Reduction of multi-fold MB transforms

Formulas of the previous subsection are in some sense consequence of the ladder-like topology of the diagram. We did not make any integration in the complex planes of MB parameters.

Consider the diagram on the l.h.s. of Fig. (8). We can repeat the trick of the previous section and obtain the following MB representation for it

$$
\begin{aligned}
\frac{1}{p_3^2}\int Dr \oint dz_2 dz_3 \left(\frac{(p_1+r)^2}{p_3^2}\right)^{z_2}\left(\frac{(p_2-r)^2}{p_3^2}\right)^{z_3} \frac{D^{(z_2,z_3)}[1+\varepsilon_2,1+\varepsilon_1,1+\varepsilon_3]}{[(p_1+r)^2]^{1+\varepsilon_2}[(p_2-r)^2]^{1+\varepsilon_1}[r^2]^{1+\varepsilon_3}} = \\
\frac{1}{p_3^2}\oint_C Dr\ dz_2 dz_3 \frac{1}{[p_2^3]^{z_2+z_3}} \frac{D^{(z_2,z_3)}[1+\varepsilon_2,1+\varepsilon_1,1+\varepsilon_3]}{[(p_1+r)^2]^{1+\varepsilon_2-z_2}[(p_2-r)^2]^{1+\varepsilon_1-z_3}[r^2]^{1+\varepsilon_3}} = \\
\frac{1}{p_3^2}\oint_C dz_2 dz_3 \frac{1}{[p_3^2]^{z_2+z_3}} J(1+\varepsilon_1-z_3,1+\varepsilon_2-z_2,1+\varepsilon_3) D^{(z_2,z_3)}[1+\varepsilon_2,1+\varepsilon_1,1+\varepsilon_3] = \\
\frac{1}{(p_3^2)^2}\oint_C dz_2 dz_3\ du\ dv\ x^u\ y^v D^{(u,v)}[1+\varepsilon_1-z_3,1+\varepsilon_2-z_2,1+\varepsilon_3] \times \\
D^{(z_2,z_3)}[1+\varepsilon_2,1+\varepsilon_1,1+\varepsilon_3] \qquad (5)
\end{aligned}
$$

This formula is derived in analogy with MB transforms of UD functions in the previous section. In detail, the procedure looks like follows. First, we calculate the MB transform of the leftmost triangle integral, this yields another triangle integral with indices depending on the complex variables of the previous MB transform. This procedure will be used in all the constructions below.

General strategy of the present investigation consists in the decomposition of the r.h.s. in terms of ε_i will produce ψ functions and its derivatives. In the r.h.s. on Fig. (8) we cannot put all the value of ε_i immediately without decomposing it terms of ε_i and observing that poles in ε_i disappear. Thus, instead of Laurent series we obtain a Taylor series. Comparing the coefficients in front of the different terms of decomposition in terms of ε_i we can derive an infinite number of new relations for two-fold MB integrals over complex parameters z_2 and z_3.

4.1 Description of the momentum integral

Notation to be used

$$
\int_n (\varepsilon_1,\varepsilon_2,\varepsilon_3)
$$

corresponds to the momentum integral with incoming momenta p_1,p_2,p_3 as they are depicted on the l.h.s. of Fig.(8) but with n loops and with amputated external legs. The ε-terms $\varepsilon_1,\varepsilon_2,\varepsilon_3$ appear in the indices of lines for the first eight propagators on the left side of the diagram, namely, in the indices $1+\varepsilon_1, 1+\varepsilon_2, 1+\varepsilon_1$ on the upper side of the diagram, $1+\varepsilon_2, 1+\varepsilon_1, 1+\varepsilon_2$ on the lower side of the diagram, and $1+\varepsilon_3$ on the first two rungs. The rest of lines have indices equal to 1. These positions of the indices are indicated in the third line of Fig.(1).

4.2 Two-fold MB transform for two-rung ladder

As a first step to our formulas, we reproduce Eq.(25) of Ref.[1]. This is already done in the previous section, however we make it again just to introduce the notation that will be used for higher rung diagram.

We consider a diagrammatic relation that can be obtained by integral convolution of the leftmost external point of the diagrams in the first line of fig. (1) with the line that has index $2-\varepsilon_3$. In such a way Dirac δ-function is produced which eliminates one of the integrations and the leftmost point on the l.h.s. is converted to the external vertex, that is, it is not a vertex of integration longer. The formula which can be used for this purpose is identity (1).

Then, we take a Fourier transform of each one of the diagrams in the diagrammatic relation obtained in that way. By keeping all the factors that appear after making the Fourier transform on both the parts of this new diagrammatic equation, we come to

$$J^3 \frac{\Gamma(1+\varepsilon_3)}{\Gamma(1-\varepsilon_3)} \frac{1}{(p_1^2)^{1-\varepsilon_1}(p_2^2)^{1-\varepsilon_2}} \int_2 (-\varepsilon_1, -\varepsilon_2, -\varepsilon_3) =$$
$$J\left[\frac{J}{\varepsilon_2\varepsilon_3} \frac{\Gamma(1+\varepsilon_3)}{\Gamma(1-\varepsilon_3)} \frac{1}{(p_1^2)^{1-\varepsilon_1} p_2^2 (p_3^2)^{1+\varepsilon_3}} J(1,1,1+\varepsilon_1) + \right.$$
$$\frac{J}{\varepsilon_1\varepsilon_2} \frac{\Gamma(1+\varepsilon_3)}{\Gamma(1-\varepsilon_3)} \frac{1}{(p_1^2)^{1-\varepsilon_1}(p_2^2)^{1-\varepsilon_2}(p_3^2)^{1+\varepsilon_3}} J(1,1,1-\varepsilon_3) +$$
$$\left. \frac{J}{\varepsilon_1\varepsilon_3} \frac{\Gamma(1+\varepsilon_3)}{\Gamma(1-\varepsilon_3)} \frac{1}{p_1^2 (p_2^2)^{1-\varepsilon_2}(p_3^2)^{1+\varepsilon_3}} J(1,1,1+\varepsilon_2) \right].$$

After a simple algebra we derive an equation

$$\int_2 (-\varepsilon_1, -\varepsilon_2, -\varepsilon_3) = \frac{J^{-1}}{(p_3^2)^{1+\varepsilon_3}} \left[\frac{1}{\varepsilon_2\varepsilon_3} \frac{1}{(p_2^2)^{\varepsilon_2}} J(1,1,1+\varepsilon_1) + \right.$$
$$\left. \frac{1}{\varepsilon_1\varepsilon_2} J(1,1,1-\varepsilon_3) + \frac{1}{\varepsilon_1\varepsilon_3} \frac{1}{(p_1^2)^{\varepsilon_1}} J(1,1,1+\varepsilon_2) \right],$$

from which by changing the signs of all the values ε_i we obtain

$$\int_2 (\varepsilon_1, \varepsilon_2, \varepsilon_3) = \frac{J}{(p_3^2)^{1-\varepsilon_3}} \left[\frac{1}{\varepsilon_2\varepsilon_3} \frac{1}{(p_2^2)^{-\varepsilon_2}} J(1,1,1-\varepsilon_1) + \right.$$
$$\left. \frac{1}{\varepsilon_1\varepsilon_2} J(1,1,1+\varepsilon_3) + \frac{1}{\varepsilon_1\varepsilon_3} \frac{1}{(p_1^2)^{-\varepsilon_1}} J(1,1,1-\varepsilon_2) \right]. \quad (6)$$

This is exactly Eq.(25) of Ref.[1]. This equation has been derived diagrammatically in the previous section. In that case the Fourier transform of the first line of fig. (1) has been done. This approach to reproduce Eq.(25) of Ref.[1] can be considered as a cross-check of the procedure used in the previous section to produce fig. (8).

At this moment we start to use the matter developed in the previous section. First of all, we rewrite the r.h.s. of Eq.(6) in the MB transformed representation

$$\int_2 (\varepsilon_1, \varepsilon_2, \varepsilon_3) = \frac{J}{(p_3^2)^{1-\varepsilon_3}} \left[\frac{1}{\varepsilon_2\varepsilon_3} \frac{(p_2^2)^{\varepsilon_2}}{(p_3^2)^{1-\varepsilon_1}} \oint_C du\ dv\ x^u\ y^v D^{(u,v)}[1-\varepsilon_1] + \right.$$
$$\left. \frac{1}{\varepsilon_1\varepsilon_2} \frac{1}{(p_3^2)^{1+\varepsilon_3}} \oint_C du\ dv\ x^u\ y^v D^{(u,v)}[1+\varepsilon_3] + \frac{1}{\varepsilon_1\varepsilon_3} \frac{(p_1^2)^{\varepsilon_1}}{(p_3^2)^{1-\varepsilon_2}} \oint_C du\ dv\ x^u\ y^v D^{(u,v)}[1-\varepsilon_2] \right], \quad (7)$$

where we have introduced a notation

$$D^{(u,v)}[1+\nu] \equiv D^{(u,v)}[1,1,1+\nu].$$

Eq.(7) can be written as

$$\int_2(\varepsilon_1,\varepsilon_2,\varepsilon_3) = \frac{J}{(p_3^2)^2}\left[\frac{1}{\varepsilon_2\varepsilon_3}\left(\frac{p_2^2}{p_3^2}\right)^{\varepsilon_2}\oint_C du\ dv\ x^u\ y^v D^{(u,v)}[1-\varepsilon_1]+\right.$$

$$\left.\frac{1}{\varepsilon_1\varepsilon_2}\oint_C du\ dv\ x^u\ y^v D^{(u,v)}[1+\varepsilon_3] + \frac{1}{\varepsilon_1\varepsilon_3}\left(\frac{p_1^2}{p_3^2}\right)^{\varepsilon_1}\oint_C du\ dv\ x^u\ y^v D^{(u,v)}[1-\varepsilon_2]\right] =$$

$$\frac{J}{(p_3^2)^2}\left[\frac{1}{\varepsilon_2\varepsilon_3}\oint_C du\ dv\ x^u\ y^{v+\varepsilon_2} D^{(u,v)}[1-\varepsilon_1]+\right.$$

$$\left.\frac{1}{\varepsilon_1\varepsilon_2}\oint_C du\ dv\ x^u\ y^v D^{(u,v)}[1+\varepsilon_3] + \frac{1}{\varepsilon_1\varepsilon_3}\oint_C du\ dv\ x^{u+\varepsilon_1}\ y^v D^{(u,v)}[1-\varepsilon_2]\right] =$$

$$\frac{J}{(p_3^2)^2}\oint_C du\ dv\ x^u\ y^v\left[\frac{D^{(u,v-\varepsilon_2)}[1-\varepsilon_1]}{\varepsilon_2\varepsilon_3} + \frac{D^{(u,v)}[1+\varepsilon_3]}{\varepsilon_1\varepsilon_2} + \frac{D^{(u-\varepsilon_1,v)}[1-\varepsilon_2]}{\varepsilon_1\varepsilon_3}\right] \equiv$$

$$\frac{1}{(p_3^2)^2}\oint_C du\ dv\ x^u\ y^v M_2^{(u,v)}[\varepsilon_1,\varepsilon_2,\varepsilon_3]. \qquad (8)$$

