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Tensorial spacetime geometries and background-independent quantum field theory

Dissertation
zur Erlangung des akademischen Grades
"doctor rerum naturalium"
(Dr. rer. nat.)
in der Wissenschaftsdisziplin "Theoretische Physik"

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät
der Universität Potsdam

von
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Potsdam, den 20.12.2012

Max-Planck-Institut
für Gravitationsphysik
Albert-Einstein-Institut

Universität Potsdam

Published online at the
Institutional Repository of the University of Potsdam:
URL http://opus.kobv.de/ubp/volltexte/2013/6573/
URN urn:nbn:de:kobv:517-opus-65731
http://nbn-resolving.de/urn:nbn:de:kobv:517-opus-65731

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Acknowledgements

I would like to thank my supervisor Dr. Frederic P. Schuller for the opportunity to work with him on general tensorial spacetimes. It was always interesting and exciting. I would also like to thank him for all the intense, political discussions we had in our spare time and for the Frederic P. Schuller coffee fund. Next, I would like to thank my colleagues Sergio Rivera and Christof Witte for all the interesting discussions and a very pleasant work environment. A huge part of the work presented in this thesis I did with Sergio in particular. It was a pleasure to work with you.

I would also like to thank Robert Oeckl, Daniele Colosi and Max Dohse for the wonderful time I had in Morelia, Mexico working with them. In particular, I would like to thank Daniele Colosi for the pleasure of working with him on two articles after I returned to Germany. It is not always easy to keep a collaboration alive over such a great distance. Thank you for your endurance.

I also would like to thank Frank Hellmann and Ralf Banisch for the pleasure of working with them on the GBF. Although our results are not part of this thesis, I learned a lot about the GBF when we were working on the subject and I organized the introduction to the GBF in this thesis like the the one in our paper.

Further, I would like to thank Prof. Dr. Martin Wilkens for being my official supervisor at the University of Potsdam. I am also very thankful for the productive time at the Max Planck Institute which financed my PhD via the IMPRS for Geometric Analysis, Gravitation and String Theory. I would like to thank Stefan Fredenhagen for his fantastic work as the IMPRS coordinator.

I would like to thank my parents Jens-Uwe and Andrea Rätzel and my brother Oliver Rätzel for their love and unconditional support over all these years. Finally, I would like to thank all my friends. Having their friendship helped me through the hard times and made many things easier.

Chapter 1

Introduction

Since physics is about predicting the future, a basic condition for every physical theory should be that it is predictive. For a field theory on a differentiable manifold $\mathcal{M}$ this condition can be translated into the condition that there must be a well-posed initial data problem for that field theory. Interestingly, for matter fields like the Klein-Gordon and the Dirac field that couple to a Lorentzian metric, at least locally, the existence of a well-posed initial data problem is already ensured precisely due to the fact that the metric is Lorentzian.
However, in recent years, there have been a number of attempts to establish modified spacetime backgrounds [1–7]. Most of them were motivated from approaches to quantum gravity [8–18], but many were also motivated from other particular physical and mathematical models [19–39]. Besides the need for testing grounds for quantum gravity results, the interest in spacetime backgrounds beyond Lorentzian geometry stems partly from the diversity of observations made over the last few years suggesting that there is something wrong with our understanding of the matter content of the universe or gravitational dynamics or both. There are observations of the gravitational lensing of galaxies [40], high redshift supernovae [41, 42] and the cosmic microwave background [43, 44] that suggest that an overwhelming 83% of the matter and 95% of the total mass-energy in the universe is of unknown type. Even worse, one has to assume that this unknown 83% of the total matter in the universe is not interacting with the electromagnetic field to match the observations, which earned it the name "dark matter" [45]. This can be interpreted as a hint of a new particle physics [46] or a new gravitational physics [47]. Considering modified backgrounds, one would of course expect the new particle physics and gravitational physics to arise at the same time: On the one hand, the restriction to Lorentzian spacetimes also severely restricts the type of matter fields that can be considered [48], and this restriction is strongly used in particle physics [49]. On the other hand, general relativity is fundamentally based on the metric concept [50].

The main scope of this thesis is now to investigate what can be said about general tensorial backgrounds given by a tensor field G. We want to find out how they are restricted in principle independently of their physical motivation. This will be done by investigating a tensorial test matter field coupled to the tensorial structure G. Notably, this is exactly how Einstein was led to

Lorentzian geometry in [51] where the test matter field theory he was considering was Maxwell electrodynamics. In this thesis, we will start only from the fundamental physical requirement that the corresponding field theory be predictive and that there be a well-defined notion of observers and positive energy. From that we will find that there are four conditions that must be fulfilled by every tensorial background in order to be a viable spacetime structure which will then be called a tensorial spacetime structure.

These conditions heavily restrict possible modifications from the class of Lorentzian spacetimes. More precisely, the central result we will obtain in this thesis is that a tensorial spacetime structure G must give rise to a cotangent bundle function P that is a reduced, hyperbolic, time-orientable and energy-distinguishing homogeneous polynomial in each fiber. The first chapter of this thesis will define and justify these conditions. They will all have their essential basis in the application of well-known results from the theory of partial differential equations, real algebraic geometry and convex analysis, to questions one considers in classical physics like the aforementioned initial data problem and the definition of the kinematics and dynamics of massless and massive particles. In particular, we will see that the conditions we find are sufficient to define dispersion relations and actions for massless and massive particles and duality theories between momenta and velocities for massless and massive particles, generalizing those known from Lorentzian spacetimes. Furthermore, we will be able to define observer frames and inertial, non-rotating laboratories. Such notions would be necessary to interpret spacetime objects (like the electromagnetic field strength) in terms of quantities that can be measured in laboratories (in the case of the field strength, the electric and magnetic field).

As remarked above, the restriction to Lorentzian spacetimes also restricts the type of matter fields that can be considered. Hence, another part of this thesis will deal with the question of which matter fields can be considered on general tensorial spacetimes. In particular, we will investigate massive matter fields that couple directly to the cotangent bundle function P. This will be done, however, for the case of a flat spacetime (a notion that will be clarified in Chapter 4). In that case, it will prove possible to define generalizations of the Klein-Gordon and Dirac fields. The Dirac field is of particular interest for establishing full quantum electrodynamics on general tensorial spacetimes.

However, classical matter field theory on general tensorial spacetimes is only a first step towards a total liberation of modern physics from the Lorentzian metric. First, we need a prescription for how to quantize matter fields and second, we need a dynamical theory for the spacetime itself. The first point is approached in the second part of this thesis and the second point was investigated in [52] where the authors derived a system of linear partial differential equations that must be fulfilled by the constraints governing the dynamics of the geometry. It is particularly interesting that - following the arguments of [52] - in a Lorentzian spacetime we would inevitably arrive at the Einsteinian theory of gravity. This was already obtained much earlier in [53] using the approach that was generalized in [52] for general tensorial spacetimes.

In the second part of this thesis, I will introduce a background-independent quantum field theory approach called general boundary formulation (GBF). We will obtain that the geometric quantization scheme used for the GBF can be also used to quantize scalar fields on tensorial spacetimes in a very systematic way if canonical commutation relations are assumed to hold. The presentation of this application will be the last part of this thesis. It will be used to draw conclusions about the quantization of massive matter fields on tensorial spacetimes.

The GBF was developed in [54–72] because of the necessity in non-perturbative quantum gravity to overcome the conceptual restrictions imposed by the metric [54, 68, 73]: In standard quantum theory, one needs a $3 + 1$-split of the spacetime which can only be defined from the background metric. If the metric itself is quantized non-perturbatively, there is no background metric from which one could define the spacetime split.

Instead, the GBF gives a true generalization of standard quantum field theory that relies neither on a Lorentzian metric nor on a spacetime split. It is an axiomatic framework that allows one to formulate quantum field theory on general spacetime regions with general boundaries. In this sense, it resembles topological quantum field theory [74] by which it was originally inspired [54]. More specifically, the set of axioms assigns algebraic structures to geometrical structures and ensures the consistency of these assignments.

In particular, in the GBF, we can consider quantum field theories in compact spacetime regions. This may provide a way to solve the problem of locality in quantum gravity [68] which can be stated as follows: On Lorentzian spacetimes, quantum field theories are required to be microcausal which means that spacelike separated measurements can be performed independently. That enables us to consider measurements locally, i.e., without considering spacelike separated parts of the Lorentzian spacetime. Again, this definition of locality hinges on the existence of a Lorentzian metric defining the causal structure of the Lorentzian spacetime. As stated above, there is no such background metric if the metric is meant to be quantized non-perturbatively.

The central algebraic structures of the GBF are associated with the boundary of the region under consideration which earns the GBF its name together with the fact that general boundaries can be considered. This fixation on the boundary resembles that of the holographic principle, and the GBF might be seen as a particular realization of this principle [54, 68].

One central element of the GBF is a generalization of the standard notion of transition amplitude. As in the standard formulation, probabilities for physical processes can be derived from this amplitude. The basis for the corresponding probability interpretation which does not rely on a notion of time is based on the generalization of the Born rule [57, 73]. It is argued in [68] that this solves - to some extent - the measurement problem of quantum gravity [73].

Although the GBF does not rely on a notion of spacetime metric, it can be applied to cases in which such a metric is present, like for example quantum field theories on a fixed Lorentzian spacetime. On the one hand, this is a good testing ground for and gives structural insight into the GBF. On the other hand, it makes it possible to investigate a much wider class of setups then just spacelike initial and final data hypersurfaces like in the standard formulation of quantum field theory. In this context, the main scope of the GBF research is the investigation of quantum field theories on compact spacetime regions, especially, those with just one connected boundary having spacelike as well as timelike parts. Another class of regions that can be considered only in the GBF are those with timelike boundaries which we will consider also in this thesis. I would like to emphasize that neither compact spacetime regions nor those with timelike boundaries can be treated in the standard formulation of QFT. Hence, the GBF offers a completely new perspective not only on quantum gravity but also on the well-established quantum theory of matter fields.

In recent years, the GBF was applied to a variety of problems and various results have been obtained [58, 65–72]. In particular, the crossing symmetry of the S-matrix of perturbative quantum field theory is a general property of quantum field theories in the GBF. In [71] this result was explicitly obtained employing a certain spacetime region of a Lorentzian spacetime with a timelike boundary called a hypercylinder because in a particular frame it consists of the same compact ball in every time slice and is translationally symmetric in the timelike direction.

The GBF was also applied to anti-deSitter space. There, no spatially asymptotic region can be defined which obstructs the definition of an S-matrix in the standard formulation of quantum field theory. However, the definition of temporal asymptotic regions within the GBF made it possible to provide meaningful and rigorous definitions for in- and out-states and the S-matrix that also work in anti-deSitter space [65]. For this purpose, again, a hypercylinder region was employed. Using the same techniques, it should also be possible to construct a quantum field theory for a stationary black hole spacetime like that described by the Schwarzschild or more generally the Kerr metric. That will be part of a new project.

There exist two different mathematical frameworks for the GBF called representations: The first representation called the Schrödinger representation was proposed in [54–56]. The second representation is called holomorphic representation and was established based on a mathematical rigorous framework for scalar fields called geometric quantization for linear field theories in [59] and affine field theories in [63]. In [60], Oeckl showed that the amplitudes derived with the two representations coincide, and thus the decision to use one or the other depends only on their respective technical advantages for particular applications. In this thesis we will only present the holomorphic representation.

Furthermore, three different quantization prescriptions for observables in the GBF have been explored so far: one is the Feynman quantization prescription which is a mathematical rigorous version of the path integral quantization of observables [61], the second is the Berezin-Toeplitz quantization prescription and the third is the normal ordering prescription [64]. Among them, the Feynman quantization prescription is exceptional since it is used in standard quantum field theory leading to results coinciding with experimental observations. We will find the same in an application of the GBF we will present as an example in Section 6. Furthermore, in contrast to the other quantization prescriptions, the Feynman quantization prescription fulfills certain identities when several spacetime regions are combined [61]. We will have the pleasure of using some of the results from [61] in this thesis. For the sake of completeness, let us mention that beside all the results concerning quantum scalar field theory, the GBF was applied to fermionic field theories in [62] and to two-dimensional Yang-Mills theories in [58] where also regions with corners were considered.

In [75], Frank Hellmann, Ralf Banisch and I established an Unruh-DeWitt detector model in the framework of the GBF and used it to gain insight into the structure of the GBF. In particular, we were able to generalize the notion of initial and final states to a notion of incoming and outgoing states that is general enough to apply to spacetime regions with timelike hypersurfaces. We showed for specific examples that, using this interpretation, the response of the Unruh-DeWitt detector can be used to fix the vacuum state on timelike hypersurfaces.

In this thesis we will quantize of a massive scalar field in Rindler space within the GBF framework and propose an analysis of the Unruh effect from the GBF perspective. Let us remark that the Unruh effect is defined in the literature as "the equivalence between the Minkowski vacuum and a thermal bath of Rindler particles" [76]. This result is sometimes obtained in a mathematically rather sloppy way by expressing the vacuum state in Minkowski space in terms of two sets of field modes where the elements of one of them are vanishing outside of the Rindler wedge of Minkowski space (Rindler space embedded in Minkowski space) and the elements of the other are vanishing exactly on the Rindler wedge. The degrees of freedom corresponding to modes vanishing on the Rindler wedge are then traced out from the density matrix corresponding to the Minkowski vacuum, and the resulting density matrix is shown to be equivalent to a thermal state in Rindler space [76–82]. One point of critique is that in this derivation not all modes that decompose the field in Minkowski space are considered; there is always at least one mode that does not fit into the sets described above. It is argued in [83–87] that neglecting this mode corresponds to an additional boundary condition. This critique was, however, addressed in [88], and a conclusion regarding its validity was never reached. However, another point of critique comes from the result that the vacuum state of Minkowski space cannot be represented as a density matrix in the quantum theory on Rindler space [89, 90]. Hence, this derivation of the Unruh effect can only be approximately correct.

A mathematically more rigorous derivation of the Unruh effect is provided by algebraic quantum field theory (AQFT) by showing that the Minkowski vacuum state (which is in AQFT a map from the set of observables to the complex numbers) turns out to satisfy the conditions defining a thermal state in AQFT on Rindler space when restricted to the algebra of observables defined on open subsets of the right Rindler wedge of Minkowski space [89, 91]. More specifically, it is shown in [91] that ",given an arbitrary field (in general, interacting) on a manifold X, the restriction of the field to a certain open submanifold $X^{(+)}$, whose boundaries are event horizons, satisfies the Kubo-Martin-Schwinger (KMS) thermal equilibrium conditions", which additionally delivers the derivation of the Hawking effect in AQFT. However, it is argued in [90] that this result, although correct in its mathematical form, does not imply the same as the original statement of the Unruh effect. This is done by doubting the physical content of the KMS thermal equilibrium conditions. Whether this is a valid argument or not in the GBF, we do not have the tools at hand to derive the Unruh effect in this generality.

Instead, in this thesis, we will consider a Weyl observable W corresponding to a generic source term with support in the interior of the right Rindler wedge. In particular, we consider this observable because every n-point correlation function of the quantum field theory can be derived from it. Hence, by calculating the corresponding expectation value we obtain the expectation values for all n-point correlation functions. First, we will calculate the vacuum expectation value of W when quantized in the Minkowski quantization. Then, we will calculate the expectation value of the Weyl observable W in a particular thermal state in Rindler quantization. We will show that when using the Feynman quantization prescription for Weyl observables these expectation values coincide, and when using the Berezin-Toeplitz quantization prescription for Weyl observables they do not coincide. This suggests that there are some limitations on the applicability of the Berezin-Toeplitz quantization. Beside this, the application to the Unruh effect is of interest for the GBF since it is a direct application of the quantization of observables, and the first time thermal states are used in the GBF.

Finally, we will use the background-independent geometric quantization scheme that we will introduce for the holomorphic representation of the GBF for an example of field quantization on tensorial spacetimes. For that purpose, we do not need the full generality of the GBF; we are dealing with initial and final data on Cauchy hypersurfaces. However, the geometric quantization scheme suits perfectly for situations with generalized backgrounds. More specifically, we will use the geometric quantization scheme to quantize a generalization of the Klein-Gordon field on a non-metric tensorial spacetime with a dispersion relation of fourth order. That means in particular that the corresponding field equations will be of fourth order. We will find that additional solutions not corresponding to classical particles have to be included in order to obtain a microcausal theory when canonical commutation relations are imposed. We will obtain that Lorentzian spacetimes are the only tensorial spacetimes on which one can consistently establish a microcausal, unitary quantum scalar field theory fulfilling canonical commutation relations (CCRs) such that only classical interpretable particles exist. Including the non-classical modes, however, leads to mathematical problems and conceptual problems concerning the interpretation of these modes. Comparing this result to results obtained for the imaginary mass Klein-Gordon field, we will argue that different inertial observers would see a different content of non-classical particles in the same state of the field.

The outline of the thesis is the following: In Chapter 2, we start by deriving the fundamental conditions a generic tensorial spacetime must fulfill and define notions of observers and the kinematics and dynamics for massless and massive point particles. In Chapter 3, we derive general properties of tensorial spacetimes and give two examples of classes tensorial structures. By applying the identified conditions, we show how these classes must be restricted to lead to viable spacetime structures. In Chapter 4, we construct non-tensorial massive field theories. In Chapter 5, we review the GBF framework in detail and in Chapter 6 we apply the GBF to the Unruh effect. In section 7 we apply the background-independent geometric quantization framework developed for the GBF to a massive scalar field with a dispersion relation of fourth order on a tensorial spacetime.

Chapter 2

Introduction to tensorial spacetime geometries

In this chapter we will derive the conditions a general tensorial structure must fulfill in order to constitute a viable spacetime structure. For that purpose we will deal with the theory of partial differential equations, algebraic geometry and convex analysis. We will introduce a notion of observers and introduce the kinematics and dynamics for massless and massive particles on general tensorial spacetimes. The results presented in this and the subsequent chapters have already been published in [92].

2.1 Field equations

Let us assume that we are given a differentiable, n-dimensional manifold $\mathcal{M}$ and a tensor field G ("the geometry") coupling to a tensor field ϕ ("the matter"). Let us assume further that the dynamics of the matter field can be encoded in an action $S[\phi, G]$. We want to probe the tensorial geometry G by the matter field. To this end we have to assume that the field equations corresponding to the action $S[\phi, G]$ are linear. Only then we can consider solutions of arbitrarily small amplitude and the matter field ϕ can be seen as test matter. Hence, we assume that Euler-Lagrange equations corresponding to the action $S[\phi, G]$ have the form

$$\left[\sum_{n=1}^{s} Q_{IJ}(G)^{\mu_1 \ldots \mu_n}\, \partial_{\mu_1} \ldots \partial_{\mu_n}\right] \phi^J(x) = 0, \tag{2.1}$$

where the small Greek indices are the indices corresponding to the coordinates and run from 0 to $n-1$ and the $Q(G)$ are coefficient matrices and the indices I, J label the collection of tensor entries of ϕ. The assumed linearity of the field equations implies that the coefficients $Q[G]$ depend only on G but not on the field ϕ.

We impose the obvious condition of predictivity of the field equations (2.1) in the sense that they possess a well-posed Cauchy problem (or initial data problem) as it was defined by Hadamard [93], that is:

(a) for suitably chosen initial value surfaces, and consistent initial data, there exist unique solutions of the field equations (2.1) and

(b) the solutions depend continuously on the initial data.

As will be made clear in the following sections these conditions translate into a necessary condition on the cotangent bundle function $P : T^*\mathcal{M} \to \mathbb{R}$ defined by

$$P(q) := \rho(G) \det_{IJ} \left(Q_{IJ}(G)^{\mu_1 \cdots \mu_s} q_{\mu_1} \dots q_{\mu_s}\right) \tag{2.2}$$

for every covector field q, which on each cotangent space $T_x^*\mathcal{M}$ induces the so-called principal polynomial P_x. Note that only the highest order coefficient in (2.1) transforms as a tensor in all its indices and that only this coefficient is employed in the definition (2.2). The factor $\rho(G)$ is a scalar density of appropriate weight that must be constructed from the tensor background G to counter the weight of the determinant in (2.2), such as to make the cotangent bundle function P indeed a scalar function. Even with the weight fixed in this manner, this leaves ample choice for the precise form of $\rho(G)$, and only later we will have physical reason to fix this in one way or another.

If the field theory defined by the field equations in (2.1) possesses gauge symmetries, the determinant in (2.2) will vanish. In that case, one must either fix the gauge or reformulate the field theory in terms of gauge-independent quantities, identify constraint equations and evolution equations and calculate P only from the latter [94].

2.2 Hyperbolicity

The above-mentioned condition on P that is necessary for the predictivity of (2.1) is that for all $x \in \mathcal{M}$ the principal polynomial P_x is hyperbolic. A homogeneous polynomial of degree r, i.e. $P_x(\lambda q) = \lambda^r P_x(q)$ for all $\lambda \in \mathbb{R}$, is hyperbolic if there exists a covector $h \in T_x^*\mathcal{M}$ with $P_x(h) \neq 0$ such that for every momentum $q \in T_x^*\mathcal{M}$ the equation $P_x(q + \lambda h) = 0$ is only solved for real λ. The covector h is then called a hyperbolic covector with respect to P_x. Due to the homogeneity of P_x, the vanishing set of P_x defined as $\mathcal{N}_x := \{q \in T_x^*\mathcal{M} | \, P_x(q) = 0\}$ is a cone, i.e. every real positive multiple λq of an element q of $\mathcal{N}_x$ is contained in $\mathcal{N}_x$. This leads to a geometrical reformulation of the algebraic definition of hyperbolicity as the property that there must exist a covector h such that every line drawn through any q in the direction of h intersects $\mathcal{N}_x$ exactly r times. To illustrate this condition we show some examples in Figure 2.1. The full condition for the predictivity of the field equations (2.1) is then properly stated in the following theorem:

Theorem 2.2.1. (Theorems 2.1 and 3.1 of [95], and Theorem 1.2.1 of [96]) *Assume that the Cauchy problem for the field equations (2.1) is well-posed in a region of spacetime. Then the principal symbol P defines a homogeneous hyperbolic polynomial P_x at every point x of the considered region. More precisely,*

$$P_x : T_x^*M \to \mathbb{R}\,, \qquad q \mapsto P_x(q) = P(x, q) \tag{2.3}$$

must be a homogeneous hyperbolic polynomial. Moreover, suitable initial value surfaces must have co-normals which are hyperbolic with respect to P_x.

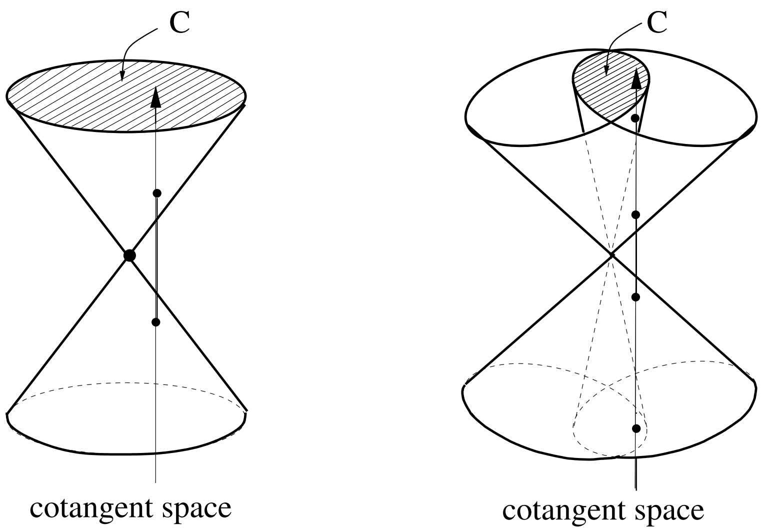

Figure 2.1: The vanishing set $\mathcal{N}_x$ of a homogeneous, hyperbolic polynomial P_x of degree r is a cone and we can find h such that every line drawn through any q in the direction of h intersects $\mathcal{N}_x$ exactly r times. Here, we see the vanishing sets of a second-degree and a fourth-degree polynomial as well as one of their hyperbolicity cones $C(P_x, h)$, respectively. It is of particular interest that the vanishing set of a hyperbolic polynomial always consists of hypersurfaces of dimension $n-1$ which we will show later in Lemma 2.3.1.

Thus our physical requirement of predictivity translates into the following mathematical condition:

Condition 1 (Hyperbolicity): We require the principal symbol P of the field equations (2.1) to give rise to a hyperbolic polynomial P_x in every cotangent space $T_x^*\mathcal{M}$.

An interesting example for a class of hyperbolic polynomials are the polynomials $P_x(q) = g_x^{\mu\nu} q_\mu q_\nu$ induced by the inverse g^{-1} of a Lorentzian metric g on a four dimensional manifold $\mathcal{M}$. By choosing an appropriate basis ϵ^μ in $T_x^*\mathcal{M}$ such that $g_x^{-1}(\epsilon^\mu, \epsilon^\nu) = \mathrm{diag}(1,-1,-1,-1)$ (which is always possible for a Lorentzian metric) we find that $P_x(q) = \tilde{q}_0^2 - \tilde{q}_1^2 - \tilde{q}_2^2 - \tilde{q}_3^2$ where $\tilde{q}_\nu = q_\mu \epsilon^\mu_\nu$. Choosing h such that $\tilde{h} = (1,0,0,0)$ we find that for an arbitrary q we have $P_x(q+\lambda h) = (\tilde{q}_0 + \lambda)^2 - \tilde{q}_1^2 - \tilde{q}_2^2 - \tilde{q}_3^2 = 0$ has the two real solutions $\lambda_{1,2} = \sqrt{\tilde{q}_1^2 + \tilde{q}_2^2 + \tilde{q}_3^2} - \tilde{q}_0$. So that the P_x are recognized to be hyperbolic, indeed.

It is of particular interest that if there exists one hyperbolic covector h for the homogeneous polynomial P_x, then there exists a whole convex cone of hyperbolic covectors $C(P_x, h)$. Convexity convexity means that for every h_1 and h_2 in $C(P_x, h)$ and every $0 \leq \lambda \leq 1$, the whole line $\lambda h_1 + (1-\lambda) h_2$ lies in $C(P_x, h)$. The cone $C(P_x, h)$ is called a hyperbolicity cone, is bounded by elements of $\mathcal{N}_x$ and can be specified as the set of all $q \in T_x^*\mathcal{M}$ such that for all $\lambda \geq 0$ we have $P_x(q + \lambda h) \neq 0$.

A polynomial $P_x = P_1 \dots P_l$ is hyperbolic with respect to some covector h if and only if all of its factors are hyperbolic with respect to h. Moreover, the hyperbolicity cone is $C(P_x, h) =$

$C(P_1, h) \cap \cdots \cap C(P_l, h)$ [1].

In some circumstances it will be more convenient to make statements about hyperbolicity and hyperbolicity cones for every cotangent space at the same time. Hence, we say in the following that the principal symbol P is hyperbolic if every P_x is hyperbolic and there exists a smooth covector field h such that P_x is hyperbolic with respect to $h(x)$. If P is smooth, we have a smooth distribution of hyperbolicity cones $C(P_x, h)$. We call this distribution $C(P, h)$.

The above definition of the hyperbolicity cone is somewhat implicit. A constructive definition is afforded by the following theorem:

THEOREM 2.2.2. (Theorem 5.3 of [98]) *Let the polynomial P_x be hyperbolic with respect to h and $P_x(h) > 0$. Then a generic covector v lies in the hyperbolicity cone $C(P_x, h)$ if and only if it fulfills the $r = \deg P$ inequalities*

$$\det H_i(v, h) > 0 \qquad \textit{for all } i = 1, \ldots, r\,, \tag{2.4}$$

where the bilinear maps $H_1, H_2, \ldots, H_r$ are constructed as

$$H_i(v,h) = \begin{bmatrix} h_1 & h_3 & h_5 & \ldots & h_{2i-1} \\ h_0 & h_2 & h_4 & \ldots & h_{2i-2} \\ 0 & h_1 & h_3 & \ldots & h_{2i-3} \\ 0 & h_0 & h_2 & \ldots & h_{2i-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & h_i \end{bmatrix}_{i\times i} \qquad \textit{where } h_j \textit{ is set to } 0 \textit{ for } j > i \tag{2.5}$$

from the coefficients of the expansion

$$P_x(v + \lambda h) = h_0(v,h)\,\lambda^r + h_1(v,h)\,\lambda^{r-1} + \cdots + h_r(v,h)\,. \tag{2.6}$$

For any given h the condition that $P_x(h) > 0$ can be always fulfilled by multiplying P_x with -1 if necessary.

We are now ready to identify the next condition a tensorial spacetime has to satisfy. The alert reader might have noticed that the vanishing set $\mathcal{N}_x$ for the polynomial P_x in the Lorentzian metric example above is exactly the set of lightlike momenta. It turns out that this is the correct interpretation for the distribution of vanishing sets $\mathcal{N}$ for every hyperbolic principal symbol P, as we will learn from considering the geometrical optical limit.

2.3 The geometric optical limit

We will show in the following that the cotangent bundle function P provides the dispersion relation obeyed by wavelike solutions of the field Equation (2.1) in the geometric optical limit. To see this, consider solutions taking the form of a formal series

$$\phi^N(x,\lambda) = e^{i\frac{S(x)}{\lambda}} \sum_{j=0}^{\infty} \varphi_j^N(x)\lambda^j\,, \tag{2.7}$$

[1] These and more properties of hyperbolic polynomials and their hyperbolicity cones can be found in [97].

where $\phi_j^N(x)$ is a tuple of functions for each j and S/λ is the phase of the wave-like solution and considered to be real. The function $S(x)$ is known as the eikonal function in the literature. The geometrical optical limit is now obtained by considering only small λ, which can be thought of as the short wavelength limit of the wavelike solutions considered [99–102]. Using the ansatz (2.7) in the field Equation (2.1), we obtain

$$e^{i\frac{S(x)}{\lambda}}\lambda^{-s}\left[Q_{MN}(G)^{\mu_1\cdots\mu_s}\partial_{\mu_1}S(x)\cdots\partial_{\mu_s}S(x)\,\varphi_0^N(x)+\sum_{j=1}^{\infty}v_{Mj}(x)\lambda^j\right]=0\,, \tag{2.8}$$

where each of the $v_{Mj}(x)$ terms depends on some of the matrix coefficients Q of the differential Equation (2.1), on the coefficients $\phi_j^N(x)$ of the expansion (2.7) and on the eikonal function S and its derivatives of lower than the highest order s. For ϕ^N to be a solution to the field equations (2.1), the expression in (2.8) must vanish for every order of λ separately. This leads to the conclusion that the eikonal function must fulfill the equation

$$\det_{MN}\Big(Q_{MN}(G)^{i_1\cdots i_s}(x)\partial_{i_1}S(x)\cdots\partial_{i_s}S(x)\Big)=0\,, \tag{2.9}$$

which can be written using the principal polynomial at x as

$$P_x(dS)=0$$

. This solvability condition is known as the eikonal equation. This is why we identify $\mathcal{N}_x$ as the set of momenta of massless particles.

For the following constructions we need to impose a harmless technical condition:

Condition 2: We require that the polynomials P_x induced by the principal symbol P do not contain repeated factors when written as a product of irreducible polynomials $P_x = P_1 \dots P_l$.

We call a polynomial that fulfills this condition reduced. The condition is indeed harmless in the sense that the vanishing set of an arbitrary power of a polynomial coincides with the vanishing set of the polynomial; in other words, when canceling repeated factors, we do not lose any information about the vanishing set. On the other hand, the condition avoids degeneracies that would occur otherwise. To make this statement clearer, we will show in the following that if P_x is reduced and hyperbolic, the vanishing ideal $\mathcal{V}(\mathcal{N}_x)$ is generated by P_x, where $\mathcal{V}(\mathcal{N}_x)$ is the set of all polynomials that vanish on $\mathcal{N}_x$. In other words, every element of $\mathcal{V}(\mathcal{N}_x)$ can be decomposed as a product of polynomials containing P_x at least once. We can see this as the statement that P_x and $\mathcal{N}_x$ contain the same amount of information in the sense of algebraic geometry which is the mathematical discipline concerned about the vanishing sets of polynomials. To give the proof, we first need to define the set $\mathcal{N}_x^{\text{smooth}} \subseteq \mathcal{N}_x$ of all massless momenta at which the derivative $DP_x(q) := \frac{\partial}{\partial q}P_x(q)$ does not vanish. Geometrically, this can be seen as the set of points on the set $\mathcal{N}_x$ with non-vanishing gradient. Second, we need to give and prove two technical lemmas that we will also need in a later section.

LEMMA 2.3.1. *For a reduced, homogeneous, hyperbolic polynomial P_x, the set $\mathcal{N}_x^{smooth}$ is a dense subset of the cone $\mathcal{N}_x$ of massless momenta.*

Proof. Since the set of massless momenta $\mathcal{N}_x$ is generated from a single polynomial P , it follows from Definition 3.3.4 of [103] that the set of singular points is $\text{Sing}(\mathcal{N}_x) = \mathcal{N}_x \setminus \mathcal{N}_x^{\text{smooth}}$. But then $\dim \text{Sing}(\mathcal{N}_x) < \dim \mathcal{N}_x = \dim \mathcal{M} - 1 = n - 1$, where the inequality is Proposition 3.3.14 of [103] and the equality follows from the hyperbolicity of P [104]. Thus we know that the singular set is at most of dimension $n - 2$. Further, we know from the first remark in 3.4.7 of [105] that $\text{Sing}(\mathcal{N}_x)$, being a real algebraic set, can be expressed as a finite union of analytic semi-algebraic manifolds S_i and that every such manifold has a finite number of connected components. From the propositions 2.8.5 and 2.8.14 of [103] we thus obtain that $\dim \text{Sing}(\mathcal{N}_x) = \max(\dim(S_i)) = \max(d(S_i))$, where $d(A_i)$ is the topological dimension of the semi-algebraic submanifold $S_i \subset T_x^*\mathcal{M}$. Since $\dim \text{Sing}(\mathcal{N}_x) \leq n - 2$ we conclude that $\text{Sing}(\mathcal{N}_x)$ consists of only finitely many sub-manifolds of $\mathbb{R}^n$ of topological dimension less than or equal to $n - 2$. Thus its complement $\mathcal{N}_x^{\text{smooth}} = \mathcal{N}_x \setminus \text{Sing}(\mathcal{N}_x)$ is dense in $\mathcal{N}_x$.

