THE SPHERICAL HARMONICS METHOD, 1.
(GENERAL DEVELOPMENT OF THE THEORY)

CRT - 340 (REVISED)

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Summary

A method of obtaining approximate solutions of the transport equation is presented in a form applicable in principle to any geometry. The approximation will give good results in cases where the angular distribution is not very anisotropic. The basis of the approximation is to expand the density per unit solid angle \( \psi(\mathbf{r}, \hat{\mathbf{n}}) \) in spherical harmonic tensors formed from \( \hat{\mathbf{n}} \), the unit vector in the direction of velocity, and to break off the expansion. A differential equation whose degree increases with the order of the approximation is obtained for the total density \( \psi^{(0)}(\mathbf{r}) \). This equation has the form

\[
\prod_{\ell} (\nu_{\ell}^2 - \nu_{\ell_1}^2) \psi^{(0)}(\mathbf{r}) = 0,
\]

where the numbers \( \nu_{\ell} \) depend on the order of the approximation and on the value of the parameter \( \alpha \) of the medium, but not at all on the geometry.

When the equation for the total density is an ordinary equation, we simulate the physical condition of continuity of \( \psi(\mathbf{r}, \hat{\mathbf{n}}) \) at a boundary in a multi-medium problem by requiring that the spherical harmonic moments of \( \psi(\mathbf{r}, \hat{\mathbf{n}}) \) which we retain be continuous; and this determines the constants in the solution for
\( \psi^{(0)}(\mathbf{r}) \). The form of the solution for the total density and the necessary moments in an approximation of general order is given explicitly for plane and spherical symmetry; and for cylindrical symmetry the solution is given for two low-order approximations.

In a later report (CRT-338, Revised) the application of the method to several problems involving plane and spherical symmetry will be discussed in detail and the results of a number of examples already worked will also be given.
1. Introduction

In this report we discuss a method of obtaining approximate solutions to the transport equation

$$\mathbf{j} \cdot \nabla \left[ \mathbf{j} \psi (\mathbf{r}, \hat{n}) \right] + \psi (\mathbf{r}, \hat{n}) = (1-a) \psi^{(0)} (\mathbf{r}) / 4\pi + q(\mathbf{r}, \hat{n}).$$  \hspace{1cm} (1.1)

(The notation is discussed below.) For the transport equation with plane symmetry,

$$\mu \frac{\partial \psi (z, \mu)}{\partial z} + \psi (z, \mu) = \frac{1-a}{2} \psi^{(0)} (z) + q(z, \mu),$$  \hspace{1cm} (1.2)

this method amounts to the familiar device of expanding $\psi (z, \mu)$ (and $q(z, \mu)$) in Legendre polynomials:

$$\psi (z, \mu) = \frac{1}{2} \sum_{k=0}^{\infty} (2k+1) \psi_k (z) P_k (\mu),$$  \hspace{1cm} (1.3)

where $\psi_k (z) = \int \psi (z, \mu) P_k (\mu) d\mu$; and obtaining values for the first few terms of the series on the assumption that the remaining terms are negligible. (The simplest diffusion approximation is obtained by keeping only the first two terms of (1.3)).

This type of approximation will, of course, work best in cases where the angular distribution of neturons is not too violently anisotropic, or, at least, is very anisotropic only over relatively small parts of the region considered. It may then be effectively applied to multi-medium problems where all the media are some number (say 1 or 2) of mean free paths thick, and where the medium

* The content of this report was worked out in the first instance by Dr. J. LeCaine, Dr. P.R. Wallace and the author, in collaboration. The author has been responsible for the preparation of the report; and in this he has had the assistance of suggestions by Dr. G. Placzek.
within which we wish to study the density is not too heavily capturing; but it cannot be expected to give reliable results easily if applied to thin shells or to situations where a \( \delta \)-function or very marked anisotropy is everywhere present in the angular distribution.

In the method presented here the plane symmetry mentioned above is included as a special case; for we expand the function \( \psi(\mathbf{r}, \mathbf{n}) \) in spherical harmonic tensors formed from the vector \( \mathbf{n} \) before specifying the particular geometry. (These tensors are equivalent to the usual surface spherical harmonics of \( \mathbf{n} \).)

In the present report we give first the general development of the theory, in a form in which no particular symmetry is assumed; so that the formulae up to the end of § 5 are valid for any geometry. In the remaining sections, §§ 6, 7, 8, we specialize the geometry to plane, spherical and cylindrical symmetry. The first two cases have, of course, been handled before, without resorting to tensors. (These cases are relatively simple in virtue of the fact that \( \psi(\mathbf{r}, \mathbf{n}) \) depends on only one angular variable, so that the expansion in spherical harmonics reduces to an expansion in Legendre polynomials.) For a consideration of either one of these two cases exclusively the tensor notation would only be an encumbrance. But the present approach becomes worthwhile for one interested in applying the approximation to several geometries; and, in the case of the cylinder, any less formally systematic method would seem to be out of the question.

In the plane and spherical cases it is quite feasible (at least when the source term is very simple) to carry out
approximations which amount to keeping up to 9 terms in the series (1.3). In the cylinder it becomes very onerous to go beyond the stage analogous to keeping 4 terms in (1.3).

Several constants appear repeatedly in the computations involved in actually applying the method: tables of these constants are given in CRT-338 Revised, "The Spherical Harmonic Method II" by C. Mark.

In Appendix A we recall, for convenience, definitions from the tensor calculus and we define the spherical harmonic tensors.

In Appendix B we give most of the mechanical work required in handling the spherical case. In Appendix C we give some of the mechanical detail required for the cylindrical case. In Appendix D we give a proof of a general lemma which is formally useful in establishing the completeness of the solutions given in the particular geometries.

In a following report (CRT-338, Revised) we shall discuss the technique of applying the method to several particular types of problem. In the cases of plane and spherical symmetry we give also the results of such application, and these will indicate the degree of convergence to be expected of the method in various situations.

**Notation.**

For physical quantities, we use the notation of MT-4 (Placzek and Volkoff: "Notes on Diffusion of Neutrons without Change in Energy"). \( \vec{n} \) is a unit vector.
$\psi(\mathbf{r}, \mathbf{n})$ is the number of neutrons per unit volume and per unit solid angle at the point $\mathbf{r}$, travelling in the direction $\mathbf{n}$.

$\psi^{(0)} = \psi^{(0)}(\mathbf{r}) = \int d\Omega \psi(\mathbf{r}, \mathbf{n})$ is the total density of neutrons at $\mathbf{r}$. $q(\mathbf{r}, \mathbf{n})$ is the source term, giving the number of neutrons produced at $\mathbf{r}$ which travel in the direction $\mathbf{n}$ (per unit volume and per unit solid angle).

$\mu$: when (as in all the case studied in detail here) $\psi(\mathbf{r}, \mathbf{n})$ is a function of only one space variable, $\mu$ is the cosine of the angle between the direction in which that variable is measured and the direction of $\mathbf{n}$.

$l$ is the total mean free path of a neutron.

$\alpha$, when positive, is the ratio $l/l_c$, where $l_c$ is the capture mean free path (alternatively, $\alpha = 1/(N+1)$, where $N$ is the average number of collisions a neutron suffers before capture). If $\alpha$ be negative then it may be interpreted as $(1-k)l/l_c$ where $k$ is a multiplication constant greater than unity.

For tensors, we use a notation somewhat similar to that in McConnell: Applications of the Absolute Differential Calculus, Blackie, 1931.

$\Omega^i$ are contravariant components of $\mathbf{n}$.

$\mathbf{T}^i_1 \cdot \cdot \cdot _n$ are contravariant components of the symmetric spherical harmonic tensor of order $n$ formed from $\mathbf{n}$.

$D_i$ indicates the tensor operation of covariant differentiation with respect to $x^i$.

$D_{ij} \ldots$ indicates the tensor operation of successive covariant differentiation with respect to $x^i, x^j \ldots$

$g^{ij}$ is the contravariant metric tensor.
\[ \psi_{i \ldots} = \psi_{i \ldots}(\mathbf{r}) = \int d\Omega \psi(\mathbf{r}, \Omega) T^{i \ldots}. \]

We use the summation convention that if the same literal index appears both as a subscript and superscript in a particular symbol or product of symbols, we imply summation over the index; e.g., \[ u^i v_i = \mathbf{u} \cdot \mathbf{v}; \] and such indices are not to be included in determining the tensor rank of the term in which they appear.

The following symbols are used throughout with the same significance:

- \( F_n \), defined in (3.4)
- \( G_n \), defined in (3.8)
- \( v_1 \), which might be more explicitly designated \( v_1^{(m)} \), are, in the \((m-1)^{st}\) approximation (see following (3.4)), the roots of \( F_m(x^2) = 0 \). We number the \( v_1 \) so that \( v_1 \) is always an approximation to the root of the transcendental equation \( \arctanh v = 1/(1-\alpha) \), where \( \arctanh = \tanh^{-1} \), and for \( i \neq 1 \) we take \[ |v_i| < |v_{i+1}|. \]

2. Expansion of \( \psi(\mathbf{r}, \Omega) \) in Tensors.

Let \( T^{i \ldots}_{i \ldots} \) represent a component of the contravariant, symmetric spherical harmonic tensor of the \( n^{th} \) order formed from the vector \( \mathbf{r} \). By \( T^{i \ldots}_{i \ldots} \) we represent the associated covariant tensor. These tensors are easily written in terms of the components of \( \mathbf{r} \); for example, for the contravariant tensors we have:

\[ T^i = \mathbf{r}^i (i=1,2,3); \]

\[ T^{ij} = (3 \mathbf{r}^i \mathbf{r}^j - g^{ij})/2, \text{(where } g^{ij} \text{ is the metric tensor}); \]

\[ T^{ijk} = (5 \mathbf{r}^i \mathbf{r}^j \mathbf{r}^k - 5 \sum_{(3)} g^{ij} \mathbf{r}^k)/2; \]  \hspace{1cm} (2.1)

\[ T^{ijkl} = (35 \mathbf{r}^i \mathbf{r}^j \mathbf{r}^k \mathbf{r}^l - 5 \sum_{(6)} g^{ij} \mathbf{r}^k \mathbf{r}^l + 5 \sum_{(3)} g^{ij} g^{kl})/8. \]

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(1) Definitions of tensor terms and notation are given in Appendix A.
(In the above, \( \sum \) represents the sum of all the different terms of the form indicated which may be obtained by interchanging indices; the number of such terms is also shown.) These spherical harmonic tensors satisfy the recurrence relation:

\[
\Omega^{n+1}_{\mathbf{i} \mathbf{l} \cdots \mathbf{i}_{n}} = \frac{n+1}{2n+1} \left\{ \Omega^{n}_{\mathbf{i} \mathbf{l} \cdots \mathbf{i}_{n+1}} + \frac{2n-1}{n(2n+1)} \sum_{k=1}^{n} T^{i_{1} \cdots i_{k-1} i_{k+1} \cdots i_{n+1}} \right\}
\]