Here we shift the variable of integration in the complex plane. That means, the contour of integration C still passes between the left and right poles. Positions of the poles in the plane of two complex variables will be changed with that trick but their nature (left or right) cannot be changed with such a trick.

On the other hand, we have formula (5), from which we obtain

$$\oint_C dz_2dz_3\ D^{(u,v)}[1+\varepsilon_1-z_3, 1+\varepsilon_2-z_2, 1+\varepsilon_3]D^{(z_2,z_3)}[1+\varepsilon_2, 1+\varepsilon_1, 1+\varepsilon_3] =$$

$$J\left[\frac{D^{(u,v-\varepsilon_2)}[1-\varepsilon_1]}{\varepsilon_2\varepsilon_3} + \frac{D^{(u,v)}[1+\varepsilon_3]}{\varepsilon_1\varepsilon_2} + \frac{D^{(u-\varepsilon_1,v)}[1-\varepsilon_2]}{\varepsilon_1\varepsilon_3}\right] = M_2^{(u,v)}[\varepsilon_1,\varepsilon_2,\varepsilon_3]. \qquad (9)$$

This formula is valid for any u, v and presents by itself a nontrivial result which can be used in practical application of MB integration. This is the two-fold MB transform of two-loop integral, and it is valid not only in the limit of all $\varepsilon_i \to 0$, but for any ε_i. Decomposing this formula in terms of ε_i we obtain infinite number of new relations. The explicit form of $M_2^{(u,v)}[\varepsilon_1,\varepsilon_2,\varepsilon_3]$ is calculated in the next section in the limit of vanishing ε_i, the result is Eq. (17),

$$\oint_C dz_2dz_3\ D^{(u,v)}[1-z_3, 1-z_2, 1]D^{(z_2,z_3)}[1,1,1] =$$

$$\Gamma^2(1+u+v)\Gamma^2(-u)\Gamma^2(-v)\ \times$$

$$\left[\frac{1}{2}\left(\psi'(-v)+\psi'(-u)\right) - \frac{1}{2}\left(\psi(-v)-\psi(-u)\right)^2 - \frac{3}{2}\left(\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)\right)^{(2)}_{\varepsilon}\right].$$

4.3 Two-fold MB transform for three-rung ladder

We can derive a new formula for two-fold MB transformation going to higher loops in the ladder diagrams. We start to work with integral $\int_3(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ which appears on the l.h.s. in the third line of Fig.(1) and take a Fourier transform of each one of the diagrams of that line. The Fourier transform of the l.h.s. can be written as

$$\left(\frac{\Gamma(1-\varepsilon_1)}{\Gamma(1+\varepsilon_1)}\right)^3\left(\frac{\Gamma(1-\varepsilon_2)}{\Gamma(1+\varepsilon_2)}\right)^3\left(\frac{\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_3)}\right)^2\frac{1}{p_1^2}\frac{1}{p_2^2}\int_3(-\varepsilon_1,-\varepsilon_2,-\varepsilon_3)$$

and the whole diagrammatic relation becomes as

$$J^3 \frac{\Gamma(1+\varepsilon_3)}{\Gamma(1-\varepsilon_3)} \frac{1}{p_1^2 p_2^2} \int_3 (-\varepsilon_1, -\varepsilon_2, -\varepsilon_3) = J^2 \frac{1}{\varepsilon_2\varepsilon_3} \frac{\Gamma(1+\varepsilon_3)}{\Gamma(1-\varepsilon_3)} \frac{1}{(p_1^2)^{1-\varepsilon_1} p_2^2 (p_3^2)^{1-\varepsilon_2}} \int_2 (-\varepsilon_1) +$$

$$\frac{J^3}{\varepsilon_1\varepsilon_2} \frac{\Gamma(1+\varepsilon_3)}{\Gamma(1-\varepsilon_3)} \frac{1}{(p_1^2)^{1-\varepsilon_1} (p_2^2)^{1-\varepsilon_2} p_3^2} \int_2 (-\varepsilon_1, -\varepsilon_2, -\varepsilon_3) + \frac{J^2}{\varepsilon_1\varepsilon_3} \frac{\Gamma(1+\varepsilon_3)}{\Gamma(1-\varepsilon_3)} \frac{1}{p_1^2 (p_2^2)^{1-\varepsilon_2} (p_3^2)^{1-\varepsilon_1}} \int_2 (-\varepsilon_2).$$

We kept all the factors that appear after making the Fourier transform on both the parts of the diagrammatic equation. Here we use a brief notation

$$\int_n (-\varepsilon_1) \equiv \int_n (-\varepsilon_1, 0, \varepsilon_1), \qquad \int_n (-\varepsilon_2) \equiv \int_n (0, -\varepsilon_2, \varepsilon_2).$$

After a little algebra we obtain the decomposition of three-loop integral in terms of two-loop integrals,

$$\int_3 (-\varepsilon_1, -\varepsilon_2, -\varepsilon_3) = J^{-1} \left[\frac{1}{(p_1^2)^{-\varepsilon_1} (p_3^2)^{1-\varepsilon_2}} \frac{1}{\varepsilon_2\varepsilon_3} \int_2 (-\varepsilon_1) + \right.$$

$$\left. \frac{J}{(p_1^2)^{-\varepsilon_1} (p_2^2)^{-\varepsilon_2} p_3^2} \frac{1}{\varepsilon_1\varepsilon_2} \int_2 (-\varepsilon_1, -\varepsilon_2, -\varepsilon_3) + \frac{1}{(p_2^2)^{-\varepsilon_2} (p_3^2)^{1-\varepsilon_1}} \frac{1}{\varepsilon_1\varepsilon_3} \int_2 (-\varepsilon_2) \right].$$

By changing the signs of all the values ε_i we obtain an equation

$$\int_3 (\varepsilon_1, \varepsilon_2, \varepsilon_3) = J \left[\frac{1}{(p_1^2)^{\varepsilon_1} (p_3^2)^{1+\varepsilon_2}} \frac{1}{\varepsilon_2\varepsilon_3} \int_2 (\varepsilon_1) + \right.$$

$$\left. \frac{J^{-1}}{(p_1^2)^{\varepsilon_1} (p_2^2)^{\varepsilon_2} p_3^2} \frac{1}{\varepsilon_1\varepsilon_2} \int_2 (\varepsilon_1, \varepsilon_2, \varepsilon_3) + \frac{1}{(p_2^2)^{\varepsilon_2} (p_3^2)^{1+\varepsilon_1}} \frac{1}{\varepsilon_1\varepsilon_3} \int_2 (\varepsilon_2) \right].$$

The previous equation can be re-written in terms of MB transform, and we obtain the result

$$\int_3(\varepsilon_1,\varepsilon_2,\varepsilon_3) = J\left[\frac{1}{\varepsilon_2\varepsilon_3}\frac{1}{(p_1^2)^{\varepsilon_1}(p_3^2)^{3+\varepsilon_2}}\oint_C du\ dv\ x^u\ y^v M_2^{(u,v)}(\varepsilon_1)+\right.$$
$$J^{-1}\frac{1}{\varepsilon_1\varepsilon_2}\frac{1}{(p_1^2)^{\varepsilon_1}(p_2^2)^{\varepsilon_2}(p_3^2)^3}\oint_C du\ dv\ x^u\ y^v M_2^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)+$$
$$\left.\frac{1}{\varepsilon_1\varepsilon_3}\frac{1}{(p_2^2)^{\varepsilon_2}(p_3^2)^{3+\varepsilon_1}}\oint_C du\ dv\ x^u\ y^v M_2^{(u,v)}(\varepsilon_2)\right] =$$
$$J\left[\frac{1}{\varepsilon_2\varepsilon_3}\left(\frac{p_3^2}{p_1^2}\right)^{\varepsilon_1}\frac{1}{(p_3^2)^{3-\varepsilon_3}}\oint_C du\ dv\ x^u\ y^v M_2^{(u,v)}(\varepsilon_1)+\right.$$
$$J^{-1}\frac{1}{\varepsilon_1\varepsilon_2}\left(\frac{p_3^2}{p_1^2}\right)^{\varepsilon_1}\left(\frac{p_3^2}{p_2^2}\right)^{\varepsilon_2}\frac{1}{(p_3^2)^{3-\varepsilon_3}}\oint_C du\ dv\ x^u\ y^v M_2^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)$$
$$\left.+\frac{1}{\varepsilon_1\varepsilon_3}\left(\frac{p_3^2}{p_2^2}\right)^{\varepsilon_2}\frac{1}{(p_3^2)^{3-\varepsilon_3}}\oint_C du\ dv\ x^u\ y^v M_2^{(u,v)}(\varepsilon_2)\right] =$$
$$\frac{J}{(p_3^2)^{3-\varepsilon_3}}\left[\frac{1}{\varepsilon_2\varepsilon_3}\oint_C du\ dv\ x^{u-\varepsilon_1}\ y^v M_2^{(u,v)}(\varepsilon_1) + J^{-1}\frac{1}{\varepsilon_1\varepsilon_2}\oint_C du\ dv\ x^{u-\varepsilon_1}\ y^{v-\varepsilon_2} M_2^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)\right.$$
$$\left.+\frac{1}{\varepsilon_1\varepsilon_3}\oint_C du\ dv\ x^u\ y^{v-\varepsilon_2} M_2^{(u,v)}(\varepsilon_2)\right] =$$
$$\frac{J}{(p_3^2)^{3-\varepsilon_3}}\oint_C du\ dv\ x^u\ y^v\left[\frac{1}{\varepsilon_2\varepsilon_3}M_2^{(u+\varepsilon_1,v)}(\varepsilon_1) + \frac{J^{-1}}{\varepsilon_1\varepsilon_2}M_2^{(u+\varepsilon_1,v+\varepsilon_2)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)+\right.$$
$$\left.+\frac{1}{\varepsilon_1\varepsilon_3}M_2^{(u,v+\varepsilon_2)}(\varepsilon_2)\right].$$