LEMMA 2.3.2. *If P_x is a reduced homogeneous hyperbolic polynomial with hyperbolicity cone C_x at some point $x \in \mathcal{M}$, then for all covectors $s \in T_x^*\mathcal{M}\backslash closure(C_x)$ there exists a massless covector r on the boundary ∂C_x of the hyperbolicity cone such that $s(DP_x(r)) < 0$.*

Proof. It is clear that if $y \in C_x$ and $s \notin \text{closure}(C_x)$, the line $y + \lambda s$ intersects the boundary ∂C_x at some $r_0 = y + \lambda_0 s$ for some positive λ_0. Thus $P_x(r_0) = 0$ and, since $P_x(C_x) > 0$, we have $P_x(r_0 - \epsilon s) > 0$ for sufficiently small positive ϵ. Now we must distinguish two cases: First assume that $P_x(r_0 + \epsilon s) < 0$, from which it follows that $\frac{d}{d\epsilon}P_x(r_0 + \epsilon s)|_{\epsilon=0} = s(DP_x(r_0)) < 0$, which proves the lemma with $r := r_0$; Second, assume that $P_x(r_0 + \epsilon s) > 0$ which is equivalent to $\frac{d}{d\epsilon}P_x(r_0 + \epsilon s)|_{\epsilon=0} = s(DP_x(r_0)) = 0$ which in turn holds if and only if $DP_x(r_0) = 0$ (to see the latter equivalence assume that, to the contrary, $s(DP_x(r_0)) = 0$ and $DP_x(r_0) \neq 0$; this implies that s must be tangential to ∂C_x at r_0, but since y lies in C_x and C_x is a convex cone $y + \lambda s$ could then not intersect ∂C_x at r_0, which we assumed, however). So to prove the lemma in this second case, we need to construct another $r_0' \in \partial C_x$ that satisfies the condition $s(DP_x(r_0')) < 0$. Now since the First Lemma guarantees that the set $\mathcal{N}_x^{\text{smooth}}$, on which DP_x is non-zero, lies dense in $\mathcal{N}_x$, we can find in every open neighborhood U around r_0 a vector $r_0' \in \partial C_x$ such that $DP_x(r_0') \neq 0$. We define $z := r_0' - r_0$ and $y' := y + z$. Since C_x is an open cone, y' lies in C_x if we choose a small enough neighborhood, and the line $y' + \lambda s$ intersects ∂C_x at r_0'. Finally since $r_0' \in \partial C_x$ we know that $P_x(r_0') = 0$ and $P_x(r_0' - \epsilon s) > 0$. We conclude that $s(DP_x(r_0')) < 0$. This proves the Second Lemma with $r := r_0'$.

Now we are ready for our problem of algebraic geometry. Recall that an ideal $I \subset R$ in a ring R is a subset that is closed under addition and under multiplication with an arbitrary ring element. In our case, R is the ring of real polynomials on T_x^*M in $\dim M = n$ real variables. Now on the one hand, we can consider the situation where we are given an ideal I. Then, we can define the vanishing set $\mathcal{V}(I)$ as the set of cotangent vectors that are common zeros to all polynomials in I. On the other hand, we can start from a subset S of cotangent space and consider the set $\mathcal{I}(S)$ of all polynomials in R that vanish on all members of that set S. Now, it can be shown that $\mathcal{I}(S)$ is an ideal in the ring of polynomials on cotangent space, and that one always has the inclusion

$$\mathcal{I}(\mathcal{V}(I)) \supseteq I\,. \tag{2.10}$$

The question under which conditions the left and right hand sides are equal is studied in the Nullstellensätze of algebraic geometry. While this question is relatively straightforward for polynomials over algebraically closed fields [106] like the complex numbers, for the real numbers underlying our study here, we need to apply some theorems that were originally developed in order to solve Hilbert's seventeenth problem[2]. The central result for our purposes is

Proposition 2.3.3. *Let P_x be a reduced homogeneous hyperbolic polynomial on $T^*_x\mathcal{M}$, then the following equality holds:*

$$\mathcal{I}(\mathcal{N}_x) = \langle P_x \rangle\,. \tag{2.11}$$

Proof. From Lemma 2.3.1, we know that $\mathcal{N}_x^{\text{smooth}} \neq \emptyset$ (as follows from the hyperbolicity of P_x). Here $\langle P_x \rangle$ denotes the ideal containing all polynomials that have P_x as a factor. Drawing on the said results from real algebraic geometry, this is seen as follows. Let $P_{x\,i}$ be the ith irreducible factor of P_x. Then there exists a $q \in \mathcal{N}^{\text{smooth}}(P_{x\,i})$ so that Corollary 2.9 of [107] shows that $P_{x\,i}$ generates a real ideal, i.e., $\mathcal{I}(\mathcal{N}(P_{x\,i})) = \langle P_{x\,i}\rangle$. According to corollary 2.8 of [107], the reduced polynomial P_x thus also generates a real ideal since it does not contain repeated factors. Finally, Theorem 4.5.1 of [103] yields the claim.

That the equality in Equation (2.10) holds for hyperbolic polynomials will be of importance when we try to find the vector duals of massless momenta. This will be part of the insight gained in the next section.

2.4 Massless particles

We showed in the preceding section that the principal symbol P provides the dispersion relation for massless particles as

$$P_x(q) = 0 \tag{2.12}$$

it satisfies the conditions we identified: it must be a hyperbolic and reduced polynomial. That puts us now into the position to define the dynamics of free, massless point particles as given by the Helmholtz action

$$I_0[x, q, \lambda] = \int d\tau\, [q_\alpha \dot{x}^\alpha + \lambda P(x, q)]\,. \tag{2.13}$$

[2] Hilbert's seventeenth problem is one of the 23 problems set out by Hilbert in 1900. It poses the question whether any multivariate polynomial that takes only non-negative values over the reals can be represented as a sum of squares of rational functions.

For that action, we have chosen the Hamiltonian $\lambda P(x, q)$ because it is a pure constraint which ensures via the Lagrange multiplier λ that the momenta q lie in the vanishing set $\mathcal{N}_x$ at every point x of the trajectory of a massless particle. To obtain from (2.13) a functional depending on the trajectory x and not on the momentum q we have to invert the relation

$$\dot{x} = \lambda \, DP_x(q) \tag{2.14}$$

which we obtain by varying (2.13) with respect to the momentum q for all $q \in \mathcal{N}_x^{\text{smooth}}$. We get rid of the Lagrange multiplier λ by reformulating (2.14) as

$$[DP_x(q)] = [\dot{x}] \, , \tag{2.15}$$

where $[X]$ denotes the projective equivalence class of all vectors collinear with the vector X. Since P_x is homogeneous of degree r, the gradient DP_x is homogeneous of degree $r-1$ and the function $[DP_x] : [\mathcal{N}_x^{\text{smooth}}] \to [DP_x(\mathcal{N}_x^{\text{smooth}})] \subset \mathbb{P}T_x\mathcal{M}$ mapping $[q]$ to $[DP_x(q)]$ is well defined. $[DP_x]$ will be denoted in the following as the Gauss map.

Having gotten rid of the Lagrange multiplier by rewriting Equation (2.14) in projective space, carries the advantage of having at our disposal mathematical results about the inverse of the Gauss map in algebraic geometry. It turns out that for every reduced, hyperbolic polynomial P_x there exists a homogeneous polynomial $P_x^\#$ called the dual polynomial such that the image of $\mathcal{N}_x^{\text{smooth}}$ under the Gauss map is contained in the vanishing set $\mathcal{N}_x^\#$ of $P_x^\#$. In other words, the relation

$$P_x^\#(DP_x(q)) = 0 \tag{2.16}$$

holds for all $q \in \mathcal{N}_x$. It will turn out that if P_x fulfills the additional conditions we will introduce in Section 2.5, the dual polynomial $P_x^\#$ is determined by (2.16) up to a polynomial factor. This factor can be removed up to a constant by defining the dual $P_x^\#$ as a polynomial of minimal degree fulfilling (2.16).

In this section, it remains to show that there exists a polynomial $P_x^\#$ that fulfills the condition (2.16) for any reduced, hyperbolic P_x. The proof relies on a branch of algebraic geometry known as elimination theory and is constructive. In addition, one sees that the Gauss dual can in principle be calculated algebraically[3]. We start by reformulating the condition in (2.16) as the statement that we are searching for polynomial conditions which a vector X must satisfy such that the polynomials

$$P_x(k), \quad X^{i_1} - DP_x(k)^{i_1}, \quad \ldots, \quad X^{i_d} - DP_x(k)^{i_d} \tag{2.17}$$

all vanish for some k. If the real Nullstellensatz holds, Proposition 11.10 and the Elimination Theorem in [106] tell us that X must lie in the vanishing set of a uniquely specified ideal $\mathcal{I}$ of the ring of real polynomials over $T_x\mathcal{M}$. The Elimination Theorem in [106] tells us additionally how to find the generators of $\mathcal{I}$, that is the set of polynomials $\{P_1, \ldots, P_l\}$ such that every element Q

[3]The respective algorithm is implemented in Mathematica but exhausts the capabilities of common office hardware if applied to higher degree polynomials in several variables.

of $\mathcal{I}$ can be written as $Q = \sum_i Q_i P_i$ where the Q_i are polynomials[4]. In particular, the vanishing set of $\mathcal{I}$ is equivalently defined as the maximal set on which all P_i vanish. Then, we obtain $P_x^\#$ as the sum of the squares of the generators as $P_x^\# = P_1^2 + \cdots + P_l^2$ since $P_1^2 + \cdots + P_l^2$ vanishes if and only if each of the P_i vanishes. Although the construction of a $P_x^\#$ is in principle possible via the elimination theory it is quite a difficult problem in practice. In particular, the problem of finding a $P_x^\#$ is reduced tremendously if P_x factorizes as $P_x = P_1 \ldots P_l$. In that case we define the dual as $P_x^\# = P_1^\# \ldots P_l^\#$ where for every i the polynomial $P_i^\#$ is a dual to P_i. This definition fulfills the condition (2.16) which can be seen easily from the product rule. However, we will see later an example in which dual polynomials to generically irreducible P can be constructed explicitly from the same background tensor field G from which P was constructed using knowledge about the tensor field.

Let us now return to the Gauss map

$$[DP_x] : [\mathcal{N}_x^{\text{smooth}}] \to [\mathcal{N}_x^\#]\,, [q] \mapsto [DP_x(q)]\,. \tag{2.18}$$

The inverse is obtained as the map

$$[DP_x^\#] : [\mathcal{N}_x^{\#\text{smooth}}] \to [\mathcal{N}_x]\,, [v] \mapsto [DP_x^\#(v)]\,, \tag{2.19}$$

since we then have for null covectors $k \in \mathcal{N}_x^{\text{smooth}}$ that

$$[DP_x^\#]([DP_x]([q])) = [q] \qquad \text{if} \;\; \det(DDP_x)(k) \neq 0\,, \tag{2.20}$$

so that the dual Gauss map $[DP^\#]$ acts as the inverse of the Gauss map on the images of all covectors q satisfying the above determinantal non-degeneracy condition. That the identity (2.16) holds can be seen by re-writing it as

$$P_x^\#(DP_x(q)) = Q_x(q) P_x(q) \tag{2.21}$$

for all $q \in T_x^*\mathcal{M}$. By differentiating with respect to q and applying the chain rule and then Euler's Theorem[5] on the right hand side of (2.21) we find for every covector fulfilling the non-degeneracy condition in (2.20) that

$$DP_x^\#(DP_x(q)) = \frac{Q_x(q)}{r-1} q\,. \tag{2.22}$$

The projection of (2.22) gives then Equation (2.20). Finally, we can invert the relation (2.15) between the trajectory tangent and the momentum we derived from the action in (2.13) above as

$$q = \mu DP_x^\#(\dot{x}) \tag{2.23}$$

where the factor μ covers the fact that we inverted the relation only in projective space without taking λ into account. As in the case of the hyperbolicity cones, we define the cotangent bundle

[4]Explicitly this can be constructed using Buchberger's algorithm and Gröbner bases (see [106] for an exhaustive introduction).

[5]Euler's Theorem tells us the simple fact that for any function f that is homogeneous of degree $\deg f$, the relation $Df(v)v = (\deg f) f(v)$ holds for any v in the domain of f.

function P as the tangent bundle function $P^{\#}$ given as a smooth distribution of dual polynomials such that at every $x \in \mathcal{M}$ the polynomial $P^{\#}_x$ is a dual to P_x. We call $P^{\#}$ the Gauss dual. The action (2.13) can then be re-written using $P^{\#}$ as

$$I_0[x, \mu] = \int d\tau \mu P^{\#}_x(\dot{x}) \,, \tag{2.24}$$

where the Lagrange multiplier μ serves two purposes. On the one hand, it ensures that $\dot{x}$ lies in the dual vanishing set $\mathcal{N}^{\#}_x$, and on the other hand, it covers the indeterminacy of the dual polynomial $P^{\#}_x$. With (2.15) and (2.23) we can interpret the Gauss map $[DP_x]$ and its inverse $[DP^{\#}_x]$ as associating particle momenta in $\mathcal{N}^{\text{smooth}}_x$ with particle velocities in $\mathcal{N}^{\#\text{smooth}}_x$ and vice versa. The tangent bundle function $P^{\#}$, which will be called the Gauss dual of P in the following, is hence interpreted as the tangent space geometry seen by massless particles. In the next section, this will lead us to our fourth requirement.

2.5 Time-orientability and energy-distinguishability

Let us summarize shortly which conditions we imposed on the tangent bundle function P and thus on the underlying geometric tensor G so far. First, we required that P can be identified as the principal symbol of a linear partial differential equation describing matter which turned out to be equivalent to the requirement that P give rise to a homogeneous polynomial P_x in every cotangent space. The linearity corresponded to the idea that one uses test matter to probe the geometry. Second, we required all the P_x to be hyperbolic, which was a necessary condition for the linear partial differential equations to be predictive in the sense of giving rise to a well-posed initial data problem. Hence, the first two conditions, which were already found a long time ago in [104], are inevitable from a physical perspective. Third, we required the P_x to be reduced, which meant that we took out multiple factors. This was a rather mathematical condition that ensured the applicability of theorems from algebraic geometry that we used later.

The conditions we will impose in the following are different in spirit. They cannot be deduced step-by-step from fundamental conditions. Their only justification is that at the end they turn out to lead to a consistent framework that includes observers associated with stable massive particles that can interpret spacetime using geometry. We call this property interpretability.

We start with the requirement of time-orientability: We want to have a definition of time orientation for which we need as a minimal condition that at least locally we can tell when a vector points forward in time. For this purpose, we want to be able to pick a smooth distribution $C^{\#}$ of hyperbolicity cones of $P^{\#}$ which we call the observers cones. Possible observer trajectories in our framework will then be only those curves $\gamma : [\tau_1, \tau_2] \to \mathcal{M}$ that have tangents lying in $C^{\#}_{\gamma(\tau)}$ for all $\tau \in [\tau_1, \tau_2]$. Vector fields in $C^{\#}$ will be called timelike forward-pointing. The distribution $C^{\#}$ can of course be equally defined by giving a smooth vector field T such that $T(x)$ is hyperbolic with respect to $P^{\#}_x$ at every point $x \in \mathcal{M}$, i.e., for all vector fields V the polynomial equation

$$P^{\#}_x(V(x) + \lambda T(x)) = 0 \tag{2.25}$$

has only real solutions λ. This is precisely the way a time orientation is specified in the special case of Lorentzian geometry (where $P(q) = g^{-1}(q, q)$ and $P^{(}v, v) = g(v, v)$, see Section 3.4).
A necessary condition for the existence of a $C^{\#}$ is of course the hyperbolicity of every $P^{\#}_x$. So we impose the following condition:

Condition 3 (Time orientability): We require that the Gauss dual $P^{\#}$ to P induces a hyperbolic, reduced, homogeneous polynomial $P^{\#}_x$ in every tangent space T_xM.

That the hyperbolicity of P_x does not already imply the hyperbolicity of $P^{\#}_x$ can be seen from the counterexample in Figure 2.2.

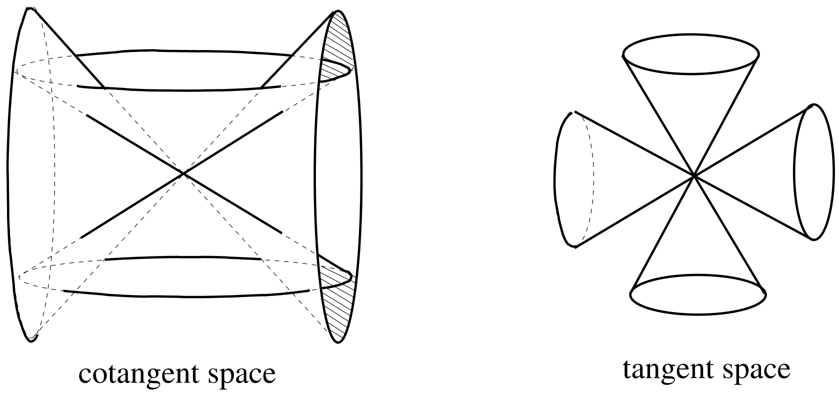

Figure 2.2: Example of a hyperbolic polynomial with non-hyperbolic dual polynomial; shown are the respective vanishing sets.

Furthermore, if $P^{\#}_x$ is reduced and hyperbolic, we find - recalling the result in 2.3.3 - that the principal ideal $\langle P^{\#}_x \rangle$ is in one-to-one correspondence with the dual vanishing set $\mathcal{N}^{\#}_x$ to every $\mathcal{N}_x$ which was defined as the algebraic closure of the image of $\mathcal{N}^{\text{smooth}}_x$ under the Gauss map. In that case, $P^{\#}_x$ is uniquely specified up to a polynomial factor by the condition in Equation (2.16) where the polynomial factor can be removed up to a constant by taking a polynomial of minimal degree fulfilling (2.16) for $P^{\#}_x$.

Let us assume in the following that we have picked an observer cone $C^{\#}$. We find that those momenta q at a point x whose energy is positive from every observer's point of view lie in the cone

$$(C^{\#}_x)^{\perp} = \{p \in T^*_x\mathcal{M} \,|\, p(v) > 0 \text{ for all } v \in C^{\#}_x\}\,. \tag{2.26}$$

The cone $-(C^{\#}_x)^{\perp}$ then obviously defines the set of momenta whose energy is negative from the perspective of every observer in $C^{\#}$. If the polynomial P_x is of the product form $P_x = P_1 \dots P_l$, we find that the positive energy cone is simply the sum of the positive energy cones coming from the duals of the factors P_i [108], i.e.

$$(C^{\#}_x)^{\perp} = (C^{\#}_{1;x})^{\perp} + \dots + (C^{\#}_{l;x})^{\perp}\,, \tag{2.27}$$

where the sum of two convex sets is just the set of all sums of any two elements of the two sets.

Having specified an observer-independent notion of positive and negative energy, we want these notions to distinguish, in particular, massless momenta of positive and negative energy. More precisely, we impose the condition:

> **Condition 4 (Energy distinguishability):** We require that we can pick a hyperbolicity cone $C^\#$ for the dual $P^\#$ to P such that $\mathcal{N}_x \setminus \{0\} \subset (C^\#_x)^\perp \cup -(C^\#_x)^\perp$ at every $x \in \mathcal{M}$.

Then we can immediately prove the following important result:

PROPOSITION 2.5.1. *For hyperbolic, time-orientable and energy-distinguishable geometries, the set of massless momenta $\mathcal{N}_x$ cannot contain any null planes in spacetime dimensions $d \geq 3$, which in turn implies that the degree of P cannot be odd.*

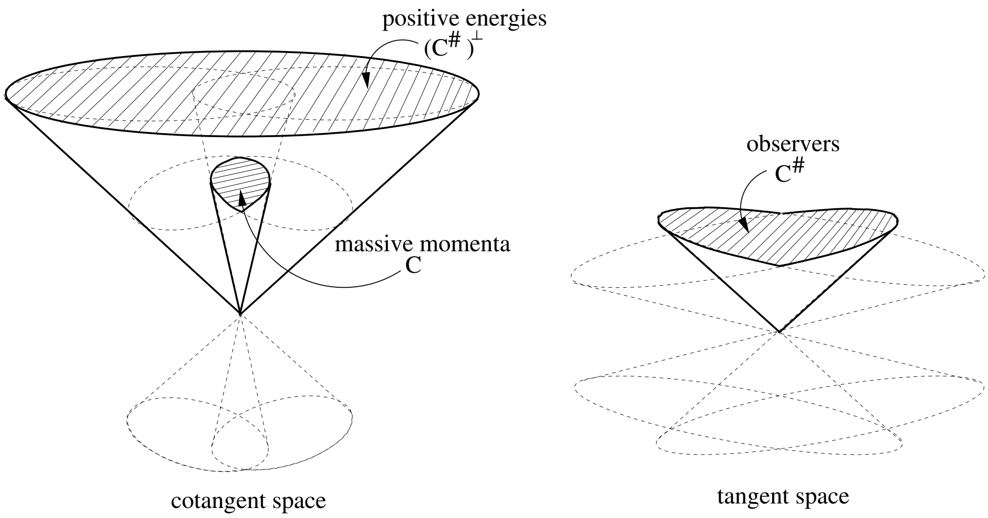

Figure 2.3: A hyperbolic, time-orientable and energy-distinguishing polynomial P_x.

Proof. First, we prove that the hyperbolicity and time-orientability of P_x implies that

$$\text{closure}(C^{\#\perp}_x) \cap -\text{closure}(C^{\#\perp}_x) = \{0\}\,. \tag{2.28}$$

Let k_0 be such that $k_0 \in \text{closure}(C^{\#\perp}_x)$ and $k_0 \in -\text{closure}(C^{\#\perp}_x)$. It follows from the definition of the dual cone that the following inequalities are true for all $x \in C^\#_x$: $x.k_0 \geq 0$ and $x.k_0 \leq 0$. If this would be true, the hyperbolicity cone $C^\#_x$ had to be a plane or a subset of a plane. That would contradict the property of $C^\#_x$ being open. Second, suppose that the vanishing set $\mathcal{N}_x$ contains a plane. From $\text{closure}(C^{\#\perp}_x) \cap -\text{closure}(C^{\#\perp}_x) = \{0\}$ it follows that $C^{\#\perp}_x \setminus \{0\}$ is a proper subset of a half-space. A proper subset of a half-space cannot contain any complete plane through the origin. Hence the existence of a null plane of P_x would obstruct the energy-distinguishing property. Third, this fact immediately restricts us to cotangent bundle functions P of even degree. For suppose $\deg P$ was odd. Then on the one hand, we would have an odd number of null sheets. On the other hand, the homogeneity of P implies that null sheets in a cotangent space come in pairs, of which one partner is the point reflection of the other. Together this implies that we would have at least one null hyperplane.

Furthermore, we can prove for time-orientable and energy-distinguishing hyperbolic polynomial geometries that P_x fulfills a property called completeness that will be crucial in the theory of massive particles and observers in the next section. A hyperbolic polynomial P_x is called complete if the lineality space

$$L(P_x) = \{a \in T_x^*M \,|\, \text{for all } y \in T_x^*M \text{ and } \lambda \in \mathbb{R} : P_x(y + \lambda a) = P_x(y)\} \tag{2.29}$$

only contains the zero covector. In other words, in order to be complete, P_x must depend on all covector components in any chosen basis. Geometrically, completeness can be read off from the closure of the hyperbolicity cones since it is equivalent to [109]

$$\text{closure}(C(P_x, h)) \ \cap \ \text{closure}(C(P_x, -h)) = \{0\}\,. \tag{2.30}$$

That completeness is already implied by the energy-distinguishing condition can be easily seen as follows: Picking up the argument given at the start of this section, we know that

$$\text{closure}(C_x^{\#\perp}) \cap -\text{closure}(C_x^{\#\perp}) \supseteq \text{closure}(C_x) \cap -\text{closure}(C_x)\,. \tag{2.31}$$

Thus, if the right hand side differs from $\{0\}$ (meaning that P_x is incomplete), the left hand side will contain non-zero covectors, too (showing that P_x is not energy-distinguishing). Because of the inclusion, this only holds in this direction. We conclude that the energy-distinguishing property already implies completeness.
From P_x being complete, we have the validity of some inequalities similar to those known from inner product spaces. First, we define the polarization tensor $P_x^{\mu_1,\dots,\mu_{\deg P}}$ such that its coefficients are those of the polynomial P_x. Second, we consider one particular hyperbolicity cone C_x and assume that P_x is positive on this cone, i.e., for all $h \in C_x$ we have $P_x(h) > 0$; Due to the definition of the hyperbolicity cone $P_x(h) \neq 0$ for all $h \in C_x$ and we can always achieve $P_x(h) > 0$ by multiplying P_x with -1 if necessary. Then, if P_x is complete we have that

1. the reverse triangle inequality

$$P_x^{1/\deg P}(q_1 + q_2) \geq P_x^{1/\deg P}(q_1) + P_x^{1/\deg P}(q_2) \tag{2.32}$$

holds for all q_1 and q_2 in the same hyperbolicity cone C_x and

2. a reverse Cauchy-Schwarz inequality

$$P_x^{\mu_1,\dots,\mu_{\deg P}} q_{1,\mu_1} \dots q_{\deg P,\mu_{\deg P}} \geq P_x(q_1)^{1/\deg P} \cdots P_x(q_{\deg P})^{1/\deg P} \tag{2.33}$$

is satisfied for all $q_1, \dots, q_{\deg P}$ in the same hyperbolicity cone C_x, where equality holds if and only if all arguments q_i are proportional to each other.

2.6 Massive point particles

Before we develop the kinematics of massive point particles in this framework, we have to pick a particular hyperbolicity cone C and call it the hyperbolicity cone in what follows. For a bi-hyperbolic and energy-distinguishing dispersion relation, there is always a hyperbolicity cone in cotangent space that is of positive energy with respect to a chosen time orientation $C^{\#}$. For let $\tilde{C}_x$ be some hyperbolicity cone of P_x, whose boundary $\partial\tilde{C}_x$ we know to be a connected set of null covectors. Now on the one hand, the complete vanishing set of P_x is contained in $(\tilde{C}^{\#}_x)^{\perp} \cup -(\tilde{C}^{\#}_x)^{\perp}$ due to the energy-distinguishing property. On the other hand, we have that (2.28) holds. Hence either $\tilde{C}_x$ or $-\tilde{C}_x$ is of positive energy. We assume that the hyperbolicity cone we selected is of positive energy. Since the sign of $P_x(q)$ is the same for all $q \in C_x$ we assume furthermore in the following that $P_x(q)$ is positive for all $q \in C_x$. Then we define the massive dispersion relation as the condition

$$P_x(q) = m^{\deg P} \tag{2.34}$$

for some fixed, real positive m, which we call the mass. Here is the point in this framework where the volume element $\rho(G)$ in the definition of P in (2.2) becomes important. Before, we were only concerned with the vanishing set, which is of course independent of this factor. But now, the quotient of two different volume elements $\rho(G)$ and $\rho'(G)$ is a scalar spacetime function which can be brought to the right hand side of (2.34), thus effectively changing the mass term m. Thus, the choice of the density ρ can be understood as a choice of conversion of mass densities from field theory to point masses in particle theory.

The form of the dispersion relation (2.34) has the advantage that the mass shells defined by different values of m foliate the whole hyperbolicity cone C_x, but at the same time are completely contained in the latter, i.e. momenta to massive particles are always hyperbolic with respect to P_x. We thus see immediately from the convexity of C_x that a single positive-energy *massless* particle cannot decay into positive energy massive particles. From the reverse triangle inequality (2.32) we see that the decay of a *massive* particle into two or more massive particles gives rise to a mass defect as it is known from massive particles in Lorentzian geometry.

Similar to the case of massless particles, we write an action for the massive particles as the Helmholtz action

$$I[x,q,\lambda] = \int d\tau \left[q_a \dot{x}^a - \lambda m \ln P(x, \frac{q}{m})\right], \tag{2.35}$$

which describes free massive particles since the massive dispersion relation $P(x,q) = m^{\deg P}$ is enforced through variation with respect to λ. We have chosen the specific form of the Lagrange multiplier term to have at our disposal the theory of Legendre duals on the open convex cones C_x, see [108]. To be more precise, the function,

$$f_x : C_x \to \mathbb{R}\,, \qquad f_x(q) = -\frac{1}{\deg P} \ln P_x(q)\,, \tag{2.36}$$

called the barrier function, is firstly guaranteed to be strictly convex, i.e., for each $\lambda \in [0,1]$ we have $f_x((1-\lambda)v + \lambda w) < (1-\lambda) f_x(v) + \lambda f_x(w)$ for all v, w in the hyperbolicity cone C_x, due to the completeness of P [109], which in turn is guaranteed by the energy-distinguishing property, as we saw in the previous chapter; secondly, near the boundary of the convex set, it behaves such that for all $q \in C_x$ and $b \in \partial C_x$

$$\lim_{\lambda\to 0^+}(D_{q-b}f_x)(b+\lambda(q-b)) = 0\,, \tag{2.37}$$

which property is known as essential smoothness in convex analysis. The important point is that strict convexity and essential smoothness together ensure that the barrier function f_x induces an invertible Legendre map

$$L_x : C_x \to L_x(C_x)\,, \qquad q \mapsto -(Df_x)(q) \tag{2.38}$$

and a Legendre dual function

$$f_x^L : L_x(C_x) \to \mathbb{R}, \quad f_x^L(v) = -L_x^{-1}(v)v - f_x(L_x^{-1}(v))\,, \tag{2.39}$$

which can be shown, again due to our four conditions, to be a strictly convex and essentially smooth function on the open convex set $L_x(C_x)$. Note that the two minus signs in (2.39) are correct, and due to our sign conventions. In fact, the inverse Legendre map is the Legendre map of the Legendre dual function f^L:

$$-Df_x^L = L_x^{-1}(v) + DL_x^{-1}(v)v + DL_x^{-1}(v)Df_x(L_x^{-1}(v)) = L_x^{-1}(v)\,. \tag{2.40}$$

In other words, the Legendre dual of the Legendre dual $(L_x(C_x), f_x^L)$ of (C_x, f_x) is again (C_x, f_x), see Theorem 26.5 of [108].

Using the above Legendre theory we are able to invert the relation between particle momentum and particle velocity $\dot{x}$ which we obtain by varying with respect to q of the action (2.35) to be $\dot{x} = (\lambda \deg P)L_x(q/m)$. We find that

$$q = mL_x^{-1}(\dot{x}/(\lambda \deg P))\,, \tag{2.41}$$

and it becomes clear now why we have chosen the particular form of the Lagrange multiplier term in the action (2.35): It enables us to use the theory of Legendre transform in the above straightforward way. We can now rewrite the action (2.35) as

$$I[x,\lambda] = -m\deg P\int d\tau\,\lambda f^L(\dot{x}/(\lambda\deg P)) = -m\deg P\int d\tau\left[\lambda f_x^L(\dot{x}) + \lambda\ln(\lambda\deg P)\right]\,, \tag{2.42}$$

where for the second equality we used the easily verified scaling property $f^L(\alpha\dot{x}) = f^L(\dot{x}) - \ln\alpha$. From the variation of the action (2.42) with respect to λ, we then obtain that

$$f^L(\dot{x}) + \ln(\lambda\deg P) + 1 = 0\,. \tag{2.43}$$

Using this twice, we have $\lambda f_x^L(\dot{x}) + \lambda\ln(\lambda\deg P)) = -\lambda = -\exp(-f_x^L(\dot{x}) - 1)/\deg P$. Noting that because of $\dot{x} \in L_x(C_x)$ we also have for $\tilde{q} := L_x^{-1}(\dot{x})$ that $\tilde{q}(\dot{x}) = 1$ and thus $f_x^L(\dot{x}) = -1 - f_x(L^{-1}(\dot{x}))$, and defining the tangent bundle function

$$P_x^* : L_x(C_x) \to \mathbb{R}\,, \qquad P_x^*(v) = P_x(L_x^{-1}(v))^{-1}\,, \tag{2.44}$$

we eliminate λ in (2.42) and finally arrive at the equivalent action

$$I[x] = m \int d\tau P^*_x(\dot{x})^{1/\deg P} \tag{2.45}$$

for a free point particle of positive mass m. While the tangent bundle function P^* is generically non-polynomial, it is elementary to see that it is homogeneous of degree $\deg P$, and for later reference we also show the useful relation

$$L_x^{-1}(v) = \frac{1}{\deg P} \frac{DP^*_x(x,v)}{P^*_x(x,v)} \,. \tag{2.46}$$

The action (2.45) is reparametrization invariant, as it should be. However, parameterizations for which $P(x, L^{-1}(x,\dot{x})) = 1$ along the curve are distinguished since they yield the simple relation

$$\dot{x} = L_x(q/m) \tag{2.47}$$

between the free massive particle velocity $\dot{x}$ and the particle momentum q everywhere along the trajectory x. This normalization is thus the generalization of the notion of proper time to hyperbolic polynomial spacetimes. Furthermore, we call the cone $L_x(C_x)$ the cone of massive particle velocities at the point x and the tangent bundle function P^* the Legendre dual of P. We conclude from the action (2.45) that the tangent bundle function P^* represents the spacetime geometry seen by massive particles. We emphasize that although P^* and $P^{\#}$, which represent the spacetime geometries seen by massive and massless particles respectively, are both certain duals to P they are generically different functions. $P^{\#}$ is polynomial and P^* generically not. In particular, calculating the inverse Legendre map L_x^{-1}, and thus P^* explicitly, is very difficult in concrete examples. Indeed, there are no non-Lorentzian examples in which L_x^{-1} has been calculated yet. In this sense, the dual geometry $(T\mathcal{M}, P^{\#}, P^*)$ is much less straightforward than (T^*M, P).

With the above definition of massive particles, we will now prove that all observers can be considered to be massive particles, i.e. that $C_x^{\#} \subseteq L_x(C_x)$. In the following, we formulate the result in the form of two lemmas since we will need them in a later section as well:

LEMMA 2.6.1. *For any reduced hyperbolic homogeneous cotangent bundle function P we have $L_x(C_x) = interior(C_x^{\perp})$.*

Proof. Since by assumption P_x is reduced, hyperbolic and homogeneous, we get from Lemma 2.3.1 and Lemma 2.3.2 the statement: for all $p \in T^*_x\mathcal{M} \setminus \text{closure}(C_x)$ there exists an $r \in \partial C_x$ such that $p.DP_x(r) < 0$. Since $p.DP_x(q)$ is a continuous function of q, we conclude that for all $p \in T^*_x\mathcal{M} \setminus \text{closure}(C_x)$ there exists a $q \in C_x$ such that $p.DP_x(q) < 0$. That implies that the set $L_x(C_x)^{\perp}$ is a subset of $\text{closure}(C_x) \setminus \{0\}$. Since $L_x(C_x)$ is convex, we get $L_x(C_x) \supseteq (\text{closure}(C_x) \setminus \{0\})^{\perp} = \text{interior}(C_x^{\perp})$. Furthermore, we know that $L_x(C_x) \subseteq C_x^{\perp}$. Since $L_x(C_x)$ is open it follows that $L_x(C_x) = \text{interior}(C_x^{\perp})$.

LEMMA 2.6.2. *For any bi-hyperbolic and energy-distinguishing cotangent bundle function P, we have $C_x^{\#} \subseteq interior(C_x^{\perp})$.*

Proof. At the beginning of this section we argued that there exists a hyperbolicity cone C_x of P_x that lies completely in $(C_x^\#)^\perp$. From $(C_x^\#)^\perp \supseteq C_x$ and the fact that $C_x^\#$ is open, we conclude that $C_x^\# \subseteq \text{interior}(C_x^\perp)$.

This result will lead us in the next section to the definition of observer frames and a candidate for the definition of parallel transport on tensorial spacetimes. Furthermore, by taking the dual on the left and right hand sides of the equation in Lemma 2.6.2, we find that $C_x \subseteq (C_x^\#)^\perp$, which tells us that all massive momenta are of positive energy as we indeed claimed before in Section 2.5, although the definition of energy-distinguishability was only formulated for the momenta of massless particles.

