

Progress in analytical calculations for the anomalous magnetic moment of the muon *

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We present results for certain classes of diagrams contributing to the anomalous magnetic moment of the muon at five-loop order. Our method is based on first constructing an approximating function for the vacuum polarization function of the photon at four loop order which later can be numerically integrated to obtain the anomalous magnetic moment of the muon.

1. Introduction

The anomalous magnetic moments of electron and muon are some of the best measured and theoretically predicted quantities. The QED corrections have recently been calculated numerically up to five loops in [1, 2]. Up to next-to-next-to-leading order complete analytical results are available [3, 4]. At four-loop order only partial results exist, contributions from corrections to the vacuum polarization function of the photon have been calculated in [5], contributions due to light lepton loops in [6] and due to heavy leptons in [7]. Recently, in Ref. [8] some five-loop corrections have been calculated by using the leading term in the high-energy expansion of the vacuum polarization function of the photon. Since this approach lead to a surprisingly

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large deviation from the numerical result, the method has been improved in Ref. [9]. In the following we will review the main ingredients of the analysis in Ref. [9] and discuss the results.

2. Setup of the Calculation and Results

The QED corrections to the anomalous magnetic moment a_μ can be calculated in perturbation theory and can thus be written in form of a power series in the fine structure constant α

$$a_\mu = \sum_{k=1}^{\infty} \left(\frac{\alpha}{\pi}\right)^k a_\mu^{(2k)}, \quad (1)$$

where $a_\mu^{(2k)}$ can be further decomposed – following the conventions in Ref. [1] – as

$$a_\mu^{(2k)} = A_1^{(2k)} + A_2^{(2k)}(m_e/m_\mu) + A_2^{(2k)}(m_\tau/m_\mu) + A_3^{(2k)}(m_e/m_\mu, m_\tau/m_\mu). \quad (2)$$

$A_1^{(2k)}$ contains the universal contributions, which in case of the muon anomalous magnetic moment only contain muon loops. The diagrams contributing to $A_2^{(2k)}(m_e/m_\mu)$ and $A_2^{(2k)}(m_\tau/m_\mu)$ have at least one electron or tau loop, respectively. In $A_3^{(2k)}(m_e/m_\mu, m_\tau/m_\mu)$ contributions from diagrams with both electron and tau loops are collected. In this paper we are mainly interested in contributions to $A_2^{(2k)}(m_e/m_\mu)$ without any muon loops.

The contributions to the anomalous magnetic moment of the muon due to photon polarization effects can be calculated (cf. Fig. 1) by using [10]

$$a_\mu = \frac{\alpha}{\pi} \int_0^1 dx (1-x) \frac{1}{1 + \Pi(s_x)}, \quad s_x = -\frac{x^2}{1-x} m_\mu^2, \quad (3)$$

where Π denotes the vacuum polarization function as defined in Eq. (4). This formula can be obtained by considering the one-loop result for $g-2$ for the case of a heavy photon in combination with the dispersion relation for $\Pi(q^2)$. The classes of diagrams accessible by this method are shown in Fig. 2. Thus, we have to find a suitable approximation for $\Pi(q^2)$ which in turn can be integrated to obtain a_μ .

We define the vacuum polarization function $\Pi(q^2)$ as usual by

$$(q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi(q^2) = ie^2 \int dx \langle 0 | e^{iqx} T j^\mu(x) j^\nu(0) | 0 \rangle, \quad (4)$$

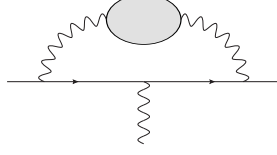


Fig. 1. Prototype diagram

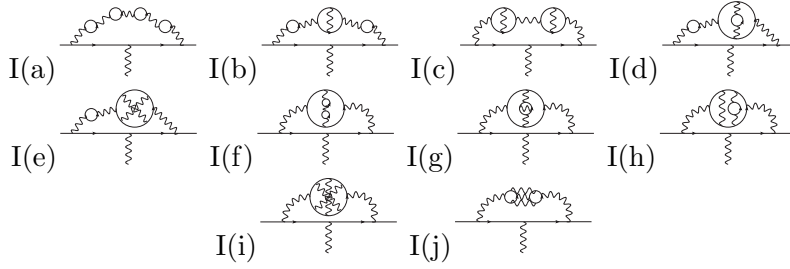


Fig. 2. Classes of diagrams accessible by the used method.