Here we shift the variable of integration in the complex plane. That means, the contour of integration C still passes between the left and right poles. Positions of the poles in the plane of two complex variables will be changed but their nature (to belong to the set of left poles or to the set of the right poles) cannot be changed with such a trick.

On the other side, in complete analogy with Eq.(5) we obtain

$$\int_3(\varepsilon_1,\varepsilon_2,\varepsilon_3) =$$
$$\frac{1}{(p_3^2)^2}\int\ Dr\ \oint dz_2dz_3\left(\frac{(p_1+r)^2}{p_3^2}\right)^{z_2}\left(\frac{(p_2-r)^2}{p_3^2}\right)^{z_3}\frac{M_2^{(z_2,z_3)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)}{[(p_1+r)^2]^{1+\varepsilon_1}[(p_2-r)^2]^{1+\varepsilon_2}r^2} =$$
$$\frac{1}{(p_3^2)^2}\int\ Dr\ \oint_C\ dz_2dz_3\frac{1}{[p_3^2]^{z_2+z_3}}\frac{M_2^{(z_2,z_3)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)}{[(p_1+r)^2]^{1+\varepsilon_1-z_2}[(p_2-r)^2]^{1+\varepsilon_2-z_3}r^2} =$$
$$\oint_C\ dz_2dz_3\frac{1}{[p_3^2]^{2+z_2+z_3}}J(1+\varepsilon_2-z_3,1+\varepsilon_1-z_2,1)M_2^{(z_2,z_3)}(\varepsilon_1,\varepsilon_2,\varepsilon_3) =$$
$$\frac{1}{(p_3^2)^{3-\varepsilon_3}}\oint_C\ dz_2dz_3\ dudv\ x^uy^vD^{(u,v)}[1+\varepsilon_2-z_3,1+\varepsilon_1-z_2,1]M_2^{(z_2,z_3)}(\varepsilon_1,\varepsilon_2,\varepsilon_3). \qquad (10)$$

We derive

$$\oint_C dz_2 dz_3 D^{(u,v)}[1+\varepsilon_2 - z_3, 1+\varepsilon_1 - z_2, 1] M_2^{(z_2,z_3)}(\varepsilon_1,\varepsilon_2,\varepsilon_3) = \quad (11)$$

$$J\left[\frac{1}{\varepsilon_2\varepsilon_3} M_2^{(u+\varepsilon_1,v)}(\varepsilon_1) + \frac{J^{-1}}{\varepsilon_1\varepsilon_2} M_2^{(u+\varepsilon_1,v+\varepsilon_2)}(\varepsilon_1,\varepsilon_2,\varepsilon_3) + \frac{1}{\varepsilon_1\varepsilon_3} M_2^{(u,v+\varepsilon_2)}(\varepsilon_2)\right] = M_3^{(u,v)}[\varepsilon_1,\varepsilon_2,\varepsilon_3]$$

This formula is valid for any u, v and presents by itself a nontrivial result which can be used in practical application of MB integration. This is the two-fold MB transform of three-loop integral, and it is valid not only in the limit of all $\varepsilon_i \to 0$, but for any ε_i. Decomposing this formula in terms of ε_i we obtain infinite number of new relations. The limit $\varepsilon_i \to 0$ of $M_3^{(u,v)}[\varepsilon_1,\varepsilon_2,\varepsilon_3]$ is calculated in the next section.

4.4 Two-fold MB transform for four-rung ladder

In the next paragraphs of this section we consider the case of four-loop momentum triangle ladder diagram. It looks like the diagram on the l.h.s. of the third line of Fig.(1) but with one more rung. To get that graphical representation, the diagrams in the third line of fig. (1) are integrated with three more propagators, index of each propagator is equal to 1. The Fourier transform of the l.h.s. of the diagrammatic relation obtained in such a way contains the integral $\int_4(\varepsilon_1,\varepsilon_2,\varepsilon_3)$,

$$\left(\frac{\Gamma(1-\varepsilon_1)}{\Gamma(1+\varepsilon_1)}\right)^3 \left(\frac{\Gamma(1-\varepsilon_2)}{\Gamma(1+\varepsilon_2)}\right)^3 \left(\frac{\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_3)}\right)^2 \frac{1}{p_1^2}\frac{1}{p_2^2}\int_4(-\varepsilon_1,-\varepsilon_2,-\varepsilon_3),$$

and we derive the identity

$$J^3\frac{\Gamma(1+\varepsilon_3)}{\Gamma(1-\varepsilon_3)}\frac{1}{p_1^2p_2^2}\int_4(-\varepsilon_1,-\varepsilon_2,-\varepsilon_3) = J^2\frac{1}{\varepsilon_2\varepsilon_3}\frac{\Gamma(1+\varepsilon_3)}{\Gamma(1-\varepsilon_3)}\frac{1}{p_1^2p_2^2(p_3^2)^{1-\varepsilon_2}}\int_3(-\varepsilon_1) +$$
$$\frac{J^3}{\varepsilon_1\varepsilon_2}\frac{\Gamma(1+\varepsilon_3)}{\Gamma(1-\varepsilon_3)}\frac{1}{p_1^2p_2^2p_3^2}\int_3(-\varepsilon_1,-\varepsilon_2,-\varepsilon_3) + \frac{J^2}{\varepsilon_1\varepsilon_3}\frac{\Gamma(1+\varepsilon_3)}{\Gamma(1-\varepsilon_3)}\frac{1}{p_1^2p_2^2(p_3^2)^{1-\varepsilon_1}}\int_3(-\varepsilon_2).$$

After a little algebra we obtain the decomposition of four-loop integral in terms of three-loop integrals,

$$\int_4(-\varepsilon_1,-\varepsilon_2,-\varepsilon_3) = J^{-1}\left[\frac{1}{(p_3^2)^{1-\varepsilon_2}}\frac{1}{\varepsilon_2\varepsilon_3}\int_3(-\varepsilon_1) + \frac{J}{p_3^2}\frac{1}{\varepsilon_1\varepsilon_3}\int_3(-\varepsilon_1,-\varepsilon_2,-\varepsilon_3) + \right.$$
$$\left.\frac{1}{(p_3^2)^{1-\varepsilon_1}}\frac{1}{\varepsilon_1\varepsilon_3}\int_3(-\varepsilon_2)\right],$$

from which by changing the signs of all the values ε_i we obtain an equation

$$\int_4(\varepsilon_1,\varepsilon_2,\varepsilon_3) = J\left[\frac{1}{(p_3^2)^{1+\varepsilon_2}}\frac{1}{\varepsilon_2\varepsilon_3}\int_3(\varepsilon_1) + \frac{J^{-1}}{p_3^2}\frac{1}{\varepsilon_1\varepsilon_3}\int_3(\varepsilon_1,\varepsilon_2,\varepsilon_3) + \right.$$
$$\left.\frac{1}{(p_3^2)^{1+\varepsilon_1}}\frac{1}{\varepsilon_1\varepsilon_3}\int_3(\varepsilon_2)\right].$$

This equation can be re-written in terms of MB transform

$$\int_4(\varepsilon_1,\varepsilon_2,\varepsilon_3) = J\left[\frac{1}{\varepsilon_2\varepsilon_3}\frac{1}{(p_3^2)^{1+\varepsilon_2}(p_3^2)^{3+\varepsilon_1}}\oint_C du\ dv\ x^u\ y^v M_3^{(u,v)}(\varepsilon_1)+\right.$$
$$J^{-1}\frac{1}{\varepsilon_1\varepsilon_2}\frac{1}{(p_3^2)^{4-\varepsilon_3}}\oint_C du\ dv\ x^u\ y^v M_3^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)+$$
$$\left.\frac{1}{\varepsilon_1\varepsilon_3}\frac{1}{(p_3^2)^{1+\varepsilon_1}(p_3^2)^{3+\varepsilon_2}}\oint_C du\ dv\ x^u\ y^v M_3^{(u,v)}(\varepsilon_2)\right] =$$
$$\frac{J}{(p_3^2)^{4-\varepsilon_3}}\oint_C du\ dv\ x^u\ y^v\left[\frac{1}{\varepsilon_2\varepsilon_3}M_3^{(u,v)}(\varepsilon_1)+\frac{J^{-1}}{\varepsilon_1\varepsilon_2}M_3^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)+\frac{1}{\varepsilon_1\varepsilon_3}M_3^{(u,v)}(\varepsilon_2)\right] \equiv$$
$$\frac{1}{(p_3^2)^{4-\varepsilon_3}}\oint_C du\ dv\ x^u\ y^v M_4^{(u,v)}[\varepsilon_1,\varepsilon_2,\varepsilon_3]$$