(2.2)

\[
\Gamma^{4}_{\mathbf{i} \mathbf{i}_{n+1}} = \frac{2}{n(2n+1)} \sum_{1 \leq k < j \leq n} T^{i_{1} \cdots i_{k-1} i_{k+1} \cdots i_{j-1} i_{j+1} \cdots i_{n+1}} \Gamma^{k}_{i j}.
\]

The components of these tensors form an orthogonal system on the surface of the unit sphere. They (or rather, the physical components \( T^{\mathbf{i} \mathbf{l} \cdots \mathbf{i}_{n}} \)) are linear combinations of the usual surface spherical harmonics \( P_{n}(m)(\mu) \cos \theta \phi \). Examples of these combinations are given in the following table. For this purpose we adopt the abbreviations

\[
P_{n}(m)(\mu) \cos m \phi = C_{n}^{m}, \quad P_{n}(m)(\mu) \sin m \phi = S_{n}^{m}, \quad P_{n}(\mu) = P_{n}.
\]

(2.3)

If we number the axes so that \( \theta(= \cos^{-1} \mu) \) is measured from the direction of \( x^{3} \) and so that the azimuthal angle \( \phi \) is measured from the direction of \( x^{2} \), then we have

for \( T^{i} \):

\[
T^{1} = S_{1}^{1}, \quad T^{2} = C_{1}^{1}, \quad T^{3} = P_{1}.
\]

(2.4)

In general,

\[
\text{for } T^{i_{1} \cdots i_{n}} \quad T^{33 \cdots 3} = P_{n};
\]

(2.5)

and in particular,
for $T_{ij}$: $T_{i1}^{11} = -(2P_2 + C_2^2)/4$, $T_{22}^{22} = -(2P_2 - C_2^2)/4$;

for $T_{ijk}$: $T_{i113}^{11} = -(6P_3 + C_3^2)/12$, $T_{22}^{22} = -(6P_3 - C_3^2)/12$; \hfill (2.6)

for $T_{ijkl}$: $T_{i111}^{11} = (72P_4 + 8C_4^2 + C_4^4)/192$, $T_{1133}^{11} = -(12P_4 + C_4^2)/24$;

for $T_{ijkm}$: $T_{i1113}^{11} = (360P_5 + 24C_5^2 + C_5^4)/960$, $T_{1133}^{11} = -(20P_5 + C_5^2)/40$.

For plane or spherical symmetry we shall require only the moments with respect to $P_3$ as in (2.5). The table above, along with the relations (proved in Appendix A)

\[ \sum_{i=1}^{3} T_{i1k}^{i1} m = 0, \] \hfill (2.7)

is sufficient to express in terms of surface spherical harmonic moments all the components we shall require in the cylindrical case.

We now define the moments with respect to these tensors:

\[ \psi_{11}^{i1} = \psi_{i1}^{i1} n_{i1}^{i1} = \int \psi_{11}^{i1} T_{i1}^{i1} n_{i1}^{i1} \, d\Omega, \] \hfill (2.8)

where the integration is over the unit sphere about the point $P$. Let us represent scalar products such as $\psi_{ij}^{i} T_{ij}$ by $\psi_{(2)}^{i} T_{(2)}^{i}$. Then, with the help of equations (A.9) and (A.10) one can establish the relation

\[ T^{(k)}_{(k)}(\Omega) T^{(k)}_{(k)}(\Omega') = (2k)! P_{k}(\Omega, \Omega')/2^{k} (k)_{2}. \]

If we now substitute from this in the usual spherical harmonic expansion

\[ \psi(\vec{r}, \vec{\Omega}) = (1/4\pi) \sum_{k=0}^{\infty} (2k+1) \int d\Omega' P_{k}(\vec{\Omega}, \vec{\Omega}') \psi(\vec{r}, \vec{\Omega}') \]

we obtain the expansion of $\psi(\vec{r}, \vec{\Omega})$ in terms of spherical harmonic tensors in the form...
\[ \psi(\vec{r},\vec{n}) = \left( \frac{1}{4\pi} \right) \sum_{k=0}^{\infty} \frac{(2k+1)2^k(k!)^2 \psi^{(k)}(\vec{r}) T(k)}{(2k)!} \]  

(2.9)

Before making use of these definitions in the transport equation (1.1), we suppose that the unit of length in the medium being considered is so adjusted that the "\( l^{-1}\)div" of the cm.-system of coordinates of (1.1) becomes merely "div" in the new system. The transport equation will then be

\[ \text{div} \left\{ \hat{n} \psi(\vec{r},\vec{n}) \right\} + \psi(\vec{r},\vec{n}) = (1-a)\psi^{(0)}(\vec{r})/4\pi. \]  

(2.10)

If we use the symbol \( D_j \) to indicate the operation of covariant differentiation, then \( \text{div} \left\{ \hat{n} \psi \right\} \) will be written as \( D_j \left\{ \hat{n} \psi \right\} \), so that (2.10) may be rewritten in our notation as

\[ D_j \left\{ \hat{n} \psi(\vec{r},\vec{n}) \right\} + \psi(\vec{r},\vec{n}) = (1-a)\psi^{(0)}(\vec{r})/4\pi. \]  

(2.11)

We are now in a position to replace this integro-differential equation by an infinite system of linear differential equations in the moments (2.8). For this purpose we multiply (2.11) by \( i_1, i_1 i_2, i_1 i_2 i_3, \ldots \) in turn, use the recurrence relation (2.2), and then integrate over \( \Omega \), to obtain the following set of equations:

(2) From here on we omit the source term \( q(\vec{r},\vec{n}) \) of (1.1) since it introduces no essential difference: to include it merely requires an additional term \( (2k+1)q \frac{i_1 \cdots i_n(\vec{r})}{i_1 \cdots i_n(\vec{r})} \) on the right in equations (2.12). In particular, if the source is isotropic the only change is the additional term \( q_0(\vec{r}) \) in the first equation of (2.12) Some problems have been done including an isotropic source term, and references will be made to these in CRT-338, Revised.
The basis of the approximation is to assume that after some stage the terms in the expansion (2.9) become negligible. If, in accordance with this assumption, we suppose that \( \psi^{i_1 \ldots i_n} \equiv 0 \) for \( n \geq N \), then (2.12) reduces to a finite system of equations which, in principle at least, could be solved. Successive approximations would be obtained by keeping terms of higher and higher order in \( \psi^{i_1 \ldots i_n} \).


Equations (2.12) are equations in the tensor components. We can immediately obtain a set of scalar equations from them by applying the operators \( D_i \) to the first, \( D_i D_j \) to the next, and so on. Recalling that \( D_i \psi = -\alpha \psi^{(0)} \), \( D_i D_j \psi = \nabla^2 \psi^{(0)} \) and \( D_i D_j \psi = \nabla^2 \psi^{(1)} \), the Laplacian, we obtain the equations:

\[
D_1 \psi^{(1)} = -\alpha \psi^{(0)},
\]
\[
2D_2 \psi^{(2)} + \nabla^2 \psi^{(0)} + 3D_1 \psi^{(1)} = 0,
\]
\[
3D_3 \psi^{(3)} + 2\nabla^2 \left( D_1 \psi^{(1)} \right) + 5D_2 \psi^{(2)} = 0,
\]
\[
4D_4 \psi^{(4)} + 3\nabla^2 \left( D_2 \psi^{(2)} \right) + 7D_3 \psi^{(3)} = 0,
\]

etc.
and in general
\[ n D(n)\psi^{(n)} + (n-1)v^2 \left\{ D(n-2)\psi^{(n-2)} \right\} + (2n-1) D(n-1)\psi^{(n-1)} = 0. \]
From equations (3.1) we may express \( D(n)\psi^{(n)} \), \( n=1,2,3, \ldots \)
in terms of scalar operations on \( \psi^{(o)} \). These expressions are:

\[ D(1)\psi^{(1)} = -a\psi^{(o)}, \]
\[ D(2)\psi^{(2)} = \left( -v^2/2 + 3a/2 \right)\psi^{(o)}, \]
\[ D(3)\psi^{(3)} = \left\{ (5+4a)v^2/6 - 5a/2 \right\}\psi^{(o)}, \]
\[ D(4)\psi^{(4)} = \left\{ 3v^4/8 - 5(7+11a)v^2/24 + 35a/8 \right\}\psi^{(o)}, \]
\[ D(5)\psi^{(5)} = \left\{ -(161+64a)v^4/120 + 7(3+7a)v^2/8 - 63a/8 \right\}\psi^{(o)}, \]
\[ D(6)\psi^{(6)} = \left\{ -5v^6/16 + 21(14+11a)v^4/80 - 7(11+34a)v^2/16 + 231a/16 \right\}\psi^{(o)}, \]

and in general,
\[ D(k)\psi^{(k)} = F_k(v^2)\psi^{(o)} \tag{3.3}\]
where \( F_k \) is a polynomial of degree \( \left\lfloor \frac{k}{2} \right\rfloor \),

\[ F_0 = 1, \quad F_1 = -a, \quad \text{and} \]
\[ n F_n(v^2) + (n-1)v^2 F_{n-2}(v^2) + (2n-1) F_{n-1}(v^2) = 0. \tag{3.4}\]

What we shall call the \((m-1)\)st approximation is obtained by assuming that \( \psi^{(m)} = 0 \). Thus for the \((m-1)\)st approximation we have at once the differential equation for \( \psi^{(o)} \):

\[ F_m(v^2)\psi^{(o)} = 0. \tag{3.5}\]

We shall consider mainly approximations of so-called "odd" order in which \( m \) is even (say \( m=2n \)). This approximation is also referred to as the \( \psi_{2n-1} \)-approximation, since the moments of order \( 2n-1 \) are the last to be retained in the expansion (2.9). The operator \( F_{2n}(v^2) \) may be factored in the form
\[ F_{2n}(\nabla^2) = (\text{const}) \prod_{i=1}^{n} (\nabla^2 - \nabla_i^2); \quad (3.6) \]

so that the fundamental solutions with which we shall approximate \( \psi(\phi) \) are solutions of the equations

\[ \left\{ \nabla^2 - \nabla_i^2 \right\} f(\frac{\phi}{r}) = 0 \quad (3.7) \]

for the particular geometry considered. It is worthy of note that while the form of the fundamental solutions will depend on the geometry (exponentials, for plane case; Bessel functions, for cylinder; etc.) the parameters \( \nu_i \) entering the arguments of the solutions are independent of the geometry and depend only on the order of the approximation and the value of \( \alpha \) in the medium considered.