To summarize, we have defined tensorial spacetimes as given by tensor fields G coupling to matter such that the resulting cotangent bundle function P defined in Equation (2.2) induces a homogeneous, hyperbolic, reduced polynomial P_x in every cotangent space $T_x^*\mathcal{M}$ and additionally fulfills the properties of time-orientability and energy-distinguishability. The first three properties were derived from first principles, and the other three were shown to lead to a consistent framework containing observers that can interpret spacetime in geometrical terms coinciding with their notion of positive and negative energy. In particular, we identified the two generically different tangent bundle functions $P^\#$ and P^* describing the spacetime geometry seen by massless and massive particles respectively. Additionally, we constructed the Gauss and the Legendre maps to map massless and massive particles to the corresponding velocities and we showed that these duality maps are mathematically well-defined and invertible. In the next chapter we will use these results to derive further properties of a general tensorial spacetime.

Chapter 3

General properties and examples for tensorial spacetimes

In this section we will investigate general properties of a tensorial spacetime. First, we will define observer frames and investigate their transformation behavior. Second, we will show the relation of tensorial spacetime geometry to Finsler geometry and give a candidate for a parallel transport and finally we will show how superluminal motion emerges in tensorial spacetimes and how vacuum Cherenkov radiation nevertheless imposes a "soft speed limit".

3.1 Observer frames and observer transformations

We found in the last section that the spacetime geometry seen by massive particles is represented by the tangent bundle function P^*. Furthermore, we found that observer trajectories can be identified with those of infraluminal massive particles in our framework. This leads us directly to the definition of observer frame in tensorial spacetimes: Let e_0 be the tangent of an observers trajectory at the point $x \in \mathcal{M}$ parameterized in proper time, i.e. $P^*_x(e_0) = 1$. Via the inverse Legendre map e_0 can be mapped to the covector $\epsilon^0 = L^{-1}(e_0)$. Now, the spatial slice seen by the observer e_0 is the set of vectors annihilated by ϵ^0. Let us choose a basis e_a of vectors in this subset then we obtain a frame $e_\alpha = (e_0, e_a)$ and the dual frame $\epsilon^\alpha = (\epsilon^0, \epsilon^a)$ defined such that $\epsilon^\alpha(e_\beta) = \delta^\alpha_\beta$. In particular, we have for the co-frame

$$\begin{aligned} P(\epsilon^0, \ldots, \epsilon^0) &= 1 \qquad (3.1) \\ P(\epsilon^0, \ldots, \epsilon^0, \epsilon^\alpha) &= 0\,. \qquad (3.2) \end{aligned}$$

Depending on various possible measurement prescriptions for spatial distances, it would be quite useful to have a prescription for the normalization of the spatial frame elements e_a especially for the interpretation of experimental situations. Unfortunately, there is no distinguished measurement prescription but we will be able to carry on without any.

The co-frame ϵ^α can be used to obtain a space-time split of spacetime quantities. An example would be the dispersion relation itself that can be written as

$$P_x(E\epsilon^0 + \vec{p}_a\epsilon^a) = m^{\deg P} \qquad (3.3)$$

for $m \geq 0$ to obtain a non-covariant dispersion relation of the form $E = E(\vec{p})$ which can be expanded as

$$E(\vec{p}) = \sum_{i=0}^{\infty} c^{a_1 \cdots a_i} p_{a_1} \cdots p_{a_i} \,. \tag{3.4}$$

Usually the latter is the form in which generalized dispersion relations are obtained in quantum gravity [10, 12, 18]. It is obvious from the constructions above that to find the covariant dispersion relation from a non covariant one is rather involved, since one has to specify the observer who sees the space-time split that leads to the non-covariant form. However, beginning with the covariant form of a generalized dispersion relation the non-covariant form can be useful since it relates more directly to measurable quantities.

We may consider the relation between two observer frames at the same point $x \in \mathcal{M}$. For that purpose, we will investigate infinitesimal transformations between covectors on the same mass shell in following. Let us consider $\epsilon'^0 = \epsilon^0 + \delta\epsilon^0 \in C_x$ where $\delta\epsilon^0$ is an infinitesimal variation of ϵ^0. We assume that $\delta\epsilon^0$ is given such that ϵ^0 and ϵ'^0 satisfy the mass shell condition $P_x(\epsilon_0') = m^{\deg P}$. Note that this condition can also be written using the hyperbolic barrier function f_x defined in Equation (2.36) as

$$f_x(\epsilon^0) = f_x(\epsilon^0 + \delta\epsilon^0) \,. \tag{3.5}$$

Expanding the right hand side to first order at ϵ_0, we obtain the condition

$$L_x^\alpha(\epsilon_0)\delta\epsilon^0_\alpha = 0 \,, \tag{3.6}$$

where $L_x(\epsilon_0)$ is the action of the Legendre map L_x on ϵ_0. This equation is holds if

$$\delta\epsilon^0_\alpha = \omega_{\alpha\beta} L_x^\beta(\epsilon^0) \,, \tag{3.7}$$

where $\omega_{\alpha\beta}$ is antisymmetric and contains $n(n-1)/2$ parameters specifying the infinitesimal transformations which are generically non-linear due to the generic dependence of the Legrende map on ϵ^0. If we now set $m = 1$ to achieve the proper time normalization of the observer frames, we find that

$$\delta e_0^\alpha = \omega_{\beta\gamma} P_x^{0\cdots 0\alpha\beta}\, e_0^\gamma \,, \tag{3.8}$$

with $P_x^{0\cdots 0\alpha\beta} := P_x(\epsilon^0, \cdots, \epsilon^0, \epsilon^\alpha, \epsilon^\beta)$, must hold.
Additionally, we would like to obtain the transformation behavior of the spatial frame components e_a. Let us consider the infinitesimal variation $e_a' = e_a + \delta e_a$. For the spatial components we have by definition:

$$(\epsilon^0 + \delta\epsilon^0)(e_a + \delta e_a) = 0 \,. \tag{3.9}$$

By using (3.7) we obtain that this condition is fulfilled if they transform as

$$\delta e_\alpha^\mu = \omega_{\nu\rho} P_x^{0\cdots 0\mu\nu}\, e_\alpha^\rho \,. \tag{3.10}$$

Equations (3.10) and (3.8) can now be expressed in the single equation

$$e'^\mu_\sigma = \left[\delta^\mu_\kappa + \omega_{\gamma\nu} P_x^{0\cdots 0\mu[\nu}\delta^{\gamma]}_\kappa\right] e^\kappa_\sigma \,. \tag{3.11}$$

These equations show the infinitesimal transformations connecting neighboring observer frames. Due to the definition $P_x^{0\cdots0\alpha\beta} = P_x(\epsilon^0, \cdots, \epsilon^0, \epsilon^\alpha, \epsilon^\beta)$ they are generically non-linear. Only in the case of a $\deg P = 2$ polynomial P_x corresponding to the Lorentzian metric case, they become linear. Then, the quantities $P_x^{\mu[\nu}\delta_\kappa^{\gamma]}$ in Equation (3.11) correspond to the generators of the Lorentz group.

We need to emphasize, that the above transformations are not uniquely specified by (3.6) and (3.9). That could be fixed by imposing a normalization condition on the spatial components of the frame as it is done in the definition of the Lorentz transformations using the metric. However, as explained above, there is no distinguished normalization condition at hand.

3.2 Lorentzian Finsler geometry and parallel transport

In a next step we can now extend the definition of observer frames to whole worldlines by considering a parallel transport which will give us a notion of freely falling non-rotating observer frame. For instance, this is needed if we want to determine the electric and magnetic field strengths seen by such an observer for a given electromagnetic field strength two-form F. Since the trajectories of observers are those of massive particles, their motion is governed by the action

$$I[x] = m \int d\tau P_x^*(\dot{x})^{1/\deg P} . \tag{3.12}$$

Using the reparametrization invariance to set $P_x^*(\dot{x}) = 1$ along the curve, it is straightforward to derive the equations of motion and, using in the following standard techniques of Finsler geometry [110], to bring them to the form

$$\ddot{x}^\alpha + \Gamma^\alpha(x, \dot{x}) = 0 \tag{3.13}$$

with the geodesic spray coefficients

$$\Gamma^\alpha(x, v) = \frac{1}{2} g_{(x,v)}^{\alpha\mu} \left(\frac{\partial g_{(x,v)\,\mu\gamma}}{\partial x^\beta} + \frac{\partial g_{(x,v)\,\beta\mu}}{\partial x^\gamma} - \frac{\partial g_{(x,v)\,\beta\gamma}}{\partial x^\mu} \right) v^\beta v^\gamma . \tag{3.14}$$

These in turn are constructed from the tangent space metrics g_{e_0} defined by

$$g_{(x,e_0)}(u, v) = \frac{1}{2} \left. \frac{\partial^2 P_x^*(e_0 + su + tv)^{2/\deg P}}{\partial s \partial t} \right|_{s=t=0} , \tag{3.15}$$

whose inverses appearing in the expression (3.14) are guaranteed to exist from the completeness of the cotangent bundle function P. Indeed, for $e_0 = L(\epsilon^0)$, an explicit expression for the metric (3.15) in terms of f^L is given by

$$g_{(x,e_0)\,\alpha\beta} = P_x^{*2/degP}(e_0) \left(-(DDf_x^L(e_0))_{\alpha\beta} + 2L_x^{-1}{}_\alpha(e_0) L_x^{-1}{}_\beta(e_0) \right) , \tag{3.16}$$

and for its inverse in terms of f_x by

$$g_{(x,\epsilon^0)}^{\alpha\beta} = P_x{}^{2/degP}(\epsilon^0) \left(-(DDf_x(\epsilon^0))^{\alpha\beta} + 2L_x{}^\alpha(\epsilon^0) L_x{}^\beta(\epsilon^0) \right) , \tag{3.17}$$

where $(DDf_x(\epsilon^0))^{\alpha\beta}(DDf_x^L(L(\epsilon^0)))_{\beta\gamma} = \delta^\alpha_\gamma$. Remarkably, the Finsler metric (3.16) is automatically Lorentzian. We show this as follows: Consider the cotangent frame ϵ^α defined above, then from expression (3.17) it follows that

$$g^{\alpha\beta}_{(x,\epsilon^0)}\epsilon^0_\alpha\epsilon^0_\beta = {P_x}^{2/degP}(\epsilon^0) > 0, \tag{3.18}$$

$$g^{\alpha\beta}_{(x,\epsilon^0)}\epsilon^0_\alpha\epsilon^a_\beta = 0. \tag{3.19}$$

But since any covector $\vec{p}$ on the spatial hyperplane defined by $L_x(\epsilon^0)$ can be written as $\vec{p} = p_a\epsilon^a$, we have

$$g^{\alpha\beta}_{(x,\epsilon^0)}p_a\epsilon^a_\alpha p_b\epsilon^b_\beta = -P_x^{2/degP}(\epsilon^0)(DDf_x(\epsilon^0))^{\alpha\beta}p_a\epsilon^a_\alpha p_b\epsilon^b_\beta < 0, \tag{3.20}$$

where the last inequality follows from the positive definiteness of the Hessian of f_x (see Theorems 4.2 and remark 4.3 of [109]). Thus we conclude that the metric (3.17) and hence its inverse (3.16) are Lorentzian.

The form of Equation (3.13) suggests the identification of a parallel transport on the manifold $\mathcal{M}$ that gives rise to an identification of geodesics minimalizing the action functional and autoparallels, i.e. curves γ for which the tangent vector field $\dot\gamma$ is parallely transported along the curve.

For that purpose we define the derivative operators

$$\delta_\nu = \frac{\partial}{\partial x^\nu} - \Gamma^\mu{}_\nu(x,v)\frac{\partial}{\partial v^\mu}\,, \qquad \text{where} \quad \Gamma^\nu{}_\mu(x,v) := \frac{\partial\Gamma^\nu(x,v)}{\partial v^\mu}\,. \tag{3.21}$$

Then, we define the Chern-Rund connection coefficients as follows:

$$\Gamma^\nu{}_{\mu\kappa}(u,v) = \frac{1}{2}g^{\nu\sigma}_{(x,v)}\left(\delta_\mu g_{(x,v)\,\sigma\kappa} + \delta_\kappa g_{(x,v)\,\mu\sigma} - \delta_\sigma g_{(x,v)\,\nu\kappa}\right)\,, \tag{3.22}$$

which shows full formal analogy to the Levi-Civita connection in metric geometry. Under a change of coordinates $x = x(\tilde{x})$, the coefficients in (3.22) transform precisely as the coefficients of a linear connection would due to the use of the δ_ν operators. It is then straightforward to see that for any vector $w \in L_x(C_x)$ and vector field u on $\mathcal{M}$, we can define a new vector field with components

$$(\nabla_w u)^\nu = w^\alpha\partial_\alpha u^\nu + \Gamma(x,w)^\nu{}_{\mu\kappa}w^\mu u^\kappa\,. \tag{3.23}$$

We find that $\nabla_w(u+v) = \nabla_w u + \nabla_w v$ and $\nabla_w(fu) = (wf)u + f\nabla_w u$ for any function f and vector fields u, v which means that ∇_w acts as a derivation on vector fields. Thus, by imposing the Leibniz rule

$$\nabla_w(S\otimes T) = (\nabla_w S)\otimes T + S\otimes(\nabla_w T) \tag{3.24}$$

we can extend ∇_w to act on arbitrary tensor fields S, T on M. The derivation ∇_w is not linear in its directional argument w and leads to what is often called a non-linear connection in the literature. Nevertheless, the non-linear covariant derivative ∇ achieves the reformulation of the geodesic Equation (3.13) as the autoparallel equation

$$\nabla_{\dot\gamma}\dot\gamma = 0\,, \tag{3.25}$$

for the curve γ which is the necessary condition for a parallel transport to be associated with observers.

The non-linear connection ∇ provides sufficient structure for the discussion of freely falling non-rotating frames. The key technical observation is that for a frame field $e_0, \dots, e_{d-1}$ that is parallely transported along the first frame vector e_0,

$$\nabla_{e_0} e_\alpha = 0\,, \tag{3.26}$$

we have the conservation equation

$$e_0^\mu \partial_\mu \left(g_{e_0}(e_\alpha, e_\beta)\right) = 0\,. \tag{3.27}$$

This means in particular that any normalization imposed on spacetime frames using the metric (3.16) is preserved along the worldline of a freely falling observer. In turn, (3.26) establishes a consistent notion of freely falling and non-rotating observer frames. This can then be used to model inertial laboratories.

3.3 “Superluminal” propagation and the vacuum Cherenkov process

We found in Section 2.6 that the inclusion $C_x^\# \subseteq L_x(C_x)$ holds. Since in the generic case $C_x^\#$ is indeed a proper subset of $L_x(C_x)$ and the boundary of the hyperbolicity cone $C_x^\#$ consists of the velocities of massless particles in a generic tensorial spacetime there are massive particles moving faster than some massless particles, namely those in $L_x(C_x) \setminus \text{closure}(C_x^\#)$. We call this phenomenon superluminal propagation in the following. We will prove that although these particle velocities exist in principle, particles moving with this velocities will eventually radiate off massless particles and due to the related loss of energy, will finally move with velocities only in $C_x^\#$. Furthermore, particles with velocities in the observer cone $C_x^\#$, among them all observer, are stable under this process. The described radiation process will be called vacuum Cherenkov process in the following, since it resembles the process of radiation of photons from massive particles moving in matter with a speed faster than the speed of light in that environment.

Let us first specify the process in momentum space: Given a particle with momentum p we consider the process in which this particle is split into two particles of which one is massive with momentum q and the other is massless with momentum $p-q$ (see Figure 3.1). We do not bother here about how this process works on a more fundamental scale regarding any field theoretic description of the particles involved. We only care about the question whether this process is at all kinematically possible or not. As a result we obtain the following proposition:

Proposition 3.3.1. *The Cherenkov process as described above is forbidden if and only if the ingoing momentum p lies in the stability cone $L_x^{-1}(C_x^\#)$.*

Proof. First of all we get from the Third and Fourth Lemma that every observer corresponds to a massive momentum, $C_x^\# \subseteq L_x(C_x) = \text{interior}(C_x^\perp)$, so that $L_x^{-1}(C_x^\#)$ is well defined and

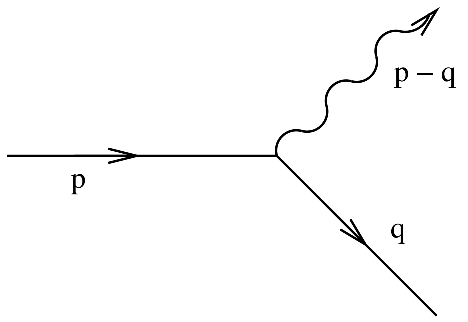

Figure 3.1: Vertex for the vacuum Cherenkov process. A particle of momentum p and mass m radiates off a particle of the same mass m and momentum q and a massless particle of momentum $p-q$.

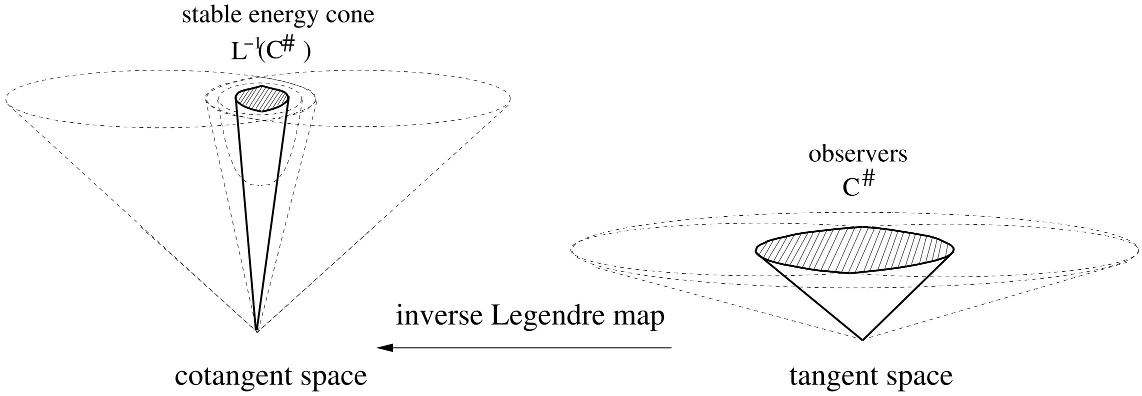

Figure 3.2: Stability cone: if and only if an observer can ride on a particle, the particle cannot lose energy by a vacuum Cerenkov process

always lies within C_x. It is now easy to see that a massive particle of mass m and positive energy momentum p may only radiate off a positive energy massless particle if there exists a positive energy massless momentum $r \in N_x^+$ such that $r(L_x(p)) > 0$. For consider the function

$$u(\lambda) := -\ln P_x\left(\frac{p-\lambda r}{m}\right) . \tag{3.28}$$

Since for any positive λ, the covector $-\lambda r \in -(C_x^\#)^\perp$ lies in some half-space of the cotangent bundle, while $p \in C_x \subset (C_x^\#)^\perp$ lies in the corresponding other half, we conclude that for some $\lambda_0 > 1$ the line $p-\lambda r$ will necessarily intersect the boundary of C_x, so that $\lim_{\lambda\to\lambda_0} u(\lambda_0) = +\infty$. Further, from Theorems 4.2 and remark 4.3 of [109], we know that for a complete hyperbolic P_x the Hessian of the barrier function $-\ln P_x$ is positive definite. Hence, we find that $u''(\lambda) > 0$ everywhere on its domain. Now first assume that the massive particle of momentum p decays into a massive particle of the same mass and of momentum $p-r$ and a massless particle of momentum r, thus respecting energy-momentum conservation. Then we have from the equality of masses for the ingoing and outgoing massive particles that $u(0) = u(1) = 0$. But because $u''(\lambda) > 0$, the only way for the analytic function u to take the same finite values at $\lambda = 0$ and $\lambda = 1$ while tending to $+\infty$ at some $\lambda_0 > 1$ is to have $0 > u'(0) = -r(L_x(p))$. Conversely, assume that $r(L_x(p)) > 0$ for some $r \in \mathcal{N}_x^+$. Then $u'(0) < 0$ and we conclude by the mean value theorem that there must be a (because of $u''(\lambda) > 0$ unique) λ_1 with $0 < \lambda_1 < \lambda_0$ such that $u(\lambda_1) = 0$, i.e., there is an outgoing particle of the same mass such that the process occurs.

Let us illustrate the vacuum Cherenkov process a little more in the $1+1$ dimensional case for a fourth degree hyperbolic geometry. The extension of the results to the physical $3+1$ dimensional case and to cases of higher degree is then just a computational and not a conceptual challenge. Consider a fourth degree hyperbolic, time-orientable and energy-distinguishing polynomial P_x

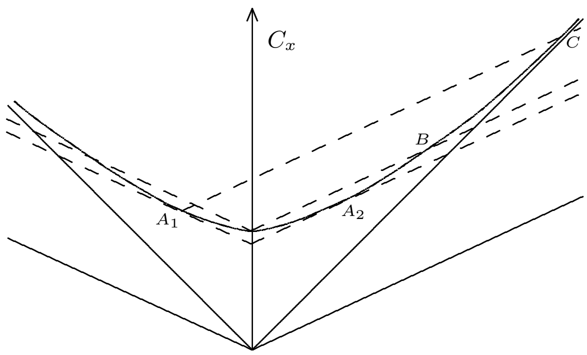

Figure 3.3: Slow Massless momenta (outer), fast massless momenta (inner cone), mass-shell and points defining decay regions with different properties.

such that its vanishing set $\mathcal{N}_x$ is represented by the straight lines in Figure 3.3 and the curve inside the hyperbolicity cone is the convex mass-shell given by $P_x(q) = m^4$. Due to the definition of the Gauss map from the gradient DP_x the outer lines in Figure 3.3 correspond to faster particles and the inner lines to slower particles. When the outer left line of massless momenta is now shifted to the point A_1 we find that it is tangential to the mass-shell at this point which means that there will be no second intersection with the mass-shell and thus A_1 is the left boundary of the cone stable under the Cherenkov process. The right boundary A_2 is analogously constructed using the right outer line of massless momenta. All particles with momenta outside of the cone defined by the lines intersecting the origin and A_1 and A_2, respectively, can undergo the Cherenkov process. When we now take the outer cone of massless momenta and shift it to the minimum of the mass shell, we find that its intersection with the ladder defines again a qualitatively different set of particle momenta, namely those that change their direction when undergoing the Cherenkov process (see point B). This is of course a frame-dependent statement, but is of crucial importance. Consider that we have prepared a source of hypothetical superluminal particles and we can adjust the energy of the particles generated by that source. If we would put a detector just distant enough from the source and we would increase the energy just above the point B we would immediately start to have a considerably lower detection rate of particles from the source than before. When we shift the outer cone further to the point A_1, we obtain the point C which is the right boundary of the set of momenta of particles that would achieve a stable state when undergoing the Cherenkov process. Instead, particles with momenta above that point arrive at a momentum outside the region marked by A_1 and would eventually undergo the Cherenkov process a second time. This way, regions with any higher number of possible Cherenkov processes and reflections would appear when conducting this construction further. For the cases discussed, the result of the source/detection experiment is illustrated in Figure 3.4. To know how far we

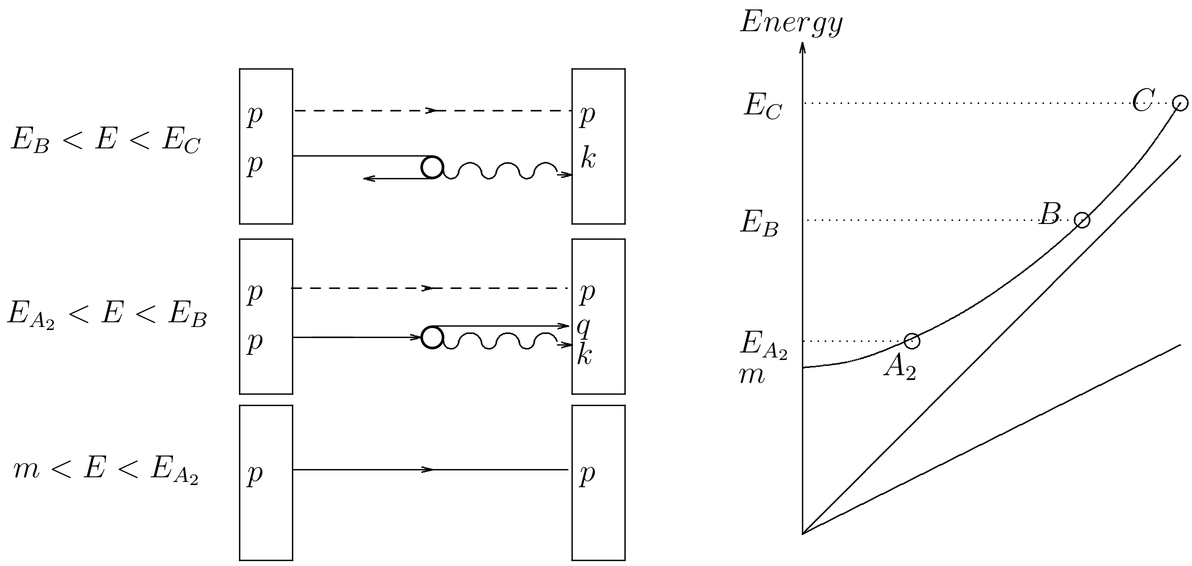

Figure 3.4: Detection of superluminal and infraluminal particles according to their energy.

have to put the detector from the source (depending on the difference between the outer and the inner cone of massless momenta it could even exceed the radius of the visible part of the universe) we would need to calculate decay rates of the Cherenkov process. For that we need to develop the quantum theory for particles satisfying hyperbolic, time-orientable and energy-distinguishing dispersion relations with $\deg P > 2$. The development of Chapters 5 to 7 are in no small part motivated by this question among orthers.

3.4 Example: Lorentzian geometry from Maxwell electrodynamics

In this and the following section we consider two concrete examples of tensorial spacetimes that can be derived from matter field equation just in the way we introduced tensorial spacetimes in Section 2. The first example, treated in this section, will be standard Maxwell electrodynamics and the resulting geometry will be Lorentzian geometry. The second example, treated in the following section, will be area metric electrodynamics and the resulting geometry will be described by polynomials of fourth degree.

Maxwell electrodynamics is usually defined by stipulating the following action for the one-form field A:

$$S[A,g] = -\frac{1}{4}\int d^4x\,\sqrt{|\det(g)|}\,g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}, \tag{3.29}$$

where $F = dA$ is the field strength, d is the exterior derivative and $g^{\mu\nu}$ are the components of the inverse of the metric g^{-1} in some coordinate-induced frame. We assume the manifold $\mathcal{M}$ to be orientable and to be equipped with a canonical volume form $(\omega_g)_{\mu\nu\rho\sigma} = \sqrt{|\deg(g)|}\epsilon_{\mu\nu\rho\sigma}$, where ϵ is the totally antisymmetric Levi-Cevita symbol with $\epsilon_{0123} = 1$. Variation of the action (3.29) with respect to the field A gives rise to two sets of Euler-Lagrange equations,

$$dF = 0 \quad \text{and} \quad dH = 0\,, \tag{3.30}$$

where the field induction H is the tensor density defined as

$$H_{\mu\nu} = -\frac{1}{2}\sqrt{|\det(g)|}\epsilon_{\mu\nu\rho\sigma}g^{\rho\alpha}g^{\sigma\beta}F_{\alpha\beta}\,. \tag{3.31}$$

Let us assume that we are given a suitable initial data hypersurface Σ_0 such that there are coordinates $x^\alpha = (t, x^a)$ and Σ lies at $t = 0$. Then, we define the electric and magnetic fields as

$$E_a = F(\partial_t, \partial_a) \quad \text{and} \quad B^a = \left(\sqrt{|\det(g)|}\right)^{-1}\epsilon^{0abc}F(\partial_b, \partial_c)\,, \tag{3.32}$$

respectively. These three-component fields cover the full information contained in the field strength. The set of equations (3.30) however, consists of eight equations. This and the fact that two of the eight equations do not contain any time derivative tells us that there are only six evolution equations. They are given by the spatial components of the Euler-Lagrange equations in (3.30) as the first order system

$$\left(A^{\alpha M}{}_N\,\partial_\alpha + C^M{}_N\right)u^N = 0\,, \tag{3.33}$$

where the E and B are contained in the vector $u^N = (E_a, B^a)$ and the matrices A^α are given as

$$A^{0M}{}_N = \begin{bmatrix} g^{00}g^{mn} - g^{0m}g^{0n} & 0 \\ 0 & \delta^m_n \end{bmatrix}, \tag{3.34}$$

and

$$A^{aM}{}_N = \begin{bmatrix} -2(g^{0(m}g^{n)a} - g^{0a}g^{mn}) & -\frac{1}{2}\sqrt{|\det g|}\epsilon_{0ncd}(g^{cm}g^{da} - g^{ca}g^{dm}) \\ (\sqrt{|\det g|})^{-1}\epsilon^{0mna} & 0 \end{bmatrix}, \tag{3.35}$$

where the indices $M, N = 1, \ldots, 6$ are identified with the indices $m, n = 1, \ldots, 3$ respectively as $M = m + 3(I - 1)$ and $N = n + 3(J - 1)$ where I is the number of the row and J the number of the column of the matrix on the right hand side of Equation (3.34) and (3.35).

The matrices C will not be relevant for us since we are only concerned with the principal symbol, which is defined from the highest derivative part of (3.33) (see [101] for the exact expressions). With the definition (2.2), we obtain from (3.33) the reduced principal symbol $\tilde{P}_x(q) = q_0 g_x^{-1}(q, q)$. One can show [99] that $q_0 = 0$ is inconsistent with the constraint equations. What remains is then $P_x(q) = g_x^{-1}(q, q)$. The required hyperbolicity of the principal symbol translates then directly into the condition that g_x must be of Lorentzian signature as we prove in the following: Let us assume that there exists a covector h such that P_x is hyperbolic with respect to h, i.e. for all q the equation $P_x(q + \lambda h) = \lambda^2 g_x^{-1}(h, h) + 2\lambda g_x^{-1}(h, q) + g_x^{-1}(q, q) = 0$ has only real roots. But then the discriminant of this equation is positive, i.e.,

$$(g_x^{-1}(h, q))^2 - g_x^{-1}(h, h)g_x^{-1}(q, q) > 0\,.$$

We now take a covector basis $\{\epsilon^0, \epsilon^\alpha\}$ with $\epsilon^0 = h$ and such that $g_x^{-1}(\epsilon^0, \epsilon^\alpha) = 0$. Thus $g_x^{-1}(\epsilon^0, \epsilon^0) > 0$ and the above discriminant condition is written as $q_\alpha q_\beta g_x^{-1}(\epsilon^\alpha, \epsilon^\beta) < 0$ for all q_α, q_β, which already proves that g_x^{-1} must be of Lorentzian signature.
Using then the Routh-Hurwitz condition in theorem 2.2.2 we obtain the hyperbolicity cone to h as

$$C(P_x, h) = \{q \in T_x^*\mathcal{M} \,|\, g_x^{-1}(h, q) > 0 \text{ and } g_x^{-1}(q, q) > 0\} \,. \tag{3.36}$$

The vanishing set is the usual light cone $\mathcal{N}_x = \{q \in T_x^*\mathcal{M} |\, g_x^{\alpha\beta} q_\alpha q_\beta = 0\}$ which in this case coincides with the smooth part of the vanishing set $\mathcal{N}_x^{\text{smooth}}$.
For the metric polynomial $P_x(q) = g_x^{-1}(q, q)$ it is easy to see that the duality condition (2.21) is fulfilled for the polynomial $P_x^\#(v) := g_x(v, v)$ which is again a hyperbolic polynomial since g_x is of Lorentzian signature if and only if g^{-1} is. Picking a hyperbolic vector field e^0 for $P^\#$ corresponds then to the usual definition of a time orientation on the Lorentzian spacetime $(\mathcal{M}, g)$ and we obtain $C_x^\# := \{w \in T_x\mathcal{M} \,|\, g_x(e^0, w) > 0 \text{ and } g_x(w, w) > 0\}$. The massless point particle action is given as

$$S[x, \mu] = \int d\tau \, \mu \, g_x(\dot{x}, \dot{x}) \,, \tag{3.37}$$

which is the well-known action for massless point particles on the Lorentzian spacetime $(\mathcal{M}, g)$.
That P_x is energy-distinguishing also already follows from g_x being of Lorentzian signature. This can be seen as follows: Let v be a generic element of $C_x^\#$. The definition of $C_x^\#$ can be written using the hyperbolic vector v as $C_x^\# = \{w \in T_x\mathcal{M} \,|\, g_x(v, w) > 0 \text{ and } g_x(w, w) > 0\}$. On the one hand, this tells us that $g_x(\cdot, v)$ is a covector in $(C_x^\#)^\perp$. On the other hand, $g_x(\cdot, C_x^\#) = C_x$ and $(C_x^\#)^\perp \cup \{0\}$ closed. But the closure$(C_x) \cup -$closure(C_x) contains the full light cone.

The massive dispersion relation is given by the expression $g^{-1}(q, q)_x = m^2$, and the Legendre map is $L_x(q) := g^{-1}(\cdot, q)/g^{-1}(q, q)$ which has the inverse $L_x^{-1}(v) {=} g(\cdot, v)/g(v, v)$. We obtain the tangent bundle function $P_x^*(v) := g(v, v)$, which gives rise to the action

$$S[x] = \int d\tau \, m \sqrt{g_x(\dot{x}, \dot{x})} \tag{3.38}$$

governing the motion of massive particles. We find then that a frame for the observer defined by the vector e^0 is defined by the conditions

$$g_x(e^0, e^0) = 1 \quad \text{and} \quad g_x(e^0, e^a) = 0 \,. \tag{3.39}$$

Furthermore, in the Lorentzian case a particular normalization of the spatial frame components is distinguished, namely the condition $g_x(e^a, e^b) = -\delta^{ab}$ which leads to $g_x(e^\alpha, e^\beta) = \eta^{\alpha\beta}$ where η is the Minkowski metric.
In particular, we find in the case of Maxwell electrodynamics that $P^\#$ and P^* coincide, although, technically, P^* is of course only defined on the image of the Legrende map $L(C) = C^\#$. But as a polynomial, we can easily consider its extension to all of $T\mathcal{M}$. In the next section, we will consider an example in which $P^\#$ will differ from P^*.