On the other hand, in analogy to Eq.(5),

$$\int_4(\varepsilon_1,\varepsilon_2,\varepsilon_3) =$$
$$\frac{1}{(p_3^2)^{3-\varepsilon_3}}\int Dr\ \oint dz_2 dz_3\left(\frac{(p_1+r)^2}{p_3^2}\right)^{z_2}\left(\frac{(p_2-r)^2}{p_3^2}\right)^{z_3}\frac{M_3^{(z_2,z_3)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)}{(p_1+r)^2(p_2-r)^2r^2} =$$
$$\frac{1}{(p_3^2)^{3-\varepsilon_3}}\int Dr\ \oint_C dz_2 dz_3\frac{1}{[p_3^2]^{z_2+z_3}}\frac{M_3^{(z_2,z_3)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)}{[(p_1+r)^2]^{1-z_2}[(p_2-r)^2]^{1-z_3}r^2} =$$
$$\frac{1}{(p_3^2)^{3-\varepsilon_3}}\oint_C dz_2 dz_3\frac{1}{[p_3^2]^{z_2+z_3}}J(1-z_3,1-z_2,1)M_3^{(z_2,z_3)}(\varepsilon_1,\varepsilon_2,\varepsilon_3) =$$
$$\frac{1}{(p_3^2)^{4-\varepsilon_3}}\oint_C dz_2 dz_3\ dudv\ x^u y^v D^{(u,v)}[1-z_3,1-z_2,1]M_3^{(z_2,z_3)}(\varepsilon_1,\varepsilon_2,\varepsilon_3). \qquad (12)$$

We obtain

$$\oint_C dz_2 dz_3 D^{(u,v)}[1-z_3,1-z_2,1]M_3^{(z_2,z_3)}(\varepsilon_1,\varepsilon_2,\varepsilon_3) =$$
$$J\left[\frac{1}{\varepsilon_2\varepsilon_3}M_3^{(u,v)}(\varepsilon_1)+\frac{J^{-1}}{\varepsilon_1\varepsilon_2}M_2^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)+\frac{1}{\varepsilon_1\varepsilon_3}M_2^{(u,v)}(\varepsilon_2)\right] = M_4^{(u,v)}[\varepsilon_1,\varepsilon_2,\varepsilon_3]. \qquad (13)$$

This formula is valid for any u, v and presents by itself a nontrivial result which can be used in practical application of MB integration. This is the two-fold MB transform of four-loop momentum integral, and it is valid not only in the limit of all $\varepsilon_i \to 0$, but for any ε_i. Decomposing this formula in terms of ε_i we obtain infinite number of new relations.

5 One-fold MB transforms

In this section we collect useful formulas for one-fold MB transforms.

$$\oint_C dz\ x^z \Gamma^2(-z)\Gamma^2(1+z) = -\frac{\ln x}{1-x}$$

$$\oint_C dz\ x^z \left\{\Gamma^2(-z)\Gamma(z+1)\Gamma^*(z)\right\} = -\left(\frac{1}{2}\ln^2 x + 2\zeta(2)\right) + \ln(1-x)\ln x + \mathrm{Li}_2(x)$$

$$\oint_C dz\ x^z \Gamma^2(-z)\Gamma^2(1+z)\psi(1+z) =$$

$$\frac{1}{(1-x)}\left[-\psi(1)\ln x + \ln(1-x)\ln x - \zeta(2) + \mathrm{Li}_2(x)\right]$$

It is possible to work with MB transforms making use of such tricks like derivation with respect to parameter and integration by parts in the complex plane. Here we demonstrate several examples. First of all, after the derivation with respect to x of

$$\frac{1}{1+x} = \oint_C dz\ x^z \Gamma(-z)\Gamma(1+z)$$

we obtain

$$\frac{x}{(1+x)^2} = \oint_C dz\ x^z \Gamma(1-z)\Gamma(1+z).$$

The result can be checked directly by counting of residues. Now, let us check that the integral of the total derivative is equal to zero. Indeed, we have

$$0 = \oint_C dz\ \frac{d}{dz}\{x^z\Gamma(-z)\Gamma(1+z)\} = \ln x \oint_C dz\ x^z\Gamma(-z)\Gamma(1+z) +$$

$$\oint_C dz\ x^z\Gamma(-z)\Gamma(1+z)(-\psi(-z)+\psi(1+z)) =$$

$$\frac{\ln x}{1+x} - \oint_C dz\ x^z\frac{\Gamma(-z)\Gamma(1+z)}{z} + \oint_C dz\ x^z\Gamma(-z)\Gamma(1+z)(\psi(1+z)-\psi(1-z)). \tag{14}$$

We calculate all the integrals on the r.h.s. explicitly and show that their sum is zero. The direct calculation of residues leads to

$$\oint_C dz\ x^z\Gamma(-z)\Gamma(1+z)\psi(1+z) = \psi(1) - x\psi(2) + x^2\psi(3) + \dots$$

$$\oint_C dz\ x^z\Gamma(-z)\Gamma(1+z)\psi(1-z) = \psi(1) - x\psi(1) + x^2\psi(2) - x^3\psi(3) + \dots - \ln x + \frac{\ln x}{1+x}$$

$$\oint_C dz\ x^z\frac{\Gamma(-z)\Gamma(1+z)}{z} = \ln x - \ln(1+x)$$

Substituting these results in (14) we reproduce zero in the l.h.s. of it.

6 Explicit results of two-fold MB transform

In order to use formulas obtained for the two-fold integration explicitly, we need to calculate the r.h.s. of the equations in the limit of vanishing ε_i. First, we consider the result for $\int_2(\varepsilon_1,\varepsilon_2,\varepsilon_3)$. There are two ways to derive the result in this limit for the r.h.s.

6.1 Belokurov-Usyukina decomposition for higher rung ladder

The first way to calculate that limit is to use the representation after shifting complex variables of integration in the complex plane, that is we have to calculate the limit of Eq.(8),

$$\frac{D^{(u,v-\varepsilon_2)}[1-\varepsilon_1]}{\varepsilon_2\varepsilon_3} + \frac{D^{(u,v)}[1+\varepsilon_3]}{\varepsilon_1\varepsilon_2} + \frac{D^{(u-\varepsilon_1,v)}[1-\varepsilon_2]}{\varepsilon_1\varepsilon_3},$$

when all the ε-terms are vanishing. By making use of Eq.(8) and the explicit form of the MB transform D defined in (4), we write this expression as

$$\begin{aligned} &\frac{1}{\varepsilon_1\varepsilon_2}\frac{\Gamma(-u)\Gamma(-v)\Gamma(\varepsilon_1+\varepsilon_2-u)\Gamma(\varepsilon_1+\varepsilon_2-v)\Gamma^2(1-\varepsilon_1-\varepsilon_2+u+v)}{\Gamma(1-\varepsilon_3)\Gamma(1+\varepsilon_3)} \\ -&\frac{1}{\varepsilon_2(\varepsilon_1+\varepsilon_2)}\frac{\Gamma(-u)\Gamma(\varepsilon_2-v)\Gamma(\varepsilon_1-u)\Gamma(\varepsilon_1+\varepsilon_2-v)\Gamma^2(1-\varepsilon_1-\varepsilon_2+u+v)}{\Gamma(1-\varepsilon_1)\Gamma(1+\varepsilon_1)} \\ -&\frac{1}{\varepsilon_1(\varepsilon_1+\varepsilon_2)}\frac{\Gamma(\varepsilon_1-u)\Gamma(-v)\Gamma(\varepsilon_1+\varepsilon_2-u)\Gamma(\varepsilon_2-v)\Gamma^2(1-\varepsilon_1-\varepsilon_2+u+v)}{\Gamma(1-\varepsilon_2)\Gamma(1+\varepsilon_2)} \end{aligned} \tag{15}$$

We do not write factors

$$\frac{1}{\Gamma(1-\varepsilon_i)\Gamma(1+\varepsilon_i)} \tag{16}$$

in the next formulas. However, we recover contributions of them at the end of the derivation. To calculate the limit of vanishing ε_i, we need to decompose (15) in terms of ε_1 and ε_2. The last factor is common, and we omit it,

$$\begin{aligned} &\frac{1}{\varepsilon_1\varepsilon_2}\Gamma(-u)\Gamma(-v)\Gamma(\varepsilon_1+\varepsilon_2-u)\Gamma(\varepsilon_1+\varepsilon_2-v) \\ -&\frac{1}{\varepsilon_2(\varepsilon_1+\varepsilon_2)}\Gamma(-u)\Gamma(\varepsilon_2-v)\Gamma(\varepsilon_1-u)\Gamma(\varepsilon_1+\varepsilon_2-v) \\ -&\frac{1}{\varepsilon_1(\varepsilon_1+\varepsilon_2)}\Gamma(\varepsilon_1-u)\Gamma(-v)\Gamma(\varepsilon_1+\varepsilon_2-u)\Gamma(\varepsilon_2-v) = \\ &\frac{1}{\varepsilon_1\varepsilon_2}\Gamma(-u)\Gamma(-v)\Gamma(\varepsilon_1+\varepsilon_2-u)\Gamma(\varepsilon_1+\varepsilon_2-v) \\ -&\frac{1}{\varepsilon_2\varepsilon_1}\left(1-\frac{\varepsilon_2}{\varepsilon_1}\right)\Gamma(-u)\Gamma(\varepsilon_2-v)\Gamma(\varepsilon_1-u)\Gamma(\varepsilon_1+\varepsilon_2-v) - \frac{1}{\varepsilon_1^2}\Gamma^2(\varepsilon_1-u)\Gamma^2(-v) = \\ &\frac{1}{\varepsilon_1\varepsilon_2}\Gamma(-u)\Gamma(\varepsilon_1+\varepsilon_2-v)\left[\Gamma(-v)\Gamma(\varepsilon_1+\varepsilon_2-u)-\Gamma(\varepsilon_2-v)\Gamma(\varepsilon_1-u)\right] \\ +&\frac{1}{\varepsilon_1^2}\Gamma(-u)\Gamma(-v)\Gamma(\varepsilon_1-u)\Gamma(\varepsilon_1-v) - \frac{1}{\varepsilon_1^2}\Gamma^2(\varepsilon_1-u)\Gamma^2(-v) = \\ &\Gamma^2(-u)\Gamma^2(-v)\left[\frac{1}{2}\left(\psi'(-v)+\psi'(-u)\right) - \frac{1}{2}\left(\psi(-v)-\psi(-u)\right)^2\right]. \end{aligned}$$

Recovering the contribution of factors (16), we obtain

$$\lim_{\varepsilon_1\to 0,\varepsilon_2\to 0} M_2^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3) = \Gamma^2(1+u+v)\Gamma^2(-u)\Gamma^2(-v)\times$$
$$\left[\frac{1}{2}\left(\psi'(-v)+\psi'(-u)\right) - \frac{1}{2}\left(\psi(-v)-\psi(-u)\right)^2 - \frac{3}{2}\left(\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)\right)^{(2)}_{\varepsilon}\right]. \quad (17)$$

The last term is a value of Riemann zeta function multiplied by the first UD function. This result can be represented in other forms. For example, by integrating by parts in the complex u and v planes we reproduce result (21) of the next subsection. However, this form is important since it stands in the right hand side of Eq. (11) in the limit of vanishing ε_i.