In the "even" approximation, in which the moments of order \( 2n \) are the last ones retained, there is a new feature introduced. The operator \( F_{2n}(\nabla^2) \) is of the same degree as \( F_{2n}(\nabla^2) \) in (3.6); so that if (3.5) is an ordinary differential equation (i.e., if \( \psi(\phi)(\frac{\phi}{r}) \) is a function of only one space-coordinate) there will apparently be the same number of constants in the solution for \( \psi(\phi) \) when \( m = 2n+1 \) as when \( m = 2n \). We are, however, retaining more moments in the \( \psi_{2n} \)-approximation than in the \( \psi_{2n-1} \)-approximation. As we shall see when we specialize the geometry, there are just enough moments in the \( \psi_{2n-1} \)-approximation to completely determine the constants when we apply boundary conditions; and hence there would seem to be too many moments for this purpose in the \( \psi_{2n} \)-approximation.

A natural way of clearing up this apparent difficulty is
the following. (3) When we consider the special geometries, we shall find that if we take \( f(v, r) \), where \( \left\{ v^2 - v^2 \right\} f(v, r) = 0 \), for the fundamental solution for the density, and if we deduce from this the form of the higher moments, then \( \psi(k) \) will have the factor

\[
G_k(v) = F_k(v^2)/v^k
\]

(3.8). If we assume \( \psi(m) = 0 \) then \( v \) must be a root of \( G_m(v) = 0 \). In the odd approximation these roots are just the numbers (3.6); but in the even approximation (\( m = 2n + 1 \)) we have, in addition to the \( 2n \) roots of \( F_{2n+1}(v^2) = 0 \), a root with infinite absolute value. In the even approximations we therefore add this infinite root to the zeros of \( F_{2n+1} \). This amounts to allowing a \( \delta \)-function in these approximations.

4. The Characteristic Roots of the \( \psi(n-l) \) - Approximation.

When \( \alpha \) is positive the roots (\( iv_1 \)) of the equation

\[
F_n(v^2) = 0
\]

(4.1)

are all real and, with the usual exception of the smallest, are all larger than unity in size. This smallest root is always an approximation to the root of the equation

\[
\text{arctanh} \psi = 1/(1-\alpha),
\]

(4.2)

the approximation improving as \( n \) increases. If \( \alpha \) is negative the first root is pure imaginary (and is still an approximation to the

(3) This method of handling the even approximations was suggested to the author by Dr. R. E. Marshak.
root of (4.2) the other roots being real and larger than unity as before.

These statements follow immediately from the fact that the roots of (4.1) are the same as the roots of the equation

\[ 1 = (1-a) \sum_{i=1}^{n} \frac{p_i}{2(1-x\mu_i)}, \quad (4.3) \]

where the \( \mu_i \) are the \( n \) roots of \( P_n(\mu) = 0 \) and the \( p_i \) are the Christoffel numbers. The Christoffel numbers (the weight numbers in the Gaussian mechanical quadrature formula) are all positive; and have the property that if \( f(\mu) \) be any polynomial in \( \mu \) of degree up to \( 2n-1 \), then

\[ \int_{-1}^{1} f(\mu) d\mu = \sum_{i=1}^{n} p_i f(\mu_i). \]

From this it follows that for small \( x \) the right side of (4.3) expanded in powers of \( x \) agrees with the expansion of \( (1-a) \arctan x/x \) up to the term involving \( x^{2(n-1)} \). Hence the smallest root of (4.3), or (4.1), is an approximation to the to the root of (4.2). The fact that the \( p_i \) are positive, and that \( |\mu_i| < 1 \), guarantees that the remaining roots of (4.3) are real and larger than unity in size.

The equivalence of the roots of (4.1) and (4.3) may be seen by the following. If we collect the terms of (4.3) on the left side and multiply the resulting equation through by \( C_n \prod_{i=1}^{n} (1-x\mu_i) \), where \( P_n(\mu) = C_n \mu^n + \ldots \), and designate the left side of the equation so obtained by \( L_n \), it is then possible to show that
In view of (3.4) this establishes that $L_{2n} = (-1)^n F_n$, so that (4.3) and (4.1) will have the same roots.

From this we have incidentally that

$$F_n(x^2) = (-1)^n x^n p_n(1/x) \left[ 1 - (1-a) \sum_{i=1}^{n} p_i / 2(1-x \mu_i) \right],$$  

(4.4)

where $p_i$ and $\mu_i$ are related to $p_n$, as in (4.3).

It is interesting to note that equation (4.3), with $n$ even, is the equation used by G. C. Wick to determine the exponents in his method of approximation to solutions of the transport equation in problems involving plane boundaries. (See: Wick - Über Ebene Diffusionsprobleme, Zeit, f, Phys., Vol. 121, pp. 702-718, 1943; and also S. Chandrasekhar, On the Radiative Equilibrium of a Stellar Atmosphere, II, Astrophysical Journal, July 1944.)

5. Equations for the Tensor Components.

In § 3 we obtained a differential equation for the density $s^{(0)}(\mathbf{r})$. In geometries where $s^{(0)}(\mathbf{r})$ is a function of only one space coordinate (the only geometries we shall consider in detail) equations (3.6) are ordinary differential equations. To determine the density completely, we shall have to impose some boundary conditions; and in this connection it is useful to have expressions for the components of the tensors of higher order. Equations for these components may be obtained from the equations (2.12) by an elimination process quite similar to that used to obtain equations for $s^{(0)}$. Thus to obtain an equation for $s^1$ we assume that the form of $s^{(0)}$ is already known, and starting with the second equation of (2.12) we eliminate all terms involving tensors of the second
or higher order. For $\psi_{ij}$ we start with the third equation of 
(2.12) and assume that $\psi^{(o)}$ and $\psi^1$ are known. In high-order 
approximations this elimination is very tedious: and fortunately 
it is quite unnecessary for problems with plane or spherical 
symmetry (see §§ 6 and 7): but it has, so far at least, been 
necessary in the case of the cylinder. We therefore include the 
equations for the various tensor components in a $\psi_3$- and also a 
$\psi_5$-approximation for any geometry. These are:

In the $\psi_3$-approximation:

$$15 \left\{ 3v^2 - 7 \right\} \psi^1 = - \left\{ 9v^2 - 5(7+2a) \right\} D^i \psi^{(o)},$$

$$15 \left\{ v^2 - 7 \right\} \psi^{ij} = 15 \sum_{(2)} D^i \psi^{ij} + \left\{ 3 D^i \psi^{ij} - (v^2-10a)g^{ij} \right\} \psi^{(o)},$$

$$21 \psi^{ijk} = 2 \sum_{(3)} g^{ij} D^k \psi^{(o)} - 5 \sum_{(3)} D^i \psi^{jk}.$$  \hfill (5.1)

In the $\psi_5$-approximation:

$$35 \left\{ 5v^4 - 30v^2 + 33 \right\} \psi^1 = - \left\{ 25v^4 - 14(21+4a)v^2 + 35(11+4a) \right\} D^i \psi^{(o)},$$

$$35 \left\{ v^4 - 18v^2 + 33 \right\} \psi^{ij} = 210 \left\{ v^2 - 3 \right\} \sum_{(2)} D^i \psi^{ij}$$

$$+ \left\{ 30v^2 - 21(9+a) \right\} D^{ij} - (10v^4 - 21(3+7a)v^2 + 420a)g^{ij} \right\} \psi^{(o)},$$

$$63 \left\{ 3v^2 - 11 \right\} \psi^{ijk} = -35 \left\{ v^2 - 9 \right\} \sum_{(3)} D^i \psi^{ijk} + 21 \sum_{(3)} \left\{ 5D^{ij} - (2v^2 - 9)g^{ij} \right\} \psi^{(o)},$$

$$63 \left\{ v^2 - 11 \right\} \psi^{ijk} = 189 \sum_{(4)} D^i \psi^{jkl} + 5 \sum_{(6)} \left\{ 7D^i \psi^{(o)} \right\} \psi^{jkl}$$

$$+ 42 \sum_{(12)} g^{ij} D^k \psi^{ij} + \sum_{(6)} \left\{ 10 D^k \psi^{jkl} - (v^2 - 21a)g^{jkl} \right\} \psi^{(o)},$$
55^j_{k\ell m} = 2 \sum_{(10)} g^{ij}_{\ell} D_{\lambda} \psi^{n\ell m} - 9 \sum_{(5)} g^{m}_{\ell} D_{\lambda} \psi^{j\ell m}.

6. **Specialization to Plane Symmetry.**

In this case \(\psi(\vec{r},\vec{n})\) is a function of only one space coordinate, \(z\) (distance from the plane, to which we assign the index 3 in our tensor notation), and of the one angular coordinate \(\mu (\cosine\ of\ the\ angle\ between\ \vec{n}\ and\ the\ direction\ of\ z)\). All the spherical harmonic moments vanish except those with respect to \(P_n(\mu)\); so that all the tensor components which do not vanish are proportional to the components \(\psi^{33\cdots3}\). Only covariant derivatives with index 3 are different from zero, and these become ordinary derivatives with respect to \(z\).

For convenience we shall use \(\psi_n(z)\) in place of \(\psi^{33\cdots3}(z)\).

Equation (3.3) then becomes

\[
d^k\psi_k(z)/dz^k = F_k(d^2/dz^2)\psi(z). \tag{6.1}
\]

In the \(\psi_{2n-1}\)-approximation the differential equation for \(\psi_0\) will be, from (3.5) and (3.6),

\[
\prod_{1=1}^{n} \left(\frac{d^2}{dz^2} - \nu^2_i\right) \psi_0(z) = 0, \tag{6.2}
\]

so that

\[
\psi_0(z) = \sum_{i=1}^{n} \left\{ A_i \exp(\nu_i z) + B_i \exp(-\nu_i z) \right\}, \tag{6.3}
\]

where we take \(\nu_i > 0\). Substituting (6.3) in (6.1), integrating \(k\) times, and using (3.8), we obtain (on omitting all constants of integration)

\[
\psi_k(z) = \sum_{i=1}^{n} G_k(\nu_i) \left\{ A_i \exp(\nu_i z) + (-1)^k B_i \exp(-\nu_i z) \right\}, \tag{6.4}
\]
(k=0, 1, 2, ..., 2n-1). That the constants of integration are to be omitted may be seen by assuming that they are present and, on substituting in (2.12), proving that each vanishes. It may also be seen by noticing that in the plane case the equations (2.12) are a system of 2n equations in 2n functions \( \psi_0, \ldots, \psi_{2n-1} \). The system of solutions (6.4) contains 2n arbitrary constants, and it is shown in Appendix D that (6.4) is the complete solution.