3.5 Example: area metric geometry from pre-metric electrodynamics

Area metric geometry was introduced in [111, 112] and can be derived in our framework from a generalization of Maxwell electrodynamics that can be thought of as a special case of pre-metric electrodynamics [24–26]. Consider a tensor $G_{\mu\nu\rho\sigma}$ having the symmetries

$$G_{\mu\nu\rho\sigma} = -G_{\nu\mu\rho\sigma} = G_{\nu\mu\sigma\rho} \quad \text{and} \quad G_{\mu\nu\rho\sigma} = G_{\rho\sigma\mu\nu}\,. \tag{3.40}$$

We call such an object an area metric. Its inverse is defined via the relation

$$(G^{-1})^{\alpha\beta\mu\nu} G_{\mu\nu\rho\sigma} = 4\,\delta^{[\alpha}_{\rho}\delta^{\beta]}_{\sigma}\,. \tag{3.41}$$

To write the generalization of the Maxwell action (3.29) there is still one element missing namely a volume element since $\sqrt{|\det(g)|}$ is still formulated in terms of the metric. It was shown in [94] that a good generalization for the volume element constructed from the area metric G only is the scalar density of weight one $f(G) = |\det(\text{Petrov}(G))|^{1/6}$ where Petrov$G$ is defined as the 6×6 matrix

$$\text{Petrov}(G) = \begin{bmatrix} G_{0101} & G_{0102} & G_{0103} & G_{0123} & G_{0131} & G_{0112} \\ & G_{0202} & G_{0203} & G_{0223} & G_{0231} & G_{0212} \\ \ddots & & G_{0303} & G_{0323} & G_{0331} & G_{0312} \\ & \ddots & & G_{2323} & G_{2331} & G_{2312} \\ & & \ddots & & G_{3131} & G_{3112} \\ & & & \ddots & & G_{1212} \end{bmatrix}. \tag{3.42}$$

Then, we obtain the action

$$S[A,G] = -\frac{1}{8}\int dx^4 f(G)\,\left[F_{\alpha\beta}F_{\gamma\delta}\,G^{\alpha\beta\gamma\delta}\right], \tag{3.43}$$

where $f(G) = |\det \text{Petrov}(G)|^{1/6}$.

We can now follow the same steps as in the case of Maxwell electrodynamics to investigate area metric electrodynamics from the perspective of tensorial spacetime geometries. As a first step, we have to obtain the field equations from the action (3.43). By varying with respect to the field A and using the definition of the field strength, we obtain the field equations as

$$dF = 0 \quad \text{and} \quad dH = 0\,, \tag{3.44}$$

where as H is called the field induction and is related to the field strength by

$$H_{\alpha\beta} = -\frac{1}{4} f(G)\,\epsilon_{\alpha\beta\mu\nu} G^{\mu\nu\rho\sigma} F_{\rho\sigma}\,. \tag{3.45}$$

By introducing coordinates $x^a = (t, x^\alpha)$ such that $t = 0$ provides an initial data surface Σ we can perform a space-time split of the equation. For that purpose, we first define the electric and the magnetic fields, E_a and B^a respectively[1], as

$$E_a := F(\partial_t, \partial_a) \quad \text{and} \quad B^a := f^{-1}(G)\epsilon^{0abc}F(\partial_b, \partial_c)\,. \tag{3.46}$$

[1]Formulating the Euler-Lagrange equations in terms of these gauge independent variables E_a and B^a is the only way to deal with gauge theories when deriving the principal symbol.

The Euler-Lagrange equations in (3.44) represent a set of eight partial differential equations for the six field components E_a and B^a. However, we find, as in the metric case, that the zero components of both equations do not contain any time derivative and hence, represent constraint equations giving restrictions on the initial data on Σ. The spatial components of (3.44) represent the evolution equations and can be written as the set of first order equations

$$\left(A^{bM}{}_N\,\partial_b + C^M{}_N\right) u^N = 0\,, \tag{3.47}$$

where $u^N := (E_\alpha, B^\alpha)$ and the 6×6 matrices A^b are given as

$$A^{0M}{}_N = \begin{bmatrix} G^{0m0n} & 0 \\ 0 & \delta^m_n \end{bmatrix}, \tag{3.48}$$

and

$$A^{aM}{}_N = \begin{bmatrix} -2\,G^{0(mn)a} & -\frac{1}{2}f(G)\epsilon_{0ncd}\,G^{cdma} \\ f(G)^{-1}\epsilon^{0mna} & 0 \end{bmatrix}. \tag{3.49}$$

The matrices $C^M{}_N$ depend on the area metric tensor G and the volume element $f(G)$ but since they appear only in the lower order terms of (3.47), they are not relevant for the construction of the principal symbol which is to be calculated from the determinant of $A^\alpha q_\alpha$. Before computing the determinant, we decompose the area metric G as

$$\mathrm{Petrov}(G)^{[\alpha\beta][\gamma\delta]} = \begin{bmatrix} M & K \\ K^T & N \end{bmatrix}, \tag{3.50}$$

with antisymmetric index pairs $[01], [02], [03], [23], [31], [12]$. The matrices M, K, N are 3×3 matrices related to the area metric through

$$\begin{aligned} M^{ab} = G^{0a0b}\,, \quad K^a_b &= \frac{1}{2}\epsilon_{0bmn}G^{0amn}\,, \\ N_{ab} &= \frac{1}{4}\epsilon_{0amn}\epsilon_{0bsd}G^{mnsd}, \end{aligned} \tag{3.51}$$

Using now that for any $n\times n$ matrices A, B, C, D

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - BC) \quad \text{if} \quad CD = DC, \tag{3.52}$$

we can write the determinant of $A^\alpha q_\alpha$ as the determinant of a 3×3 matrix as

$$f^2(G)\det(A^\alpha q_\alpha) = \det\left(G^{0m0n}q_0^2 - 2G^{0(mn)a}q_a q_0 + G^{manc}q_a q_c\right) = -q_0^2 P_x(q)\,, \tag{3.53}$$

where $P_{Gx}(q)$ on the right hand side of the equation above is known as the Fresnel polynomial and is cast into the covariant expression

$$P_x(q) = -\frac{1}{24}(\omega_{G_x})_{mnpq}(\omega_{G_x})_{rstu}G_x^{mnr(a}G_x^{b|ps|c}G_x^{d)qtu}q_a q_b q_c q_d\,, \tag{3.54}$$

with $(\omega_{G_x})_{mnpq} = f(G)\epsilon_{mnpq}$.

Remark that the factor $f^2(G)$ in front of the determinant in Equation (3.53) makes P_{Gx} a scalar function as required. One can also show, as in the metric case, that $q_0 = 0$ is inconsistent with the constraint equations, such that we finally find that the massless dispersion relation on an area metric background as derived from the action (3.43) is given by $P_x(q) = 0$. This result has been obtained first by Hehl, Rubilar and Obukhov [113],[114] in the context of pre-metric electrodynamics.

To carry on with the discussions of the properties of area metric geometries from the perspective of general tensorial spacetimes we will now introduce a classification of area metrics. First, an area metric can of course be classified in different equivalence classes under $GL(4)$-transformations, i.e. two area metrics H and G are in the same equivalence class if there exists a $GL(4)$-transformation t relating H and G as

$$G^{\alpha\beta\gamma\delta} = t^{\alpha}{}_{\mu} t^{\beta}{}_{\nu} t^{\gamma}{}_{\rho} t^{\delta}{}_{\sigma} H^{\mu\nu\rho\sigma} \,. \tag{3.55}$$

However, in [94] it was shown that these infinitely many equivalence classes can be ordered in 23 meta-classes. Let us consider the endomorphism J in the space of two-forms at x defined in components as

$$J_{\gamma\delta}{}^{\alpha\beta} := G^{\alpha\beta\mu\nu}(\omega_g)_{\mu\nu\gamma\delta} \,. \tag{3.56}$$

The meta-classes of area metrics can now be labeled by symbols of the type $[A\bar{A}\ldots BCD]$ with A, B, C, D integers called Segré types encoding the size of the Jordan blocks of J and whether the eigenvalues of the corresponding blocks of J are real or complex. Here, an integer A followed by $\bar{A}$ tells us that J contains a Jordan block of size A with a complex eigenvalue of J and another Jordan block of the same size containing the complex conjugate of that eigenvalue. Otherwise, J contains a Jordan block of size B, C, D containing real eigenvalues of J. As an example the meta-class $[1\bar{1}1\bar{1}1\bar{1}]$ contains six different complex eigenvalues with three of them just the complex conjugate of the other three. The resulting classification is provided by theorem 4.3 of [94]. The equivalence classes in every meta-class can be parameterized by real parameters and can be specified by giving one area metric in its Petrov form containing these parameters as entries called normal form in the following. We obtain

- three meta-classes where the Jordan blocks of the corresponding endomorphism J only have complex eigenvalues $\sigma_i \pm i\tau_i$,
- four meta-classes with real Jordan blocks in J of at most size one
- and 16 meta-classes with at least one real Jordan block in J of size greater or equal two.

It was proven in Lemma 5.1 of [94] that the vanishing set of the principal polynomial (3.54) for the meta-classes $VIII$ to $XXIII$ contain at least one plane. From proposition 2.5.1 follows then that the meta-classes $VIII$ to $XXIII$ cannot give rise to time-orientable geometries. Hence, they are excluded from the consideration immediately. We will show only one metaclass explicitly, namely the metaclass with the Segré type $[1\bar{1}\ 1\bar{1}\ 1\bar{1}]$ which we gave the number I since it contains the class of Lorentzian metrics. One element G of the corresponding equivalence class under $GL(4)$-transformations is then given as

$$\mathrm{Petrov}(G) = \begin{bmatrix} -\tau_1 & 0 & 0 & \sigma_1 & 0 & 0 \\ 0 & -\tau_3 & 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & -\tau_2 & 0 & 0 & \sigma_2 \\ \sigma_1 & 0 & 0 & \tau_2 & 0 & 0 \\ 0 & \sigma_3 & 0 & 0 & \tau_3 & 0 \\ 0 & 0 & \sigma_2 & 0 & 0 & \tau_1 \end{bmatrix}, \tag{3.57}$$

The validity of our four conditions for the principal symbol of the remaining 7 meta-classes depends on the values of the parameters in the meta-classes and how they appear in the normal form. Due to the complexity of the problem, there is no result yet telling us for which values of the parameters which of the 7 meta-classes lead to hyperbolic principal polynomials. However, it was possible to find the Gauss dual $P^{\#}$ for the first 7 meta-classes using the inverse area metric as

$$P^{\#}_x(q) = -\frac{1}{24}(\omega^{-1}_{G_x})_{mnpq}(\omega^{-1}_{G_x})_{rstu}(G_x)_{mnr(a}(G_x)_{b|ps|c}(G_x)_{d)qtu}v^a v^b v^c v^d\,, \tag{3.58}$$

where the inverse of ω_{G_x} is defined analogously to the inverse of the metric. To find the Legendre P^* is much harder and there does not exist any examples for $\deg P \geq 4$ yet. This is of course partly due to the non-polynomial nature of P^*_x.

Chapter 4

Non-tensorial classical field theories on tensorial spacetimes

In the previous sections, we showed how tensorial spacetimes relate to the massless dispersion relation $P_x(q) = 0$ which arises from the corresponding matter field theories in the geometric optical limit. The massive dispersion relation

$$P_x(q) = m^r \tag{4.1}$$

with $r = \deg P$ and the action (2.35) governing the motion of free massive particles were then postulated. It was shown that these postulates led to a consistent identification of the trajectories of observers with those of massive particles. Moreover, the specific form of the massive dispersion relation allowed for the definition of observer frames and parallel transport. This was interpreted as the interpretability of tensorial spacetimes in purely geometrical terms. In this section, we will consider field theories with the dispersion relation (4.1) on a flat tensorial spacetime: We assume that there exists a coordinate system in which the components of the polarization tensor $P_x^{\mu_1\ldots\mu_{\deg P}}$ are constant all over the n dimensional manifold $\mathcal{M}$ and $\mathcal{M} \cong \mathbb{R}^n$. This is to some extent the generalization of Minkowski space to tensorial spacetimes. Then we can identify $T_x\mathcal{M}$ and $\mathcal{M}$ for every point $x \in \mathcal{M}$. In the following, we will always assume to be working in such a coordinate system and we will use P equivalently for P_x and x equivalently for tangent vectors and points in $\mathcal{M}$.

We will consider classical field theories (also non-tensorial ones) leading to massive dispersion relations. In particular, we will show that there exists a well-posed initial data problem for those theories and we will find generalizations of scalar and fermionic fields. The work presented in this chapter was done in collaboration with Frederic P. Schuller and Sergio Rivera.

4.1 Scalar field equations

As the first and easiest case we will investigate a real scalar field $\phi : \mathcal{M} \to \mathbb{R}$. This will also be the case for which we will investigate the quantum field theory in a later section. For the field equation

$$\left[(-1)^{r/2} P^{\mu_1\ldots\mu_r}\partial_{\mu_1}\ldots\partial_{\mu_r} - m^r\right]\phi = 0\,, \tag{4.2}$$

we find that $\phi = e^{-i\,p.x}$ is a solution only if p satisfies the massive dispersion relation (4.1). A possible action for that equation is given by

$$S_{\mathcal{M}}[\phi] = \int_{\mathcal{M}} d^n x\, \mathcal{L}(\phi, \partial_\mu \phi, \ldots, \partial_{\mu_1} \ldots \partial_{\mu_{r/2}} \phi)\,, \tag{4.3}$$

(note that $r = \deg P$ is always even, see proposition 2.5.1) with

$$\mathcal{L}(\phi, \partial_\mu \phi, \ldots, \partial_{\mu_1} \ldots \partial_{\mu_{r/2}} \phi) = P^{\mu_1 \ldots \mu_r} \frac{1}{2} (\partial_{\mu_1} \cdots \partial_{\mu_{r/2}} \phi)(\partial_{\mu_{r/2+1}} \cdots \partial_{\mu_r} \phi) - \frac{1}{2} m^r \phi^2\,. \tag{4.4}$$

This can be generalized to a complex field $\Phi : \mathcal{M} \to \mathbb{C}$ as

$$\mathcal{L}(\phi, \partial_\mu \Phi, \ldots, \partial_{\mu_1} \ldots \partial_{\mu_{r/2}} \Phi) = P^{\mu_1 \ldots \mu_r} \frac{1}{2} (\partial_{\mu_1} \cdots \partial_{\mu_{r/2}} \Phi^*)(\partial_{\mu_{r/2+1}} \cdots \partial_{\mu_r} \Phi) - \frac{1}{2} m^r \Phi^* \Phi\,. \tag{4.5}$$

such that both, Φ and Φ^*, satisfy the scalar field Equation (4.2). The action (4.3) is invariant under a local $U(1)$ transformation which gives rise to the current

$$j^\mu(\Phi) = i \sum_{l=0}^{r/2-1} (-1)^l P^{\mu\nu_1 \cdots \nu_{r-1}} \quad [(\partial_{\nu_1} \cdots \partial_{\nu_l} \Phi^*) \left(\partial_{\nu_{l+1}} \cdots \partial_{\nu_{r-1}} \Phi\right) \tag{4.6}$$

$$- (\partial_{\nu_1} \cdots \partial_{\nu_l} \Phi) \left(\partial_{\nu_{l+1}} \cdots \partial_{\nu_{r-1}} \Phi^*\right)]\,. \tag{4.7}$$

This current satisfies the continuity equation $\partial_\mu j^\mu = 0$ if Φ is a solution to the field equations[1]. By promoting the global $U(1)$ symmetry to a local one we obtain the minimal coupling Lagrangian

$$\begin{aligned} \mathcal{L}(\Phi, \partial_\mu \phi, \ldots, \partial_{\mu_1} \ldots \partial_{\mu_{r/2}} \Phi) \quad = \quad & P^{\mu_1 \ldots \mu_r} \frac{1}{2} (D_{\mu_1} \cdots D_{\mu_{r/2}} \Phi^*)(D_{\mu_{r/2+1}} \cdots D_{\mu_r} \Phi) \\ & - \frac{1}{2} m^r \Phi^* \Phi\,. \end{aligned} \tag{4.8}$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative associated with the Abelian gauge field A_μ. Then, e is the charge associated with the field Φ. As we saw in Section 3.4, area metric electrodynamics gives rise to a fourth order dispersion relation $P(q) = m^4$ for massive point particles. By considering the corresponding massive scalar field theory, we would obtain a theory of area metric electrodynamics with charges described by the equation 4.2. For the sake of completeness, we will present the current for the coupling to the electromagnetic field also for the field equations of lower order than r, which we will consider in the following.

4.2 Field equations of first order

For a systematic study of field theories of derivative order lower than r giving rise to the massive dispersion relation (4.1), we start with the lowest possible order equation

$$(-i\Gamma^\mu \partial_\mu + m\,\mathbb{I})\,\Phi = 0\,. \tag{4.9}$$

For plane wave solutions of the form $\Phi_p = v(p) e^{ip.x}$ with $v(p) \in V$ we find as a solvability condition that

$$\det\left(\Gamma^\mu p_\mu + m\mathbb{I}\right) = 0\,, \tag{4.10}$$

[1] The real version of the current in (4.6) is exactly what we will find for the current in (5.21) in Section 5.

from which we can determine conditions on the Γ^μ precisely by requiring that (4.10) coincides with the massive dispersion relation (4.1) to some integer power α. We can write this as the condition that

$$\det\left(\Gamma^\mu p_\mu + m\mathbb{I}\right) = c\left(P(p) - m^s\right)^\alpha \tag{4.11}$$

for some hyperbolic, time-orientable and energy-distinguishing polynomial P, where $s = \deg P$ and c is a nonzero complex number. That this equation should also hold for $p = 0$ justifies subsequently the assumption that the coefficient matrix following m must be invertible which we used to set it to be the identity without loss of generality in Equation (4.9).

Theorem 4.2.1. *For (4.9) to have (4.1) as its dispersion relation, the following conditions on the coefficient matrices Γ^a must hold*

- *(i)* $\{\Gamma^{\mu_1}, \dots, \Gamma^{\mu_r}\} = r! P^{\mu_1,\dots,\mu_r}\mathbb{I}$ *where* $\{A_1, \dots, A_l\} = \sum_{\pi \in S_l} A_{\pi(1)} \dots A_{\pi(l)}$ *with* S_l *the set of all permutations of* l *elements and*
- *(ii)* $Tr(\{\Gamma^{\mu_1}, \dots, \Gamma^{\mu_l}\}) = 0$ *for all* $l = 1, \dots, r-1$.

Thus for a covariant dispersion relation of degree $r = \deg P$, the coefficient matrices Γ^μ satisfy a $(\deg P)$-ary algebra expressed in condition (i) which is a generalization of the binary Dirac algebra, and some supplementary conditions (ii).

Proof. First, we are going to derive conditions for the more general form of the Equation (4.9) as

$$\left((-i)^s \Gamma^{\mu_i \cdots \mu_s} \partial_{\mu_1} \cdots \partial_{\mu_s} + m^s\,\mathbb{I}\right)\Phi = 0\,. \tag{4.12}$$

As in the special case above, for (4.12) to lead to the dispersion relation (4.1), the condition

$$\det\left(\Gamma^{\mu_i \cdots \mu_s} p_{\mu_1} \cdots p_{\mu_s} + m^s \mathbb{I}\right) = c\left(P(p) - m^r\right)^\alpha \tag{4.13}$$

must hold for some complex constant c and some integer α. By setting $p = 0$ we find that $c = (-1)^\alpha$ and that $\alpha = sd/r$ with d the dimension of the vector space V in which the coefficient matrices $\Gamma^{\mu_i \cdots \mu_s}$ are represented and $r = \deg P$. With $\Gamma(p) = \Gamma^{\mu_i \cdots \mu_s} p_{\mu_1} \cdots p_{\mu_s}$, we arrive at the equation

$$\det\left(\mathbb{I} + m^{-s}\Gamma(p)\right) = \left(1 - m^{-r}P(p)\right)^{\frac{sd}{r}}\,. \tag{4.14}$$

Using the identity $\det(A) = \exp(\mathrm{Tr}(\log(A)))$ we can compare the left and right hand side of the equation using the Taylor series expansions of $\log(1 + x)$ at $x = 0$ and obtain

$$\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} m^{-sj}\,\mathrm{Tr}\left((\Gamma(p))^j\right) = -\frac{sd}{r}\sum_{j=1}^{\infty}\frac{1}{j} m^{-rj} P^j\,, \tag{4.15}$$

which can be only fulfilled if $r/s \in \mathbb{N}$. In that case, we obtain that for $\rho \in \mathbb{N}$ must hold

$$(-1)^{\rho+1}\mathrm{Tr}\left((\Gamma(p))^\rho\right) = \begin{cases} -dP^{\rho s/r}(p) & \text{for}\ \rho s/r \in \mathbb{N} \\ 0 & \text{else} \end{cases} \tag{4.16}$$

Defining now $N := \rho s/r$ we find that

$$\mathrm{Tr}\left(\left(\frac{(-\Gamma(p))^{r/s}}{P(p)}\right)^N\right) = D \tag{4.17}$$

for all $N \in \mathbb{N}$. This is only possible if $(-\Gamma(p))^{r/s} = P(p)\mathbb{I}$. Since this must hold for all p we find that

- (i) $\Gamma^{(\mu_1,\ldots,\mu_s} \ldots \Gamma^{\mu_{r/s-s+1}\cdots\mu_{r/s})} = (-1)^{r/s} P^{\mu_1,\ldots,\mu_r}\mathbb{I}$ and
- (ii) $\mathrm{Tr}(\Gamma^{(\mu_1,\ldots,\mu_s} \ldots \Gamma^{\mu_{(l-1)s+1}\cdots\mu_{ls})}) = 0$ for all $l = 1,\ldots,r/s-1$

which reduces to the expressions in the theorem for the case of $s = 1$. □

Example: First order field equation on Lorentzian spacetime

Let us consider the metric dispersion relation $\eta^{ab}q_a q_b - m^2 = 0$ where $\eta = \mathrm{diag}(1,-1,-1,-1)$ and $n = 4$ which is $3+1$-dimensional Minkowski space. Since $r = 2$ in this case, theorem 4.2.1 tells us that we have to find matrices $\gamma(q) = \gamma^\mu q_\mu$ satisfying

$$\{\gamma^\mu, \gamma^\mu\} = 2\eta^{\mu\nu} \quad \text{and} \quad \mathrm{Tr}(\gamma^\mu) = 0 \quad . \tag{4.18}$$

This is the standard Dirac algebra and a set of matrices γ^μ fulfilling the above conditions is the set of Dirac matrices which are given in the Weyl representation as

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix} , \tag{4.19}$$

where $\sigma^\mu := (\mathbb{I}_{2\times2}, \sigma^m)$ and $\bar{\sigma}^\mu := (\mathbb{I}_{2\times2}, -\sigma^m)$, with σ^m the Pauli matrices satisfying

$$\sigma^{(a}\bar{\sigma}^{b)} = \bar{\sigma}^{(a}\sigma^{b)} = \eta^{ab}\mathbb{I}_{2\times2} \, . \tag{4.20}$$

Example: First order field equation on bimetric spacetimes

The second example we investigate is the case of a fourth order hyperbolic polynomial spacetimes such that the polynomial P factorizes as $P(q) = g^{-1}(q,q)h^{-1}(q,q)$ where g^{-1} and h^{-1} are Lorentzian metrics. We denote such a spacetime as bi-metric in the following. A principal polynomial that factorizes in two hyperbolic polynomials of degree two arises for instance in the case of area metric electrodynamics when the area metric is of meta-class I and the parameters fulfill the conditions $\sigma_1 = \sigma_2 = \sigma_3$ and $\tau_1 = \tau_2$. However, not all bi-metric spacetimes arise from area metric spacetimes.

Let us now employ theorem 4.2.1 to find matrices Γ^μ that give rise to the massive dispersion relation $g^{-1}(q,q)h^{-1}(q,q) = m^4$ via the field equations (4.9). We find that they have to fulfill the quartary algebra

$$\{\Gamma^\mu, \Gamma^\nu, \Gamma^\gamma, \Gamma^\delta\} = 4\Gamma^{(\alpha}\Gamma^\beta\Gamma^\gamma\Gamma^{\delta)} = 4g^{(\alpha\beta}h^{\gamma\delta)}\mathbb{I} \, , \tag{4.21}$$

and the supplementary trace conditions

$$\mathrm{Tr}(\Gamma^\alpha) = \mathrm{Tr}(\{\Gamma^\alpha, \Gamma^\beta\}) = \mathrm{Tr}(\{\Gamma^\alpha, \Gamma^\beta, \Gamma^\gamma\}) = 0\,. \tag{4.22}$$

Since g^{-1} and h^{-1} are inverse Lorentzian metrics, we can find frames e and f such that

$$g^{\gamma\delta} = \eta^{\alpha\beta} e^\gamma_\alpha e^\delta_\beta \quad \text{and} \quad h^{\gamma\delta} = \eta^{\alpha\beta} f^\gamma_\alpha f^\delta_\beta\,. \tag{4.23}$$

In terms of these frames we can give a representation satisfying the above conditions with the 16×16 matrices

$$\Gamma^\alpha = \left[\begin{array}{cccc|cccc}
0 & 0 & 0 & e^\alpha_\beta\sigma^\beta & 0 & 0 & 0 & 0\\
0 & 0 & f^\alpha_\beta\sigma^\beta & 0 & 0 & 0 & 0 & 0\\
f^\alpha_\beta\bar\sigma^\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & e^\alpha_\beta\bar\sigma^\beta & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & e^\alpha_\beta\sigma^\beta & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & f^\alpha_\beta\sigma^\beta\\
0 & 0 & 0 & 0 & 0 & f^\alpha_\beta\bar\sigma^\beta & 0 & 0\\
0 & 0 & 0 & 0 & e^\alpha_\beta\bar\sigma^\beta & 0 & 0 & 0
\end{array}\right], \tag{4.24}$$

where σ^α and $\bar\sigma^\alpha$ are again the Pauli matrices.

Action principle and coupling to electrodynamics

The conditions we identified in theorem (4.2.1) to ensure that the wave equations (4.9) give rise to a massive dispersion relation of the form (4.1) do not ensure that these field equations are the Euler-Lagrange equations obtained from an action. However, that would be desirable, for instance, to learn something about symmetries and coupling terms to other fields or to derive objects like the energy momentum tensor. The additional conditions that have to be satisfied to ensure the existence of an action are given in the following proposition.

PROPOSITION 4.2.2. *The set of equations (4.9) can be derived from the scalar action functional*

$$S[\Phi] = \int d^n x\, \bar\Phi \left[i\,(\Gamma)^\mu \partial_\mu - m^r\right] \Phi\,, \tag{4.25}$$

where $\bar\Phi_N = \Phi^\dagger_M \Gamma^M{}_N$*, if the matrix* Γ *is such that*

$$(\Gamma^\dagger)^{-1} (\Gamma^\mu)^\dagger \Gamma^\dagger = \Gamma^\mu\,. \tag{4.26}$$

Moreover, the action is real if, in addition, the matrix Γ *is Hermitian, i.e.,* $\Gamma^\dagger = \Gamma$.

Proof. Variation of the action above with respect to $\bar\Phi$ trivially reproduces equations (4.9). But we have to make sure that the variation of the action with respect to Φ gives rise to the same field equations. But we obtain

$$(\delta_\Phi \mathcal{L})^\dagger = 0 \qquad \Rightarrow \qquad (i)^r (\Gamma^\mu)^\dagger \Gamma^\dagger \partial_\mu \Phi - m^r \Gamma^\dagger \Phi = 0\,.$$

Multiplying the last expression by $(\Gamma^\dagger)^{-1}$, we indeed obtain that Equation (4.26) must hold. Concerning the reality of the action, using integration by parts and assuming that Equation (4.26) is satisfied, we obtain

$$S^\dagger[\Phi] = \int d^n x\, \Phi^\dagger \Gamma^\dagger \left[i\, (\Gamma)^\mu \partial_\mu - m^r \right] \Phi \,.$$

Thus, $S^\dagger[\Phi, \bar{\Phi}] = S[\Phi, \bar{\Phi}]$ if $\Gamma = \Gamma^\dagger$, which proves the proposition.

From the action in 4.25 we can now derive a current as we did for the scalar field above. The action (4.25) is invariant under a global $U(1)$ transformation and Noether's theorem leads to the expression

$$j^\mu(\Phi) = i \left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) \Phi \right] = i\bar{\Phi}\, \Gamma^\mu \Phi \tag{4.27}$$

which satisfies the continuity equation $\partial_\mu j^\mu(\Phi) = 0$.
The minimal coupling to the electromagnetic field A is obtained by promoting the global $U(1)$ symmetry of the action to a local one. Then, we find the Lagrangian density

$$\mathcal{L}(\Phi) = \bar{\Phi} \left[i\Gamma^\mu D_\mu - m \right] \Phi \,, \tag{4.28}$$

where $D_\mu = \partial_\mu + ieA_\mu$ and e is the charge associated with the field ψ. Given an action S_{EM} for the one form field A and expanding the covariant derivative in the expression above, we find the Lagrangian density

$$\mathcal{L}(\Phi, A) = \bar{\Phi} \left[i\Gamma^\mu \partial_\mu - m \right] \Phi + iej^\mu A_\mu \, + \mathcal{L}_{EM}, \tag{4.29}$$

for the full electrodynamics.

Example: First order field equation on Lorentzian spacetime

Let us again consider the metric dispersion relation $\eta^{ab} q_a q_b - m^2 = 0$ in $3+1$-dimensional Minkowski space with the matrices Γ^μ we identified in Equation (4.19). It is then easily verified that $\Gamma = \gamma^0$ satisfies the conditions in proposition 4.2.2. Hence, we recover the metric Dirac equation as obtained from the real Lagrangian density

$$\mathcal{L} = i\bar{\Phi}\gamma^\mu \partial_\mu \Phi - m\bar{\Phi}\Phi \,. \tag{4.30}$$

Example: First order field equation on bimetric spacetimes and coupling to bimetric electrodynamics

Let us now consider again the case of $P(q) = g^{-1}(q,q) h^{-1}(q,q)$. The reader might have wondered why we considered the 16-dimensional representation of the Γ matrices in (4.24) while already each one of the diagonal blocks satisfies the conditions of theorem 4.2.1 separately. The reason for that is that we need both blocks to find a matrix Γ satisfying the condition of proposition

4.2.2. We find that

$$\Gamma = \left[\begin{array}{cccc|cccc} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ \hline 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0 \end{array}\right] \tag{4.31}$$

satisfies

$$\Gamma^\dagger = \Gamma, \quad \Gamma^{-1} = \Gamma, \quad \text{and} \quad \Gamma\,\Gamma^{a\dagger}\,\Gamma = \Gamma^a\,.$$

Defining $\bar{\Phi} = \Phi^\dagger\Gamma$, the Dirac equation in (4.9) with the matrices Γ^μ above is equivalent to the Euler-Lagrange equations to the action

$$S[\Phi] = \int d^n x\,\bar{\Phi}\,[i\Gamma^\mu\partial_\mu - m]\,\Phi\,. \tag{4.32}$$

Restricting attention to area metrics G giving rise to bimetric dispersion relations $P_G = g^{-1}h^{-1}$ (which have been identified in [115]), we obtain the Lagrangian

$$\mathcal{L}(\Phi, A) = \bar{\Phi}\,[i\Gamma^\mu\partial_\nu - m]\,\Phi + iej^\mu A_\mu - \frac{1}{8}\,G^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,, \tag{4.33}$$

which describes the interaction of the electromagnetic field A and the Dirac field ψ on a bimetric area metric spacetime. Using the current in (4.27) one might proceed with the quantization of this Lagrangian in order to study, for instance, refinements to quantum electrodynamics on hyperbolic, time-orientable and energy-distinguishing spacetimes [116].

There is still an important point that must be discussed, namely the freedom in the choice of frames e and f for the construction of the matrices Γ^μ and the question how far the action (4.32) is invariant under this choice. First note, that we can perform two independent Lorentz transformations

$$e'^{\mu}_{\nu} = \Lambda^\mu{}_\rho(\alpha)e^\rho_\nu \qquad \text{and} \qquad f'^{\mu}_{\nu} = \Lambda^\mu{}_\rho(\beta)f^\rho_\nu \tag{4.34}$$

on the frames e and f maintaining the property that they bring the Lorentzian metrics g^{-1} and h^{-1} to the Minkowski form (where α and β stand for the parameters of each transformation). We thus have the freedom $SO(1,3)\times SO(1,3)$ in choosing the matrices Γ^μ.
More precisely, we can consider transformed matrices Γ'^μ with frames f' and e' related to the old frames by equations (4.34) which still satisfy the bimetric quartic dispersion relation, and they can therefore be equally used in order to write our Dirac equation. At the level of the matrices these transformations can be implemented using the Pauli matrices as

$$\Gamma'^\mu = T(\alpha,\beta)\Gamma^\mu S^\dagger(\alpha,\beta) \tag{4.35}$$

where

$$T(\alpha,\beta)=\left[\begin{array}{cccc|cccc} A_L(\alpha) & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & A_L(\beta) & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & A_R(\beta) & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & A_R(\alpha) & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & A_L(\alpha) & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & A_L(\beta) & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & A_R(\beta) & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_R(\alpha) \end{array}\right] \tag{4.36}$$

and

$$S(\alpha,\beta)=\left[\begin{array}{cccc|cccc} A_R(\beta) & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & A_R(\alpha) & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & A_L(\beta) & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & A_L(\alpha) & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & A_R(\alpha) & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & A_R(\beta) & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & A_L(\alpha) & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_L(\beta) \end{array}\right] \tag{4.37}$$

where

$$\begin{aligned} A_L(\alpha) &= e^{-\frac{1}{4}\alpha_{\mu\nu}\sigma^\mu\overline{\sigma}^\nu} \\ A_R(\alpha) &= e^{-\frac{1}{4}\alpha_{\mu\nu}\overline{\sigma}^\mu\sigma^\nu}\,. \end{aligned} \tag{4.38}$$

where $\alpha_{\mu\nu}$ is the antisymmetric matrix associated to the parameters of the Lorentz transformation denoted with α and analogously for $A_L(\beta)$ and $A_R(\beta)$. Solutions of the equations of motion to the transformed Γ-matrices are given by

$$\Phi' = (S^\dagger)^{-1}\Phi \tag{4.39}$$

when ψ is a solution to the untransformed matrices. We find that the kinematical term in the action is invariant under simultaneous transformation of Γ-matrices and solutions if

$$\Gamma((S^\dagger)^{-1})^\dagger\Gamma T = \mathbb{I}\,, \tag{4.40}$$

which is always fulfilled. In contrast, for the mass term we find the condition

$$\Gamma((S^\dagger)^{-1})^\dagger\Gamma(S^\dagger)^{-1} = \mathbb{I}\,, \tag{4.41}$$

which is only fulfilled if $\alpha=\beta$.

We notice that for any two arbitrary Lorentz transformations $\Lambda(\alpha)$ and $\Lambda(\beta)$, there exist two Lorentz transformations λ and λ' such that

$$\Lambda(\alpha) = \lambda\lambda' \qquad \text{and} \qquad \Lambda(\beta) = \lambda^{-1}\lambda'\,. \tag{4.42}$$

Indeed, this is achieved by taking $\lambda = \sqrt{\Lambda(\alpha)\Lambda^{-1}(\beta)}$ and $\lambda' = \sqrt{\Lambda(\alpha)\Lambda^{-1}(\beta)}\Lambda(\beta)$. The above statement about the invariance of the action tells us that every transformation where λ' is arbitrary and λ is the identity leads to a unitarily equivalent representation of the quartary algebra

(4.21). If λ is not the identity we arrive by the transformation (4.42) at a different action if the mass is non-zero. Hence, via applying the transformation labeled by λ to the matrices Γ^μ we obtain a six parameter family of first order field theories fulfilling the conditions of theorem 4.2.1 and proposition 4.2.2.