6.2 Integration by parts in the complex plane

In the second way we used integration by parts in the complex plane like that described in the previous section. Indeed, as we have derived in Eq. (8)

$$\frac{1}{\varepsilon_1\varepsilon_2}\oint_C du\ dv\ x^u\ y^v D^{(u,v)}[1+\varepsilon_3] + \frac{1}{\varepsilon_2\varepsilon_3}y^{\varepsilon_2}\oint_C du\ dv\ x^u\ y^v D^{(u,v)}[1-\varepsilon_1] +$$
$$\frac{1}{\varepsilon_1\varepsilon_3}x^{\varepsilon_1}\oint_C du\ dv\ x^u\ y^v D^{(u,v)}[1-\varepsilon_2] =$$
$$\oint_C du\ dv\ x^u\ y^v\left[\frac{1}{\varepsilon_1\varepsilon_2}D^{(u,v)}[1+\varepsilon_3] + \frac{1}{\varepsilon_2\varepsilon_3}y^{\varepsilon_2}D^{(u,v)}[1-\varepsilon_1] + \frac{1}{\varepsilon_1\varepsilon_3}x^{\varepsilon_1}D^{(u,v)}[1-\varepsilon_1]\right]. \quad (18)$$

Our purpose is to take a limit $\varepsilon_1\to 0$, $\varepsilon_2\to 0$ for this expression. By using Eq. (8) and definition (4), we write the integrand explicitly

$$\frac{1}{\varepsilon_1\varepsilon_2}D^{(u,v)}[1+\varepsilon_3] + \frac{1}{\varepsilon_2\varepsilon_3}y^{\varepsilon_2}D^{(u,v)}[1-\varepsilon_1] + \frac{1}{\varepsilon_1\varepsilon_3}x^{\varepsilon_1}D^{(u,v)}[1-\varepsilon_1] =$$
$$\Gamma(-u)\Gamma(-v)\left[\frac{1}{\varepsilon_1\varepsilon_2}\frac{\Gamma(\varepsilon_1+\varepsilon_2-u)\Gamma(\varepsilon_1+\varepsilon_2-v)\Gamma^2(1-\varepsilon_1-\varepsilon_2+u+v)}{\Gamma(1+\varepsilon_1+\varepsilon_2)\Gamma(1-\varepsilon_1-\varepsilon_2)} + \right.$$
$$-\frac{1}{\varepsilon_2(\varepsilon_1+\varepsilon_2)}y^{\varepsilon_2}\frac{\Gamma(\varepsilon_1-u)\Gamma(\varepsilon_1-v)\Gamma^2(1-\varepsilon_1+u+v)}{\Gamma(1+\varepsilon_1)\Gamma(1-\varepsilon_1)}$$
$$\left.-\frac{1}{\varepsilon_1(\varepsilon_1+\varepsilon_2)}x^{\varepsilon_1}\frac{\Gamma(\varepsilon_2-u)\Gamma(\varepsilon_2-v)\Gamma^2(1-\varepsilon_2+u+v)}{\Gamma(1+\varepsilon_2)\Gamma(1-\varepsilon_2)}\right]$$

The brackets in this formula can be re-written in the following form

$$
\frac{1}{\varepsilon_1\varepsilon_2}\left[\frac{\Gamma(\varepsilon_1+\varepsilon_2-u)\Gamma(\varepsilon_1+\varepsilon_2-v)\Gamma^2(1-\varepsilon_1-\varepsilon_2+u+v)}{\Gamma(1+\varepsilon_1+\varepsilon_2)\Gamma(1-\varepsilon_1-\varepsilon_2)}\right.
$$

$$
\left.-y^{\varepsilon_2}\frac{\Gamma(\varepsilon_1-u)\Gamma(\varepsilon_1-v)\Gamma^2(1-\varepsilon_1+u+v)}{\Gamma(1+\varepsilon_1)\Gamma(1-\varepsilon_1)}\right]
$$

$$
-\frac{1}{\varepsilon_1^2}\left[\frac{\Gamma(\varepsilon_1-u)\Gamma(\varepsilon_1-v)\Gamma^2(1-\varepsilon_1+u+v)}{\Gamma(1+\varepsilon_1)\Gamma(1-\varepsilon_1)}-x^{\varepsilon_1}\Gamma(-u)\Gamma(-v)\Gamma^2(1+u+v)\right]+o(\varepsilon_2)=
$$

$$
\frac{1}{\varepsilon_1}\Gamma(-u)\Gamma(-v)\Gamma^2(1+u+v)\left[\psi(-u)+\psi(-v)-2\psi(1+u+v)-\ln y\right]+
$$

$$
\left(\frac{\Gamma(\varepsilon_1-u)\Gamma(\varepsilon_1-v)\Gamma^2(1-\varepsilon_1+u+v)}{\Gamma(1+\varepsilon_1)\Gamma(1-\varepsilon_1)}\right)^{(2)}_{\varepsilon_1}-\ln y\left(\Gamma(\varepsilon_1-u)\Gamma(\varepsilon_1-v)\Gamma^2(1-\varepsilon_1+u+v)\right)'_{\varepsilon_1}+
$$

$$
\frac{1}{\varepsilon_1}\Gamma(-u)\Gamma(-v)\Gamma^2(1+u+v)\left[\psi(-u)+\psi(-v)-2\psi(1+u+v)-\ln x\right]+
$$

$$
+\frac{1}{2}\left(\frac{\Gamma(\varepsilon_1-u)\Gamma(\varepsilon_1-v)\Gamma^2(1-\varepsilon_1+u+v)}{\Gamma(1+\varepsilon_1)\Gamma(1-\varepsilon_1)}\right)^{(2)}_{\varepsilon_1}-\frac{1}{2}\ln^2 x\Gamma(-u)\Gamma(-v)\Gamma^2(1+u+v)=
$$

$$
\frac{3}{2}\left(\frac{\Gamma(\varepsilon_1-u)\Gamma(\varepsilon_1-v)\Gamma^2(1-\varepsilon_1+u+v)}{\Gamma(1+\varepsilon_1)\Gamma(1-\varepsilon_1)}\right)^{(2)}_{\varepsilon_1}
$$

$$
-\ln y\left(\Gamma(\varepsilon_1-u)\Gamma(\varepsilon_1-v)\Gamma^2(1-\varepsilon_1+u+v)\right)'_{\varepsilon_1}-\frac{1}{2}\ln^2 x\Gamma(-u)\Gamma(-v)\Gamma^2(1+u+v).
$$

First of all, we demonstrate that all the poles in ε_i disappear. The corresponding contribution taking into account the factor $\Gamma(-u)\Gamma(-v)$ has the form

$$
\frac{1}{\varepsilon_1}\oint_C du\ dv x^u y^v\Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v)\left[(\psi(-u)+\psi(-v)-2\psi(1+u+v)-\ln y)\right.
$$

$$
\left.+\ (\psi(-u)+\psi(-v)-2\psi(1+u+v)-\ln x)\right]=
$$

$$
\frac{1}{\varepsilon_1}\oint_C du\ dv x^u y^v\Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v)\left[(2\psi(-u)-2\psi(1+u+v)-\ln x)\right.
$$

$$
\left.+\ (2\psi(-v)-2\psi(1+u+v)-\ln y)\right]=
$$

$$
-\frac{1}{\varepsilon_1}\oint_C du\ dv(\partial_u+\partial_v)\{x^u y^v\Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v)\}=0.
$$

Here we applied integration by part technique, developed in the previous section, for the plane of the complex variables. Thus, we have derived the following result for the initial integral (18)

$$
\oint_C du\ dv\ x^u\ y^v\Gamma(-u)\Gamma(-v)\left[\frac{3}{2}\left(\frac{\Gamma(\varepsilon_1-u)\Gamma(\varepsilon_1-v)\Gamma^2(1-\varepsilon_1+u+v)}{\Gamma(1+\varepsilon_1)\Gamma(1-\varepsilon_1)}\right)^{(2)}_{\varepsilon_1}\right.
$$

$$
\left.-\ln y\left(\Gamma(\varepsilon_1-u)\Gamma(\varepsilon_1-v)\Gamma^2(1-\varepsilon_1+u+v)\right)'_{\varepsilon_1}-\frac{1}{2}\ln^2 x\Gamma(-u)\Gamma(-v)\Gamma^2(1+u+v)\right].
$$