If \( a=0 \) we have \( \psi_1 = 0 \). The functions (6.4) become

\[
\begin{align*}
\psi_0(z) &= A_1 z + B_1 + \sum_{i=2}^{n} \left[ A_i \exp(v_1 z) + B_i \exp(-v_1 z) \right], \\
\psi_1(z) &= -A_1/3, \\
\psi_k(z) &= \sum_{i=2}^{n} G_k(v_i) \left[ A_i \exp(v_i z) + (-)^k B_i \exp(-v_i z) \right], \quad k=2, 3, \ldots, 2n-1.
\end{align*}
\]

In the \( \psi_{2n} \)-approximation, the only modifications to equations (6.3) to (6.5) are that the \( v_1 \) are now the zeros of \( F_{2n+1} \) and we add to \( \psi_k(z) \) the term \( A_0 G_k(v_0) \exp(-|v_0 z|) \) in which \( |v_0| = \infty \).

In applying the approximation to, say, a problem in which we have two media with a plane boundary, we set up the solution (6.4) in each medium, apply the appropriate conditions at the outer boundary of each medium (which might, for example, be infinity); and at the interface we require the physically obvious continuity of the \( \psi_k \). These conditions lead to a set of linear equations for the constants \( A_i \) and \( B_i \). The whole labour of applying the approximation is in the solution of such linear equations. In CRT-338 Revised this question will be discussed in detail in
connection with several specific problems, and it will be shown that in some problems similar to the Milne problem considerable simplifications in the sets of linear equations are available.

In applying the approximation it must be kept in mind that in writing (2.10), on which all our subsequent formulae are based, we used the mean free path as a unit of length. This means that in (6.3) to (6.5) \( z \) is measured in the mean free path of the medium in which the solution is set up. To change back to a cm-system it is only necessary to replace \( z \) by \( y/\ell \) throughout (\( y \) in cm).

7. Spherical Symmetry.

As in the plane, \( \Psi(\vec{r}, \vec{\mu}) \) is dependent on only one space-coordinate: \( r \), and one angular coordinate: \( \mu \). All the non-zero tensor components are proportional to \( \mu^{33 \ldots 3} \), which we shall write as \( \Psi_\mu(r) \). Covariant derivatives are not, however, equivalent to ordinary derivatives. It is shown in Appendix B that equation (3.3) reduces to

\[
(D + \frac{2}{r})(D + \frac{3}{r}) \ldots (D + \frac{k+1}{r}) \Psi_k(r) = F_k(\nu^2) \Psi_0(r) \quad (7.1)
\]

where \( D = d/dr \).

In the \( \Psi_{2n-1} \)-approximation the differential equation for \( \Psi_0(r) \) will be, from (3.5) and (3.6), \( \prod_{i=1}^{n} (\nu^2 - \nu_i^2) \Psi_0(r) = 0 \); so that

\[
\Psi_0(r) = \sum_{i=1}^{n} \left[ A_i \exp(\nu_i r)/\nu_i r + B_i \exp(-\nu_i r)/(-\nu_i r) \right], \quad (\nu_i > 0) \quad (7.2)
\]

Using this in (7.1) shows that to obtain \( \Psi_k(r) \) we have to solve equations of the type
(D+2/r)(D+3/r) ... \left\{D+(k+l)/r \right\} f_k(r) = F_k(\eta^2) \exp(\eta r)/\eta r

where \eta may be positive or negative.

It is easily verified that the solution of this equation

if of the form

\[ f_k(r) = G_k(\eta) H_k(\eta r) \]  \hspace{1cm} (7.4)

where, with \eta = \eta r,

H_0(x) = \exp(x)/x, H_1(x) = (1-1/x)H_0(x),

H_n(x) = H_{n-2}(x) - (2n-1)H_{n-1}(x)/x \hspace{1cm} (7.5)

From (7.1) to (7.4) we now have, in the spherical case,

\[ \psi_k(r) = \sum_{i=1}^{n} G_k(\nu_i) \left\{ A_i H_k(\nu_i r) + (-1)^k B_i H_k(-\nu_i r) \right\}, \hspace{1cm} (7.6) \]

for \( k = 0, 1, 2, \ldots, 2n-1 \).

(4) As in the case of the plane, no constants of integration need be retained in the solution of (7.3). This is shown in Appendix D.

(5) For completeness, this recursion formula is established directly in Appendix B. The function \( H_k(x) \) appearing here are closely related to Bessel functions, as is evident from the recursion relation (7.5), as well as the relation

\[ \left\{ d/dx+(k+1)/x \right\} H_k(x) = H_{k-1}(x), \] which is immediately available from (7.3). In fact \( H_k(x) = -\sqrt{2/(\pi x)} K_{k+1/2}(-x) \), where \( K_\nu(x) \) is the modified Bessel function of the second kind and of order \( \nu \),

\( K_\nu(x) \sim \sqrt{\pi/2x} \exp(-x) \) for \( x \) large.
In the special case of no capture \((a=0)\) we have \(v_1 = 0\). Then

\[
\psi_0(r) = A_1 + B_1/r + \sum_{i=2}^{\infty} \left\{ A_i H_0(v_1 r) + B_i H_0(-v_1 r) \right\},
\]

\[
\psi_1(r) = B_1/3r^2,
\]

\[
\psi_k(r) = B_1 k^i/(2k+1)r^{k+1} + \sum_{i=2}^{\infty} G_k(v_1)\left\{ A_i H_k(v_1 r) + (-1)^k B_i H_k(-v_1 r) \right\}
\]

for \(k = 2, 3, \ldots, 2n-1\).

As in the plane case, the even approximation will differ from (7.6) and (7.7) only by having the additional term

\[
A_0 G_k(v_0) H_k(-|v_0 r|),
\]

in which we take \(v_0 = \infty\). If we are considering a medium extending from \(r = a\) to \(\infty\), then in applying boundary conditions we treat this not as a term in \(A_0\), but in the new (finite) quantity \(\tilde{A} = A_0 H_0(-|v_0 a|)\). This additional term will then behave as a \(\delta\)-function, giving no contribution for \(r > a\).


In this case \(\psi(\vec{r}, \vec{u})\) is a function of only one space coordinate, \(r\) (distance from the axis of the cylinder, to which we assign the index 3); but it is a function of two angular coordinates. For these we use \(\theta(= \cos^{-1} \mu)\), the angle between the direction of \(\vec{u}\) and the direction in which \(r\) is measured, and \(\phi\), the complement of the angle between the projection of \(\vec{u}\) on a plane normal to the radius of the cylinder and the direction of the axis of the cylinder. This means that if we number our space variables so that \((x^1, x^2, x^3) \equiv (z, \phi, r)\) then the angle \(\phi\) is measured in the \(x^1, x^2\) plane, and measured from the direction of \(x^2\). It is clear that \(\psi(\vec{r}, \vec{u})\) at any point will be even in \(\phi\); so
that all moments of $\psi(\vec{r}, \vec{n})$ with respect to $\sin m\phi$ will vanish. Furthermore, $\psi(\vec{r}, \vec{n})$ is even in the angle $(\pi/2 - \phi)$; so that moments of $\psi(\vec{r}, \vec{n})$ with respect to $\cos(2m+1)\phi$ will also vanish. Thus the only non-vanishing moments of $\psi(\vec{r}, \vec{n})$ must involve even powers of $\sin \phi$ and $\cos \phi$. We have numbered the axes so that the physical components of $\Omega^i$ are

$$\Omega^1 = \sin \theta \sin \phi, \quad \Omega^2 = \sin \theta \cos \phi, \quad \Omega^3 = \cos \theta.$$ 

Hence the only non-vanishing components of $\psi_{\prod_{i=1}^n}^{i_1...i_n}$ must have an even number of both $1$'s and $2$'s in their indices. Expressions for these components in terms of spherical harmonic moments are available from (2.6).

In the cylindrical case there will $\left[\frac{n}{2}\right] + 1$ independent components of order $n$, and altogether $n(n+1)$ independent components to consider in the $\psi_{2n-1}$-approximation. In this $\psi_{2n-1}$-approximation the differential equation (3.5) for the density is of order $2n$, so that there will be only $2n$ arbitrary constants in the solution for the density. The complete solution of the system of equations (2.12) will, however, involve $n(n+1)$ constants; so that there are $n(n-1)$ constants which must appear in various of the higher-order moments but do not appear in the density. To determine these moments it will be necessary to have more than the $2n$ scalar equations (3.2) (which were sufficient in the case of the plane); and we have used the equations in § 5.

The differential equation for the density is

$$\sum_{i=1}^{n} (\nabla^2 - v_i^2)\psi_0 = 0.$$  \hspace{1cm} (8.1)
In cylindrical coordinates the form of the operator $\nabla^2$ applied to a scalar is $D^2 + (1/r)D$, where $D = \frac{d}{dr}$. Thus from (8.1) we obtain

$$\psi_0(r) = \sum \left\{ A_i I_0(\nu_i r) + B_i K_0(\nu_i r) \right\}, \quad (8.2)$$

where $I_0$ and $K_0$ are the modified Bessel functions of the first and second kinds, respectively for large $x$, $I_0(x) \sim \exp(x)/\sqrt{2\pi x}$, $K_0(x) \sim \sqrt{\pi/2x}\exp(-x)$.

We may now use (8.2) in the equations of § 5 to obtain expressions for $\psi_3$, $\psi_{11}$, $\psi_{33}$, and so on. In doing so it will be necessary to test the solutions of each order to see which are extraneous. It is also necessary to recall that the form of the operator $\nabla^2$ as an ordinary differential operator depends on the tensor character of the quantity on which it operates. The required forms are given in (C.1), Appendix C.

This very tedious process has been carried through in the case of a $\psi_3$ - approximation, and also a $\psi_5$ - approximation. The form of the components in the $\psi_{2n-1}$-approximation is not immediately apparent from these examples; and it scarcely seems fruitful to investigate this general form since it is improbable that the labour of applying the $\psi_7$-(or even the $\psi_5$- ) approximation in the form in which it is at present available will ever appear to be worthwhile. The results in the $\psi_3$ - and the $\psi_5$ - cases are given below. Rather than giving the tensor components themselves we give combinations of the components which have a more convenient form.