Let me summarize: We defined the massive scalar field theory and generalized Dirac equations on tensorial spacetimes. In particular, we found conditions the coefficient matrices of the field equation for a generalized Dirac equation must fulfill to have the massive dispersion relation (2.34) and to derivable via an action principle. We also gave charged currents for all these massive matter field theories that can be used to couple them to electrodynamics.
To establish quantum electrodynamics on tensorial spacetimes and to deal with the massive field theories further, we would now like to have a quantization procedure these fields. That would be of particular interest for the calculation of decay rates in the vacuum Cherenkov process. Furthermore, having a quantum theory of matter fields at hand, other results could be derived that would help to compare them with experimental results of earth bound particle physics or observations in astroparticle physics.
To investigate the possibility for such a quantization scheme is the aim of the next chapters.

Chapter 5

Background independent QFT: introduction to the general boundary formulation

In this chapter we will introduce the general boundary formulation (GBF). In the first section, we will start from standard quantum mechanics to motivate the setup we will use; Hilbert spaces of maps over phase spaces. This extends then naturally to quantum field theory. Finally, we will give the axiomatic basis of the GBF.

In the second section, we will present the probability interpretation of the GBF and the generalized Born rule. In the third section, we will present the quantization of observables in the GBF.

In the fourth section, we will start from an action for a scalar field on a general background and derive its kinematics using notions of symplectic geometry. In Section 5.5 to 5.10, we will review the holomorphic quantization scheme of the GBF introduced in [59] which we will use in Chapter 7 to quantize a scalar field in non-metric tensorial spacetime. In Section 5.5, we will introduce Kähler polarizations and positive complex structures and explain that there is a one-to-one correspondence between them. In the framework of the holomorphic representation, we will introduce coherent states and a notion of vacuum state. We will explain what unitary evolution means in that context and we will show how Weyl observables are quantized and ladder operators give rise to the one particle sector of the Hilbert space.

5.1 General Boundary Formulation: Axiomatic framework

The general boundary formulation (GBF) was developed mainly to cover the problem of time coming up when one tries to quantize gravity [73]. It can be stated the following way: The standard formulation of quantum theory relies on a notion of time evolution. The space-time split is defined by the metric as we explained in Section 1. In General Relativity however, gravity is encoded precisely in the metric. Hence, if one wants to start quantizing the metric to quantize gravity without having a background metric in advance, i.e. non-perturbatively, there is no space-time split defined that could be used for the quantum theory. Additionally, the GBF may provide a way to solve the problem of locality in quantum gravity [68] which can be stated as follows: On Lorentzian spacetimes, quantum field theories are required to be microcausal which means that spacelike separated measurements can be performed independently. That enables us to consider measurements locally, i.e. without considering spacelike separated parts of the Lorentzian spacetime. Again, this definition of locality hinges on the existence of a Lorentzian metric defining the causal structure of the Lorentzian spacetime.

Now, the GBF is a true generalization of quantum theory that neither relies on a metric nor on a spacetime split. It is an axiomatic framework that resembles that of topological quantum field theory [74] from which it was originally inspired. It is based on a generalization of the notion of transition amplitudes from which conditional probabilities can be derived using a generalized Born rule. This retreat to conditional probabilities is argued to solve to some extend the measurement problem of quantum gravity [73].

Let me mention shortly that although generically no Lorentzian metric is needed in the GBF, Lorentzian spacetimes represent a good testing ground for the GBF framework. Results of the GBF in Lorentzian spacetimes are in particular that the crossing symmetry of the S-matrix of perturbative quantum field theory is a general property of scalar quantum field theories in the GBF [71] and that in the GBF, one can provide meaningful and rigorous definitions for in- and out-states and the S-matrix also in Anti-deSitter space [65]. Using the same techniques, it should also be possible to construct a quantum field theory for a stationary black hole spacetime. Besides the mentioned results dealing with scalar fields, the GBF was applied to fermionic field theories in [62] and to two-dimensional Yang-Mills theories in [58] where also regions with corners were considered.

In the following, we will motivate the mathematical structure of the GBF by shortly reviewing the standard formulation of quantum mechanics and quantum field theory. Then we will introduce the axioms of the GBF.

In the standard formulation of quantum theory, the states of a physical system are represented as maps ψ from a suitable restricted subspace of the phase space of the classical system to the complex numbers forming a complex Hilbert space $\mathcal{H}$. For instance, in the case of a non-relativistic point particle in flat space the subspace of the phase space used for the position representation of states is the set of all points in space which is modeled by $\mathbb{R}^3$. A physical process is then modeled as the unitary evolution starting at a state $\psi_1 \in \mathcal{H}$ associated with an instant of time t_1 to a state $\psi_2 \in \mathcal{H}$ associated with an instant of time t_2, if no measurement occurs and a projection otherwise. Eventually, the theory leads to the prediction of transition amplitudes $\rho(\psi_1, \overline{\psi_2}) = \langle \psi_2, U(t_2, t_1)\psi_1 \rangle$ via an evolution operator U and the inner product on the Hilbert space $\mathcal{H}$. From that we obtain the probability density for the above evolution[1] as the modulus squared.

Now in Quantum Field Theory (QFT) the systems under consideration are fields and the config-

[1] Here, the bar denotes the complex conjugate.

uration space corresponds to initial data given on an entire oriented[2] Cauchy hypersurface[3] Σ. Again the theory leads to the prediction of amplitudes $\rho(\psi_1, \overline{\psi_2})$ that give rise to the probability for an evolution between states ψ_1 and ψ_2. Like the configuration spaces at times t_1 and t_2 , also the states ψ_1 and ψ_2 correspond to Cauchy hypersurfaces at t_1 and t_2, respectively. For a global hyperbolic spacetime there is a foliation of the spacetime by Cauchy hypersurfaces. Let the foliation parameter be t. Then, we find for every pair of Cauchy hypersurfaces Σ_1 and Σ_2 of that foliation a region M such that the boundary ∂M is the union of Σ_1 with $\overline{\Sigma}_2$ where $\overline{\Sigma}_2$ is the same hypersurface as Σ_2 but with the opposite orientation. Furthermore, we can construct the "boundary Hilbert space" $\mathcal{H}_{\partial M} = \mathcal{H}_1 \otimes \mathcal{H}_2^*$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are two copies of the Hilbert space $\mathcal{H}$ associated with the hypersurfaces Σ_1 and Σ_2 respectively and $\mathcal{H}_2^*$ is the dual Hilbert space to $\mathcal{H}_2$.

Now, in the GBF these constructions are generalized in an axiomatic way. To write down the axioms, we first define some notions that will be used frequently in this thesis. Let $\mathcal{M}$ be an n-dimensional differentiable manifold.

- A region is an oriented n-dimensional submanifold of $\mathcal{M}$, possibly with boundary.
- A hypersurface is an oriented $(n-1)$-dimensional submanifold of $\mathcal{M}$ without boundary.
- If Σ is a hypersurface, then $\overline{\Sigma}$ denotes the same manifold with opposite orientation.
- The symbol $\otimes$ denotes the tensor product of vector spaces and the symbol $\hat{\otimes}$ denotes the completed tensor product of Hilbert spaces.

Orientation of a d-dimensional manifold is meant here in the usual sense namely that there exists a nowhere vanishing d-form on that manifold. In the case of a hypersurface Σ, this translates directly to the notion of orientation we introduced above (the choice of a nowhere vanishing normal vector co-vector field).

Since the GBF is still work in progress, I will present here the most recent set of axioms that can be found in [59]. It differs from preceding versions mostly in formulations and technical details such as the inclusion or exclusion of boundaries with corners.

(T1) Associated to each hypersurface Σ is a complex separable Hilbert space $\mathcal{H}_\Sigma$, called the state space of Σ. We denote its inner product by $\langle\cdot,\cdot\rangle_\Sigma$.

(T1b) Associated to each hypersurface Σ is a anti-linear isometry $\iota_\Sigma : \mathcal{H}_\Sigma \to \mathcal{H}_{\overline{\Sigma}}$, i.e. $\iota(\lambda\psi) = \overline{\lambda}\iota(\psi)$ for all $\psi \in \mathcal{H}_\Sigma$ and $\lambda \in \mathbb{C}$. This map is an involution, in the sense that $\iota_{\overline{\Sigma}} \circ \iota_\Sigma$ is the identity in $\mathcal{H}_\Sigma$.

[2] The orientation of a hypersurface Σ is the arbitrary choice of which nowhere vanishing normal co-vector fields to Σ we call positive and which negative oriented with respect to Σ.

[3] We have a Cauchy hypersurface if initial data given on that hypersurface lead to a well posed initial data problem as it was defined by Hadamard, that is the solutions depend uniquely and continuously on the initial data.

(T2) Suppose the hypersurface Σ decomposes into a disjoint union of hypersurfaces $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_n$. Then, there is an isometric isomorphism of Hilbert spaces $\tau_{\Sigma_1,\ldots,\Sigma_n;\Sigma} : \mathcal{H}_{\Sigma_1} \hat{\otimes} \cdots \hat{\otimes} \mathcal{H}_{\Sigma_n} \to \mathcal{H}_\Sigma$. The composition of the maps τ associated with two consecutive decompositions is identical to the map τ associated to the resulting decomposition.

(T2b) The involution ι is compatible with the above decomposition. That is $\tau_{\overline{\Sigma}_1,\ldots,\overline{\Sigma}_n;\overline{\Sigma}} \circ (\iota_{\Sigma_1} \hat{\otimes} \cdots \hat{\otimes} \iota_{\Sigma_n}) = \iota_\Sigma \circ \tau_{\Sigma_1,\ldots,\Sigma_n;\Sigma}$.

(T4) Associated with each region M is a linear map from a dense subspace $\mathcal{H}^\circ_{\partial M}$ of the state space $\mathcal{H}_{\partial M}$ of its boundary ∂M (which carries the induced orientation) to the complex numbers, $\rho_M : \mathcal{H}^\circ_{\partial M} \to \mathbb{C}$. This is called the amplitude map.

For completeness, we will give the following three additional axioms, which however will not be used explicitly in this thesis.

(T3x) Let Σ be a hypersurface. Consider the boundary $\partial\hat{\Sigma}$ of the associated "empty region" $\hat{\Sigma}$ (it can be seen as a limit case of a region) to be given by the disjoint union $\partial\hat{\Sigma} = \overline{\Sigma} \cup \Sigma'$, where Σ' denotes a second copy of Σ. Then, $\tau_{\overline{\Sigma},\Sigma';\partial\hat{\Sigma}}(\mathcal{H}_{\overline{\Sigma}} \otimes \mathcal{H}_{\Sigma'}) \subseteq \mathcal{H}^\circ_{\partial\hat{\Sigma}}$. Moreover, $\rho_{\hat{\Sigma}} \circ \tau_{\overline{\Sigma},\Sigma';\partial\hat{\Sigma}}$ restricts to a bilinear pairing $(\cdot,\cdot)_\Sigma : \mathcal{H}_{\overline{\Sigma}} \times \mathcal{H}_{\Sigma'} \to \mathbb{C}$ such that $\langle \cdot,\cdot \rangle_\Sigma = (\iota_\Sigma(\cdot),\cdot)_\Sigma$.

(T5a) Let M_1 and M_2 be regions and $M := M_1 \cup M_2$ be their disjoint union. Then $\partial M = \partial M_1 \cup \partial M_2$ is also a disjoint union and $\tau_{\partial M_1,\partial M_2;\partial M}(\mathcal{H}^\circ_{\partial M_1} \otimes \mathcal{H}^\circ_{\partial M_2}) \subseteq \mathcal{H}^\circ_{\partial M}$. Moreover, for all $\psi_1 \in \mathcal{H}^\circ_{\partial M_1}$ and $\psi_2 \in \mathcal{H}^\circ_{\partial M_2}$

$$\rho_M \circ \tau_{\partial M_1,\partial M_2;\partial M}(\psi_1 \otimes \psi_2) = \rho_{M_1}(\psi_1)\rho_{M_2}(\psi_2)\,. \tag{5.1}$$

(T5b) Let M be a region whose boundary decomposes into a disjoint union $\partial M = \Sigma_1 \cup \Sigma \cup \overline{\Sigma'}$, where Σ' is a copy of Σ. Let M_1 denote the gluing of M with itself along Σ, $\overline{\Sigma'}$ and suppose that M_1 is a region. Note $\partial M_1 = \Sigma_1$. Then, $\tau_{\Sigma_1,\Sigma,\overline{\Sigma'};\partial M}(\psi \otimes \xi \otimes \iota_\Sigma(\xi)) \in \mathcal{H}^\circ_{\partial M}$ for all $\psi \in \mathcal{H}^\circ_{\partial M_1}$ and $\xi \in \mathcal{H}_\Sigma$. Moreover, for any ON-basis $\{\xi\}_{i\in I}$ of $\mathcal{H}_\Sigma$, we have for all $\psi \in \mathcal{H}^\circ_{\partial M_1}$

$$\rho_{M_1}(\psi) \cdot c(M;\Sigma,\overline{\Sigma'}) = \sum_{i\in I} \rho_M \circ \tau_{\Sigma_1,\Sigma,\overline{\Sigma'};\partial M}(\psi \otimes \xi_i \otimes \iota_\Sigma(\xi_i))\,, \tag{5.2}$$

where $c(M;\Sigma,\overline{\Sigma'}) \in \mathbb{C} \setminus \{0\}$ is called the gluing anomaly factor and depends only on the geometric data.

The requirements formulated in the axioms ensure that all the tools we need for the prediction of experimental situations are at hand.

5.2 Probability interpretation and generalized Born rule

As explained above in the standard formulation of QFT probabilities arise as the squared modulus of transition amplitudes. For instance, the probability for the transition of the system under a time evolution U from the state ψ_{t_1} at an instance of time t_1 to ψ_2 at an instance of time

t_2 is given as $P(\psi_1, \psi_2) = |\rho(\psi_1, \overline{\psi_2})|^2 = |\langle \psi_2, U(t_2, t_1)\psi_1\rangle|^2$. This probability is a conditional probability in the sense that it gives the probability to obtain the system in the state ψ_2 at t_2 given that it was in the state ψ_1 at t_1. In an experimental situation, we can say that ψ_1 encodes the knowledge about the preparation and ψ_2 encodes the knowledge about the measurement. This is now generalized to the GBF case the following way: Given a spacetime region M preparation and measurement are encoded via specifying closed subspaces $\mathcal{S}$ and $\mathcal{A}$ of the boundary Hilbert space $\mathcal{H}_{\partial M}$ respectively. Then, the probability to find the system in a state in $\mathcal{A}$ given that it is prepared in a state in $\mathcal{A}$ is given as

$$P(\mathcal{A}|\mathcal{S}) := \frac{||\rho_M \circ P_{\mathcal{S}} \circ P_{\mathcal{A}}||^2}{||\rho_M \circ P_{\mathcal{S}}||^2}\,, \tag{5.3}$$

where $P_{\mathcal{S}}$ and $P_{\mathcal{A}}$ are the orthogonal projectors onto the subspaces $\mathcal{S}$ and $\mathcal{A}$ respectively. For the expression in (5.3) to be meaningful, the composed map $\rho_M \circ P_{\mathcal{S}}$ must be continuous on $\mathcal{H}_{\partial M}$ and non-vanishing at least on an open subset of $\mathcal{H}_{\partial M}$. These conditions present of course a restriction on the possible choices for $\mathcal{S}$, but are sufficiently restrictive to obtain a probability interpretation without imposing an impossible situation. If $\mathcal{S}$ fits these requirements, $\rho_M \circ P_{\mathcal{S}}$ is an element of the dual Hilbert space $\mathcal{H}^*_{\partial M}$ and the norm $||\cdot||$ is the one on $\mathcal{H}^*_{\partial M}$. The identity in (5.3) is called the generalized Born rule[4].

From (5.3), we recover the probability of the standard formulation as follows: Restricting the possible choices of $\mathcal{A}$ to subsets of $\mathcal{S}$ - which means that we only ask questions that take fully into account what we already know about the state of the system - we find that (5.3) becomes

$$P(\mathcal{A}|\mathcal{S}) = \frac{||\rho_M \circ P_{\mathcal{A}}||^2}{||\rho_M \circ P_{\mathcal{S}}||^2}\,. \tag{5.4}$$

Starting from an orthonormal basis in $\mathcal{A}$ which we extend first to an orthonormal basis of $\mathcal{S}$ we obtain after another extension an orthonormal basis $\{\xi_i\}_{i\in\mathbb{N}}$ of $\mathcal{H}_{\partial M}$. Using Riesz' Theorem we find that the element of $\psi_{\rho,\mathcal{A}} \in \mathcal{H}_{\partial M}$ that corresponds to the dual element $\rho_M \circ P_{\mathcal{A}}$ via the inner product on $\mathcal{H}_{\partial M}$ is given as

$$\psi_{\rho,\mathcal{A}} = \sum_{\xi_i \in \mathcal{A}} \overline{\rho_M(\xi_i)}\xi_i\,, \tag{5.5}$$

and analogously for $\rho_M \circ P_{\mathcal{S}}$. The norm of $\rho_M \circ P_{\mathcal{A}}$ in $\mathcal{H}^*_{\partial M}$ can then be written as the norm of $\psi_{\rho,\mathcal{A}}$ in $\mathcal{H}_{\partial M}$ and we can rewrite the expression in Equation (5.4) as

$$P(\mathcal{A}|\mathcal{S}) = \frac{\sum_{\xi_j \in \mathcal{A}} |\rho_M(\xi_j)|^2}{\sum_{\xi_i \in \mathcal{S}} |\rho_M(\xi_i)|^2}\,. \tag{5.6}$$

Now, let us consider a region M with boundary $\partial M = \Sigma_1 \cup \overline{\Sigma_2}$ with Σ_1 and $\overline{\Sigma_2}$ disjoint Cauchy hypersurfaces that are part of a foliation of the spacetime $\mathcal{M}$ parameterized by t. In that case we use the isometric isomorphism of axiom (T2) to identify the boundary Hilbert space $\mathcal{H}_{\partial M}$ with $\mathcal{H}_{\Sigma_1}\hat{\otimes}\mathcal{H}_{\overline{\Sigma_2}}$. Let us assume further that we are given a unitary map $U(t_2, t_1)$ such that

[4]For a much more elaborate introduction of the probability interpretation of the GBF we direct the interested reader to [117].

$\rho_M(\psi_1 \otimes \iota(\psi_2)) = \langle \psi_2, U(t_2, t_1)\psi_1 \rangle_{\Sigma_2}$. We will explain in Section 5.8 more precisely what unitary evolution means in the GBF framework.

Now, we want to know the probability for the transition from the normalized state $\psi_1 \in \mathcal{H}_{\Sigma_1}$ to the normalized state $\psi_2 \in \mathcal{H}_{\Sigma_2}$. Hence, we set $\mathcal{S} = \{\psi_1 \otimes \xi | \, \xi \in \mathcal{H}_{\overline{\Sigma_2}}\}$ and $\mathcal{A} = \{\psi_1 \otimes \iota(\psi_2)\}$ and we obtain

$$P(\mathcal{A}|\mathcal{S}) = \frac{|\rho_M(\psi_1 \otimes \iota(\psi_2))|^2}{\sum_{\xi_i \in \mathcal{S}} |\rho_M(\xi_i)|^2} = |\langle \psi_2, U(t_2, t_1)\psi_1 \rangle_{\Sigma_2}|^2 \,, \tag{5.7}$$

which recovers the standard expression for the transition amplitude $P(\psi_1, \psi_2)$.

5.3 Observable maps and expectation values

Beside transition amplitudes, expectation values of observables are central elements of the standard formulation of QFT to compare the results of the theory with experiments. Assuming that the system is in the state ψ, the expectation value for a measurement of an observable $\hat{O} : \mathcal{H} \to \mathcal{H}$ is given as $\langle \hat{O} \rangle = \langle \psi, \hat{O}\psi \rangle$. In the GBF, this definition is generalized in a way more closer to algebraic quantum field theory, where observables are always associated with open subsets of spacetime. These definitions are again given in an axiomatic form [61, 64]. We present these axioms here for the sake of completeness although we will only use them implicitly by using results derived from them.

(O1) Associated to each spacetime region M is a real vector space $\mathcal{O}_M$ of linear maps $\mathcal{H}^\circ_{\partial M} \to \mathbb{C}$, called *observable maps*. In particular, $\rho_M \in \mathcal{O}_M$.

(O2a) Let M_1 and M_2 be regions and $M = M_1 \cup M_2$ be their disjoint union. Then, there is an injective bilinear map $\diamond : \mathcal{O}_{M_1} \times \mathcal{O}_{M_2} \hookrightarrow \mathcal{O}_M$ such that for all $O_1 \in \mathcal{O}_{M_1}$ and $O_2 \in \mathcal{O}_{M_2}$ and $\psi_1 \in \mathcal{H}^\circ_{\partial M_1}$ and $\psi_2 \in \mathcal{H}^\circ_{\partial M_2}$,

$$O_1 \diamond O_2(\psi_1 \otimes \psi_2) = O_1(\psi_1) O_2(\psi_2). \tag{5.8}$$

This operation is required to be associative in the obvious way.

(O2b) Let M be a region with whose boundary decomposes into a disjoint union $\partial M = \Sigma_1 \cup \Sigma \cup \overline{\Sigma'}$ and M_1 given as in (T5b). Then, there is a linear map $\diamond_\Sigma : \mathcal{O}_M \to \mathcal{O}_{M_1}$ such that for all $O \in \mathcal{O}_M$ and any orthonormal basis $\{\xi_i\}_{i \in I}$ of $\mathcal{H}_\Sigma$ and for all $\psi \in \mathcal{H}^\circ_{\partial M_1}$,

$$\diamond_\Sigma(O)(\psi) \cdot c(M; \Sigma, \overline{\Sigma'}) = \sum_{i \in I} O(\psi \otimes \xi_i \otimes \iota_\Sigma(\xi_i)). \tag{5.9}$$

This operation is required to commute with itself and with (O2a) in the obvious way.

The gluing operations defined in (O2a) and (O2b) as well as their iterations and combinations are referred to in the literature as compositions of observables or the combination of measurements [64]: "Combination is here to be understood as in classical physics, when the product of observables is taken.".

In the GBF, expectation values for operators are given with respect to a closed subset $\mathcal{S} \subseteq \mathcal{H}_{\partial M}$ encoding the knowledge about the system. We have

$$\langle O \rangle_{\mathcal{S}} := \frac{\langle \rho_M \circ P_{\mathcal{S}}, O \rangle}{||\rho_M \circ P_{\mathcal{S}}||^2} \,. \tag{5.10}$$

For this expression to be meaningful, we need again that $\rho_M \circ P_{\mathcal{S}}$ is a continuous map that differs from zero. Setting $O = \rho_M \circ P_{\mathcal{A}}$ and assuming that $\mathcal{A} \subseteq \mathcal{S}$ we obtain Equation (5.3) as a special case of Equation (5.10).

We recover the expression $\langle O \rangle_\Sigma = \langle \psi, \hat{O}\psi \rangle_\Sigma$ of the standard formulation of QFT when considering an empty region $\hat{\Sigma}$ with $\partial\hat{\Sigma} = \Sigma \cup \overline{\Sigma'}$ where Σ' is another copy of Σ. Assuming a unitary time evolution $U : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$, we find that $\mathcal{H}_{\partial\hat{\Sigma}} = \mathcal{H}_\Sigma \hat{\otimes} \overline{\mathcal{H}_{\Sigma'}}$. For every $\psi \in \mathcal{H}_\Sigma$ and $\psi' \in \mathcal{H}_{\Sigma'}$, we set

$$O(\psi \otimes \iota(\psi')) = \langle \psi', \hat{O}\psi \rangle_\Sigma \,. \tag{5.11}$$

The preparation of the system in the state ψ corresponds to the subset $\mathcal{S} = \{\psi \otimes \iota(\xi) | \, \xi \in \mathcal{H}_{\Sigma_2}\}$. From axiom (T3x) we know that in the case of an empty region, we can express the amplitude via the inner product and find

$$\rho_{\hat{\Sigma}} \circ P_{\mathcal{S}}(\xi \otimes \iota(\psi')) = \langle \psi', P_\psi \xi \rangle_\Sigma \tag{5.12}$$

where P_ψ is the orthogonal projection onto ψ. Let $\{\xi_i\}_{i\in\mathbb{N}}$ be an orthonormal basis in $\mathcal{H}_\Sigma$ chosen such that $\xi_1 = \psi$ then we get

$$||\rho_{\hat{\Sigma}} \circ P_{\mathcal{S}}||^2 = \sum_{i,j=1}^{\infty} |\rho_{\hat{\Sigma}} \circ P_{\mathcal{S}}(\xi_i \otimes \iota(\xi_j))|^2 = \sum_{i,j=1}^{\infty} |\langle \xi_j, P_\psi \xi_i \rangle_\Sigma|^2 = 1 \,. \tag{5.13}$$

On the other hand, we find that

$$\begin{aligned} \langle \rho_{\hat{\Sigma}} \circ P_{\mathcal{S}}, O \rangle_\Sigma &= \sum_{i,j=1}^{\infty} \overline{\rho_{\hat{\Sigma}} \circ P_{\mathcal{S}}(\xi_i \otimes \iota(\xi_j))} O(\xi_i \otimes \iota(\xi_j)) \\ &= \sum_{i,j=1}^{\infty} \langle P_\psi \xi_i, \xi_j \rangle_\Sigma \langle \xi_j, \hat{O}\xi_i \rangle_\Sigma = \langle \psi, \hat{O}\psi \rangle_\Sigma \,. \end{aligned} \tag{5.14}$$

The last ingredient we need to define is a vacuum state, in analogy to the standard formulation. However, we will postpone this step to a point after the introduction of a particular representation of the core axioms called the holomorphic representation. Then, we will see much more easily how the vacuum state is introduced and which properties it has to fulfill. In the following, we will start the introduction of the holomorphic representation of GBF by deriving the symplectic geometry of the phase space of scalar field theories. We restrict our consideration to scalar field theories since we will only deal with those in this thesis. The GBF can be applied to a much wider range of field theories[5].

[5] See for example [58] and [62] for the GBF of two dimensional Yang-Mills or four dimensional fermionic field theory, respectively.

5.4 Symplectic geometry of scalar field theories

In this section we will introduce the classical phase space structure of scalar field theories. We will introduce the phase space as a symplectic vector space which is not particularily related to the GBF (this perspective is widely used in the literature [118]) but will be convienient for the implementation of the field quantization in the next section. Let us assume that we are given an action

$$S_M(\phi) := \int_M \Lambda(\phi, \partial_\mu\phi, \dots, \partial_{\mu_1}\dots\partial_{\mu_r}\phi, x)\,, \tag{5.15}$$

for the scalar field ϕ in a region $M \subseteq \mathcal{M}$ where $\Lambda(\phi, \partial_\mu\phi, \dots, \partial_{\mu_1}\dots\partial_{\mu_r}\phi, x)$ is the n-form

$$\Lambda(\phi, \partial_\mu\phi, \dots, \partial_{\mu_1}\dots\partial_{\mu_r}\phi, x) = \Lambda_{\nu_1\cdots\nu_n}(\phi, \partial_\mu\phi, \dots, \partial_{\mu_1}\dots\partial_{\mu_r}\phi, x) dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_n}\,, \tag{5.16}$$

containing the Lagrangian density $\Lambda_{\nu_1\cdots\nu_n}$ depending on the field ϕ and all its derivatives reaching the order r and the integral measure. Performing a variation $\phi \to \phi + s\delta\phi$, where s is a small, real parameter, we find that to first order in s, we can write Λ as a sum of an exact n-form dj and a term depending on $\delta\phi$ but not on any derivative of $\delta\phi$. We obtain

$$\delta S_M(\phi) := s\int_M (dj + F_{E-L}\delta\phi) \tag{5.17}$$

where the term F_{E-L} is the n-form given in coordinates as

$$F_{E-L} = (F_{E-L})_{\nu_1\cdots\nu_n} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_n}\,, \tag{5.18}$$

with components

$$(F_{E-L})_{\nu_1\cdots\nu_n} = \frac{\partial\Lambda_{\nu_1\cdots\nu_n}}{\partial\phi} + \sum_{i=1}^{n}(-1)^i\partial_{\mu_1}\dots\partial_{\mu_i}\frac{\partial\Lambda_{\nu_1\cdots\nu_n}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_i}\phi)} = 0\,, \tag{5.19}$$

which are to the Euler-Lagrange equations. The $(n-1)$-form j is called the symplectic current [119] and is defined as

$$j = j^\mu{}_{\mu\mu_1\dots\mu_{n-1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{n-1}} \tag{5.20}$$

where the components

$$j^\mu{}_{\nu_1\dots\nu_n} = \frac{\partial\Lambda_{\nu_1\dots\nu_n}}{\partial(\partial_\mu\phi)}\delta\phi + \sum_{i=1}^{n-1}\left(\frac{\partial\Lambda_{\nu_1\dots\nu_n}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_i}\partial_\mu\phi)} + \lambda^{\mu_1\dots\mu_i\mu}{}_{\nu_1\dots\nu_n}\right)\partial_{\mu_1}\dots\partial_{\mu_i}\delta\phi \tag{5.21}$$

with

$$\lambda^{\mu_1\dots\mu_j}{}_{\nu_1\dots\nu_n} = -\partial_\nu\left(\frac{\partial\Lambda_{\nu_1\dots\nu_n}}{\partial(\partial_{\mu_1}\dots\partial_{\mu_j}\partial_\nu\phi)} + \lambda^{\mu_1\dots\mu_j\nu}{}_{\nu_1\dots\nu_n}\right) \tag{5.22}$$

for $j < r$ and $\lambda^{\mu_1\dots\mu_j} = 0$ for $j \geq r$ form a covariant tensor field that is completely antisymmetric in the lower indices. We denote by L_Σ the real vector space of germs[6] at Σ of solutions to the

[6] For functions f and g on a neighborhood U_0 of Σ in $\mathcal{M}$, define $f \sim_\Sigma g$ if there is a neighborhood $U \subseteq U_0$ of Σ such that $f|_U = g|_U$. The equivalence classes of $\sim_\Sigma$ are called *germs at* Σ. In particular, $f \sim_\Sigma g$ implies that all derivatives of f and g at Σ are equal.

Euler-Lagrange equations (5.19). By identifying the variation $\delta\phi$ with an element of L_Σ, we obtain the symplectic potential associated with a hypersurface Σ

$$(\theta_\Sigma)_\phi(\delta\phi) := \int_\Sigma j\,. \tag{5.23}$$

From the symplectic potential we obtain the bilinear map

$$[\cdot,\cdot]_\Sigma : L_\Sigma \times L_\Sigma \to \mathbb{R}, \quad [\phi,\xi]_\Sigma := (\theta_\Sigma)_\phi(\xi)\,, \tag{5.24}$$

and an anti-symmetric bilinear map by anti-symmetrization

$$\omega_\Sigma : L_\Sigma \times L_\Sigma \to \mathbb{R}, \quad \omega_\Sigma(\phi,\phi') = \frac{1}{2}[\phi,\phi']_\Sigma - \frac{1}{2}[\phi',\phi]_\Sigma\,, \tag{5.25}$$

which we call the symplectic form. We assume in the following that ω_Σ is always non-degenerate. For a hypersurface decomposing into a disjoint union of hypersurfaces, $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_n$ we then have that $L_\Sigma = L_{\Sigma_1} \oplus \cdots \oplus L_{\Sigma_n}$, and $\theta_\Sigma = \theta_{\Sigma_1} + \ldots + \theta_{\Sigma_n}$. For a region $M \subseteq \mathcal{M}$ we define $\omega_{\partial M}$ as the symplectic form associated to the boundary ∂M of M. Furthermore, we define L_M as the vector space of global solutions to the Euler-Lagrange equations in the region M.

To give an example, let us consider a free Klein-Gordon field on a Lorentzian spacetime $(\mathcal{M}, g)$. In this case we have

$$\begin{aligned}&\Lambda(\phi,\partial_\mu\phi,\ldots,\partial_{\mu_1}\ldots\partial_{\mu_n}\phi,x)\\ &= \tfrac{1}{2}\sqrt{|\det g(x)|}\,\epsilon_{\mu_1\ldots\mu_n}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_d n}\left(g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x) - m^2\phi^2(x)\right)\end{aligned} \tag{5.26}$$

where $\epsilon_{\mu_1\ldots\mu_n}$ is the totally antisymmetric Levi-Cevita symbol and we find that the symplectic potential is given as

$$(\theta_\Sigma)_\phi(\delta\phi) = \int d\sigma_\mu(y)\,\left(\delta\phi\, g^{\mu\nu}\partial_\nu\phi\right)(X(y)) \tag{5.27}$$

where $d\sigma$ is the integral measure on Σ given as

$$\begin{aligned} d\sigma_\mu(y) &= \sqrt{|\det g(X(y))|}\,\epsilon_{\mu\mu_1\ldots\mu_{n-1}}X^{\mu_1}{}_{,a_1}(y)\cdots X^{\mu_{n-1}}{}_{,a_{n-1}}(y)\,dy^{a_1}\wedge\cdots\wedge dy^{a_{n-1}}\\ &= \epsilon\sqrt{|\det g^{(n-1)}(X(y))|}\,dy^1\cdots dy^{n-1}\,n_\mu\,, \end{aligned} \tag{5.28}$$

where n is the unit normal covector field to Σ with positive orientation, X the embedding function defining the hypersurface Σ in $\mathcal{M}$ and $g^{(n-1)}$ the induced metric

$$g^{(n-1)}_{ab} = g_{\mu\nu}X^\mu_{,a}X^\nu_{,b}\,, \tag{5.29}$$

on Σ, where $X_{,a}$ denotes the partial derivative $\frac{\partial X}{\partial y_a}$ and $\epsilon = +1$ or $\epsilon = -1$ (depending on the orientation of Σ with respect to the chosen coordinate system). The symplectic form $\omega_{\partial M}$ for a region M is then given as

$$\omega_{\partial M}(\phi,\phi') = \frac{1}{2}\int_{\partial M} d\sigma_\mu(y)\left(g^{\mu\nu}\left(\phi\partial_\nu\phi' - \phi'\partial_\nu\phi\right)\right)(y)\,. \tag{5.30}$$

for elements $\phi', \phi \in L_{\partial M}$. By partial integration, we find that for solutions ϕ and ϕ' to the Euler-Lagrange equations, the symplectic form vanishes.

In the first part of this thesis, I will restrict all the applications of the GBF to the case of the free Klein-Gordon field. Although the theory I will present in the following is as general as described above, this special case suffices to show the conceptual questions that arise when abandoning the metric from the framework.

5.5 Holomorphic quantization of symplectic geometries

So far, two quantization schemes have been implemented in the GBF which transform a classical field theory into a quantum field theory that satisfies the axioms of Section 5.1: the Schrödinger representation derived from the Feynman path integral quantization prescription [54–58, 117] and the holomorphic representation [59] which is a quantization scheme inspired by geometric quantization [118]. In [61] it was shown that their is a one-to-one correspondence between the two quantization schemes which justifies the term "representation" for them. In this thesis I will use the holomorphic representation only which I will introduce in the following.