with the current $j^\mu = \bar{\psi}\gamma^\mu\psi$ and write it as an expansion in the fine-structure constant α

$$\Pi(q^2) = \frac{\alpha}{\pi}\Pi^{(1)}(q^2) + \left(\frac{\alpha}{\pi}\right)^2 \Pi^{(2)}(q^2) + \left(\frac{\alpha}{\pi}\right)^3 \Pi^{(3)}(q^2) + \left(\frac{\alpha}{\pi}\right)^4 \Pi^{(4)}(q^2) + \mathcal{O}(\alpha^5). \quad (5)$$

In the following we will collect the available results for the low- and high-energy region and the threshold region, which will later be used to construct an approximating function. For details on the calculation of the listed results, please refer to Ref. [9]. In the following n_h and si label contributions from lepton loops and singlet diagrams.

In the low-energy limit the polarization function can be expanded in a power series in $z = q^2/(4m_q^2) < 1$

$$\Pi_{le}^{(n)} = \sum_{k=1}^{\infty} \Pi_{le,k}^{(n)} z^k. \quad (6)$$

For $\Pi_{le}^{(4)}$ one obtains

$$\begin{aligned} \Pi_{le}^{(4)} = & z (n_h^2(0.066 \mathbf{si} + 0.571) + 0.112n_h^3 + 0.834n_h) \\ & + z^2 (0.025n_h^3 + n_h^2(0.140 \mathbf{si} + 0.366) + 2.230n_h) \\ & + z^3 (0.012n_h^3 + n_h^2(0.126 \mathbf{si} + 0.277) + 3.396n_h). \end{aligned}$$

In the high-energy region we write the result in the form

$$\Pi_{\text{he}}^{(n)} = \sum_{k=0}^{\infty} \Pi_{\text{he},k}^{(n)} z^{-k} \quad (7)$$

with

$$\begin{aligned} \Pi_{\text{he}}^{(4)} = & n_h^3 \left(-0.009 \log^3(-4z) + 0.019 \log^2(-4z) - 0.076 - 0.086 \log(-4z) \right) \\ & + n_h^2 \left(-0.021 \log^2(-4z) + \log(-4z)(0.496 \mathbf{si} - 0.258) \right. \\ & \left. + (0.638 - 1.619 \mathbf{si}) \right) + n_h(0.180 \log(-4z) - 1.972) \\ & + \left[n_h^2 \left((-2.546 \mathbf{si} - 0.015) + 0.188 \log^3(-4z) - 0.938 \log^2(-4z) \right. \right. \\ & \left. \left. + 2.414 \log(-4z) \right) + n_h^3 \left(-0.028 \log^3(-4z) + 0.181 \log^2(-4z) \right. \right. \\ & \left. \left. - 0.666 \log(-4z) + 0.684 \right) + n_h \left(0.141 \log^2(-4z) - 0.281 \log^3(-4z) \right. \right. \\ & \left. \left. + 1.265 - 2.048 \log(-4z) \right) \right] / z \end{aligned}$$

The polarization function in the threshold region can be written as

$$\Pi_{\text{thr}}^{(n)} = 16\pi^2 \sum_{k=2-n}^{\infty} \Pi_{\text{thr},k}^{(n)} (\sqrt{1-z})^k. \quad (8)$$

with the four-loop contribution

$$\begin{aligned} \Pi_{\text{thr}}^{(4)} = & \frac{14.640n_h}{1-z} + \frac{-184.800n_h - 70.130n_h \log(1-z)}{\sqrt{1-z}} + 8.278n_h \log^2(1-z) \\ & + \log(1-z) (-185.400n_h - 3.553n_h^2) - 6.220n_h \log^3(1-z) + C, \end{aligned}$$

where C denotes an unknown constant.