In the $\psi_3$ - approximation

$$\psi_0 = \sum_{i=1}^3 \left\{ A_i I_0(\nu_i r) + B_i K_0(\nu_i r) \right\}. \quad (8.3)$$
\[
\psi^3 = \sum_{i=1}^{3} G_1(\nu_i) \left\{ A_1 I_1(\nu_i r) - B_1 K_1(\nu_i r) \right\},
\]
(8.3)

\[
\psi^{11} = \sum_{i} - G_2(\nu_i) \left\{ A_1 I_0(\nu_i r) + B_1 K_0(\nu_i r) \right\}/2 + M I_0(\beta r) + N K_0(\beta r),
\]

\[
\psi^{11} + 2\psi^{33} = \sum_{i} 3 G_2(\nu_i) \left\{ A_1 I_2(\nu_i r) + B_1 K_2(\nu_i r) \right\}/2 + M I_2(\beta r) + N K_2(\beta r),
\]

\[
\psi^{113} = \sum_{i} - G_3(\nu_i) \left\{ A_1 I_3(\nu_i r) - B_1 K_3(\nu_i r) \right\}/2 - 5(M I_1(\beta r) - N K_1(\beta r))/\beta,
\]

\[
3\psi^{113} + 4\psi^{333} = \sum_{i} 5 G_3(\nu_i) \left\{ A_1 I_3(\nu_i r) - B_1 K_3(\nu_i r) \right\}/2
- 5(M I_3(\beta r) - N K_3(\beta r))/\beta.
\]

Where \( \beta = \sqrt{3/7} \).

Since, in cylindrical coordinates, \( h(1) = h(3) = 1 \), there is no difference between physical components and tensor components for the quantities appearing in (8.3). The same is true in (8.4), below.

In the \( \psi_5 \) - approximation,

\[
\psi_0 = \sum_{i=1}^{3} \left\{ A_1 I_0(\nu_i r) + B_1 K_0(\nu_i r) \right\},
\]

\[
\psi^3 = \sum_{i} G_1(\nu_i) \left\{ A_1 I_1(\nu_i r) - B_1 K_1(\nu_i r) \right\},
\]
(8.4)

\[
\psi^{11} = \sum_{i} - G_2(\nu_i) \left\{ A_1 I_0(\nu_i r) + B_1 K_0(\nu_i r) \right\}/2
+ \sum_{j=1}^{2} \left\{ M_j I_0(\beta_j r) + N_j K_0(\beta_j r) \right\},
\]
\[
\psi^{11} + 2\psi^{33} = \sum_i \frac{3}{2} G_2(\nu_i) \left\{ A_i I_2(\nu_{ir}) + B_i K_2(\nu_{ir}) \right\}
+ \sum_j \left\{ M_j I_2(\beta_{jr}) + N_j K_2(\beta_{jr}) \right\},
\]

\[
\psi^{113} = \sum_i \frac{5}{2} G_3(\nu_i) \left\{ A_i I_1(\nu_{ir}) - B_i K_1(\nu_{ir}) \right\}
- \sum_j \frac{5}{3\beta_j} \left\{ M_j I_1(\beta_{jr}) - N_j K_1(\beta_{jr}) \right\},
\]

\[
3\psi^{113} + 4\psi^{333} = \sum_i \frac{11}{2} G_3(\nu_i) \left\{ A_i I_3(\nu_{ir}) - B_i K_3(\nu_{ir}) \right\}
- \sum_j \frac{11}{2\beta_j} \left\{ M_j I_3(\beta_{jr}) - N_j K_3(\beta_{jr}) \right\},
\]

\[
\psi^{1111} = \sum_i \frac{11}{8} G_4(\nu_i) \left\{ A_i I_0(\nu_{ir}) + B_i K_0(\nu_{ir}) \right\}
+ \sum_j \frac{5}{12\beta_j^2} \left\{ M_j I_0(\beta_{jr}) + N_j K_0(\beta_{jr}) \right\}
+ P I_0(\gamma r) + Q K_0(\gamma r),
\]

\[
\psi^{1111} + 2\psi^{1133} = \sum_i \frac{11}{8} G_4(\nu_i) \left\{ A_i I_2(\nu_{ir}) + B_i K_2(\nu_{ir}) \right\}
- \sum_j \frac{11}{12\beta_j^2} \left\{ M_j I_2(\beta_{jr}) + N_j K_2(\beta_{jr}) \right\}
+ P I_2(\gamma r) + Q K_2(\gamma r),
\]

\[
\psi^{1111} + 3\psi^{1133} + 3\psi^{3333} = \sum_i \frac{35}{8} G_4(\nu_i) \left\{ A_i I_4(\nu_{ir}) + B_i K_4(\nu_{ir}) \right\}
- \sum_j \frac{35}{12\beta_j^2} \left\{ M_j I_4(\beta_{jr}) + N_j K_4(\beta_{jr}) \right\}
+ P I_4(\gamma r) + Q K_4(\gamma r).
\]
\[
\psi_{11113} = \sum_{i} 3 G_{5}(v_{1}) \left\{ A_{1} I_{1}(v_{1}r) - B_{1} K_{1}(v_{1}r) \right\}/8
\]
\[ - \sum_{j} 7(\beta_{j}^{2}-7) \left\{ M_{j} I_{1}(\beta_{j}r) - N_{j} K_{1}(\beta_{j}r) \right\}/4\beta_{j}
\]
\[ - 9 \left\{ P_{I_{1}}(\gamma r) - Q_{K_{1}}(\gamma r) \right\}/5\gamma,
\]
\[
3\psi_{11113} + 4\psi_{11333} = \sum_{i} 7 G_{5}(v_{1}) \left\{ A_{1} I_{3}(v_{1}r) - B_{1} K_{3}(v_{1}r) \right\}/8
\]
\[ + \sum_{j} 7(\beta_{j}^{2}-7) \left\{ M_{j} I_{3}(\beta_{j}r) - N_{j} K_{3}(\beta_{j}r) \right\}/4\beta_{j}
\]
\[ - 27 \left\{ P_{I_{3}}(\gamma r) - Q_{K_{3}}(\gamma r) \right\}/5\gamma,
\]
\[
5\psi_{11113} + 20\psi_{11333} + 16\psi_{33333} = \sum_{i} 63 G_{5}(v_{1}) \left\{ A_{1} I_{5}(v_{1}r) - B_{1} K_{5}(v_{1}r) \right\}/8
\]
\[ + \sum_{j} 105(\beta_{j}^{2}-7) \left\{ M_{j} I_{5}(\beta_{j}r) - N_{j} K_{5}(\beta_{j}r) \right\}/4\beta_{j}
\]
\[ - 9 \left\{ P_{I_{5}}(\gamma r) - Q_{K_{5}}(\gamma r) \right\}/\gamma,
\]
where \( \beta_{j}(j=1,2) \) are the positive roots of \( \beta^{4} - 18\beta^{2} + 33 = 0 \), and \( \gamma = \sqrt{11} \).

In the special case of no capture \( (a=0) \) we have \( v_{1}=0 \).

The equations (8.3) and (8.4) remain as they are except for the terms from \( i=1 \) in the first sum. These terms must be replaced as follows: in both the \( \psi_{3} \)- and \( \psi_{5} \)-approximation, (8.3) and (8.4), the terms with \( i=1 \)

- in \( \psi_{0} \) are to replaced by \( A_{1} - B_{1} \log r \)
- in \( \psi_{3} \) by \( B_{1}/3r \)
- in \( \psi_{11} + 2\psi_{33} \) by \( 2B_{1}/5r^{2} \)
- in \( 3\psi_{113} + 4\psi_{333} \) by \( 8B_{1}/7r^{3} \)
in \( \psi^{1111} + 8\psi^{1133} + 8\psi^{3333} \) by \( 16B_1/3r^4 \),
in \( 5\psi^{1111} + 20\psi^{1133} + 16\psi^{3333} \) by \( 324B_1/11r^5 \),
and in all the other terms by \( 0 \).

**APPENDIX A**

**Definition of Tensors.**

In this appendix we recall, for convenience, the required definitions from tensor calculus. We also sketch the development of the properties of the spherical harmonic tensors.

Suppose that the variables \( x^i \) are transformed into variables \( \bar{x}^i \) by the general non-singular transformation

\[
\bar{x}^i = f^i(x^1, x^2, x^3), \tag{A.1}
\]

where \( f^1, f^2, f^3 \) may be arbitrary functions of the \( x^i \). From (A.1) and the inverse transformation we get linear relations between the differentials:

\[
d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^j} dx^j, \quad dx^i = \frac{\partial x^i}{\partial \bar{x}^j} d\bar{x}^j. \tag{A.2}
\]

If, under the transformation (A.1), a set of quantities \( u^i \) transform so that \( \bar{u}^i = (\partial \bar{x}^i/\partial x^j) u^j \), then the \( u^i \) are called the contravariant components of a vector \( \bar{u} \). Contravariance, that is, this particular type of behaviour under transformation, will be indicated by a superscript. If a set of quantities \( v_i \) transform under (A.1) by the law \( \bar{v}_i = (\partial \bar{x}^j/\partial x^i) v_j \), then the \( v_i \) are the covariant components of a vector \( \bar{v} \). For higher-order sets of quantities: \( u^{ij} \) are contravariant components of a tensor of rank two if, under (A.1), \( \bar{u}^{ij} = (\partial \bar{x}^i/\partial x^m)(\partial \bar{x}^j/\partial x^n) u^{mn} \); \( u^i \) are mixed components of a tensor if \( \bar{u}^i = (\partial \bar{x}^i/\partial x^m)(\partial x^n/\partial \bar{x}^j) u^m \); and so on.
The (symmetric) covariant metric tensor $g_{ij}$ is defined by the line element

$$ds^2 = g_{ij} dx^i dx^j.$$ 

If $u^i$ and $u_i$ are related by the equation $u^i = g_{ij} u^j$, then $u^i$ and $u_i$ are said to be "associated"; they are, respectively, the contravariant and covariant components of the same vector. Similarly, if $u_{ij} = g_{im} g_{jn} u^m$ then $u_{ij}$ and $u^{ij}$ are, respectively, contravariant and covariant components of the same tensor. By $g^{ij}$ we indicate the contravariant metric tensor, the tensor associated with $g_{ij}$. If $g$ is the determinant $|g_{ij}|$, and $G_{ij}$ the cofactor of $g_{ij}$ in $g$, then $g^{ij} = \frac{g_{ij}}{g}$. If a Cartesian coordinate system, since $g_{ij} = 0 (i \neq j)$, $G_{ii} = 1$, there is no distinction between contravariant and covariant components.

Tensor Differentiation.