To obtain a quantum field theory from the symplectic geometry as we derived it in Section 5.1, we need an additional datum. In the holomorphic representation, this is the choice of a compatible and positive complex structure, i.e. a linear map $J_\Sigma : L_\Sigma \to L_\Sigma$ such that $J_\Sigma^2 = -\mathrm{id}_\Sigma$, $\omega_\Sigma(J_\Sigma\cdot, J_\Sigma\cdot) = \omega_\Sigma(\cdot,\cdot)$ and $\omega_\Sigma(\cdot, J_\Sigma\cdot)$ positive definite. That turns L_Σ into a complex Hilbert space and allows the definition of coherent states that form the dense subset required in axiom (T4). It was shown in [61] that the choice of complex structure corresponds to the choice of a vacuum state in the Schrödinger representation which is why we will use the two terms synonymously throughout this thesis.

Holomorphic quantization is closely related to the canonical quantization prescription known from quantum field theory textbooks in the following sense: Canonical quantization proceeds from the classical phase space by expanding the classical solutions in Fourier modes. The Fourier modes are then divided into positive and negative frequency modes and the coefficients in front of the positive frequency modes are considered to be particle creation operators and the coefficients in front of the negative energy modes as annihilation operators. Here, the notion of positive and negative frequency is usually meant with respect to the spectrum of a nowhere vanishing, timelike, hypersurface orthogonal Killing vector field $\frac{\partial}{\partial t}$ that acts on the modes as a derivative operator. Clearly, such a vector field does not exist for generic hypersurfaces Σ, and certainly not if the hypersurface contains a timelike direction. But, especially this is the case we would like to consider in the GBF. In the holomorphic quantization prescription, we define the complexification $L_\Sigma^{\mathbb{C}}$ as the set of all complex linear combinations of elements of L_Σ and pick a subset $\mathcal{P}_\Sigma \subseteq L_\Sigma^{\mathbb{C}}$ (like the set of positive energy modes in canonical quantization), that must fulfill the following properties

- $\mathcal{P}_\Sigma$ is a complex polarization, which means that for the symplectic complement $\mathcal{P}_\Sigma^\perp$, defined as the set of all $\xi \in L_\Sigma^{\mathbb{C}}$ such that $\omega_\Sigma(\xi,\eta) = 0$ for all $\eta \in \mathcal{P}_\Sigma$, we have $\mathcal{P}_\Sigma^\perp = \mathcal{P}_\Sigma$ (that is, $\mathcal{P}_\Sigma$ is Lagrangian).

- $\mathcal{P}_\Sigma$ is a Kähler polarization, that is $\mathcal{P}_\Sigma$ is a complex polarization such that $\mathcal{P}_\Sigma \cap \overline{\mathcal{P}_\Sigma} = \{0\}$, where the bar denotes complex conjugation in the usual sense.

In this case, we can write $L_\Sigma^{\mathbb{C}} = \mathcal{P}_\Sigma \oplus \overline{\mathcal{P}_\Sigma}$. Complex Kähler polarizations are in one-to-one correspondence with complex structures compatible to ω_Σ. We say $J_\Sigma : L_\Sigma \to L_\Sigma$ is a complex structure compatible with ω_Σ if $J_\Sigma^2 = -\mathrm{id}_\Sigma$ and $\omega_\Sigma(J_\Sigma\cdot, J_\Sigma\cdot) = \omega_\Sigma(\cdot,\cdot)$. Given such a complex structure, define first its extension to the complexification as

$$J_\Sigma(\phi_R + i\phi_I) = J_\Sigma\phi_R + iJ_\Sigma\phi_I \tag{5.31}$$

for all ϕ_R and ϕ_I in L_Σ. Then, define the subset

$$\mathcal{P}_\Sigma \;:=\; \{\phi - iJ_\Sigma\phi,\, \phi \in L_\Sigma\} = \{\phi \in L_\Sigma^{\mathbb{C}} | J_\Sigma\phi = i\phi\}\,.$$

$\mathcal{P}_\Sigma$ and its complex conjugate $\overline{\mathcal{P}_\Sigma} = \{\phi + iJ_\Sigma\phi,\, \phi \in L_\Sigma\} = \{\phi \in L_\Sigma^{\mathbb{C}} | J_\Sigma\phi = -i\phi\}$ are complex polarizations - called the holomorphic and the antiholomorphic polarization - which can be seen using the compatibility condition above in the following way: let ϕ_1 and ϕ_2 be in $\mathcal{P}_\Sigma$, then

$$\omega_\Sigma(\phi_1,\phi_2) = \omega_\Sigma(J_\Sigma\phi_1, J_\Sigma\phi_2) = -\omega_\Sigma(\phi_1,\phi_2)\,, \tag{5.32}$$

which means that $\mathcal{P}_\Sigma \subseteq \mathcal{P}_\Sigma^\perp$. Let us assume further that for generic $\phi_1 \in L_\Sigma$ the element $\phi_2 \in L_\Sigma$ is such that $\omega_\Sigma(\phi_1,\phi_2) \neq 0$ which always can be found since ω_Σ is non-degenerate. Then, we find that $\phi_1 - iJ_\Sigma\phi_1 \in \mathcal{P}_\Sigma$ and $\phi_2 + iJ_\Sigma\phi_2 \in \overline{\mathcal{P}_\Sigma}$ and

$$\omega_\Sigma(\phi_1 - iJ_\Sigma\phi_1, \phi_2 + iJ_\Sigma\phi_2) = 2\omega_\Sigma(\phi_1,\phi_2) + 2i\omega_\Sigma(\phi_1, J_\Sigma\phi_2) \neq 0\,,$$

from which we conclude that $\mathcal{P}_\Sigma = \mathcal{P}_\Sigma^\perp$. Furthermore, $\mathcal{P}_\Sigma$ and its complex conjugate $\overline{\mathcal{P}_\Sigma}$ fulfill the Kähler property $\mathcal{P}_\Sigma \cap \overline{\mathcal{P}_\Sigma} = \{0\}$ by definition. On the other hand, if $\mathcal{P}_\Sigma$ and $\overline{\mathcal{P}_\Sigma}$ are given Kähler polarizations, we can decompose any $\phi \in L_\Sigma^{\mathbb{C}}$ uniquely into $\phi = \psi + \psi'$ with $\psi \in \mathcal{P}_\Sigma$ and $\psi' \in \overline{\mathcal{P}_\Sigma}$. Then J_Σ defined by $J_\Sigma\phi := i\psi - i\psi'$ is a complex structure compatible with ω_Σ. This can be seen the following way: let ϕ_1 and ϕ_2 be in $L_\Sigma^{\mathbb{C}}$ then

$$\begin{aligned}\omega_\Sigma(J_\Sigma\phi_1, J_\Sigma\phi_2) &= \omega_\Sigma(J_\Sigma(\psi_1+\psi_1'), J_\Sigma(\psi_2+\psi_2')) = \omega_\Sigma(i\psi_1 - i\psi_1', i\psi_2 - i\psi_2') \\ &= \omega_\Sigma(\psi_1,\psi_2') + \omega_\Sigma(\psi_1',\psi_2) = \omega_\Sigma(\psi_1+\psi_1', \psi_2+\psi_2') = \omega_\Sigma(\phi_1,\phi_2)\,.\end{aligned}$$

Given J_Σ, we have a canonical isomorphism $\pi : \mathcal{P}_\Sigma \to L_\Sigma$, $\pi(\psi) = \psi + \bar\psi$ that identifies $\mathcal{P}_\Sigma$ and L_Σ, the latter now regarded as a complex vector space. The inverse is explicitly given in terms of the complex structure as

$$\pi^{-1} = \frac{1}{2}(1 - iJ_\Sigma)\,. \tag{5.33}$$

We say that J_Σ, or equivalently $\mathcal{P}_\Sigma$, is positive if $g_\Sigma(\cdot,\cdot) := 2\omega_\Sigma(\cdot, J_\Sigma\cdot)$ is positive definite. In the following, unless otherwise stated, we shall always assume J to be a positive compatible complex structure. If J_Σ is given, then completion with respect to the bilinear map

$$\{\phi,\eta\}_\Sigma := g_\Sigma(\phi,\eta) + 2i\omega(\phi,\eta) \quad \forall \phi,\eta \in L_\Sigma \tag{5.34}$$

turns the real vector space L_Σ into a complex Hilbert space $\mathcal{H}^1_\Sigma = (L_\Sigma, \{\cdot,\cdot\}_\Sigma)$, where multiplication with i is given by applying J_Σ. In the following we will identify L_Σ with $\mathcal{H}^1_\Sigma$ and only talk about L_Σ. Later in this section we will find that $L_\Sigma = \mathcal{H}^1_\Sigma$ can be identified with the one particle sector of the Hilbert space $\mathcal{H}_\Sigma$ of holomorphic quantization required in axiom (T1) which is now constructed as the separable complex Hilbert space given by $\mathcal{H}_\Sigma := L^2_{hol}(\hat{L}_\Sigma, d\mu_\Sigma)$. The latter is the space of square-integrable holomorphic functions[7] on $\hat{L}_\Sigma$ with respect to a certain Gaussian measure $d\nu_\Sigma$, and the inner product is defined as

$$\langle\psi',\psi\rangle_\Sigma := \int_{\hat{L}_\Sigma} \psi(\phi)\overline{\psi'}(\phi) d\nu_\Sigma(\phi). \tag{5.35}$$

Here, $\hat{L}_\Sigma$ is a certain extension of L_Σ that was introduced by Oeckl in [59] to deal with technical difficulties which arise because L_Σ is infinite-dimensional; $\hat{L}_\Sigma$ is the algebraic dual of the topological dual of L_Σ. In this thesis, I will use the results by Oeckl et al. without presenting the constructions in detail. In the following, we will use L_Σ equivalently for $\hat{L}_\Sigma$. The measure $d\nu_\Sigma$ can be thought of as a mathematically rigorous version of the heuristic expression

$$\langle\psi',\psi\rangle_\Sigma = \int_{L_\Sigma} \psi(\phi)\overline{\psi'}(\phi) \exp\left(-\frac{1}{2} g_\Sigma(\phi,\phi)\right) d\mu_\Sigma(\phi). \tag{5.36}$$

where $d\mu_\Sigma$ is a translation invariant measure which, in fact, does not exist. The reader interested in a deeper understanding of the mathematical details is recommended to read [59].
Now, the amplitude map ρ_M of axiom (T4) for a region M with boundary ∂M is given by

$$\rho_M(\psi) = \int_{L_{\tilde{M}}} \psi(\phi) d\nu_{\tilde{M}}(\phi) \tag{5.37}$$

where $\psi \in \mathcal{H}_{\partial M}$, $L_{\tilde{M}} \subset L_{\partial M}$ is the subspace of global solutions on M restricted to ∂M, and $\nu_{\tilde{M}}$ is the Gaussian measure constructed in [59] analogously to $d\nu_\Sigma$. To construct the involution of axiom (T1b) relating oppositely oriented manifolds explicitly, we can take $\mathcal{H}_{\overline{\Sigma}} := L^2_{antihol}(L_\Sigma, d\mu_\Sigma)$, that is, the antiholomorphic functions. When viewed as functions on L_Σ we then find that

$$\iota_\Sigma \psi := \overline{\psi}\,. \tag{5.38}$$

for all $\psi \in \mathcal{H}_\Sigma$. We will see in the next section, when considering the explicit construction of coherent states, that we can identify $L^2_{antihol}(L_\Sigma, d\mu_\Sigma)$ with $L^2_{hol}(L_{\overline{\Sigma}}, d\mu_\Sigma)$, where $L_{\overline{\Sigma}}$ is the same real linear space as L_Σ, but equipped with the structures $\omega_{\overline{\Sigma}} = -\omega_\Sigma$ and $J_{\overline{\Sigma}} = -J_\Sigma$.

5.6 Coherent States

When dealing with the GBF, coherent states are of crucial importance. This is because their finite linear combinations form a dense subset in the Hilbert space $\mathcal{H}_{\partial M}$[8]. Explicit expressions for results of amplitude maps as well as of observable maps, that will be introduced later, where

[7]We say a function $f : L \to \mathbb{C}$ is holomorphic if it is continuous, bounded on every ball and holomorphic in the usual sense at every point of L.

[8]For the proof see Theorem 3.15 of [61]

derived for coherent states in [59] and [61] respectively. We can use these expressions to calculate the amplitudes and expectation values of observables for all states in $\mathcal{H}_{\partial M}$.
For every given element $\xi \in L_\Sigma$, we obtain a coherent state $K_\xi \in \mathcal{H}_\Sigma$ as the holomorphic function

$$K_\xi(\phi) := \exp\left(\frac{1}{2}\{\xi,\phi\}_\Sigma\right) \qquad \forall \phi \in L_\Sigma\,. \tag{5.39}$$

A crucial property of the coherent states is that they have the reproducing kernel property

$$\langle K_\xi, \psi\rangle_\Sigma = \psi(\xi)\ \forall\psi \in \mathcal{H}_\Sigma \tag{5.40}$$

with respect to the inner product defined in Equation (5.35).
If Σ decomposes into n disjoint components, $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_n$, then for any element $\xi \in L_\Sigma$ there exists a unique decomposition $\xi = \xi_1 + \cdots + \xi_n$ with $\xi_i \in L_{\Sigma_i}$. Then we have that

$$K_\xi = K_{\xi_1} \otimes \cdots \otimes K_{\xi_n} =: K_{(\xi_1,\ldots,\xi_n)}\,, \tag{5.41}$$

because the sesquilinear form $\{\cdot,\cdot\}$ decomposes for any $\phi \in L_\Sigma$ as $\{\xi,\phi\}_\Sigma = \sum_i\{\xi_i,\phi_i\}_{\Sigma_i}$. With the reproducing kernel property we find that also the inner product (5.35) factorizes as $\langle K_\xi, K_{\xi'}\rangle_\Sigma = \prod_i \langle K_{\xi_i}, K_{\xi_i'}\rangle_{\Sigma_i}$. Recalling that the coherent states form a dense subset in the Hilbert space $\mathcal{H}_\Sigma$ the map $\tau_{\Sigma_1,\ldots,\Sigma_n;\Sigma} : \mathcal{H}_{\Sigma_1}\hat{\otimes}\cdots\hat{\otimes}\mathcal{H}_{\Sigma_n} \to \mathcal{H}_\Sigma$ required in axiom (T2) becomes just an identification of Cauchy series. Then, also axiom (T2b) follows immediately from Equation (5.38). If Σ decomposes into two components, $\Sigma = \Sigma_1 \cup \bar{\Sigma}_2$, then for any $\xi \in L_{\Sigma_1}$ and $\xi' \in L_{\Sigma_2}$, we obtain with (5.38) the useful identity

$$K_{(\xi,\xi')} = K_\xi \otimes \overline{K_{\xi'}} \in \mathcal{H}_\Sigma\,. \tag{5.42}$$

Now, we can also use the coherent states to prove that the action of the involution ι leads to the identification $J_{\overline{\Sigma}} = -J_\Sigma$. If we reverse the orientation of Σ, $\Sigma \to \overline{\Sigma}$, then K_ξ is mapped to the antiholomorphic function $\iota_\Sigma K_\xi = \overline{K_\xi}$. When comparing this expression with the RHS of (5.39) we see that $\overline{K_\xi}$ is the same function as K_ξ defined with $\omega_{\overline{\Sigma}} = -\omega_\Sigma$ and $g_{\overline{\Sigma}} = g_\Sigma$. Therefore, $J_{\overline{\Sigma}} = -J_\Sigma$ follows as

$$\overline{\{\xi,\phi\}}_\Sigma = g_\Sigma(\xi,\phi) - i\omega_\Sigma(\xi,\phi) = g_{\overline{\Sigma}}(\xi,\phi) + i\omega_{\overline{\Sigma}}(\xi,\phi) = \{\xi,\phi\}_{\overline{\Sigma}}\,. \tag{5.43}$$

As explained above, the main advantage of using coherent states is the explicit expression for the amplitude derived in [59]. It is given as

$$\rho_M(K_\tau) = \exp\left(\frac{1}{4}g_{\partial M}(\tau^R,\tau^R) - \frac{1}{4}g_{\partial M}(\tau^I,\tau^I) - \frac{i}{2}g_{\partial M}(\tau^R,\tau^I)\right)\,, \tag{5.44}$$

where $\tau = \tau^R + J_{\partial M}\tau^I$ with $\tau^R, \tau^I \in L_{\tilde{M}}$. This decomposition is always uniquely possible since $L_{\partial M} = L_{\tilde{M}} \oplus J_{\partial M}L_{\tilde{M}}$[9]. Since the expression in (5.44) is well defined and the coherent states form a dense subset in the Hilbert space $\mathcal{H}_{\partial M}$ we find that the requirement in axiom (T4) is fulfilled.

[9]For the proof see Proposition 4.2 in [59]

Proofs for the validity of the axioms (T3x), (T5a) and (T5b) in the holomorphic representation where given in [59]. Axiom (T3x) is proven using the identity of the measure defined for the amplitude map and the inner product for the empty region in Axiom (T3x). Axiom (T5a) follows there directly from the construction of the measure $d\nu_{\tilde{M}}$ which becomes a product measure for disjoint regions. In contrast, to prove axiom (T5b), another assumption must be made namely the integrability of the map $\xi \mapsto \rho_M(K_0 \otimes K_\xi \otimes \iota_\Sigma(K_\xi))$ with respect to $(\hat{L}_\Sigma, \nu_\Sigma)$. We will not bother about this additional condition in this thesis.

5.7 Vacuum states

For every hypersurface Σ there is a distinguished coherent state: The state $\psi_{\Sigma;0} = K_0 = 1$, the constant function with value 1 on L_Σ, which is the coherent state associated with the vector $0 \in L_\Sigma$. It fulfills the following properties usually called vacuum axioms in the literature [59]:

- (V1) For each hypersurface Σ there is a distinguished state $\psi_{\Sigma;0} \in \mathcal{H}_\Sigma$, called the vacuum state.
- (V2) The vacuum state is compatible with the involution. That is, for any hypersurface Σ, $\psi_{\Sigma;0} = \iota_\Sigma(\psi_{\Sigma;0})$.
- (V3) The vacuum state is compatible with decompositions. Suppose the hypersurface Σ decomposes into components $\Sigma_1 \cup \cdots \cup \Sigma_n$. Then $\psi_{\Sigma;0} = \tau_{\Sigma_1,\ldots,\Sigma_n;\Sigma}(\psi_{\Sigma_1;0} \otimes \cdots \otimes \psi_{\Sigma_n;0})$.
- (V5) The amplitude of the vacuum state is unity. That is, for any region M, $\rho_M(\psi_{\partial M;0}) = 1$.

This list reproduces the ordering of [117]. The axiom (V4) appearing there is redundant here because it is already implied by (V5) when using the construction of empty regions. The requirements posed in the vacuum axioms lead to properties of the vacuum state that one associates in the standard formulation of QFT with the vacuum state. For instance, from (V1), (V2) and (V5) together with (T3x) follows that $\psi_{\Sigma;0}$ is normalized. Furthermore, a state fulfilling the above requirements is stable under unitary evolution in the sense of the following proposition that was proven in [59]:

PROPOSITION 5.7.1. *[Proposition 2.1. of [59]] Let M be a region such that its boundary decomposes into a disjoint union $\partial M = \Sigma \cup \overline{\Sigma'}$. Assume moreover that there is a unitary operator $U : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$ such that*

$$\rho_M \circ \tau_{\Sigma,\overline{\Sigma'};\partial M}(\psi \otimes \iota_{\Sigma'}(\psi')) = \langle \psi', U\psi \rangle_{\Sigma'} \quad \forall \psi \in \mathcal{H}_\Sigma, \psi' \in \mathcal{H}_{\Sigma'} \,. \tag{5.45}$$

Then, $U\psi_{\Sigma;0} = \psi_{\Sigma;0}$.

Proof: Since $\psi_{\Sigma;0}$ is normalized, so is $U\psi_{\Sigma;0}$. But its inner product with the also normalized state $\psi_{\Sigma';0}$ is 1. So it must be identical to $\psi_{\Sigma';0}$.□

From this proposition we can draw another conclusion. For each pair of hypersurfaces Σ and Σ' which are given such that there exist a region M such that $\partial M = \Sigma \cup \overline{\Sigma'}$ and an evolution map

U connecting the two hypersurfaces as in proposition 5.7.1 given a vacuum state on Σ we obtain a vacuum state on Σ' via U. Hence, given a spacetime possessing a foliation of hypersurfaces Σ_t and an evolution U with respect to this foliation it suffices to give a vacuum state $\psi_{\Sigma_{t_0},0}$ on one hypersurface Σ_{t_0} to obtain a vacuum state on every hypersurface of the foliation. This brings us directly to the question of when a unitary evolution map U exists and how it can be constructed in the GBF.

5.8 Time evolution and Unitarity

Let M be a region such that $\partial M = \Sigma \cup \overline{\Sigma'}$ is the disjoint union of two hypersurfaces. We can consider canonical projections $r : L_{\tilde{M}} \to L_\Sigma$ and $r' : L_{\tilde{M}} \to L_{\Sigma'}$[10]. Assume these maps are homeomorphisms. The composition $T := r' \circ r^{-1} : L_\Sigma \to L_{\Sigma'}$ maps solutions at Σ into solutions at Σ', hence it can be seen as a generalized notion of time evolution. We call T unitary if

$$J_\Sigma \circ T = T \circ J_{\Sigma'} . \tag{5.46}$$

In [59] it was shown that if T is unitary, then there is a unitary map $U : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$, $\Psi \mapsto \Psi \circ T^{-1}$ fulfilling Equation (5.45). In particular we have

$$UK_{\Sigma,\xi} = K_{\Sigma',T\xi} \,. \tag{5.47}$$

Given a map T we can also say that the complex structure $J_{\partial M}$ is compatible with the evolution if Equation (5.46) is fulfilled. We call a complex structure that is compatible with the evolution T giving rise to a unitary evolution a unitary complex structure. A generalization of this definition of unitarity of complex structures can be given that avoids the assumption of the existence of the evolution map T. Using the reproducing kernel property of the coherent states we can define a map $U^{-1} : \mathcal{H}_{\Sigma'} \to \mathcal{H}_\Sigma$ such that (5.45) is fulfilled as [59]

$$(U^{-1}\Psi)(\phi) := \rho_M(K_{\Sigma,\phi} \otimes \overline{\Psi}) \,. \tag{5.48}$$

for all $\Psi \in \mathcal{H}_{\Sigma'}$. We can extend the subspace $L_{\tilde{M}}$ to the complexification $L^{\mathbb{C}}_{\tilde{M}} \subset L^{\mathbb{C}}_{\partial M}$, such that for $\xi \in L^{\mathbb{C}}_{\tilde{M}}$ we have $\mathfrak{Re}(\xi), \mathfrak{Im}(\xi) \in L_{\tilde{M}}$. We then have that if (5.46) holds every element $\xi \in L^{\mathbb{C}}_{\tilde{M}}$ that projects to a $-i$ eigenfunction of J_Σ, meaning $J_\Sigma r(\xi) = -ir(\xi)$, projects to a $-i$ eigenfunction on Σ', $J_{\Sigma'} r'(\xi) = -ir'(\xi)$, and vice versa. Reversely we will say that a complex structure is "unitary" exactly if this condition is satisfied.

If furthermore the maps r and r' are homeomorphisms we can identify $L^{\mathbb{C}}_{\tilde{M}}$ with $L^{\mathbb{C}}_\Sigma$ and $L^{\mathbb{C}}_{\Sigma'}$. If the complex structure then is unitary in the above sense we find that $J_{\Sigma'} \circ T = T \circ J_\Sigma$ since $L^{\mathbb{C}}_{\tilde{M}}$ is the set of all $(\xi, T\xi) \in L^{\mathbb{C}}_{\partial M}$.

[10] For $\phi \in L_{\tilde{M}}$ a global solution on M, $r(\phi)$ is the germ of ϕ at Σ, in other words $r(\phi)$ is obtained by forgetting ϕ everywhere but in a small neighborhood around Σ. Same for r'.

5.9 Quantization of Weyl observables

The last step in the quantization process is the quantization of observables. An extremely useful class of observables are those given as

$$W(\phi) = \exp(iD(\phi)) \tag{5.49}$$

where D is an element of the set of linear observables $\mathcal{C}_M^{\text{lin}}$ associated to the region M. Observables of the type (5.49) are called Weyl observables in the literature. They have the advantage that every polynomial observable can be constructed from their derivatives. Hence, by quantizing Weyl observables we obtain the quantization of all polynomial observables and in particular linear observables. The quantization of Weyl observables was developed in [61] using two different quantization schemes the Feynman quantization and the Berezin-Toeplitz quantization. We will present the basic ideas and results for both schemes in the following. The observable map for an observable O will be denoted as ρ_M^O in the Feynman quantization and as $\rho_M^{\blacktriangleleft O \blacktriangleright}$ in the Berezin-Toeplitz quantization.

Feynman quantization

Using the path intregral we can immediately write down a heurist expression for the observable map of an observable $O(\phi)$ as

$$\rho_M^O(\psi) = \int_{K_M} \Psi(\phi|_{\partial M}) O(\phi) \exp(iS_M(\phi)) d\mu(\phi)\,. \tag{5.50}$$

where Ψ is a wave functional in the path integral formulation and S_M is the action. As it is widely known this expression is mathematically not well defined due to the non-existence of the measure $d\mu$. However, in [60] and [61] expressions like (5.50) was given meaning establishing the Schrödinger representation of GBF for linear and affine field theories. So let us carry on by considering W to be a Weyl observable as in (5.49). Then, we find that the path integral in (5.50) can be rewritten as the path integral

$$\rho_M^W(\psi) = \int_{K_M} \Psi(\phi|_{\partial M}) \exp(iS_M^W(\phi)) d\mu(\phi)\,, \tag{5.51}$$

where $S_M^D(\phi) = S_M(\phi) + D(\phi)$ which can be considered as an action for an affine field theory. In [61] this insight was taken seriously by the author. Using the Schrödinger representation of the GBF, the observable map ρ_M^W for a given Weyl observable $W = \exp(iD)$ with respect to the action S_M was defined as the amplitude map with respect to the action S_M^D. This construction gave rise to Proposition 4.3 in [61]. I will give the result in the following in a slightly reformulated way using the coherent states defined in (5.39).

Proposition 5.9.1 ((Proposition 4.3 of [61])). *Let M be a region, $D \in \mathcal{C}_M^{lin}$, $W := \exp(i\,D)$, and $\tau \in L_{\partial M}$. We define $\hat{\tau} \in L_{\partial M}^{\mathbb{C}}$ as $\hat{\tau} := \tau^{\mathrm{R}} - i\tau^{\mathrm{I}}$, where $\tau = \tau^{\mathrm{R}} + J_{\partial M}\tau^{\mathrm{I}}$ and $\tau^{\mathrm{R}}, \tau^{\mathrm{I}} \in L_{\tilde{M}}$. Then we obtain that*

$$\rho_M^W(K_\tau) = \rho_M(K_\tau) W(\hat{\tau})\, \rho_M^W(K_0)\,. \tag{5.52}$$

where

$$\rho_M^W(K_0) = \exp\left(\frac{i}{2}D(\eta_D) - \frac{1}{2}g_{\partial M}(\eta_D, \eta_D)\right), \tag{5.53}$$

and η_D *is the unique element of* $J_{\partial M} L_{\tilde{M}}$ *fulfilling the condition* $D(\xi) = 2\omega_{\partial M}(\xi, \eta_D)$ *for all* $\xi \in L_{\tilde{M}}$.

It follows that

$$\rho_M^W(K_0) = \exp\left(\frac{i}{2}D(\eta_D - iJ_{\partial M}\eta_D)\right), \tag{5.54}$$

and we find that $\eta_D - iJ_{\partial M}\eta_D$ lies in the polarization $\mathcal{P}_{\partial M}$ which means that a solution ψ_F in M that induces $\eta_D - iJ_{\partial M}\eta_D$ on ∂M fulfills the generalized Feynman boundary conditions[11], i.e. $(1 + iJ_{\partial M})r_{\partial M}(\psi_F) = 0$ where $r_{\partial M}$ is the map from L_M to $L_{\tilde{M}}$ introduced in Section 5.8.

Assume that there exists a scalar density ρ such that the Lagrangian density governing the dynamics of the scalar field ϕ can be written as

$$\Lambda(\phi, \partial_\mu\phi, \ldots, \partial_{\mu_1}\ldots\partial_{\mu_r}\phi, x) = \rho(x)\epsilon_{\mu_1\ldots\mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \mathcal{L}(\phi, \partial_\mu\phi, \ldots, \partial_{\mu_1}\ldots\partial_{\mu_r}\phi, x) \tag{5.55}$$

where $\mathcal{L}$ is a scalar function and ϵ is the total antisymmetric Levi-Cevita symbol in n dimensions. We denote ρ as the volume density and define the volume measure

$$dV(x) := \rho(x) dx^{\mu_1} \cdots dx^{\mu_n} \tag{5.56}$$

Now, we can write down a certain class of Weyl observables $W = \exp(iD)$ where the linear observable D is of the form

$$D(\phi) = \int_M dV(x)\, \mu(x)\phi(x) \tag{5.57}$$

with $\mu(x)$ a test function. Then, we conclude from the fact that $\eta_D - iJ_{\partial M}\eta_D$ fulfills the Feynman boundary conditions that

$$(\eta_D - iJ_{\partial M}\eta_D)(x) = \int_M dV(x)\, G_F^J(x, x')\mu(x') \tag{5.58}$$

holds for every test function $\mu(x)$ where $G_F^J(x, x')$ is the Feynman propagator (called like this because it generates solutions fulfilling the generalized Feynman boundary conditions) with respect to the complex structure $J_{\partial M}$. We obtain for a coherent state K_τ the operator amplitude map

$$\rho_M^\mu(K_\tau) = \rho_M(K_\tau) \exp\left(i\int_M dV(x)\, \mu(x)\hat{\tau}(x) + \frac{i}{2}\int_M dV(x)dV(x')\, \mu(x)G_F^J(x, x')\mu(x')\right). \tag{5.59}$$

This expression is the GBF generalization of the generating functional known from the standard formulation of QFT. In the framework of the GBF it was first derived in [71, 72] using the Schödinger-Feynman representation of the GBF. Using the holomorphic representation it was recovered in [61].

[11] See [120] for the Feynman boundary condition on spacelike hyperplanes in Minkowski.

Unitary evolution

If the boundary of the region M decomposes as $\partial M = \Sigma_1 \cup \overline{\Sigma}_2$ and the map T defined in Section 5.8 exists and is unitary, i.e. $J_{\overline{\Sigma}_2} = T \circ J_{\Sigma_1} \circ T^{-1}$ we have that $\hat{\tau} = \tau^R - i\tau^I$ in (5.59) is given explicitly as the complex solution

$$\hat{\tau} = r_1^{-1}\left(\frac{1}{2}(1+iJ_{\Sigma_1})\tau_1 + \frac{1}{2}(1-iJ_{\Sigma_1})T^{-1}\tau_2\right) \tag{5.60}$$

for all $\tau = (\tau_1, \tau_2) \in L_{\partial M} = L_{\Sigma_1} \times L_{\overline{\Sigma}_2}$ where r_1 is the homeomorphism defined in Section 5.8. With equations (5.40), (5.45) and (5.47) we can rewrite the expression for the operator amplitude in (5.59) for the coherent state $K_{\tau_1} \otimes \overline{K_{\tau_2}} \in \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\overline{\Sigma}_2}$ as

$$\begin{aligned}\rho_M^{\mu}(K_{\tau_1} \otimes \overline{K_{\tau_2}}) \quad &= \quad \exp\left(\frac{1}{2}\{T(\tau_1), \tau_2\}_{\Sigma_2}\right) \exp\left(i\int_M dV(x)\,\mu(x)\hat{\tau}(x)\right) \times \\ &\quad \times \exp\left(\frac{i}{2}\int_M dV(x)dV(x')\,\mu(x)G_F^J(x,x')\mu(x')\right). \end{aligned} \tag{5.61}$$

To find an expression for G_F^J from (5.58) we have to find η_D. For this purpose, we first define an orthonormal basis for $\mathcal{P}_{\Sigma_1}$. Let $\pi : \mathcal{P}_{\Sigma_1} \to L_{\Sigma_1}$ be the canonical linear isomorphism, with $\pi(\phi) = \phi + \bar{\phi}$. We use this isomorphism to define an inner product on $\mathcal{P}_{\Sigma_1}$:

$$\{\cdot,\cdot\}_{\mathcal{P}_{\Sigma_1}} := \{\pi(\cdot), \pi(\cdot)\}_{\Sigma_1}. \tag{5.62}$$

Using linearity of π and the fact that $\{\cdot,\cdot\}_{\Sigma_1}$ is an inner product on L_{Σ_1}, it is immediate to show that $\{\cdot,\cdot\}_{\mathcal{P}_{\Sigma_1}}$ is an inner product on $\mathcal{P}_{\Sigma_1}$. Now, assume that we have given an orthonormal basis $\mathcal{B}_{\mathcal{P}_{\Sigma_1}}$ of $\mathcal{P}_{\Sigma_1}$ with respect to this inner product (5.62). We shall write $\mathcal{B}_{\mathcal{P}_{\Sigma_1}} = \{e_\xi^+\}_{\xi\in\mathcal{I}}$ for a suitable index set $\mathcal{I}$.
We find that $\eta_D = J_{\partial M}(r_1\eta, r_2\eta)$ where $J_{\partial M} = (J_{\Sigma_1}, -T\circ J_{\Sigma_1}\circ T^{-1})$ and

$$\eta(x) := \int dV(x')\mu(x')\sum_{\sigma\in\mathcal{I}}\left(e_\sigma^+(x')e_\sigma^-(x) + e_\sigma^-(x')e_\sigma^+(x)\right). \tag{5.63}$$

This can be seen by observing that every element $\zeta \in L_M$ can be parameterized as

$$\zeta = \sum_{\sigma\in\mathcal{I}}\left(\zeta_\sigma e_\sigma^+(x) + c.c\right)$$

and it follows that for all $\zeta, \zeta' \in L_M$ holds

$$\begin{aligned}\{\zeta,\zeta'\}_{\Sigma_1} \quad &= \quad i\omega_{\Sigma_1}((1+iJ_{\Sigma_1})\zeta, (1-iJ_{\Sigma_1})\zeta') = \sum_{\sigma,\sigma'\in\mathcal{I}}\overline{\zeta_\sigma}\zeta'_{\sigma'}4i\omega_{\Sigma_1}(e_\sigma^-, e_{\sigma'}^+) \\ &= \quad \sum_{\sigma,\sigma'\in\mathcal{I}}\overline{\zeta_\sigma}\zeta'_{\sigma'}\{e_\sigma^+, e_{\sigma'}^+\}_{\mathcal{P}_{\Sigma_1}} = \sum_{\sigma\in\mathcal{I}}\overline{\zeta_\sigma}\zeta'_\sigma\,.\end{aligned}$$

Finally, we find that

$$\begin{aligned}2\omega_{\partial M}(\zeta,\eta_D) \quad &= \quad g_{\partial M}(\zeta,\eta) = 2g_{\Sigma_1}(\zeta,\eta) = 2\mathfrak{Re}(\{\zeta,\eta\}_{\Sigma_1}) \\ &= \quad \sum_{\sigma\in\mathcal{I}}\left(\zeta_\sigma\overline{\eta_\sigma} + \overline{\zeta_\sigma}\eta_\sigma\right) = \int dV(x)\mu(x)\zeta(x) = D(\zeta)\,.\end{aligned} \tag{5.64}$$

Having verified that Equation (5.63) holds we obtain that the left hand side in Equation (5.58) becomes

$$\begin{aligned}(1-iJ_{\partial M})\eta_D(x_1,x_2) &= i(1-iJ_{\partial M})(r_1\eta, r_2\eta)(x_1,x_2)\\ &= 2i\int d^4x'\mu(x')\sum_{\sigma\in\mathcal{I}}(e^-_\sigma(x')e^+_\sigma(x_1), e^+_\sigma(x')e^-_\sigma(x_2)) \qquad (5.65)\end{aligned}$$

where x_1 and x_2 are coordinates in neighborhoods of Σ_1 and Σ_2 respectively. These are the boundary conditions that the Feynman propagator G_F^J must fulfill. If now a foliation of the region M exists such that $M = [\xi_1, \xi_2]\times\Sigma$, we find that

$$G_F^J(x,x') = 2i\sum_{\sigma\in\mathcal{I}}(\theta(\xi(x')-\xi(x))e^+_\sigma(x)e^-_\sigma(x') + \theta(\xi(x)-\xi(x'))e^-_\sigma(x)e^+_\sigma(x'))\,, \qquad (5.66)$$

where θ is the Heaviside step function.