As to the last two terms we can consider the transformations

$$\oint_C du\ dv\ x^u\ y^v \Gamma(-u)\Gamma(-v) \Big[-\ln y \left(\Gamma(\varepsilon_1 - u)\Gamma(\varepsilon_1 - v)\Gamma^2(1-\varepsilon_1+u+v) \right)'_{\varepsilon_1}$$
$$-\frac{1}{2}\ln^2 x \Gamma(-u)\Gamma(-v)\Gamma^2(1+u+v) \Big] =$$
$$-\ln y \oint_C du\ dv\ x^u\ y^v \left(\psi(-u)+\psi(-v)-2\psi(1+u+v) \right) \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v)$$
$$-\frac{1}{2}\ln^2 x \oint_C du\ dv\ x^u\ y^v \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v) =$$
$$\frac{1}{2}\ln y \oint_C du\ dv\ x^u\ y^v \left(\partial_u + \partial_v \right) \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v)$$
$$-\frac{1}{2}\ln^2 x \oint_C du\ dv\ x^u\ y^v \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v) =$$
$$-\frac{1}{2}\ln y (\ln x + \ln y) \oint_C du\ dv\ x^u\ y^v \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v)$$
$$-\frac{1}{2}\ln^2 x \oint_C du\ dv\ x^u\ y^v \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v).$$

Thus, the result for integral (18) is

$$\frac{3}{2}\oint_C du\ dv\ x^u\ y^v \Gamma(-u)\Gamma(-v) \left(\frac{\Gamma(\varepsilon_1 - u)\Gamma(\varepsilon_1 - v)\Gamma^2(1-\varepsilon_1+u+v)}{\Gamma(1+\varepsilon_1)\Gamma(1-\varepsilon_1)} \right)^{(2)}_{\varepsilon_1}$$
$$-\frac{1}{2}\left(\ln^2 x + \ln x \ln y + \ln^2 y \right) \oint_C du\ dv\ x^u\ y^v \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v). \tag{19}$$

This formula can be developed further and transformed to the form without derivatives of Euler ψ function in the integrand,

$$\oint_C du\ dv\ x^u\ y^v \Gamma(-u)\Gamma(-v) \left(\Gamma(\varepsilon_1 - u)\Gamma(\varepsilon_1 - v)\Gamma^2(1-\varepsilon_1+u+v) \right)^{(2)}_{\varepsilon_1} =$$
$$\oint_C du\ dv\ x^u\ y^v \left[-\partial_u \psi(-u) - \partial_v \psi(-v) + \partial_u \psi(1+u+v) \right.$$
$$\left. +\partial_v \psi(1+u+v) + \left(\psi(-u)+\psi(-v)-2\psi(1+u+v) \right)^2 \right] \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v) =$$
$$\oint_C du\ dv\ x^u\ y^v \left[\left(\ln x - 2\psi(-u) + 2\psi(1+u+v) \right) \left(\psi(-u) - \psi(1+u+v) \right) + \right.$$
$$+ \left(\ln y - 2\psi(-v) + 2\psi(1+u+v) \right) \left(\psi(-v) - \psi(1+u+v) \right)$$
$$\left. +\left(\psi(-u)+\psi(-v)-2\psi(1+u+v) \right)^2 \right] \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v) =$$
$$\oint_C du\ dv\ x^u\ y^v \left[\ln x \left(\psi(-u) - \psi(1+u+v) \right) + \ln y \left(\psi(-v) - \psi(1+u+v) \right) \right.$$
$$\left. -\left(\psi(-u) - \psi(-v) \right)^2 \right] \Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v),$$

where we applied the integration by parts technique in the complex planes which is developed in the previous section. Thus, together with second term of (19), the result can be written in terms of powers of logarithms and MB transforms with ψ function but not with its derivatives, that is

$$\frac{3}{2}\ln x \oint_C du\ dv\ x^u\ y^v \left(\psi(-u)-\psi(1+u+v)\right)\Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v)+$$
$$\frac{3}{2}\ln y \oint_C du\ dv\ x^u\ y^v \left(\psi(-v)-\psi(1+u+v)\right)\Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v)-$$
$$-\frac{3}{2}\oint_C du\ dv\ x^u\ y^v(\psi(-u)-\psi(-v))^2\Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v)$$
$$-\frac{1}{2}\left(\ln^2 x+\ln x\ln y+\ln^2 y\right)\oint_C du\ dv\ x^u\ y^v\Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v). \qquad (20)$$

The first two terms are proportional to the first UD function. This can be checked by integration by parts procedure in the complex u and v planes. After integrating by parts and recovering the contribution of factors (16), the previous expression is transformed to

$$\left(\frac{1}{4}\ln^2\frac{x}{y}-\frac{3}{2}\left(\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)\right)^{(2)}_{\varepsilon}\right)\oint_C du\ dv\ x^u\ y^v\Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v)-$$
$$-\frac{3}{2}\oint_C du\ dv\ x^u\ y^v\left(\psi(-u)-\psi(-v)\right)^2\Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v). \qquad (21)$$

This result is related to formula (17) via integration by parts in the complex planes of u and v.

6.3 Explicit two-fold MB transform for three-rung ladder

We need to derive the same limit for $M_3^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)$. We show that the result is reduced to Euler ψ function and its derivatives. According to Eq. (11)

$$\frac{1}{\varepsilon_1\varepsilon_2}M_2^{(u+\varepsilon_1,v+\varepsilon_2)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)+\frac{J}{\varepsilon_2\varepsilon_3}M_2^{(u+\varepsilon_1,v)}(\varepsilon_1)+\frac{J}{\varepsilon_1\varepsilon_3}M_2^{(u,v+\varepsilon_2)}(\varepsilon_2)=$$
$$\frac{1}{\varepsilon_1\varepsilon_2}M_2^{(u+\varepsilon_1,v+\varepsilon_2)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)-\frac{J}{\varepsilon_2(\varepsilon_1+\varepsilon_2)}M_2^{(u+\varepsilon_1,v)}(\varepsilon_1)-\frac{J}{\varepsilon_1(\varepsilon_1+\varepsilon_2)}M_2^{(u,v+\varepsilon_2)}(\varepsilon_2)=$$
$$\frac{1}{\varepsilon_1\varepsilon_2}M_2^{(u+\varepsilon_1,v+\varepsilon_2)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)-\frac{J}{\varepsilon_2\varepsilon_2}\left(1-\frac{\varepsilon_2}{\varepsilon_1}\right)M_2^{(u+\varepsilon_1,v)}(\varepsilon_1)-\frac{1}{\varepsilon_1^2}M_2^{(u,v)}=$$
$$\frac{1}{\varepsilon_1\varepsilon_2}M_2^{(u+\varepsilon_1,v+\varepsilon_2)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)-\frac{J}{\varepsilon_2\varepsilon_1}M_2^{(u+\varepsilon_1,v)}(\varepsilon_1)+\frac{1}{\varepsilon_1^2}M_2^{(u+\varepsilon_1,v)}(\varepsilon_1)-\frac{1}{\varepsilon_1^2}M_2^{(u,v)}=$$
$$\frac{1}{\varepsilon_1\varepsilon_2}\left[M_2^{(u+\varepsilon_1,v+\varepsilon_2)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)-JM_2^{(u+\varepsilon_1,v)}(\varepsilon_1)\right]+\frac{1}{\varepsilon_1^2}\left[M_2^{(u+\varepsilon_1,v)}(\varepsilon_1)-M_2^{(u,v)}\right] \qquad (22)$$

We expand in terms of ε_1 the second term in which we need first and second degrees of ε_1 in Eq. (22). From Eq. (15) we obtain

$$
\begin{aligned}
& M_2^{(u+\varepsilon_1,v)}(\varepsilon_1) \approx \\
& \Gamma^2(1+u+v)\Gamma^2(-u)\Gamma^2(-v)\left[\frac{1}{2}\left(\psi'(-v)+\psi'(-u)\right)-\frac{1}{2}\left(\psi(-v)-\psi(-u)\right)^2\right]+ \\
& \Gamma^2(1+u+v)\Gamma(-u)\Gamma(-v)\left[\frac{1}{2}\varepsilon_1\left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(2)}_{\varepsilon_1}\left(\psi(-u)-\psi(-v)\right)+\right. \\
& \frac{1}{6}\varepsilon_1^2\left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(3)}_{\varepsilon_1}\left(\psi(-u)-\psi(-v)\right)+\frac{1}{6}\varepsilon_1\left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(3)}_{\varepsilon_1}+ \\
& \left.\frac{1}{24}\varepsilon_1^2\left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(4)}_{\varepsilon_1}\right].
\end{aligned}
$$

We do not write here the contributions from factors (16). They can be taken into account by replacing the terms

$$
\left(\Gamma(-\varepsilon_i-u)\Gamma(\varepsilon_i-v)\right)^{(2)}_{\varepsilon_i}, \quad \left(\Gamma(-\varepsilon_i-u)\Gamma(\varepsilon_i-v)\right)^{(3)}_{\varepsilon_i}, \quad \left(\Gamma(-\varepsilon_i-u)\Gamma(\varepsilon_i-v)\right)^{(4)}_{\varepsilon_i}
$$

with

$$
\left(\frac{\Gamma(-\varepsilon_i-u)\Gamma(\varepsilon_i-v)}{\Gamma(1-\varepsilon_i)\Gamma(1+\varepsilon_i)}\right)^{(2)}_{\varepsilon_i}, \quad \left(\frac{\Gamma(-\varepsilon_i-u)\Gamma(\varepsilon_i-v)}{\Gamma(1-\varepsilon_i)\Gamma(1+\varepsilon_i)}\right)^{(3)}_{\varepsilon_i}, \quad \left(\frac{\Gamma(-\varepsilon_i-u)\Gamma(\varepsilon_i-v)}{\Gamma(1-\varepsilon_i)\Gamma(1+\varepsilon_i)}\right)^{(4)}_{\varepsilon_i},
$$

which are the corresponding derivatives with respect to ε_i at the point $\varepsilon_i = 0$. Also we do not write the terms which contain an irrational number of the type $\zeta(n)$, which have the same origin in the denominators (16). The first term can be recognized as Eq. (17). We obtain for the second term of (22) the decomposition in terms of ε_1

$$
\begin{aligned}
& \frac{1}{\varepsilon_1^2}\left[M_2^{(u+\varepsilon_1,v)}(\varepsilon_1)-M_2^{(u,v)}\right] \approx \\
& \Gamma^2(1+u+v)\Gamma(-u)\Gamma(-v)\left[\frac{1}{2\varepsilon_1}\left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(2)}_{\varepsilon_1}\left(\psi(-u)-\psi(-v)\right)+\right. \\
& \frac{1}{6\varepsilon_1}\left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(3)}_{\varepsilon_1}+\frac{1}{6}\left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(3)}_{\varepsilon_1}\left(\psi(-u)-\psi(-v)\right)+ \\
& \left.\frac{1}{24}\left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(4)}_{\varepsilon_1}\right].
\end{aligned}
$$