We use the symbol $D_i$ to represent the tensor operation of covariant differentiation with respect to $x^i$. This operation is defined so that $D_i u^j$ is a covariant tensor of rank two, $D_i u^{jk}$ is a mixed tensor of rank three, and so on. $D_{ijk}$ represents the successive differentiations $D_i D_j D_k$, and has the tensor character indicated by the subscripts. One may also define contravariant differentiation, by the relation $D_i = g^{ij} D_j$. In general

$$D_i u^{jk \ldots} = \frac{\partial u^{jk \ldots}}{\partial x^i} + \left\{ \begin{array}{c} j \\ i \end{array} \right\} u^{sk \ldots} + \left\{ \begin{array}{c} k \\ i \end{array} \right\} u^{js \ldots} + \ldots$$

$$- \left\{ \begin{array}{c} s \\ i \end{array} \right\} u^{jk \ldots} - \left\{ \begin{array}{c} s \\ i \end{array} \right\} u^{jk \ldots} - \ldots$$

(A.3)
where \( \{ \frac{i}{j} \} \) is called a Christoffel symbol, and is given by

\[
\{ \frac{i}{j} \} = \frac{1}{2} g^{ln} \left( \frac{\partial g_{jn}}{\partial x^l} + \frac{\partial g_{kn}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^n} \right).
\]

(A.4)

These symbols, which are symmetric in \( j \) and \( k \), do not by themselves have tensor properties. In rectangular Cartesian coordinates \( \{ \frac{i}{j} \} = 0 \); so that in these coordinates covariant differentiation reduces to ordinary partial differentiation.

In cylindrical polar coordinates \( (x_1, x_2, x_3) \equiv (z, \phi, r) \)

\[
ds^2 = dz^2 + r^2 d\phi^2 + dr^2,
\]

and the only non-zero Christoffel symbols are

\[
\{ \frac{3}{2} \} = -r, \quad \{ \frac{2}{3} \} = \frac{1}{r}.
\]

(A.5)

In spherical polar coordinates \( (x_1, x_2, x_3) \equiv (\phi, \theta, r) \),

\[
ds^2 = r^2 \sin^2 \theta \ d\phi^2 + r^2 d\theta^2 + dr^2,
\]

and the non-zero Christoffel symbols are

\[
\{ \frac{2}{1} \} = -\sin \theta \ \cos \theta, \quad \{ \frac{3}{1} \} = -r \sin^2 \theta, \quad \{ \frac{3}{2} \} = -r,
\]

\[
\{ \frac{2}{3} \} = \{ \frac{1}{3} \} = \frac{1}{r}, \quad \{ \frac{1}{2} \} = \frac{\cos \theta}{\sin \theta}.
\]

(A.6)

Fundamental Property of Tensors.

It follows from the definition of a tensor that if two tensors are equal in one system of coordinates they are equal in any other. This property is important in that it allows us to investigate a tensor in some convenient system of coordinates and carry over the results to other systems of coordinates. For example, since in rectangular Cartesian coordinates the components of \( g_{ij} \) are constants we know that in these (and hence in any system of coordinates into which they may be transformed)
That is, the fundamental tensor is a constant with respect to tensor differentiation.

Again, since in Cartesian coordinates
\[ D_i D_j u^k \ell = D_j D_i u^k \ell, \]
and
\[ D_i \left[ u^j u^k \right] = u^j D_i u^k + u^k D_i u^j. \]
then covariant differentiation is commutative and the ordinary rules of differentiation apply in any coordinate system into which these coordinates may be transformed (i.e., in any Euclidean space). Also, since in Cartesian coordinates the invariant
\[ D_i u^i = \text{div} \mathbf{u}, \]  
then \( \text{div} \mathbf{u} \) is given by \( D_i u^i \) in any coordinates. The invariant operation \( D_i D^i \) is the Laplacian in Cartesian coordinates, and hence in any system of coordinates the operator
\[ D_i D^i = \nabla^2. \]  
We shall now use this convenient property of tensors to build up the Spherical Harmonic Tensors.

In a Cartesian system of coordinates we define
\[ T^{i_1 \ldots i_n} = \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^1 \ldots \partial x^n} \left( \frac{1}{r} \right) \]  
(at \( r=1 \)), \( \text{(A.9)} \)
where \( r \) is distance from the origin \( (r^2 = g_{ik} x^i x^k) \). Before considering other coordinate systems we express tensors in terms of components of a unit vector \( \hat{n} \) for which, in Cartesian coordinates, \( \hat{n}^i = x^i/r \). On carrying out the differentiation in \( \text{(A.9)} \), using \( \partial r/\partial x^j = x_j/r \) and
\[
\frac{\partial x_j}{\partial x^k} \left( \frac{\partial g_{j\ell}}{\partial x^k} \right) = g_{j\ell}, \text{ we get}
\]

\[
T_{i_1\ldots i_n} = \frac{(-1)^n}{n!} \left\{ (-1)^n (1.3.5\ldots (2n-1)) \frac{x_{i_1}x_{i_2}\ldots x_{i_n}}{r^{2n+1}} + \\
+ (-1)^{n-1} (1.3.5\ldots (2n-3)) \sum \frac{g_{i_1i_2x_{i_3}\ldots x_{i_n}}}{r^{2n-1}} \\
+ (-1)^{n-2} (1.3.5\ldots (2n-5)) \sum \frac{g_{i_1i_2i_3i_4x_{i_5}\ldots x_{i_n}}}{r^{2n-3}} \\
+ \ldots \text{ to } \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \text{ terms} \right\},
\]

where by \( \sum \) we indicate the sum over all the different terms which may be formed by rearranging subscripts. Now replacing \( x_j/r \) by \( \Omega_{i_j} \) and then setting \( r=1 \) we have, in Cartesian coordinates,

\[
T_{i_1\ldots i_n} = \frac{1}{n!} \left\{ (1.3.5\ldots (2n-1)) \Omega_{i_1}\ldots\Omega_{i_n} - (1.3.5\ldots (2n-3)) \right\} \\
\times \sum \Omega_{i_3}\ldots\Omega_{i_n} g_{i_1i_2} \\
+ (1.3.5\ldots (2n-5)) \sum \Omega_{i_5}\ldots\Omega_{i_n} g_{i_1i_2g_{i_3i_4}} - \ldots \ldots \right\}
\]

This tensor equation will remain valid in any coordinate system, and we use (A.11) as the form \( T_{i_1\ldots i_n} \) in general. We may, of course, also write

\[
T_{i_1\ldots i_n} = \frac{(-1)^n}{n!} D_{i_1\ldots i_n} \left( \frac{1}{r} \right).
\]

The associated tensor \( T_{i_1\ldots i_n} \) may be expressed by raising all the subscripts in (A.11):
The first few contravariant spherical harmonic tensors are written explicitly in (2.1).

In Cartesian coordinates the components of a tensor are dimensionally homogeneous and appropriate to the quantity represented by the tensor. This is not in general true for other orthogonal coordinate systems. To get quantities which are dimensionally correct, the so-called physical components of a tensor, it is necessary to express the components with respect to Cartesian axes which, at the point in question, are tangent to the parametric lines of the coordinate system in use. These Cartesian coordinates at a point are usually referred to as "Riemannian Coordinates". If the line-element for a system of orthogonal coordinates be

\[
ds^2 = \frac{(dx_1)^2}{h_1^2} + \frac{(dx_2)^2}{h_2^2} + \frac{(dx_3)^2}{h_3^2}
\]

\[
(g_{ii} = \frac{1}{h_i^2})
\]

(the bracket on \(h_i\) indicating that the quantity does not have tensor character), then the physical components of, say,

\[
u_k^{ij} = \frac{h_k}{h_i h_j} u_k^{ij}
\]

(The summation convention is not to be applied here.)

We are now in a position to indicate the close relationship between the physical components of the spherical harmonic tensors and the usual spherical harmonics \(Y_n^m(\cos \theta) \sin m \phi\).
First, there are just $2n+1$ linearly independent physical components of the tensor $(A_{i3})$. This follows from the fact that since the tensor is symmetric there are only $(n+1)(n+2)/2$ different components, and that the different physical components satisfy the $n(n-1)/2$ linear equations

$$
\sum_{j=1}^{3} T^{jj1\cdots i_n} = 0. \quad (A.15)
$$

To establish $(A.15)$ we use the representation $(A.12)$. We have

$$
h(i_3)\cdots h(i_n) \sum_{j=1}^{3} T^{jj1\cdots i_n} = \sum_{j=1}^{3} \frac{1}{h^2(j)} T^{jj1\cdots i_n}
$$

$$
= \frac{(-1)^n}{n} D_{13\cdots i_n} \left\{ \sum_{j=1}^{3} g_{jj} D^j D^j \left( \frac{1}{r} \right) \right\}
$$

$$
= \frac{(-1)^n}{n} D_{13\cdots i_n} \left\{ \nabla^2 \left( \frac{1}{r} \right) \right\} = 0.
$$

If now, as in (2.4), we number the axes so that

$$
\vec{n}_1 = \sin \theta \sin \phi, \quad \vec{n}_2 = \sin \theta \cos \phi, \quad \vec{n}_3 = \cos \theta;
$$

and use these in the relation

$$
T_{1i_3\cdots i_n} = \frac{1}{n} \left\{ (1.3.5\cdots(2n-1)) \vec{n}_1 \vec{n}_2 \cdots \vec{n}_n \cdots \right\},
$$

which is merely $(A.13)$ rewritten using $(A.14)$, it is evident that $T_{1i_3\cdots i_n}$ is a linear combination of the surface spherical harmonics of the $n$th order. In (2.4) to (2.6) we have given examples of these combinations, including all the components of which we have made use.
APPENDIX B

EQUATIONS FOR SPHERICAL HARMONIC MOMENTS IN THE SPHERICAL CASE

Due to the symmetry with respect to the azimuthal angle, all the non-zero tensor components of \( \psi^{i_1\ldots i_n} \) are proportional to \( \psi^3\ldots 3 \), as in the plane case. We define \( \psi^3\ldots 3 = \psi_n \). The only non-zero components are the moments with respect to those \( \psi^{i_1\ldots i_n} \) which contain an even number (or zero) of indices 1 and 2; that is \( \psi^{i_1\ldots i_n} = 0 \) if in \( i_1\ldots i_n \) there are an odd number of 1's and 2's.