To obtain an interpolation between these regions a Padé approximation has been used. We show the result for the approximating function in Fig. 3, where the envelope together with the relative error is displayed. Inserting the approximation for $\Pi^{(4)}$ in Eq. (3) the results shown in Tab.1 are obtained.

The results listed for classes I(a)–I(c) are exact since we numerically integrated the known one- and two-loop results for the vacuum polarization

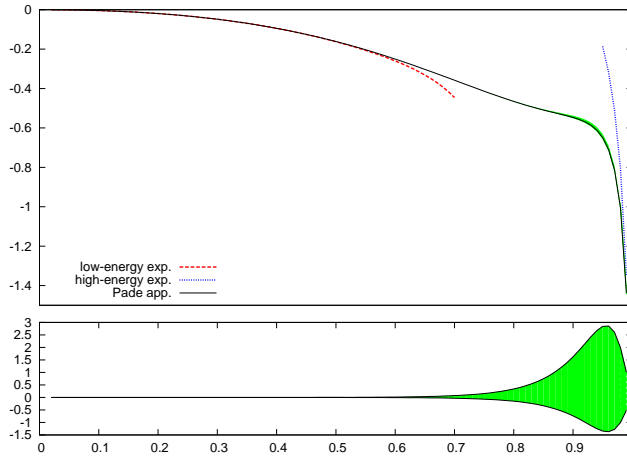


Fig. 3. Padé approximation for $\Pi^{(4)}\left(-\frac{x^2}{1-x}m_\mu^2\right)$. We show in the top half the approximants and in the bottom half the relative error with respect to the local mean of all approximants obtained.

	this work	Ref. [8]	Refs. [11, 12, 13, 14]	
I(a)	20.142 813	20.183 2	20.142 93(23)	[11]
I(b)	27.690 061	27.718 8	27.690 38(30)	[11]
I(c)	4.742 149	4.817 59	4.742 12(14)	[11]
I(d)+I(e)	6.241 470	6.117 77	6.243 32(101)(70)	[11]
I(e)	-1.211 249	-1.331 41	-1.208 41(70)	[11]
I(f)+I(g)+I(h)	4.446 8 ⁺⁶ ₋₄	4.391 31	4.446 68(9)(23)(59)	[11, 12]
I(i)	0.074 6 ⁺⁸ ₋₁₉	0.252 37	0.087 1(59)	[13]
I(j)	-1.246 9 ⁺⁴ ₋₃	-1.214 29	-1.247 26(12)	[14]

Table 1. Results for $A_2^{(10)}(m_e/m_\mu)$ with pure electronic insertions. The errors listed in the second column are estimated from the spread between different Padé approximants, which is negligible for classes I(a)–I(e). Please note that the authors of Ref. [8] only used the asymptotic form of $\Pi(s)$ and did not provide any error estimate.

function. Classes I(d) and I(e) are calculated using highly constrained Padé approximants, which have been constructed using 30 terms in the low- and high-energy expansion. Due to the vast amount of information, the results for g-2 using different approximants have very little spread and the final result is thus very precise. The situation is quite different for classes I(f)–I(j) which require the knowledge of $\Pi(q^2)$ at four-loop order. Since there is only a limited number of terms in the relevant expansions, the Padé approximation

is less precise and the precision of our result for $g-2$ is limited. In general we find good agreement with the results from Refs. [11, 12, 13, 14], but for some classes a certain tension remains.

3. Conclusions

We calculated the contribution to the anomalous magnetic moment of the muon arising from corrections to the vacuum polarization function of the photon at five-loop order. To this end we constructed an approximation of the vacuum polarization function of the photon at four-loop order based on expansion in the low- and high-energy and the threshold region. We find good agreement with the results presented in Refs. [11, 12, 13, 14].

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