Berezin-Toeplitz quantization

The Berezin-Toeplitz quantization was introduced in the GBF in [64]. The amplitude map for an observable O for this quantization scheme is given as

$$\rho_M^{\blacktriangleleft O\blacktriangleright}(\psi) := \int_{L_{\tilde{M}}} d\nu_{\tilde{M}}(\phi)\psi(\phi)O(\phi) \qquad (5.67)$$

for every $\psi\in\mathcal{H}_{\partial M}$. When applied to Weyl observables $W=\exp(iD)$ it was shown in [61] that

$$\rho_M^{\blacktriangleleft W\blacktriangleright}(K_\tau) = \rho_M(K_\tau)W(\hat{\tau})\,\rho_M^{\blacktriangleleft W\blacktriangleright}(K_0)\,. \qquad (5.68)$$

where

$$\rho_M^{\blacktriangleleft W\blacktriangleright}(K_0) = \exp\left(-g_{\partial M}(\eta_D,\eta_D)\right)\,, \qquad (5.69)$$

and η_D is the unique element of $J_{\partial M}L_{\tilde{M}}$ fulfilling the condition $D(\xi) = 2\omega_{\partial M}(\xi,\eta_D)$ for all $\xi\in L_{\tilde{M}}$. We find that the results in (5.52) and (5.68) for the two quantization schemes differ just by the expressions for the operator amplitude in the vacuum state which are related as

$$\rho_M^{\blacktriangleleft W\blacktriangleright}(K_0) = |\rho_M^W(K_0)|^2\,. \qquad (5.70)$$

We will find in Section 6.1 that due to this difference the Unruh effect cannot be derived in the Berezin-Toeplitz quantization although in the Feynman quantization it will turn out to be present as a coincidence of expectation values of local observables.

5.10 Ladder Operators and one particle states

For the interpretation of the amplitudes appearing in the GBF it will turn out to be useful to know the Fock space structure of $\mathcal{H}_\Sigma$. Especially, we are interested in the one particle sector of $\mathcal{H}_\Sigma$. To understand these structures we will construct ladder operators in the following. For this purpose, we define for any $\xi\in L_\Sigma$ the state

$$p_\xi(\phi) := \frac{1}{\sqrt{2}}\{\xi,\phi\}_\Sigma\in\mathcal{H}_\Sigma\,. \qquad (5.71)$$

which is identified as the representation of a one-particle state in the holomorphic quantization. This is due to the following properties: Every state p_ξ can be represented as the first derivative of a coherent state in the way

$$p_\xi = \sqrt{2}\frac{d}{d\alpha}K_{\alpha\xi}\,. \tag{5.72}$$

This gives rise to the identity

$$\langle p_\xi, p_{\xi'}\rangle_\Sigma = 2\frac{d^2}{d\alpha d\beta}\langle K_{\alpha\xi}, K_{\beta\xi'}\rangle = \{\xi', \xi\} \tag{5.73}$$

which is the complex conjugate of the inner product in the complex Hilbert space of solutions L_Σ. Hence, the map $p : L_\Sigma \to L_\Sigma^*$ with $\xi \mapsto p(\xi) = p_\xi$ is exactly the isometric anti-isomorphism[12] known from Riesz Theorem between the set of solutions L_Σ and the dual space L_Σ^* of continuous linear functionals on L_Σ. In other words, the set of the states (5.71) form the topological dual space L_Σ^* anti-isomorphic to the set of germs of solutions to the Euler-Lagrange equations L_Σ. For every one particle state we define a corresponding creation operator by its action on a state ψ as [64]

$$(a_\xi^\dagger\psi)(\phi) = p_\xi(\phi)\psi(\phi) = \frac{1}{\sqrt{2}}\{\xi, \phi\}_\Sigma\psi(\phi)\,. \tag{5.74}$$

In particular, applied to the vacuum state $\psi_{\Sigma;0}$ we get the one particle state, i.e. $a_\xi^\dagger\psi_{\Sigma;0} = p_\xi$. Furthermore, we define the annihilation operators on $\mathcal{H}_\Sigma$ as

$$(a_\xi\psi)(\phi) = \langle K_\phi, a_\xi\psi\rangle_\Sigma = \langle a_\xi^\dagger K_\phi, \psi\rangle_\Sigma\,. \tag{5.75}$$

When evaluated on a coherent state the action of a_ξ turns out to be

$$a_\xi K_\phi = \frac{1}{\sqrt{2}}\{\phi, \xi\}_\Sigma K_\phi$$

In particular, all annihilation operators annihilate the vacuum $\psi_{\Sigma;0} = K_0$, i.e. $a_\xi\psi_{\Sigma;0} = 0$ for all $\xi \in L_\Sigma$. By subsequent application of the ladder operators we obtain commutation relations

$$[a_\xi, a_\eta^\dagger] = \{\eta, \xi\}_\Sigma, \qquad [a_\xi, a_\eta] = 0, \qquad [a_\xi^\dagger, a_\eta^\dagger] = 0\,. \tag{5.76}$$

and the Fock structure on $\mathcal{H}_\Sigma^h$.

In the following chapters we will apply the GBF to two different setups in Lorentzian spacetimes. We start by investigating the Unruh effect in the framework of the GBF. On the one hand, this will help us to learn more about the structure and the elements of the GBF. On the other hand, the completely different viewpoint of the GBF will help to shed some light on the Unruh effect itself.

[12] An anti-isomorphism p between the complex Hilbert spaces L_Σ and L_Σ^* is a linear bijective map such that for all $\xi \in L_\Sigma$ holds $p(J_\Sigma\xi) = -ip(\xi)$. This is because in our case the complex multiplication on L_Σ is the application of J_Σ. The map p is isometric if it is compatible with the inner product in the sense of Equation (5.73).

Chapter 6

Application of background-independent QFT: Unruh effect from the GBF perspective

In this section I will present results I have worked out together with Daniele Colosi (UNAM, Campus Morelia) about the Unruh effect in the GBF which we presented in [121]. This application is of immediate importance for the GBF program since it represents a concrete application of the very recently developed quantization of observables [61]. Moreover, for the computations I will present here mixed states where used for the first time within the GBF.

In the standard formulation of QFT as well as in algebraic QFT the Unruh effect is understood as a particular relation between the vacuum state in Minkowski space and a certain thermal state in Rindler space. Rindler space can be embedded in $3+1$-dimensional Minkowski space $\mathcal{M}_{\mathrm{Mink}_3}$ as the right Rindler wedge $\mathcal{R}\times\mathbb{R}^2 \subset \mathcal{M}_{\mathrm{Mink}_3}$ where $\mathcal{R}$ is defined as $\mathcal{R} := \{x \in \mathcal{M}_{\mathrm{Mink}_1} : \eta(x,x) \leq 0,\ \tilde{x} > 0\}$ where $\eta = \mathrm{diag}(1,-1)$ the $1+1$-dimensional Minkowski metric and $\tilde{x}$ is the space component of the Cartesian coordinates on $1+1$-dimensional Minkowski space $\mathcal{M}_{\mathrm{Mink}_1}$ and the embedding is induced by the identification of the time and space component of the Cartesian coordinates of $\mathcal{M}_{\mathrm{Mink}_1}$ with the time and the first space component of $\mathcal{M}_{\mathrm{Mink}_3}$. The $1+1$-dimensional Rindler wedge $\mathcal{R}$ is covered by the Rindler coordinates (ρ,η) such that $\rho \in \mathbb{R}^+$ and $\eta \in \mathbb{R}$. If Minkowski space is considered as a model for spacetime a uniformly accelerated observer in spacetime moving along the curve $\gamma(\tau) = (1,\tau)$ parameterized with proper time τ can only communicate with points in the Rindler wedge $\mathcal{R}\times\mathbb{R}^2$. Hence, $\mathcal{R}\times\mathbb{R}^2$ can be associated with the spacetime seen by the linearly, uniformly accelerated observer and the Unruh effect is stated as follows: *"uniformly accelerated observers in Minkowski spacetime, i.e., linearly accelerated observers with constant proper acceleration also called Rindler observers, associate a thermal bath of Rindler particles also called Fulling-Rindler particles to the no-particle state of inertial observers also called the Minkowski vacuum. Rindler particles are associated with positive-energy modes as defined by Rindler observers in contrast to Minkowski particles, which are associated with positive energy modes as defined by inertial observers."*(page 2 in [76]).

This interpretation of the stated relation between the vacuum state defined in Minkowski space and the thermal state of Rindler space was first proposed by Unruh in 1976 [77]. Since then, it received a considerable amount of attention in the community because of its relation to other effects, like the particle creation from black holes (Hawking effect) and cosmological horizons. We have to emphasize however, that the Unruh effect is still a completely theoretic result. Although, there are a lot of proposals around aiming at an experimental detection of the Unruh effect [76] it was not discovered, yet. Hence, it is of extreme importance to ensure that the Unruh effect has at least a stable mathematical foundation in the framework of QFT.

In fact, the Unruh effect must be distinguished from the well known result that a uniformly accelerated Unruh-DeWitt detector responds as if submersed in a thermal bath when interacting with a quantum field in the Minkowski vacuum state [77]. In particular, the derivation of the Unruh effect in the standard formulation of QFT is done by first, identifying formally the vacuum state in Minkowski with an entangled state containing linear combinations of products of n-particle states of the field defined in the left and right Rindler wedges where the left Rindler wedge is the point reflection of the right Rindler wedge at the origin of Minkowski space. Then tracing out the degrees of freedom in the left Rindler wedge leads to a density operator in the right Rindler wedge describing a mixed thermal state at the Unruh temperature [76–82].

The derivation of the Unruh effect in algebraic QFT is much more sophisticated but works primarily along the same line of argument [91]: It is rigorously proven that the restriction of the Minkowski vacuum state to the right Rindler wedge is identical to a certain thermal state ψ in Rindler space where a thermal state in algebraic QFT is defined as a state fulfilling the KMS condition formally given as the identity of expectation values $\langle A(\tau)B\rangle_\psi = \langle BA(\tau + i\beta)\rangle_\psi$, for all observables A, B where $A(\tau)$ is the time translation of A [91]. Then, the particle content of this thermal state is investigated by expressing it as a density matrix in Rindler space which is well known to be only approximately possible (page 117-118 of [89], page 89 of [90]).

The mathematical foundation of the derivation in the standard formulation of QFT was criticized sharply by Narozhnyi et.al. in [83–87] which led to an answer by Fulling and Unruh in [88] and a reply by Narozhnyi et.al. in [122]. The central point of the criticism by Narozhnyi et.al. is that in the derivation of the Unruh effect in the standard formulation of QFT a certain term in the mode expansion of the scalar field is neglected which is equivalent to the requirement of an additional boundary condition at the origin of Minkowski space leading to a topological different spacetime.

We will obtain the same mathematical problem when investigating the GBF of the Klein-Gordon field in the Rindler wedge. However, instead of trying to establish a direct map between the Hilbert spaces associated to Minkowski space and the left and right Rindler wedge we will retreat to the identification of expectation values of local observables, i.e. observables restricted to open subsets of the Rindler wedge, in the Minkowski vacuum state and the Rindler thermal state. This result is of course rather a hint of the original Unruh effect than a proof of the effect in its full generality. However, from a physical perspective, the restriction of the set of observables

seen by the Rindler observer to the interior of the Rindler wedge is a reasonable condition since no observer should be able to measure at its spacelike infinity.
For the problems arising in the derivation of the Unruh effect in algebraic QFT we refer the interested reader to the article by Earman [90].
In the following we will investigate the Unruh effect for the $1+1$-dimensional Rindler wedge $\mathcal{R}$ embedded in $1+1$-dimensional Minkowski space. This case already contains everything we need to discuss the Unruh effect from the GBF perspective and is immediately generalizable to $3+1$ dimensions. This can be seen from the fact that the mode expansion in the additional directions in the $3+1$-dimensional Rindler wedge $\mathcal{R}\times\mathbb{R}^2$ contribute to the mode expansion in $\mathcal{R}$ like a variable mass term. The Unruh effect in $1+1$ dimensions however, is mass independent.

6.1 GBF in Minkowski space

We start with the action for the real massive Klein-Gordon field on $1+1$-dimensional Minkowski spacetime $\mathcal{M}_{\text{Mink}} = (\mathbb{R}^2, \eta = \text{diag}(1,-1))$[1] which is given by

$$S[\phi] = \frac{1}{2}\int \mathrm{d}^2x \left(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right) . \tag{6.1}$$

We consider the region $M \subset \mathcal{M}$ bounded by the disjoint union of two spacelike hypersurfaces represented by two equal time hyperplanes which we denote as $\Sigma_{1,2} : \{t = t_{1,2}\}$, i.e. $M = \mathbb{R}\times[t_1,t_2]$. The boundary of the region M corresponds to the disjoint union $\partial M = \Sigma_1 \cup \overline{\Sigma_2}$. The set of solutions of the equations of motion in the neighborhood of ∂M decomposes as a direct sum as $L_\Sigma = L_{\Sigma_1}\oplus L_{\overline{\Sigma}_2}$ where L_{Σ_1} and $L_{\overline{\Sigma}_2}$ are the sets of solutions in the neighborhood of Σ_1 and $\overline{\Sigma}_2$ respectively each equipped with the corresponding symplectic form

$$\omega_{\Sigma_i}(\phi,\phi') = \frac{1}{2}\int_{\mathbb{R}} d\tilde{x}\,\left(\phi\partial_t\phi' - \phi'\partial_t\phi\right)(\tilde{x})\,, \tag{6.2}$$

where $\omega_{\Sigma_2} = -\omega_{\overline{\Sigma}_2}$ and $\tilde{x}$ is the coordinate along the hypersurfaces.
We now proceed to the quantization according to the GBF: We will define the relevant algebraic structures we introduced in Chapter 5. We start with the complex structure associated to Minkowski space. It can be given by the expression [123]

$$J_{\Sigma_i} = \frac{\partial_t}{\sqrt{-\partial_t^2}}, \tag{6.3}$$

where $J_{\Sigma_2} = -J_{\overline{\Sigma}_2}$ which defines a unitary complex structure on L_Σ in the sense that it is compatible with the dynamics of the field, i.e. $J_{\overline{\Sigma}_2} = -T\circ J_{\Sigma_1}\circ T^{-1}$ where T is the evolution map define in Section 5.8. The corresponding Hilbert space associated with ∂M is given by the tensor product $\mathcal{H}_{\partial M} = \mathcal{H}_{\Sigma_1}\otimes\mathcal{H}_{\overline{\Sigma}_2}$, where $\mathcal{H}_{\Sigma_1}$ and $\mathcal{H}_{\overline{\Sigma}_2}$ are the Hilbert spaces associated with the hypersurface Σ_1 and $\overline{\Sigma}_2$ respectively and the inversion of the orientation is encoded by the involution $\iota : \mathcal{H}_{\Sigma_2}\to\mathcal{H}_{\overline{\Sigma}_2}$, $\psi\mapsto\overline{\psi}$ (see Section 5.6).

[1] We drop the index from now on. Whether it $3+1$- or $1+1$-dimensional Minkowski space is meant should be clear from the context.

In order to provide explicit expressions for the action of the complex structure we expand the scalar field in a complete basis of solutions of the Euler-Lagrange equations to the action (6.1)

$$\phi(\tilde{x}, t) = \int dp \, (\phi(p)\psi_p(\tilde{x}, t) + c.c.) \,, \tag{6.4}$$

where $\psi_p(\tilde{x}, t)$ are chosen to be the eigenfunctions of the boost generator, namely the boost modes[2] [3]

$$\begin{aligned}\psi_p(\tilde{x}, t) &= \frac{1}{2^{3/2}\pi} \int_{-\infty}^{\infty} \mathrm{d}q \, \exp\left(im(\tilde{x} \sinh q - t \cosh q) - ipq\right) \\ &= e^{-i\omega t} \frac{1}{2^{3/2}\pi} \int_{-\infty}^{\infty} \mathrm{d}q \, \exp\left(im\tilde{x} \sinh q - ipq\right), \end{aligned} \tag{6.5}$$

where we have introduced the operator $\omega = \sqrt{-\partial_{\tilde{x}}^2 + m^2}$. These modes are normalized as

$$\omega_{\Sigma_i}(\overline{\psi_p}, \psi_{p'}) = \delta(p - p'), \qquad \omega_{\Sigma_i}(\psi_p, \psi_{p'}) = \omega_{\Sigma_i}(\overline{\psi_p}, \overline{\psi_{p'}}) = 0. \tag{6.6}$$

The boost modes (6.5) are eigenfunctions of the complex structure (6.3), i.e. $J_{\Sigma_i}\psi_p = -i\psi_p$. Then, the symplectic form $\omega_{\Sigma_i}(\cdot, \cdot)$, the metric $g_{\Sigma_i}(\cdot, \cdot)$ and the inner product $\{\cdot, \cdot\}_{\Sigma_i}$, evaluated for two solutions $\phi, \phi' \in L_{\Sigma_i}$ $(i = 1, 2)$ take the form

$$\omega_{\Sigma_i}(\phi, \phi') = \frac{i}{2} \int_{-\infty}^{\infty} dp \, \left(\overline{\phi(p)}\phi'(p) - \phi(p)\overline{\phi'(p)}\right), \tag{6.7}$$

$$g_{\Sigma_i}(\phi, \phi') = 2\omega_{\Sigma_i}(\phi, J_{\Sigma_i}\phi') = \int_{-\infty}^{\infty} dp \, \left(\overline{\phi(p)}\phi'(p) + \phi(p)\overline{\phi'(p)}\right), \tag{6.8}$$

$$\{\phi, \phi'\}_{\Sigma_i} = g_{\Sigma_i}(\phi, \phi') + 2i\omega_{\Sigma_i}(\phi, \phi') = 2 \int_{-\infty}^{\infty} dp \, \phi(p)\overline{\phi'(p)}. \tag{6.9}$$

As explained in Chapter 5, these algebraic structures are all we need for the holomorphic representation of the GBF in Minkowski space. In the next section, we will derive the equivalent objects for the GBF of the scalar field on Rindler space.

6.2 GBF in Rindler space

For the quantization of the scalar field in Rindler spacetime we consider again the action in Equation (6.1) but restricted to the right Rindler wedge of $1 + 1$-dimensional Minkowski space given as $\mathcal{R} = \{x \in \mathcal{M}_{\text{Mink}} : \eta(x, x) \leq 0, \, \tilde{x} > 0\}$, which is covered by the Rindler coordinates (ρ, η) such that $\rho \in \mathbb{R}^+$ and $\eta \in \mathbb{R}$. The Cartesian coordinates $(t, \tilde{x})$ and the Rindler coordinates are related by the identity $t = \rho \sinh \eta$ and $\tilde{x} = \rho \cosh \eta$, on $\mathcal{R}$ and the metric of Rindler space results to be $ds^2 = \rho^2 d\eta^2 - d\rho^2$. We consider the region $R \subset \mathcal{R}$ bounded by the disjoint union of two equal-Rindler-time hyperplanes $\Sigma_{1,2}^R : \{\eta = \eta_{1,2}\}$, i.e. $R = \mathbb{R}^+ \times [\eta_1, \eta_2]$. Then, we obtain from Equation (5.30) for the symplectic form

$$\omega_{\Sigma_i^R}(\phi, \phi') = \frac{1}{2} \int_0^{\infty} \frac{d\rho}{\rho} \, \left(\phi \partial_\eta \phi' - \phi' \partial_\eta \phi\right)(\rho) \,. \tag{6.10}$$

[2] It is assumed that an infinitely small imaginary part is added to t. Moreover, the integral over p in (6.4) must be extended from $-\infty$ to $+\infty$.

[3] Usually, in the standard formulation of QFT in Minkowski, the expansion is given in the basis of plane wave solutions. However, it turns out to be more convenient for our purposes to use the boost modes. It can be shown that this expansion is equivalent to the one in the plane wave basis.

The complex structure given by the derivative with respect to the Rindler time coordinate η as

$$J_{\Sigma_i^R} = \frac{\partial_\eta}{\sqrt{-\partial_\eta^2}}\,, \tag{6.11}$$

destinguishes between positive and negative Rindler energy modes. Hence, it defines the vacuum state in Rindler space with respect to the time translation Killing vector field ∂_η (see [123, 124]). In order to repeat the construct of the quantum theory implemented in Minkowski spacetime, we start by expanding the field in a complete set of solutions of the Euler-Lagrange equations,

$$\phi^R(x) = \int_0^\infty \mathrm{d}p\, \left(\phi^R(p)\phi_p^R(x) + c.c.\right), \tag{6.12}$$

where the Fulling modes [124] ϕ_p^R read

$$\begin{aligned} \phi_p^R(x) &= \frac{(\sinh(p\pi))^{1/2}}{\pi} K_{ip}(m\rho)e^{-ip\eta} \\ &= (2\sinh(p\pi))^{1/2}e^{-p\pi/2}\,\psi_p(x), \qquad p>0, \end{aligned} \tag{6.13}$$

where K_{ip} is the modified Bessel function of the second kind, also known as Macdonald function [125]. The modes (6.13) are normalized as

$$\omega_{\Sigma_i^R}(\overline{\phi_p^R}, \phi_{p'}^R) = \delta(p-p'), \qquad \omega_{\Sigma_i^R}(\phi_p^R, \phi_{p'}^R) = \omega_{\Sigma_i}(\overline{\phi_p^R}, \overline{\phi_{p'}^R}) = 0\,. \tag{6.14}$$

and eigenfunctions of the complex structure (6.11), i.e. $J_{\Sigma_i^R}\phi_p^R = -i\phi_p^R$. The algebraic structures defined on the hypersurface Σ_i^R, considered for two solutions $\phi^R, \psi^R \in L_{\Sigma_i^R}$ result to be

$$\omega_{\Sigma_i^R}(\phi^R, \psi^R) = \frac{i}{2}\int_0^\infty \mathrm{d}p\, \left(\overline{\phi^R(p)}\psi^R(p) - \phi^R(p)\overline{\psi^R(p)}\right), \tag{6.15}$$

$$g_{\Sigma_i^R}(\phi^R, \psi^R) = \int_0^\infty \mathrm{d}p\, \left(\overline{\phi^R(p)}\psi^R(p) + \phi^R(p)\overline{\psi^R(p)}\right), \tag{6.16}$$

$$\{\phi^R, \psi^R\}_{\Sigma_i^R} = g_{\Sigma_i^R}(\phi^R, \psi^R) + 2i\omega_{\Sigma_i^R}(\phi^R, \psi^R) = 2\int_0^\infty \mathrm{d}p\, \phi^R(p)\overline{\psi^R(p)}. \tag{6.17}$$

Now, these structures allow for the implementation of the holomorphic representation within the GBF for the quantum scalar field in Rindler space.

It is important to notice that in order for the quantum theory in Rindler space to be well defined the condition $\phi^R(\rho=0,\eta)=0$ must be imposed. Indeed, when Rindler space is embedded in Minkowski space and the complex structure $J_{\Sigma_i^R}$ is expressed in Cartesian coordinates (x,t) we get the expression

$$J_{\Sigma_i^R} = \frac{\tilde{x}\partial_t + t\partial_{\tilde{x}}}{\sqrt{-(\tilde{x}\partial_t + t\partial_{\tilde{x}})^2}}. \tag{6.18}$$

which is well defined only outside of the origin of Minkowski space. Whereas the condition $\phi^R(\rho=0,\eta)=0$ is just the usual condition expressing that the field must vanish at spacelike infinity in Rindler space the condition that the field should vanish at the origin of Minkowski space is an additional condition. This condition has been discussed already by Narozhny et.al. in [83–87, 122] where the authors emphasize the relevance of the condition in the derivation of the Unruh effect in the standard formulation of QFT as well as algebraic QFT questioning the existence of the Unruh effect itself.

To avoid all the mathematical issues concerning the boundary condition at the origin of Minkowski space we will use an approach different from the standard one in the next section. In particular, we will not try to construct a map between Hilbert spaces. Inspired by some results derived within the algebraic approach to quantum field theory[4] we will compare the expectation value of a generic Weyl observable defined on a compact spacetime region in the interior of the Rindler wedge for the vacuum state in Minkowski space with that for a certain mixed state in Rindler space not making any use of the embedding of Rindler space and Minkowski space in the formalism. It will turn out that these two expectation values are equal when the observables are quantized according to the Feynman quantization prescription which can be interpreted as the appearance of the Unruh effect within the GBF. Let me remark that the coincidence of the expectation values is weaker than the original statement of the Unruh effect which implies the latter. However, as I explained above, the original statement of the Unruh effect using a map between Hilbert spaces is certainly to strong to be shown mathematically rigorously in the framework of the GBF. As the results of [83–87, 122] suggest it might be even to strong to be shown mathematically rigorously in the standard formulation of QFT.

6.3 The relation between operator amplitudes on Minkowski and Rindler space

In the following we will calculate the expectation values of a Weyl observable

$$W(\phi) = \exp\left(\int_M d^2x \mu(x)\phi(x)\right) \tag{6.19}$$

with μ supported on a compact subregion of $\mathcal{R}$ first, for the vacuum state of the GBF for the Klein-Gordon field in Minkowski space $\psi_{M;0} = K_0 \otimes \overline{K_0} \in \mathcal{H}_{\Sigma_1^R} \hat{\otimes} \mathcal{H}_{\overline{\Sigma_2^R}}$ and then, for the following mixed state in Rindler space:

$$D = \prod_p N_p^2 \sum_{n=0}^{\infty} e^{-2\pi np} \frac{2^n}{n!} \frac{\delta^n}{\delta\xi_1(p)^n} \frac{\delta^n}{\delta\xi_2(p)^n} K_{\xi_1} \otimes \overline{K_{\xi_2}}\Bigg|_{\xi_1=\xi_2=0} \tag{6.20}$$

where $K_{\xi_1} \in \mathcal{H}_{\Sigma_1^R}$ and $K_{\xi_2} \in \mathcal{H}_{\Sigma_2^R}$ are the coherent states corresponding to $\xi_i = \frac{1}{\sqrt{2}} \int_0^\infty dp\, \xi_i(p)(\phi_p^R + c.c.) \in L_{\Sigma_i^R}$ $(i = 1, 2)$ where $\xi_i(p)$ is a real function and the normalization factors are given as $N_p = (1 - e^{-2\pi p})^{1/2}$. Note that the mixed state in Equation (6.20) is equivalent to the one used in the literature of the standard formulation of the Unruh effect [76]. The expectation values of the Weyl observable (6.19) will be calculated using both the Feynman and Berezin-Toeplitz quantization scheme.

For the vacuum state in Minkowski space we obtain with (5.59) the amplitude

$$\rho_M^\mu(\psi_{M;0}) = \exp\left(\frac{i}{2}\int_M d^2x d^2x'\, \mu(x) G_F^J(x, x') \mu(x')\right) . \tag{6.21}$$

[4] We refer in particular to Fell's theorem [126] and the work of Verch [127] and of Sewell [91].

Due to the unitarity of the complex structure we can use the expression in Equation (5.66) for the Feynman propagator which gives

$$G_F^{\mathrm{Mink}}(x,x') = i\int_{-\infty}^{\infty} dp\left(\theta(t'-t)\overline{\psi_p(x)}\psi_p(x') + \theta(t-t')\psi_p(x)\overline{\psi_p(x')}\right) . \tag{6.22}$$

When x and x' are restricted to the right Rindler wedge we can express $G_F^{\mathrm{Mink}}(x,x')$ in terms of the boost modes. Using the definition of the boost modes in (6.5) and integral representations for the Bessel function

$$K_{ip}(m\rho) = \frac{1}{\cosh\left(\frac{\pi p}{2}\right)}\int_0^{\infty} dq\, \cos(m\rho\sinh q)\cos(pq) \tag{6.23}$$

$$= \frac{1}{\sinh\left(\frac{\pi p}{2}\right)}\int_0^{\infty} dq\, \sin(m\rho\sinh q)\sin(pq) \tag{6.24}$$

we find that

$$\begin{aligned}\psi_p(x) &= (2\sinh(p\pi))^{-1/2}e^{\frac{\pi p}{2}}\phi_p^R(x)\\ \psi_{-p}(x) &= (2\sinh(p\pi))^{-1/2}e^{-\frac{\pi p}{2}}\overline{\phi_p^R(x)}\,, \qquad p>0\,.\end{aligned} \tag{6.25}$$

With some rearrangements we find for the Feynman propagator

$$\begin{aligned}G_F^{\mathrm{Mink}}(x,x') &= i\int_0^{\infty} dp\frac{e^{\pi p}}{2\sinh\pi p}\left(\theta(\eta'-\eta)\overline{\phi_p^R(x)}\phi_p^R(x') + \theta(\eta-\eta')\phi_p^R(x)\overline{\phi_p^R(x')}\right) \\ &\quad +i\int_0^{\infty} dp\frac{e^{-\pi p}}{2\sinh\pi p}\left(\theta(\eta-\eta')\overline{\phi_p^R(x)}\phi_p^R(x') + \theta(\eta'-\eta)\phi_p^R(x)\overline{\phi_p^R(x')}\right)\end{aligned} \tag{6.26}$$

where we used that η is a strictly increasing function of t [5].

Let us now turn to Rindler space and the mixed state D in (6.20). The corresponding observable map for the Weyl observable (6.19) is

$$\rho_R^W(D) = \prod_p N_p^2\sum_{n=0}^{\infty} e^{-2\pi np}\frac{2^n}{n!}\frac{\delta^n}{\delta\xi_1(p)^n}\frac{\delta^n}{\delta\xi_2(p)^n}\rho_R^W(K_{\xi_1}\otimes\overline{K_{\xi_2}})\Bigg|_{\xi_1=\xi_2=0} . \tag{6.27}$$

Having a look at the expression in Equation (5.59) for the operator amplitude $\rho_R^W(K_{\xi_1}\otimes\overline{K_{\xi_2}})$ for coherent state $K_\xi = K_{\xi_1}\otimes\overline{K_{\xi_2}}$ we find that we have to calculate the following three terms to evaluate the derivatives in Equation (6.27):

- the free amplitude $\rho_R(K_\xi) = \rho_R(K_{\xi_1}\otimes\overline{K_{\xi_2}})$ can be computed using (5.44), where in the present context $\xi^R = \xi_1+\xi_2$ and $\xi^I = \xi_1-\xi_2$, leading to

$$\rho_R(K_{\xi_1}\otimes\overline{K_{\xi_2}}) = \exp\left(\frac{1}{2}\int_0^{\infty}\mathrm{d}p\,\xi_1(p)\xi_2(p)\right), \tag{6.28}$$

- the Weyl observable evaluated on the complex solution $\hat{\xi}$ given in this case by

$$\hat{\xi}(x) = \xi^R(x) - i\xi^I(x) = \frac{1}{\sqrt{2}}\int_0^{\infty}\mathrm{d}p\left(\phi_p^R(x)\xi_1(p) + \overline{\phi_p^R(x)}\xi_2(p)\right), \tag{6.29}$$

[5] As in the foregoing section, x is used as global notation for the Rindler coordinates (ρ,η).