We show that in the spherical case

\[
D_{i_1\ldots i_n} \psi^{i_1\ldots i_n} = \left( \frac{D+2}{r} \right) \left( \frac{D+3}{r} \right) \ldots \left( \frac{D+(n+1)}{r} \right) \psi_n, \quad (B.1)
\]

where \( D = d/dr \). For this purpose we note first from (2.12) that

\[
(n+1) D_j \psi^{i_1\ldots i_n} + \frac{2n+1}{n} \sum_{i=1}^{i=n} D_{i_1\ldots i_n} \psi^{i_1\ldots i_n} \quad (B.2)
\]

We specialize this equation by deciding to set \( i_1 = i_2 = \ldots = i_n = 3 \), so that the terms within each sum will be equal, and the equation becomes

\[
(n+1) D_j \psi^{i_1\ldots i_n} + (2n-1) D_{i_1 i_2} \psi^{i_1\ldots i_n} \quad (B.3)
\]

We specialize this equation by deciding to set \( i_1 = i_2 = \ldots = i_n = 3 \), so that the terms within each sum will be equal, and the equation becomes

\[
(n+1) \frac{D_{i_1\ldots i_n}}{x_j} + (2n-1) \frac{D_{i_1 i_2}}{x_j} \psi^{i_1\ldots i_n} \quad (B.4)
\]

with \( i_1 = \ldots = i_n = 3 \). Then, from (A.3),

\[
D_j \psi^{i_1\ldots i_n} = \frac{\partial}{\partial x^j} \psi^{j_1\ldots j} + \left( \frac{3}{j_j} \right) \psi^{j_1\ldots j} + n \left( \frac{3}{j_j} \right) \psi^{j_1\ldots j} \quad (B.4)
\]
On using the expressions for the Christoffel symbols in the spherical case (A.6), and the fact that $\psi_{i_1 \ldots i_n}$ is a function of $x^3$ only, (B.4) becomes

$$D_j \psi_{i_1 \ldots i_n} = (D + \frac{2}{r}) \psi_{i_1 \ldots i_n} + \left\{ 1 \right\}_{1,2} \psi_{23 \ldots 3} - \frac{n}{r} (r^2 \sin^2 \theta \psi_{113 \ldots 3} + r^2 \psi_{223 \ldots 3}).$$

(B.5)

The second term on the right of (B.5) vanishes, having only one index 2; and the terms in brackets are, by (A.14), $\psi_{113 \ldots 3} + \psi_{223 \ldots 3}$, and this is equal to $-\psi_{3 \ldots 3} = -\psi_{n-1}$ by (A.15). Hence

$$D_j \psi_{i_1 \ldots i_n} = (D + \frac{n+2}{r}) \psi_{i_1 \ldots i_n}. \quad (B.6)$$

Similarly

$$D_j \psi_{i_1 \ldots i_n} + (D + \frac{n+2}{r}) \psi_{i_1 \ldots i_n}. \quad (B.7)$$

Also

$$d_j \psi_{i_1 \ldots i_n} g^{1j} = g^{1j} D_j \psi_{i_1 \ldots i_n} - (n-1) g^{3j} \left\{ \frac{3}{3} \right\} \psi_{3 \ldots 3}.$$

Since $g^{3j} = 0$ for $j \neq 3$ and, from (A.6), $\left\{ \frac{3}{3} \right\} = 0$, we obtain

$$d_j \psi_{i_1 \ldots i_n} = D \psi_{i_1 \ldots i_n} \quad \text{when} \quad i_1 = i_2 = \ldots = i_n = 3. \quad (B.8)$$

Using (B.6) to (B.8) in (B.3) gives

$$(n+1) \left\{ D + (n+2)/r \right\} \psi_{i_1 \ldots i_n} + n \left\{ D - (n-1)/r \right\} \psi_{i_1 \ldots i_n} + (2n-1) \psi_{i_1 \ldots i_n} = 0. \quad (B.9)$$

(This equation could, of course, be obtained directly by first putting the transport equation in spherical coordinates and taking moments with respect to $P_n(\mu)$; but it has been derived here merely for the sake of completeness in the presentation. As we have stated before, the tensor approach is not justified for the spherical case alone.)
We now proceed to establish (B.1). We start from (3.1) which gives

\[ D_{n+1} \psi^{(n+1)} = - \frac{1}{n+1} \left\{ (2n+1) D_n \psi^{(n)} + n^2 D_{n+1} \psi^{(n+1)} \right\}; \]  

(B.10)

and assume that

\[ D_k \psi^{(k)} = (D+2/r)(D+3/r) \ldots \{ D+(k+1)/r \} \psi^{(k)} , \text{ for } k < n. \]  

(B.11)

This is true for

\[ D(1) \psi^{(1)} = D_1 \psi^1 = \frac{d}{dx^1} \psi^1 + \left\{ i \right\}^n \psi^n = (D + 2/r) \psi^3, \]

on using (A.6) and the fact that \( \psi^2 = 0. \)

On using (B.11) in (B.10) we get on the right the term

\[ \nabla^2 (D+2/r)(D+3/r) \ldots (D+n/r) \psi^{n-1} \]  

(B.12)

The operator \( \nabla^2 (D)D \) when applied to a scalar is of the form

\( (D+2/r)D \). Thus

the operator (B.12) = \( (D+2/r)(D+2/r) \ldots (D+n/r), \)

and this = \( (D+2/r)(D+3/r) \ldots \{ D+(n+1)/r \} \{ D+(n-1)/r \}, \)

the last form being a consequence of the relation

\( (D-k/r) \{ D+(k+1)/r \} = \{ D+(k+3)/r \} \{ D-(k+1)/r \}. \)

Now using (B.13) and (B.11) in (B.10) we obtain

\[ D_{n+1} \psi^{(n+1)} = - \frac{1}{n+1} \left\{ (2n+1) D_n \psi^{(n)} + n^2 D_{n+1} \psi^{(n+1)} \right\]; \]

and this, in view of (B.9), becomes

\[ D_{n+1} \psi^{(n+1)} = (D+2/r)(D+3/r) \ldots \{ D+(n+2)/r \} \psi^{n+1}, \]

as required. This relation is used in §7.

The Recursion Relation for the Functions \( H_k(x) \). (See (7.3) to (7.5)).

In the relation (7.3):

\[ (D+2/r)(D+3/r) \ldots \{ D+(k+1)/r \} f^k(r) = F^k(\eta^2) \exp(\eta r)/\eta r \]
we substitute $\eta r = x$, and use $D^\dagger$ for $d/dx$. This relation becomes
\[ (D^\dagger + (k+1)/x) f_k = F_k(\eta^2) \exp(x)/x \eta^k = G_k(\eta) \exp(x)/x. \]
We now assume
\[ f_k = G_k(\eta) H_k(x), \]
(where $H_0(x) = \exp(x)/x$, and $H_1(x) = (1-1/x)H_0(x)$), and it follows at once that
\[ \left\{ D^\dagger + (k+1)/x \right\} H_k(x) = H_{k-1}(x). \] (B.16)
The relation (B.9) holds for each term of the sum with which we express $\psi_n$; so that using (B.15) and replacing $\eta r$ by $x$, (B.9) gives
\[ (n+1)G_{n+1}(\eta) \left\{ D^\dagger - (n+2)/x \right\} H_{n+1}(x) + n G_n(\eta) \left\{ D^\dagger - (n-1)/x \right\} H_{n-1}(x) \]
\[ + (2n+1)G_n(\eta) H_n(x)/\eta = 0. \] (B.17)
If we use (B.16) to eliminate the derivatives in (B.17) we obtain
\[ \left\{ (n+1)G_{n+1}(\eta) + (2n+1)G_n(\eta)/\eta \right\} H_n(x) \]
\[ + n G_{n-1}(\eta) \left\{ H_{n-2}(x) - (2n-1)H_{n-1}(x)/x \right\} = 0. \] (B.18)
However, from (3.4) and (3.8) we have the relation
\[ (n+1)G_{n+1}(\eta) + (2n+1)G_n(\eta)/\eta = -n G_{n-1}(\eta); \]
and using this in (B.18) we obtain
\[ H_n(x) = H_{n-2}(x) - (2n-1)H_{n-1}(x)/x \]
as required.
APPENDIX C

Formula for the Cylindrical Case.

In the cylindrical case we have to consider only those tensor components which have an even number of 1's and 2's in their indices; and in view of relation (2.7) we may restrict our attention to components whose indices consist of only 1's and 3's. To obtain these from equations (5.1) and (5.2) we need to express various tensor derivatives in terms of ordinary derivatives. This can be done with the help of (A.3) and (A.5); and although the process may become tedious, it is quite straightforward.

We quote only the following formula which is very useful in handling (5.1) and (5.2), and which may be established easily using (A.3), (A.5) and (2.7). If $\nabla^2 \psi_{1\ldots l, 3\ldots 3}$ represents $\nabla^2 \psi_{1\ldots i_n}$ evaluated when $i_1 = i_2 = \ldots = i_r = l$ (with $r$ even) and the remaining indices are equal to 3, then

$$\nabla^2 \psi_{1\ldots l, 3\ldots 3} = (D^2 + \frac{1}{r} D - \frac{s^2}{r^2}) \psi_{1\ldots l, 3\ldots 3} - \frac{s(s-1)}{r^2} \psi_{1\ldots l, 113\ldots 3} \quad (C.1)$$

The form of (C.1) shows at once that $\psi_{1\ldots l, 3\ldots 3}$ where there are $s$ indices equal to 3 must involve the Bessel functions $I_s$ and $K_s$ and in addition the functions $I_{s-2}$ and $K_{s-2}$ which will similarly appear in $\psi_{1\ldots l, 113\ldots 3}$, etc. Hence the moment with $s$ indices 3 will be a linear combination of Bessel functions of orders $s$, $s-2$, $s-4$, and so on. This accounts for the form of the solutions given in (8.3) and (8.4).
Completeness of the Solutions (Odd Approximations)

We shall for the moment confine our attention to the odd approximations. Our purpose is to establish the completeness of the solutions given in (6.4) and (7.6) in the plane and spherical cases. This will be done in the plane case if we show that the determinant, of which the 21st and (21+1)st elements of the kth row are

\[ G_k(v_1)\exp(v_1z) \text{ and } G_k(-v_1)\exp(-v_1z), \]

is not identically zero. Let us redesignate the numbers \( v_1, v_2, \ldots, v_n \) by \( \eta_1, \eta_2, \ldots, \eta_2n \); so that the determinant in question is

\[ \left| G_k(\eta_j)\exp(\eta_jz) \right|, \quad (k = 0, 1, \ldots, 2n-1). \]

\[ (j = 1, 2, \ldots, 2n) \]

Since we may divide \( \exp(\eta_jz) \) from each column, it will suffice to show that

\[ \Delta = \left| G_k(\eta_j) \right| \neq 0. \quad (D.1) \]

In the spherical case we merely replace \( \exp(\eta_jz) \) by \( H_k(\eta_jr) \). If we suppose \( r \) to be large, then we need consider only the leading terms of \( H_k(\eta_jr) \), which are \( \exp(\eta_jr)/\eta_jr \), and independent of \( k \). Hence for the spherical case too we require only (D.1).