The derivatives are

$$
\begin{aligned}
\left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)'_{\varepsilon_1} &= \left(\psi(\varepsilon_1-v)-\psi(-\varepsilon_1-u)\right)\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v) \\
\left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(2)}_{\varepsilon_1} &= \left[\left(\psi'(-v)+\psi'(-u)\right)+\left(\psi(-v)-\psi(-u)\right)^2\right]\Gamma(-u)\Gamma(-v).
\end{aligned}
$$

Going back to the first term of Eq. (22) we derive, by using Eq. (15), that

$$
\begin{aligned}
&M_2^{(u+\varepsilon_1,v+\varepsilon_2)}(\varepsilon_1,\varepsilon_2,\varepsilon_3) = \\
&\frac{1}{\varepsilon_1\varepsilon_2}\frac{\Gamma(-\varepsilon_1-u)\Gamma(-\varepsilon_2-v)\Gamma(\varepsilon_2-u)\Gamma(\varepsilon_1-v)\Gamma^2(1+u+v)}{\Gamma(1-\varepsilon_3)\Gamma(1+\varepsilon_3)} \\
&-\frac{1}{\varepsilon_2(\varepsilon_1+\varepsilon_2)}\frac{\Gamma(-\varepsilon_1-u)\Gamma(-v)\Gamma(-u)\Gamma(\varepsilon_1-v)\Gamma^2(1+u+v)}{\Gamma(1-\varepsilon_1)\Gamma(1+\varepsilon_1)} \\
&-\frac{1}{\varepsilon_1(\varepsilon_1+\varepsilon_2)}\frac{\Gamma(-u)\Gamma(-\varepsilon_2-v)\Gamma(\varepsilon_2-u)\Gamma(-v)\Gamma^2(1+u+v)}{\Gamma(1-\varepsilon_2)\Gamma(1+\varepsilon_2)}.
\end{aligned} \tag{23}
$$

The factor $\Gamma^2(1+u+v)$ is common and we omit it for brevity in the next equations. At the end of these equalities we recover it in analogy with calculations just after Eq. (15) in the previous section. It will be recovered after taking the limit of vanishing ε-terms. We need to decompose (23) in terms of ε_2 up to its first power. The result of this decomposition up to the mentioned factor $\Gamma^2(1+u+v)$ is

$$
\begin{aligned}
&M_2^{(u+\varepsilon_1,v+\varepsilon_2)}(\varepsilon_1,\varepsilon_2,\varepsilon_3) \approx \\
&\frac{1}{\varepsilon_1}\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\Gamma(-u)\Gamma(-v)\left(\psi(-u)-\psi(-v)-\psi(1+\varepsilon_1)+\psi(1-\varepsilon_1)\right)+ \\
&\frac{1}{\varepsilon_1^2}\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\Gamma(-u)\Gamma(-v)-\frac{1}{\varepsilon_1^2}\Gamma^2(-u)\Gamma^2(-v)+ \\
&\varepsilon_2\left[\frac{1}{2\varepsilon_1}\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\left(\Gamma(\varepsilon_2-u)\Gamma(-\varepsilon_2-v)\right)^{(2)}_{\varepsilon_2}-\frac{1}{\varepsilon_1^2}\Gamma^2(-u)\Gamma^2(-v)\left(\psi(-u)-\psi(-v)\right)\right. \\
&\left.-\frac{1}{\varepsilon_1^3}\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\Gamma(-u)\Gamma(-v)+\frac{1}{\varepsilon_1^3}\Gamma^2(-u)\Gamma^2(-v)\right]+o(\varepsilon_2)
\end{aligned}
$$

Here we do not write the terms which are proportional to irrational numbers like $\zeta(n)$. They come from denominators (16). The first terms in this expression without powers of ε_2 is $M_2^{(u+\varepsilon_1,v)}(\varepsilon_1)$ and it should be subtracted in Eq. (22), thus we have, up to terms proportional to irrational numbers,

$$
\begin{aligned}
&M_2^{(u+\varepsilon_1,v+\varepsilon_2)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)-M_2^{(u+\varepsilon_1,v)}(\varepsilon_1)\approx\varepsilon_2\left[\frac{1}{\varepsilon_1^2}\left(\psi(-u)-\psi(-v)\right)\Gamma^2(-u)\Gamma^2(-v)\right. \\
&-\frac{1}{2\varepsilon_1}\left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(2)}_{\varepsilon_1}\Gamma(-u)\Gamma(-v)-\frac{1}{6}\left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(3)}_{\varepsilon_1}\Gamma(-u)\Gamma(-v) \\
&-\frac{\varepsilon_1}{24}\left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(4)}_{\varepsilon_1}\Gamma(-u)\Gamma(-v)+\frac{1}{2\varepsilon_1}\Gamma(-u)\Gamma(-v)\left(\Gamma(\varepsilon_2-u)\Gamma(-\varepsilon_2-v)\right)^{(2)}_{\varepsilon_2} \\
&+\frac{1}{2}\left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)'_{\varepsilon_1}\left(\Gamma(\varepsilon_2-u)\Gamma(-\varepsilon_2-v)\right)^{(2)}_{\varepsilon_2} \\
&\left.+\frac{\varepsilon_1}{4}\left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(2)}_{\varepsilon_1}\left(\Gamma(\varepsilon_2-u)\Gamma(-\varepsilon_2-v)\right)^{(2)}_{\varepsilon_2}-\frac{1}{\varepsilon_1^2}\Gamma^2(-u)\Gamma^2(-v)\left(\psi(-u)-\psi(-v)\right)\right].
\end{aligned}
$$

Thus, it does not contain poles in ε_1, as it has to be,

$$M_2^{(u+\varepsilon_1,v+\varepsilon_2)}(\varepsilon_1,\varepsilon_2,\varepsilon_3) - M_2^{(u+\varepsilon_1,v)}(\varepsilon_1) \approx$$

$$\varepsilon_2 \left[-\frac{1}{6} \left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(3)}_{\varepsilon_1} \Gamma(-u)\Gamma(-v) - \frac{\varepsilon_1}{24} \left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(4)}_{\varepsilon_1} \Gamma(-u)\Gamma(-v) \right.$$

$$+\frac{1}{2} \left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)'_{\varepsilon_1} \left(\Gamma(\varepsilon_2-u)\Gamma(-\varepsilon_2-v)\right)^{(2)}_{\varepsilon_2}$$

$$\left. +\frac{\varepsilon_1}{4} \left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(2)}_{\varepsilon_1} \left(\Gamma(\varepsilon_2-u)\Gamma(-\varepsilon_2-v)\right)^{(2)}_{\varepsilon_2} \right].$$

The first term in Eq. (22) is

$$\lim_{\varepsilon_2\to 0} \frac{1}{\varepsilon_1\varepsilon_2} \left[M_2^{(u+\varepsilon_1,v+\varepsilon_2)}(\varepsilon_1,\varepsilon_2,\varepsilon_3) - M_2^{(u+\varepsilon_1,v)}(\varepsilon_1) \right] \approx$$

$$\Gamma^2(1+u+v) \left[-\frac{1}{6\varepsilon_1} \left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(3)}_{\varepsilon_1} \Gamma(-u)\Gamma(-v) \right.$$

$$+\frac{1}{2\varepsilon_1} \left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)'_{\varepsilon_1} \left(\Gamma(\varepsilon_2-u)\Gamma(-\varepsilon_2-v)\right)^{(2)}_{\varepsilon_2}$$

$$-\frac{1}{24} \left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(4)}_{\varepsilon_1} \Gamma(-u)\Gamma(-v)$$

$$\left. +\frac{1}{4} \left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(2)}_{\varepsilon_1} \left(\Gamma(\varepsilon_2-u)\Gamma(-\varepsilon_2-v)\right)^{(2)}_{\varepsilon_2} \right].$$

Thus, we obtain limit for $M_3^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ when ε terms are vanishing

$$\lim_{\varepsilon_2\to 0,\varepsilon_1\to 0} M_3^{(u,v)}(\varepsilon_1,\varepsilon_2,\varepsilon_3) \approx$$

$$\lim_{\varepsilon_2\to 0,\varepsilon_1\to 0} \left(\frac{1}{\varepsilon_1\varepsilon_2} \left[M_2^{(u+\varepsilon_1,v+\varepsilon_2)}(\varepsilon_1,\varepsilon_2,\varepsilon_3) - M_2^{(u+\varepsilon_1,v)}(\varepsilon_1) \right] + \frac{1}{\varepsilon_1^2} \left[M_2^{(u+\varepsilon_1,v)}(\varepsilon_1) - M_2^{(u,v)} \right] \right) =$$

$$\lim_{\varepsilon_2\to 0,\varepsilon_1\to 0} \Gamma^2(1+u+v) \left[-\frac{1}{6\varepsilon_1} \left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(3)}_{\varepsilon_1} \Gamma(-u)\Gamma(-v) \right.$$

$$+\frac{1}{2\varepsilon_1} \left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)'_{\varepsilon_1} \left(\Gamma(\varepsilon_2-u)\Gamma(-\varepsilon_2-v)\right)^{(2)}_{\varepsilon_2}$$

$$-\frac{1}{24} \left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(4)}_{\varepsilon_1} \Gamma(-u)\Gamma(-v)$$

$$\left. +\frac{1}{4} \left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(2)}_{\varepsilon_1} \left(\Gamma(\varepsilon_2-u)\Gamma(-\varepsilon_2-v)\right)^{(2)}_{\varepsilon_2} \right]$$

$$+\Gamma^2(1+u+v)\Gamma(-u)\Gamma(-v) \left[\frac{1}{2\varepsilon_1} \left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(2)}_{\varepsilon_1} \left(\psi(-u)-\psi(-v)\right) + \right.$$

$$\frac{1}{6\varepsilon_1} \left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(3)}_{\varepsilon_1} + \frac{1}{6} \left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(3)}_{\varepsilon_1} \left(\psi(-u)-\psi(-v)\right) +$$

$$\left. \frac{1}{24} \left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(4)}_{\varepsilon_1} \right] =$$

$$\Gamma^2(1+u+v)\left[\frac{1}{4}\left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(2)}_{\varepsilon_1}\left(\Gamma(\varepsilon_2-u)\Gamma(-\varepsilon_2-v)\right)^{(2)}_{\varepsilon_2}\right]+$$
$$\Gamma^2(1+u+v)\Gamma(-u)\Gamma(-v)\left[\frac{1}{6}\left(\Gamma(-\varepsilon_1-u)\Gamma(\varepsilon_1-v)\right)^{(3)}_{\varepsilon_1}\left(\psi(-u)-\psi(-v)\right)\right] \quad (24)$$

This result is written up to terms proportional to irrational numbers, for example, of the type $\zeta(n)\Gamma^2(-u)\Gamma^2(-v)\Gamma^2(1+u+v)$, which is proportional to the MB transform of the first UD function.

7 Conclusions

In this paper we have explicitly calculated a new type of two-fold MB integrals. This type of integrals has been obtained due to recursive relations for the momentum loop integrals which are derived from Belokurov-Usyukina loop reduction method. All the integrals have been reduced to the Appell hypergeometric function corresponding to one loop momentum integral. The recursion creates an infinite system of reduction relations, which allowed us to represent the MB transform of $L+1$ momentum loop integral in terms of MB transform of L momentum loop integrals. In order to derive those relations, we needed to shift complex variables of the MB transforms on the right hand sides of the reduction relation in order to organize the same dependence on the parameters x and y as they stand on the left hand sides. For this purpose we need the representation (17) and not the representation (21).

At the first site, the results (21) and (24) are difficult to analyse, since on the right hand sides appear certain combinations of the higher order derivatives of the product of Euler gamma functions

$$\frac{d^n}{d\varepsilon^n}\Gamma(-\varepsilon-u)\Gamma(\varepsilon-v)|_{\varepsilon=0}. \quad (25)$$

These derivatives can be re-written as

$$\frac{d^n}{d\varepsilon^n}\Gamma(-\varepsilon-u)\Gamma(\varepsilon-v)|_{\varepsilon=0}=(\partial_u-\partial_v)^n\Gamma(-u)\Gamma(-v),$$

what makes them transparent for use in various integral transformations, first of all in integrations by parts.

However, the integration by parts in the plane of complex variables u and v can serve to us just to prove the coincidence of two, at first view, different representations. The reason why the representation for MB transforms of UD functions should be simple decomposition in terms of derivatives of (25) or (26) is in the recursive construction of the MB images, which has nothing to do with quantum field theory or with theory of polylogarithms. We will prove this observation further. For this purpose for us the representation (21) is more useful than the representation (17) used for derivation of the MB integrals which correspond to the Belokurov-Usyukina loop reduction method. Actually, after four rung the difference between two representation disappears. Nevertheless, we consider the integral relation that can be found in Ref. [1],

$$J(1,1,1-\varepsilon)\sim\frac{1}{2}\left[x^\varepsilon J(1-\varepsilon,1+\varepsilon,1)+y^\varepsilon J(1-\varepsilon,1+\varepsilon,1)\right]$$

and it follows that under the symbol of MB integral the identity is valid

$$D[1,1,1-\varepsilon]\simeq\frac{1}{2}\left[x^\varepsilon D[1-\varepsilon,1+\varepsilon,1]+y^\varepsilon D[1-\varepsilon,1+\varepsilon,1]\right] \quad (26)$$

We recall this identity is valid under the integration over double complex measure and cannot be consider as exact equality, this is why we write the symbol $\simeq$. In the right hand side we have dependence on x and y while on the left hand side apparently we have not.

The benefits of the representation (26) are in very simple dependence on the variable ε, namely it presents in two factors only,

$$D[1-\varepsilon, 1+\varepsilon, 1] \sim \Gamma(-u-\varepsilon)\Gamma(-v+\varepsilon)$$

The rest of factors in that MB transform does not depend on the parameters ε, so that they are not important for the derivation with respect to ε. For example, the equation for M_2 can be re-written as

$$\begin{aligned} M_2(\varepsilon_1,\varepsilon_2,\varepsilon_3) &\sim \frac{1}{\varepsilon_2\varepsilon_3} y^{\varepsilon_2} D[1-\varepsilon_1] + \frac{1}{\varepsilon_1\varepsilon_2} D[1+\varepsilon_3] + \frac{1}{\varepsilon_1\varepsilon_3} x^{\varepsilon_1} D[1-\varepsilon_2] \\ &\sim \frac{1}{\varepsilon_2\varepsilon_3} y^{\varepsilon_2} \left[x^{\varepsilon_1}\Gamma(-u-\varepsilon_1)\Gamma(-v+\varepsilon_1) + y^{\varepsilon_1}\Gamma(-u+\varepsilon_1)\Gamma(-v-\varepsilon_1)\right] \\ &+ \frac{1}{\varepsilon_1\varepsilon_2} \left[x^{-\varepsilon_3}\Gamma(-u-\varepsilon_3)\Gamma(-v+\varepsilon_3) + y^{-\varepsilon_3}\Gamma(-u+\varepsilon_3)\Gamma(-v-\varepsilon_3)\right] \\ &+ \frac{1}{\varepsilon_1\varepsilon_3} x^{\varepsilon_1} \left[x^{\varepsilon_2}\Gamma(-u-\varepsilon_2)\Gamma(-v+\varepsilon_2) + y^{\varepsilon_2}\Gamma(-u+\varepsilon_2)\Gamma(-v-\varepsilon_2)\right] \end{aligned} \quad (27)$$

Obviously, in the limit of vanishing ε_i the value of the previous expression will contain the derivatives (25) only, that is,

$$\lim_{\varepsilon_2\to 0,\varepsilon_1\to 0} M_2(\varepsilon_1,\varepsilon_2,\varepsilon_3) \sim \frac{3}{2}\left(\Gamma(-u-\varepsilon)\Gamma(-v+\varepsilon)\right)^{(2)}_{\varepsilon} + \frac{3}{2}\ln\frac{x}{y}\left(\Gamma(-u-\varepsilon)\Gamma(-v+\varepsilon)\right)'_{\varepsilon} + \frac{1}{4}\ln^2\frac{x}{y}\Gamma(-u)\Gamma(-v)$$

This expression does not have a form of (17) or (21), however it can be transformed to that form by using integration by parts in the complex planes of u and v.

In the expression for M_3, that is,

$$M_3(\varepsilon_1,\varepsilon_2,\varepsilon_3) \sim \frac{1}{\varepsilon_2\varepsilon_3} x^{-\varepsilon_1} M_2(\varepsilon_1) + \frac{J^{-1}}{\varepsilon_1\varepsilon_2} x^{-\varepsilon_1} y^{-\varepsilon_2} M_2(\varepsilon_1,\varepsilon_2,\varepsilon_3) + \frac{1}{\varepsilon_1\varepsilon_3} y^{-\varepsilon_2} M_2(\varepsilon_2)$$

we substitute a structure found in (27), and again in the limit of vanishing ε_i we obtain the decomposition in terms of the derivatives (25), that is,

$$\begin{aligned} \lim_{\varepsilon_2\to 0,\varepsilon_1\to 0} M_3(\varepsilon_1,\varepsilon_2,\varepsilon_3) &\sim \frac{5}{12}\left(\Gamma(-u-\varepsilon)\Gamma(-v+\varepsilon)\right)^{(4)}_{\varepsilon} + \\ \frac{5}{6}\ln\frac{x}{y}\left(\Gamma(-u-\varepsilon)\Gamma(-v+\varepsilon)\right)^{(3)}_{\varepsilon} &+ \frac{1}{2}\ln^2\frac{x}{y}\left(\Gamma(-u-\varepsilon)\Gamma(-v+\varepsilon)\right)^{(2)}_{\varepsilon} \\ &+ \frac{1}{12}\ln^3\frac{x}{y}\left(\Gamma(-u-\varepsilon)\Gamma(-v+\varepsilon)\right)'_{\varepsilon} \end{aligned} \quad (28)$$

By using the integration by part in the complex plane of u and v, we obtain the coincidence with Eq. (24). Following the recursive procedure for higher numbers of n, we can obtain the higher derivatives of the construction (25) multiplied by powers of $\ln\frac{x}{y}$. At present, it is difficult to calculate the coefficients in front of the decomposition terms. However, it is clear that the finiteness of the limit of vanishing ε_i has nothing to do with the polylogarithms, since instead of function Γ of Euler we can write any other smooth function in the construction (27). Thus, the infinite sum in the limit of vanishing ε_i of the quantities $M_n(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ constructed from (26) has underlying integrable structure which can be uncovered by identifying coefficients in front of the terms in the expansion (28) for higher n. That integrable structure has nothing common with MB transform or polylogarithms and is based on properties that can be studied by basic methods of mathematical analysis.

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