We now recall that the \( \eta_j \) are the roots of

\[ F_{2n}(\eta_j^2) = 0, \quad (D.2) \]

where

\[ F_0 = 1, \quad F_1 = -\alpha, \quad \text{and} \]

\[ kF_k(\eta^2) + (2k-1) F_{k-1}(\eta^2) + (k-1)\eta^2 F_{k-2}(\eta^2) = 0. \quad (D.3) \]

From (D.3) it is clear that for \( k \geq 2 \)...
\[ F_k(\eta^2) = R_k(\eta^2) + \alpha S_k(\eta^2), \]  
(D.4)

where \( R_k \) and \( S_k \) are polynomials in \( \eta^2 \) and free of \( \alpha \), and \( S_k(0) \neq 0 \).

From (D.2) and (D.4) we may write

\[ \alpha = -\frac{R_{2n}(\eta_j^2)}{S_{2n}(\eta_j^2)} \quad \text{(for all} \ j \ \text{from} \ 0 \ \text{to} \ 2n), \]  
(D.5)

where

- \( R_{2n}(\eta_j^2) \) is an even polynomial of degree \( 2n \) in \( \eta_j^2 \),
- \( S_{2n}(\eta_j^2) \) is an even polynomial of degree \( 2(n-1) \) in \( \eta_j^2 \),
- and \( S_{2n}(0) \neq 0 \).

Using (D.5) to eliminate \( \alpha \) from the \( F_k \) in (D.3), we obtain

\[ F_{0}(\eta_j^2) = \frac{S_{2n}(\eta_j^2)}{S_{2n}(\eta_j^2)}, \quad F_{1}(\eta_j^2) = \frac{R_{2n}(\eta_j^2)}{S_{2n}(\eta_j^2)}, \] 
and

\[ F_k(\eta_j^2) = T_k(\eta_j^2)/S_{2n}(\eta_j^2) \]  
(D.6)

where \( T_{2m} \) is a polynomial of degree \( n-1+m \) in \( \eta_j^2 \),

and \( T_{2n-1} \) is of the same degree as \( T_{2m} \).

We now quote a lemma, which we shall establish later, that, for the smallest \( \eta_j(j=1) \), \( F_k(\eta_j^2) \) as expressed in (D.6) has a factor of \( \eta_j^{2k} \), exactly. Since from (D.3) it is obvious that the form of \( F_k(\eta_j^2) \) is independent of \( j \), it will follow from the lemma that

\[ F_k(\eta_j^2)\eta_j^{2k} = \frac{T_k(\eta_j^2)}{S_{2n}(\eta_j^2)}, \] 

where the degree of \( T_k(\eta_j^2) \) will be \( (n-1-k/2) \) in \( \eta_j^2 \) if \( k \) is even and \( \left\lfloor n-k+1/2 \right\rfloor \) in \( \eta_j^2 \), if \( k \) is odd; and where \( T_k(0) \neq 0 \), (since the factor \( \eta_j^{2k} \) appears exactly). Using (D.7) to rewrite \( \Delta \) in (D.s1), recalling that \( G_k(\eta_j^2) = F_k(\eta_j^2)/\eta_j^k \).
Now considering the degree of \( T_k \) for \( k = 2n-1, 2n-3, \ldots \), and the fact that the coefficients of \( T_k \) are independent of \( j \), we see that the form of the terms of the rows of \( \delta \) in (D.8), starting from the bottom, are:

\[
C(0) \eta_{2n-1}^{2n-1}, C(2n-2) \eta_{2n-2}^{2n-2}, (C(2n-3) + \eta_{2n-3}^{2n-3}) \eta_{2n-3}^{2n-3}, \ldots,
\]

where the \( C \)'s are constants and \( C(0) \neq 0 \).

We may consequently reduce \( \delta \) by taking a multiple of the \((2n-1)\)st row from the \((2n-3)\)rd row, to leave it proportional to \( \eta_{2n-3}^{2n-3} \), similarly reduce the \((2n-4)\)th row by using the \((2n-2)\)nd, and so on. Then removing the constant factors, \( C_{2n-1} \), \( C_{2n-2} \), \ldots from each row, we see that, except for a non-zero constant factor, \( \delta \) is equal to the determinant \( \eta_j^k \); which is not zero since \( \eta_j \neq \eta_l (j \neq l) \).

This establishes (D.1), and hence the completeness of the solutions given in the plane and spherical cases. The fact that \( \Delta \) may be replaced by a determinant of the form \( \eta_j^k \) will also be used in CRT-338 Revised, in obtaining simplified forms of the linear equations given by the boundary conditions in particular problems with plane symmetry.

**Lemma.**

We wish to show that \( F_k(\eta_1^2) \) when expressed independently of \( a \) has the factor \( \eta_1^{2k} \) exactly. From (4.4) we have

\[
F_k(\eta_1^2) = (-1)^k \eta_1^k P_k(1/\eta_1) \left( 1 - (1-a) \sum_{i=1}^k P_i(k)/2 \left[ 1 - \eta_1 \mu_i(k) \right] \right). \tag{D.9}
\]
where the superscripts on the \( \mu_{1}^{(k)} \) and \( p_{1}^{(k)} \) indicate that they are associated with \( P_{k}(\mu) \). In (D.9) the term \( \eta_1^{k} p_{k}(1/\eta_1) \) is a polynomial in \( \eta_1 \) with a constant term different from zero. It remains to show that the bracket in (D.9) has a factor \( \eta_1^{2k} \).

Since, in the \((m-1)\)st approximation, \( \eta_1 \) is the smallest root of \( P_{m}(\eta_2) = 0 \), we have

\[
1 - (1-a) / \sum_{i=1}^{m} p_{i}(m) / \left[ 1 - \eta_1 \mu_{1}^{(m)} \right] = 0 \tag{D.10}
\]

If \( \mu_{1}^{(m)} \) is the largest positive root of \( P_{m}(\mu) = 0 \), it is clear that \( \eta_1 < 1/\mu_{1}^{(m)} \); and hence \( \eta_1^{\mu_{1}^{(m)}} < 1 \) and we may expand in (D.10) using the binomial theorem. This will give

\[
\sum_{i=1}^{m} p_{i}(m) / \left[ 1 - \eta_1 \mu_{1}^{(m)} \right] = \frac{1}{2} \sum_{l=0}^{\infty} \eta_{1}^{l} p_{1}(m) (\mu_{1}^{(m)})^{l} \tag{D.11}
\]

But for \( l \leq 2m-1 \), we have \( \int_{-1}^{1} \mu^{l} d\mu = \begin{cases} 0 & \text{if } l \text{ odd} \\ \frac{2}{2m+1} & \text{if } l \text{ even} \end{cases} \)

and hence

\[
\sum_{i=1}^{m} p_{i}(m) / \left[ 1 - \eta_1 \mu_{1}^{(m)} \right] = 1 + \eta_1^{2} / 3 + \eta_1^{4} / 5 + \ldots + \eta_1^{2(m-1)} / (2m-1) + O(\eta_1^{2m}) \tag{D.12}
\]

In (D.12) it may easily be shown that the coefficient of \( \eta_1^{2m} \) will definitely not be \( 1/(2m+1) \) but it will be \( (1-1/C_{m}^{2})/(2m+1) \), where \( C_{m} \) is the coefficient of \( \mu^{m} \) in \( P_{m}(\mu) \). From (D.12) and (D.10) we now have

\[
1-a = 1 / \left[ 1 + \eta_1^{2} / 3 + \ldots + \eta_1^{2(m-1)} / (2m-1) + O(\eta_1^{2m}) \right] \tag{D.13}
\]
Now considering the bracketed term in (D.9), since \( k < m \) we know that the largest positive root of \( P_k(\mu) \) is less than \( \mu_1(m) \), and hence \( \eta_1 \mu_1(k) < 1 \) since \( \eta_1 \mu_1(m) < 1 \). We may then expand the term in (D.9) as we did in (D.11) to obtain

\[
\begin{aligned}
&\left\{1-(1-\alpha)\sum_{i=1}^{k} p_i(k) \left[1-\eta_1 \mu_1(k)\right]\right\} = 1-(1-\alpha) \left\{1+\eta_1^2/3+\ldots+\eta_1^{2(k-1)}/(2k-1) + O(\eta_1^{2k})\right\} \\
&\quad \tag{D.14}
\end{aligned}
\]

where the coefficient of the term in \( \eta_1^{2k} \) is not \( 1/(2k+1) \). If we now substitute from (D.13) for \( (1-\alpha) \) we see at once that the right hand side of (D.14) has a factor of \( \eta_1^{2k} \) exactly. This establishes the lemma.

The Even Approximations.

Only a slight extension of the considerations used in dealing with the odd approximations is required here. In place of the \( \eta_j \) of (D.2) we have now the \( 2n \) roots of \( F_{2n+1}(\eta_j^2) = 0 \) and the additional number \( \eta_0 \), where \( |\eta_0| = \infty \). As before, we require to show

\[
\Delta = \left|G_k(\eta_j)\right| \neq 0, \quad k = 0,1,\ldots,2n, j = 0,1,\ldots,2n. \tag{D.15}
\]

The numbers \( G_k(\eta_0) \) which appear in the first column of \( \Delta \) are a set of constants which may at once be written down from (4.14), in fact \( G_k(\eta_0) \), for \( \eta_0 = \infty \), is the coefficient of the constant term in \((-1)^k P_k(\eta)\). The other columns may be expressed precisely as they were in the former case, and for (D.7) we will have

\[
F_k(\eta_j^2)/\eta_j^k = \eta_j \frac{T_k(\eta_j^2)}{S_{2n}(\eta_j^2)}. \tag{D.16}
\]
the only difference from the former case being that when \( k \) is even the total degree of the numerator on the right will be \( 2n \), instead of \( 2(n-1) \), and the degree of the denominator is now \( 2n \). It is formally obvious that in any column the coefficients of \( \eta_j^{2n} \) in the even rows are related just as are the values of \( G_{2m}(\eta_o) \); and this may be readily verified if one goes into detail. This means that if in this case we set out to reduce \( \Delta \) by subtracting multiples of the last two from the other rows containing \( \eta_j^{2n} \) we clear out the non-zero terms from the first column (except the last) at the same time as we clear out the \( \eta_j^{2n} \) from the other columns. Hence in this case \( \Delta \) is proportional to a determinant of the form

\[
\begin{vmatrix}
0 & & & \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \eta_j^k & \ddots \\
0 & \cdots & \cdots & \cdots & \cdots \\
G_{2n}(\eta_o) & \eta_j^{2n} & \eta_j & \cdots & \cdots \\
\end{vmatrix}
\]

and \((D.1)\) is true in the even approximations as well.