- the last term in the r.h.s of (5.59) contains the Feynman propagator in Rindler space which is given as

$$G_F^{\mathcal{R}}(x,x') = i\int dp\,\left(\theta(\eta'-\eta)\overline{\phi_p^R(x)}\phi_p^R(x') + \theta(\eta-\eta')\phi_p^R(x)\overline{\phi_p^R(x')}\right). \tag{6.30}$$

The observable map (6.27) can then be written as

$$\begin{aligned}\rho_R^W(D) &= \prod_p N_p^2 \sum_{n=0}^{\infty} e^{-2\pi np}\frac{\delta^n}{\delta\xi_1(p)^n}\frac{\delta^n}{\delta\xi_2(p)^n}\\ &\times \exp\left(\frac{1}{2}\,\xi_1(p)\xi_2(p) + \frac{i}{\sqrt{2}}\int \mathrm{d}^2x\,\mu(x)\left(\phi_k^R(x)\xi_1(p) + \overline{\phi_k^R(x)}\xi_2(p)\right)\right)\Bigg|_{\xi_1=\xi_2=0}\\ &\times \exp\left(\frac{i}{2}\int \mathrm{d}^2x\,\mathrm{d}^2x'\,\mu(x)G_F^{\mathcal{R}}(x,x')\mu(x')\right).\end{aligned} \tag{6.31}$$

We proceed by evaluating the first line in the r.h.s. of (6.31) by applying the general Leibniz rule

$$\frac{\mathrm{d}^n}{\mathrm{d}\gamma^n}f(\gamma)g(\gamma) = \sum_{s=0}^{n}\binom{n}{s}\frac{d^{n-s}}{d\gamma^{n-s}}f(\gamma)\frac{d^s}{d\gamma^s}g(\gamma), \tag{6.32}$$

and using the relation

$$\sum_{s=0}^{\infty}\frac{(s+n)!}{s!n!}e^{-2\pi sp} = \frac{1}{(1-e^{-2\pi p})^{n+1}}, \tag{6.33}$$

which we prove in the Appendix A. We obtain

$$\begin{aligned}& N_p^2\sum_{n=0}^{\infty} e^{-2\pi np}\frac{2^n}{n!}\frac{\delta^n}{\delta\xi_1(p)^n}\frac{\delta^n}{\delta\xi_2(p)^n}\exp\left(\frac{1}{2}\,\xi_1(p)\xi_2(p)\right)\times\\ &\times \exp\left(\frac{i}{\sqrt{2}}\int \mathrm{d}^2x\,\mu(x)\left(\phi_k^R(x)\xi_1(p) + \overline{\phi_k^R(x)}\xi_2(p)\right)\right)\Bigg|_{\xi_1=\xi_2=0}\\ &= N_p^2\sum_{n=0}^{\infty} e^{-2\pi np}\left(-\int \mathrm{d}^2x\,\mathrm{d}^2x'\,\mu(x)\mu(x')\phi_p^R(x)\overline{\phi_p^R(x')}\right)^n\frac{1}{n!}\sum_{s=0}^{\infty}\frac{(s+n)!}{s!n!}e^{-2\pi sp}\\ &= \exp\left(-\frac{e^{-\pi p}}{2\sinh(\pi p)}\int \mathrm{d}^2x\,\mathrm{d}^2x'\,\mu(x)\mu(x')\phi_p^R(x)\overline{\phi_p^R(x')}\right).\end{aligned} \tag{6.34}$$

Hence, substituting in (6.31) we obtain after some rearrangements

$$\rho_R^W(D) = \exp\left(\frac{i}{2}\int \mathrm{d}^2x\,\mathrm{d}^2x'\,\mu(x)\left[i\int_0^{\infty}dp\,\phi_p^R(x)\overline{\phi_p^R(x')}\frac{e^{-\pi p}}{\sinh(\pi p)} + G_F^{\mathcal{R}}(x,x')\right]\mu(x')\right). \tag{6.35}$$

Using the expression in Equation (6.30) for the Feynman propagator in Rindler space we find that

$$\begin{aligned}\rho_R^W(D) &= \exp\left(\frac{i}{2}\int \mathrm{d}^2x\,\mathrm{d}^2x'\,\mu(x)i\int_0^{\infty}dp\,\phi_p^R(x)\overline{\phi_p^R(x')}\frac{e^{-\pi p} + (e^{\pi p}-e^{-\pi p})\theta(\eta-\eta')}{\sinh(\pi p)}\mu(x')\right)\\ &= \exp\left(\frac{i}{2}\int \mathrm{d}^2x\,\mathrm{d}^2x'\,\mu(x)i\int_0^{\infty}dp\,\phi_p^R(x)\overline{\phi_p^R(x')}\frac{e^{-\pi p}\theta(\eta'-\eta) + e^{\pi p}\theta(\eta-\eta')}{\sinh(\pi p)}\mu(x')\right)\\ &= \exp\left(\frac{i}{2}\int \mathrm{d}^2x\,\mathrm{d}^2x'\,\mu(x)G_F^{\mathrm{Mink}}(x,x')\mu(x')\right)\end{aligned} \tag{6.36}$$

which coincides with the observable amplitude (6.21) computed in Minkowski spacetime for the vacuum state.

As we saw in Equation (5.70) the operator amplitude for the Weyl observable W in the Berezin-Toeplitz quantization scheme for the state ψ can be obtained by taking as the operator amplitude for the vacuum state the modulus square of the respective amplitude in the Feynman quantization scheme. Hence, we find

$$\rho_{\text{Mink}}^{\blacktriangleleft W \blacktriangleright}(\psi_{\partial M;0}) = \exp\left(-\frac{1}{2}\int \mathrm{d}^2x\,\mathrm{d}^2x'\,\mu(x)2\mathfrak{Im}G_F^{\text{Mink}}(x,x')\mu(x')\right) \tag{6.37}$$
$$\rho_{R}^{\blacktriangleleft W \blacktriangleright}(D) = \exp\left(-\frac{1}{2}\int \mathrm{d}^2x\,\mathrm{d}^2x'\,\mu(x)\left[\int_0^\infty dp\,\phi_p^R(x)\overline{\phi_p^R(x')}\frac{e^{-\pi p}}{\sinh(\pi p)} + 2\mathfrak{Im}G_F^{\mathcal{R}}(x,x')\right]\mu(x')\right).$$

With the result for the Feynman quantization in (6.36) we obtain that for the two amplitudes (6.37) to be equal first, the following equality must hold

$$\begin{aligned}&\int \mathrm{d}^2x\,\mathrm{d}^2x'\,\mu(x)\,2\mathfrak{Re}\int_0^\infty dp\,\phi_p^R(x)\overline{\phi_p^R(x')}\frac{e^{-\pi p}}{\sinh(\pi p)}\mu(x')\\ &= \int \mathrm{d}^2x\,\mathrm{d}^2x'\,\mu(x)\int_0^\infty dp\,\phi_p^R(x)\overline{\phi_p^R(x')}\frac{e^{-\pi p}}{\sinh(\pi p)}\mu(x')\end{aligned} \tag{6.38}$$

which can only be true if both sides vanish and second, the Feynman propagator of the Minkowski quantization and the one of the Rindler quantization must be equal when restricted to the right Rindler wedge which is obviously not true. Hence, we find that there is no ground for the Unruh effect when adopting the Berezin-Toeplitz prescription for quantizing local observables.

Remark, that for derivation of the Unruh effect in the GBF in this chapter we were considering a region bounded by two spacelike hypersurfaces. Since the boundary of the region considered itself has a boundary, namely the origin, the setup we used was already much more general than the standard formulation of QFT. In the next chapter, however, we will finally leave the realm of the standard formulation of QFT far behind by applying the GBF to regions bounded by timelike hypersurfaces. The aim will be to derive conditions for the vacuum state on timelike hypersurfaces using the response of an Unruh-DeWitt detector.

Chapter 7

Application of background-independent QFT: quantum scalar field on a tensorial spacetime

General linear electrodynamics on are metric spacetimes (as one particular example of tensorial spacetimes) was quantized and the Casimir effect in that theory was derived in [116]. Also, a classical massive particle defined in Section 2.6 was "first" quantized using standard methods of constraint quantization [128]. In this chapter, we will investigate the possibilities to establish a quantum field theory for the real massive scalar field theory defined in Section 4.1 on tensorial spacetimes. As we argued at the end of Section 3.3, quantum field theory on tensorial spacetimes would be needed to calculate decay rates for the vacuum Cherenkov process. Moreover, it would be interesting to consider elementary particle processes like for example in QED to calculate the dependence on the tensorial background. That could be one way to experimentally find deviations from Lorentzian spacetime geometries and to test the assumptions we made for the theory of tensorial spacetimes in Chapter 2. In particular, the quantized scalar field would be of interest in particle physics as the Higgs mechanism in the standard model is usually modeled using a scalar field. Furthermore, scalar field theories of higher derivative order appear in the theory of inflation [129] and quantum gravity [130–133].

Beside these physical arguments there is a good mathematical reason to consider the quantum theory of fields with higher derivative field equations, since they show generically a much better ultraviolet behavior, i.e. they are better renormalizable [131, 134, 135]. We will see this for the real scalar field in the following: Let us consider the action 4.3 for a real scalar field ϕ that was given as

$$S_{\mathcal{M}}[\phi] = \int_{\mathcal{M}} d^N x \, \left(P^{\mu_1 \dots \mu_r} \frac{1}{2} (\partial_{\mu_1} \cdots \partial_{\mu_{r/2}} \phi(x))(\partial_{\mu_{r/2+1}} \cdots \partial_{\mu_r} \phi(x)) - \frac{1}{2} m^r \phi^2(x) \right) . \tag{7.1}$$

Adding an interaction term of the type $\lambda\phi^\nu$ where $\nu \in \mathbb{N}$ with a real coupling constant λ, the action

$$S_{\mathcal{M};\mu}[\phi] = S[\phi] + \lambda \int d^N x \, \phi^\nu(x) \tag{7.2}$$

would be a valid action leading to an interacting quantum field theory. We can even look for the renormalizability of this theory using standard power counting techniques in the following way: With $\hbar = c = 1$ momenta have mass dimension 1 and every length has mass dimension -1. To get the action $S_{\mathcal{M}}[\phi]$ dimensionless, the mass dimension of the field ϕ has to be $(N - \mathrm{deg}P)/2$. To get the coupling term dimensionless, the interaction constant λ must have mass dimension $N - \nu(N - \mathrm{deg}P)/2$. To have a renormalizable theory, the mass dimension of the coupling constant ν must not be less than zero. This statement stems from the following argument that can be found in a slightly different form in [136]: Let us assume that we have regularized a formally infinite expression for a probability p that came up when we where considering the interacting theory given by (7.2) perturbatively to some order i in the coupling constant λ. Let us assume that λ is of mass dimension $\dim(\lambda)$. Let us assume further that we have done the regularization by imposing a cut-off constant κ on the momenta. Then, κ has obviously the mass dimension 1. Since the i-th order term contributing to p will be proportional to λ^i but must be non-dimensional, it must be also proportional to $\kappa^{i_1} m^{i_2}$ with $i_1 + i_2 = -i\dim(\lambda)$ since they are the only dimensionful constants appearing in the calculation. If we now assume that in the limit $m \to 0$ not all the contributions of the interaction vanish, we find that there must be cases in which the perturbative term of order i must be proportional to $\kappa^{-i\dim(\lambda)}$. Now, when removing the cut-off by taking $\kappa \to \infty$, this term would diverge for $\dim(\lambda) < 0$ which tells us that in that case the theory is non-renormalizable.

From the power counting argument we find for the action (7.2) the following necessary condition for renormalizability:

$$N - \nu(N - \mathrm{deg}P)/2 \geq 0\,. \tag{7.3}$$

Especially for $\deg P \geq N$, every interaction term of the form $\lambda\phi^\nu$ could lead to a renormalizable theory.

In the case of Lorentzian metric geometry, one can derive this statement in a mathematically very clear form, using the causal perturbation approach of Epstein and Glaser [137, 138] assuming that the free quantum field theory for ϕ is microcausal, i.e., for spacelike separated x and y we have

$$[\hat{\phi}^\nu(x), \hat{\phi}^\nu(y)] = 0\,, \tag{7.4}$$

for the field operator $\hat{\phi}$. To generalize this to other field theories we need of course a notion of spacelike separation. In the framework of tensorial spacetimes this follows from the definition of spacelike slices generalized to all hyperbolic covectors: We say that a hypersurface is spacelike if at every point x of that hypersurface its normal covector q corresponds to an observer vector, i.e. $L_x(q) \in C_x^\#$ where $C_x^\#$ is the observer cone to P_x. Since we assumed to be working in a coordinate system in which the components of the polarization tensor $P^{\mu_1\cdots\mu_r}$ are constant and $\mathcal{M} \cong \mathbb{R}^n$, we can identify $T_x\mathcal{M}$ with $\mathcal{M}$ for every point $x \in \mathcal{M}$ and define that two points x and y are spacelike separated if there exists a covector q in $L^{-1}(C^\#)$ such that $q(x - y) = 0$. This definition leads to a well defined generalization of causal perturbation theory. However, it turns out that to obtain a microcausal scalar field theory on tensorial spacetimes obeying canonical commutation relations (CCRs), additional solutions that do not correspond to classical particles have to be included in the theory. In the following section we discuss this issue for the case of a dispersion relation of fourth order.

7.1 The quantum scalar field on tensorial spacetimes fourth degree dispersion relation

We will discuss the QFT obeying CCRs in the following for the case of a hyperbolic, time orientable and energy distinguishing polynomial which takes, in some coordinate system, the form

$$P(p) = p_0^4 - 2g(\vec{p})p_0^2 + h(\vec{p})\,, \tag{7.5}$$

where g is a bi-linear map on $\mathbb{R}^{n-1}$ and h is a quadri-linear map, i.e., $h : (\mathbb{R}^{n-1})^4 \to \mathbb{R}$. This case is closely related to the dispersion relation of meta class I area metrics presented in Section 3.4.

In the following, we will quantize the higher order field theory using the background-independent geometric quantization formalism presented in Chapter 5. We will only consider regions with boundaries consisting of hypersurfaces that are spacelike in the sense of Chapter 3, i.e. such that at every point x of the hypersurface the normal lies in the observer cone $C^{\#}_x$ to the polynomial P. In this sense, we will not be using the GBF itself, but just the mathematical framework used for the holomorphic representation of the GBF.

We assume that the hyperbolic polynomial P defining the dispersion relation is of the form (7.5) for some coordinates $(x_0, \vec{x})$. Let Σ_0 be a hyperplane at x_0 then the symplectic form defined in Equation (5.25) corresponding to the action 4.3 turns out to be

$$\begin{aligned} \omega_{x_0}(\phi,\phi') &= \frac{1}{4}\int d^{N-1}x\; P^{0b_1b_2b_3}\left(\phi(x)\overleftrightarrow{\partial_{b_1}}\partial_{b_2}\partial_{b_3}\phi'(x) - \phi'(x)\overleftrightarrow{\partial_{b_1}}\partial_{b_2}\partial_{b_3}\phi(x)\right) & (7.6)\\ &= \frac{1}{4}\int d^{N-1}x\left[\phi(x)\partial_0(\partial_0^2 - g^{ij}\partial_i\partial_j)\phi'(x) - (\partial_0^2 - g^{ij}\partial_i\partial_j)\phi'(x)\partial_0\phi(x)\right. & (7.7)\\ &\quad \left.-\phi(x)'\partial_0(\partial_0^2 - g^{ij}\partial_i\partial_j)\phi(x) + (\partial_0^2 - g^{ij}\partial_i\partial_j)\phi(x)\partial_0\phi'(x)\right]. & (7.8)\end{aligned}$$

We define

$$\omega_\pm(\vec{p}) := \begin{cases} \left(g(\vec{p}) \pm \left(g(\vec{p})^2 - h(\vec{p})\right)^{1/2}\right)^{1/2} & \text{for} \quad h(\vec{p}) \geq 0 \quad \text{or} \quad s = + \\ -i\tilde{\omega}_-(\vec{p}) & \text{for} \quad h(\vec{p}) < 0 \quad \text{and} \quad s = - \end{cases}, \tag{7.9}$$

where the energies are given as

$$\tilde{\omega}_-(\vec{p}) := \left(-g(\vec{p}) + \left(g(\vec{p})^2 - h(\vec{p})\right)^{1/2}\right)^{1/2}. \tag{7.10}$$

for $h(\vec{p}) < 0$. Then, we obtain a parameterization of the solutions to the scalar field equations (4.2) as

$$\phi(x) = \int \frac{d^{N-1}p}{(2\pi)^{N-1}} \sum_{s=\pm} \frac{1}{C(s,\vec{p})D_p^0 P(\omega_{s,p},\vec{p})}\left(\phi_s(\vec{p})\tilde{\varphi}_{s,p}(x) + c.c.\right). \tag{7.11}$$

where

$$\tilde{\varphi}_{s,p}(x) := \tilde{\varphi}_{0,s,p}(x_0)e^{-i\vec{p}\cdot\vec{x}}\,, \tag{7.12}$$

$$C(s,\vec{p}) := \begin{cases} 1 & \text{for} \quad h(\vec{p}) \geq 0 \quad \text{or} \quad s=+ \\ -i & \text{for} \quad h(\vec{p}) < 0 \quad \text{and} \quad s=- \end{cases}, \tag{7.13}$$

and

$$\tilde{\varphi}_{0,s,p}(x_0) := \begin{cases} e^{-i\omega_s(\vec{p})x_0} & \text{for} \quad h(\vec{p}) \geq 0 \quad \text{or} \quad s=+ \\ \cosh(\tilde{\omega}_-(\vec{p})x_0) + i\sinh(\tilde{\omega}_-(\vec{p})x_0) & \text{for} \quad h(\vec{p}) < 0 \quad \text{and} \quad s=- \end{cases}. \tag{7.14}$$

We call the two different types of solutions in Equation (7.9) propagating and evanescent waves, respectively. Using the translation invariance of the symplectic form, we obtain

$$\begin{aligned} 0 &= \partial_0 \omega_{x_0}(e^{i(\omega^*_{s'}(\vec{p}')x_0 + \vec{p}'\cdot\vec{x})}, e^{-i(\omega_s(\vec{p})x_0+\vec{p}\cdot\vec{x})}) && (7.15)\\ &= -i(\omega_s(\vec{p}) - \omega^*_{s'}(\vec{p}'))\omega_{x_0}(e^{i(\omega_{s'}(\vec{p}')x_0 + \vec{p}'\cdot\vec{x})}, e^{-i(\omega_s(\vec{p})x_0+\vec{p}\cdot\vec{x})}) && (7.16)\end{aligned}$$

from which we conclude that solutions with energies such that $\omega_s(\vec{p}) \neq \omega^*_{s'}(\vec{p}')$ are orthogonal. We find that

$$\omega_{x_0}(\varphi^*_{s',p'}, \varphi_{s,p}) = \frac{i}{4} C(s,\vec{p}) D^0_p P(\omega_s(\vec{p}),\vec{p})\delta_{s,s'}(2\pi)^{N-1}\delta^{(N-1)}(\vec{p}-\vec{p}') \tag{7.17}$$

and for general solutions

$$\omega_{x_0}(\phi,\phi') = \frac{i}{4}\int \frac{d^{N-1}p}{(2\pi)^{N-1}} \sum_{s=\pm} \frac{1}{C(s,\vec{p})D^0_p P(\omega_{s,p},\vec{p})} \left(\overline{\phi_s(\vec{p})}\phi'_s(\vec{p}) - c.c.\right) \tag{7.18}$$

Now we use the complex structure

$$J := \begin{cases} \frac{\partial_0}{\sqrt{-\partial_0^2}} & \text{for} \quad h(\vec{p}) \geq 0 \quad \text{or} \quad s=+ \\ \mathcal{P}\frac{\partial_0}{\sqrt{\partial_0^2}} & \text{for} \quad h(\vec{p}) < 0 \quad \text{and} \quad s=- \end{cases}, \tag{7.19}$$

where $\mathcal{P}\phi(x_0,\vec{x}) = \phi(-x_0,\vec{x})$. We obtain that J acts on coefficients $\phi_s(\vec{p})$ as

$$J\phi_s(\vec{p}) = -i\phi_s(\vec{p})\,,$$

and obtain the symmetric bilinear form $g_{x_0}(\phi,\phi') = 2\omega_{x_0}(\phi, J\phi')$ as

$$g_{x_0}(\phi,\phi') = 2\omega_{x_0}(\phi,J\phi') = \frac{1}{2}\int \frac{d^{N-1}p}{(2\pi)^{N-1}} \sum_{s=\pm} \frac{1}{C(s,\vec{p})D^0_p P(\omega_{s,p},\vec{p})} \left(\overline{\phi_s(\vec{p})}\phi'_s(\vec{p}) + c.c.\right) \tag{7.20}$$

and the inner product defined in (5.34) as $\{\phi',\phi\}_{x_0} = g_{x_0}(\phi,\phi') + 2i\omega_{x_0}(\phi,J\phi')$ turns out to be

$$\{\phi,\phi'\}_{x_0} = \int \frac{d^{N-1}p}{(2\pi)^{N-1}} \sum_{s=\pm} \frac{1}{C(s,\vec{p})D^0_p P(\omega_{s,p},\vec{p})} \phi_s(\vec{p})\overline{\phi'_s(\vec{p})} \tag{7.21}$$

Then we define the space of germs L_{Σ_0} at Σ_0 to the field equations (4.2) and the holomorphic Hilbert space $\mathcal{H}_{\Sigma_0}$ as in Section 5.5. By defining creation and annihilation operators on Σ_0 as explained in Section 5.10, we obtain the canonical commutation relations

$$[a_\xi, a^\dagger_\eta] = \{\eta,\xi\}_{x_0} \quad [a_\xi, a_\eta] = 0 = [a^\dagger_\xi, a^\dagger_\eta]\,. \tag{7.22}$$

for a_ξ and $a^\dagger_\eta$ corresponding to the real solutions $\xi, \eta \in L_{\Sigma_0}$ respectively. We can then define a field operator on Σ_0 to quantize interaction terms. As in canonical quantization we write

$$\phi(x) = \sum_{s=\pm} \int \frac{d^{N-1}p}{(2\pi)^{N-1}} \frac{\sqrt{2}}{C(s,\vec{p}) D^0_p P(\omega_s(\vec{p}), \vec{p})} \left(a_{\tilde{\varphi}_{s,p}+c.c} \tilde{\varphi}_{s,p}(x) + h.c. \right) , \tag{7.23}$$

using that the action of the annihilation operator on a coherent state is given as

$$\left(a_\xi \psi_\eta \right)(\phi) = \frac{1}{\sqrt{2}} \{\phi, \xi\}_{x_0} \psi_\eta(\phi) . \tag{7.24}$$

Now we can calculate the commutator

$$\begin{aligned}
[\phi(x), \phi(y)] &= \sum_{s,s'=\pm} \int \frac{d^{N-1}p d^{N-1}p'}{(2\pi)^{2(N-1)}} \frac{2}{C(s,\vec{p}) D^0_p P(\omega_s(\vec{p}), \vec{p}) C(s', \vec{p}') D^0_p P(\omega_{s'}(\vec{p}'), \vec{p}')} \\
&\quad \left(\{\tilde{\varphi}_{s',p'} + c.c., \tilde{\varphi}_{s,p} + c.c.\} \tilde{\varphi}_{s,p}(x) \tilde{\varphi}^*_{s',p'}(y) - c.c. \right) \\
&= \sum_{s=\pm} \int \frac{d^{N-1}p}{(2\pi)^{N-1}} \frac{2}{C(s,\vec{p}) D^0_p P(\omega_{s,p}, \vec{p})} \left(\tilde{\varphi}_{s,p}(x) \tilde{\varphi}^*_{s,p}(y) - c.c. \right) .
\end{aligned}$$

For $s =$ "$-$" and $h(\vec{p}) < 0$ we find, using the parity invariance of the dispersion relation, that

$$\begin{aligned}
&\tilde{\varphi}_{-,p}(x) \tilde{\varphi}^*_{-,p}(y) + \tilde{\varphi}_{-,-p}(x) \tilde{\varphi}^*_{-,-p}(y) - c.c. \\
&= 2i \sinh \tilde{\omega}_-(\vec{p})(x_0 - y_0) e^{-i\vec{p}(\vec{x}-\vec{y})} - c.c. \\
&= -i \left(e^{-i\omega_-(\vec{p})(x_0-y_0) - i\vec{p}(\vec{x}-\vec{y})} - e^{i\omega_-(\vec{p})(x_0-y_0) + i\vec{p}(\vec{x}-\vec{y})} + c.c. \right) .
\end{aligned}$$

Recalling that if $-i\tilde{\omega}_-(\vec{p})$ is a solution also $i\tilde{\omega}_-(-\vec{p})$ is a solution, we find that

$$[\phi(x), \phi(y)] \;=\; \frac{i}{2}(E_+(x,y) - E_-(x,y)) . \tag{7.25}$$

where $E_+(x,y)$ and $E_-(x,y)$ are the advanced and retarded operators, respectively, given by Gårding in [139] as

$$E_\pm(x) = \int_{\mathbb{R}} \frac{d^N p}{(2\pi)^N} \frac{e^{\pm i\,(p - i\eta).x}}{P(p - i\eta) - m^{degP}} \tag{7.26}$$

with η an arbitrary but fixed element of the hyperbolicity cone C. It turns out that E_+ and E_- have a causal support which can be seen by choosing η large enough such that all the roots of $P(p - i\eta)$ lie above the real axis. Using now that if $\omega(\vec{p})$ is a root of $P(p) - m^{degP} = 0$ then $-\omega(-\vec{p})$ is also a root, we enumerate the roots as $\omega_1(\vec{p}), \ldots, \omega_{s/2}(\vec{p})$ such that $\omega_n(\vec{p}) \neq -\omega_n(-\vec{p})$. Then we find that Gårding's propagator can be written as

$$E_\pm(x) = \pm i\theta(\pm x_0) \sum_{n=1}^{s/2} \int_{-\infty}^{\infty} \frac{d^{N-1}p}{(2\pi)^{N-1}} \frac{1}{D^0_p P(\omega_n(\vec{p}), \vec{p})} \left[e^{i\omega_n(\vec{p})t + i\vec{p}\vec{x}} - e^{-i\omega_n(\vec{p})t - i\vec{p}\vec{x}} \right] . \tag{7.27}$$

Hence, we end up with a microcausal theory. This is due to the choice of the complex structure. When investigating the bi-linear form g_{x_0}, however, we find that it is not positive definite. For $s =$ "$-$" we have that

$$\frac{1}{4} C(s,\vec{p}) D^0_p P(\omega_{s,p}, \vec{p}) = \begin{cases} \omega_-(\vec{p})^3 - \omega_-(\vec{p}) g(\vec{p}) & \text{for} \quad h(\vec{p}) \geq 0 \\ \tilde{\omega}_-(\vec{p})^3 + \tilde{\omega}_-(\vec{p}) g(\vec{p}) & \text{for} \quad h(\vec{p}) < 0 \end{cases} . \tag{7.28}$$

Since g_{x_0} turns up in the Gaussian measure on the Hilbert space of holomorphic quantization, we interpret this as the statement that the states with $s =$ "$-$" and $h(\vec{p}) > 0$ are not normalizable. That field theories with higher derivative field equations lead to these kinds of states is well known in the literature [132, 140–142]. In particular, in [142] it is argued that this problem also appears for some massive gauge theories of higher derivative order.

When excluding the non-classical $s =$ "$-$" solutions in the commutator we obtain

$$\begin{aligned}[\phi(x), \phi(y)] &= \int \frac{d^{N-1}p}{(2\pi)^{n-1} D^0_p P(\omega, \vec{k})} \left(e^{-i\omega(\vec{p})(x^0-y^0)-i\vec{p}.(\vec{x}-\vec{y})} - c.c.\right) \\ &= \int_{C(P)} \frac{d^N p}{(2\pi)^N} 2\pi\delta\left(P(k) - m^{\deg P}\right)\left(e^{-i(x-y).p} - c.c.\right), \end{aligned} \tag{7.29}$$

where the integration is restricted to the hyperbolicity cone $C(P)$. In the case of a polynomial P induced by a Lorentzian metric, microcausality would now follow directly with an easy argument by Peskin and Schroeder [136] using the Lorentz invariance of the expression in (7.29).
In general (7.29) is not zero outside of $C^\perp(P) \cup -C^\perp(P)$, which can be seen from the massless limit of (7.29) in the following example:
Example: Let us consider a $1+1$ dimensional tensorial spacetime defined by the hyperbolic polynomial $P(p_0, p) = (p_0^2 - p^2)(p_0^2 - 2p^2)$ providing the dispersion relation $P(p_0, p) = 0$ for a massless particle. Now we take into account only the energy solutions lying on the inner cone, i.e. $p_0 = \pm\sqrt{2}|p|$, which is obviously the massless limit of those solutions fulfilling $P(p_0, p) = m^4$ and lying on the hyperbolicity cone defined by $h = (1, 0)$. We obtain that

$$[\phi(x), \phi(0)] = \alpha \int_{-\infty}^{\infty} \frac{dp}{p^3} \cos(px) \sin(\sqrt{2}px_0)\,, \tag{7.30}$$

where α is a numerical factor. One then finds, for example, that for the point $(x_0, x) = (1/3, 1)$ outside the dual of the hyperbolicity cone, $[\phi(x_0, x), \phi(0, 0)] = -\alpha\frac{\sqrt{2}\pi}{3}$.

Hence, to obtain a microcausal theory, the ghost modes have to be included although they lead to indefinite norm states usually called ghosts. In particular, we obtain the following statement: Lorentzian spacetimes are the only tensorial spacetimes on which one can consistently establish a microcausal, unitary quantum scalar field theory fulfilling canonical commutation relations such that only classical interpretable particles exist.
To discuss the quantization of ghost modes further, we would need to use more elaborate techniques dealing with negative norm states which however we could not compellingly apply before the completion of this thesis. One way to cover the problem of negative definiteness might be to construct Krein spaces. This works basically by splitting the Hilbert space into a direct sum of one part on which the inner product is negative and one on which the inner product is positive. On the negative part a positive inner product is then defined from the original inner product on the whole space by just multiplication with -1. If the direct sum of the such defined inner product spaces is a Hilbert space we define it as the total Hilbert space of the quantum theory.

This was done in the GBF in [62] for the case of the Dirac field and there should be no obstacle to do the same in our case. However, in [62] the Fock space quantization scheme is used which is slightly different from the holomorphic one we considered above. To construct the QFT using Krein spaces and to calculate the amplitude and observable maps on these spaces would be an interesting task but was not done before the completion of this thesis.

7.2 Comparison with the Klein-Gordon field of imaginary mass

We saw in the foregoing section when constructing quantum field theories on hyperbolic polynomial spacetimes one faces difficulties that are connected to the existence of solutions to the higher order field equations that cannot be interpreted as the quantized version of classical particles. In the case of a real scalar field with field equations of fourth order, we showed that all solutions have to be included to obtain a microcausal QFT if it is supposed to lead to canonical commutation relations. Using a background-independent quantum field theory, it turned out that a microcausal QFT can be constructed if the problem of negative norm states is solved. This problem resembles closely what is known about the QFT for the Klein-Gordon field with imaginary mass. In that case the field equations are of the form:

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - m^2\phi = 0\,, \tag{7.31}$$

with $\eta = \mathrm{diag}(1, -1, -1, -1)$. The solutions that appear are exactly those in Equation (7.9) for $s = -$ and ω_- the solution of the dispersion relation (7.5) with $g(\vec{p}) = \sum_\alpha p_\alpha^2$ and $h(p) = g(p)^2$. There where several attempts to construct a meaningful quantum field theory for the scalar field with imaginary mass [143–153]. However, it turned out to be difficult: For instance, in [143] and [144] evanescent modes are excluded and quantization schemes are presented for the propagating modes only which leads to a violation of microcausality as we saw above.
In [146] the author constructs a microcausal quantum field theory. The author argues that solutions to (7.31) cannot be localized and thus, cannot be interpreted as particles.

It was argued in [150] that all the problems concerning the causality and Lorentz invariance of the quantum field theory for the imaginary mass Klein-Gordon field (7.31) are resolved when the propagator of the field is (7.26) for $P(p) = g(p, p)$ and g a Lorentzian metric. This is a special case of the result we found in (B).
In the algebraic approach to quantum field theory, the propagator (7.26) can be taken as a basis for the quantization as it was shown for the case of the Klein-Gordon field in [154, 155] as follows: Let $\mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ denote the space of real smooth test function on $\mathcal{M}$ and $\mathcal{C}_0^\infty(\mathcal{M}, \mathbb{R})$ the set of those elements of and $\mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ with compact support. Then, we can understand the propagator $E = E_+ - E_-$ with

$$E_\pm(x) = \int_{\mathbb{R}} \frac{d^N p}{(2\pi)^N} \frac{e^{\pm i\,(p-i\eta).x}}{P(p-i\eta) - m^{degP}} \tag{7.32}$$

for hyperbolic P as the integral kernel for a map $\mathcal{E}$ from $\mathcal{C}_0^\infty(\mathcal{M},\mathbb{R})$ to $\mathcal{C}^\infty(\mathcal{M},\mathbb{R})$. With $\mathcal{E}$ we can define the symplectic form

$$\omega(\mathcal{E}f,\mathcal{E}g) = \int_{\mathcal{M}} d^n x\, f(x)\,\mathcal{E}(h)(x) \tag{7.33}$$

which turns the image $\mathcal{R} := \mathcal{E}(\mathcal{C}_0^\infty(\mathcal{M},\mathbb{R}))$ into a symplectic space. Then, we can define the algebra of Weyl observables W by specifying the Weyl relation

$$W(\phi)W(\phi') = e^{-i\omega(\phi,\phi')/2}W(\phi+\phi')\,, \tag{7.34}$$

for all $\phi,\phi' \in \mathcal{R}$ which is the exponential form of the usual CCR algebra. This way we arrive at a completely well defined algebra of observables. However, the problem comes again up when one tries to find a vacuum state. In [152] it was argued that "Lorentz symmetry breaking must occur" when a vacuum state is chosen for the algebra of observables corresponding to (7.31). The same statement can be found in [153]. If this is true and the imaginary mass Klein-Gordon field would exist in nature these violations could be detectable experimentally in principal. Different inertial observers should see a different particle content of imaginary mass Klein-Gordon particles in the same state of the field. In some sense this is similar to the Unruh and Hawking effect where, however, this difference is between inertial and accelerated observers.

The above mentioned results for the imaginary mass Klein-Gordon field suggest that if we would be able to perform the quantization of the massive scalar field with fourth order dispersion relation compellingly, it is very likely that different non-accelerated observers would see different particle contents in the same state of the quantum field. This should be detectable in principle. In particular, there must could be a preferred frame in which the state of the quantum field is seen as the vacuum state, i.e., without any particles.

Furthermore, having a consistent quantum field theory for higher order scalar fields at hand, we would still have the problem to interpret the solutions that do not correspond to classical particles.

Chapter 8

Conclusions

One central aim of this thesis was to investigate the conditions that a tensor field G needs to satisfy in order to provide a viable spacetime structure with well-defined notions of observers and well-posed matter field dynamics living on it. The investigation was explicitly aimed at geometries beyond Lorentzian metrics. For that purpose, we started by investigating tensorial matter fields governed by linear partial differential equations on a differentiable manifold $\mathcal{M}$. Here, the linearity of the field equations encodes exactly what is meant when one speaks of test matter; the dynamics do not depend on the amplitude of the field. Standard PDE theory identifies the principal symbol P corresponding to the linear field equation as the object encoding the predictability of the latter. In particular, for the corresponding field equations to have a well-posed initial data problem, the cotangent bundle function P must induce a homogeneous, hyperbolic polynomial in every cotangent fiber. We identified this property as a first condition on P and went on to identify three additional conditions, namely that P must be reduced, time-orientable and energy-distinguishing. Tensorial structures G giving rise to such a hyperbolic, reduced, time-orientable and energy-distinguishing P were called tensorial spacetime geometries.

These conditions on P translate directly to conditions on the tensorial background structure G. To make this point clear and to show the distinction between the fundamental geometry G and the geometry defined by polynomial P, we presented two examples of Maxwell electrodynamics corresponding to metric geometry as well as area metric electrodynamics corresponding to area metric geometry. We found that the above conditions restrict the metric geometry to being Lorentzian, and in the case of area metric geometry 16 out of 23 meta-classes of area metrics are excluded.

For a tensorial spacetime structure, we were then able to prove the existence of a certain dual tangent bundle function $P^{\#}$ to P called the Gauss dual containing the same amount of information as P. Using the duality between P and $P^{\#}$, we were able to derive the dispersion relation and the action for massless point particles and to define observers. In particular, the set of observers turned out to be a convex cone that does not contain any massless velocities, but has a boundary of massless velocities. Hence, observers are always slower than light. Additionally, all observers agree on a notion of positive and negative energy for all massless particles.

For a reduced, time-orientable and energy-distinguishing P we where then also able to define the dispersion relation and an action for massive momenta. These were governed by another dual tangent bundle function P^* called the Legendre dual. We found that trajectories of observers can be identified with those of massive particles in the framework of general tensorial spacetimes. By giving the tangent space geometry $(P^{\#}, P^*)$, we solved the problem of finding a valid generalization of Finsler geometry including the dynamics of massless particles. The twist required here is that the geometry must be established by one function P on the cotangent bundle, rather than the tangent bundle where one starts in the standard approach to Finsler geometry. Moreover, one must restrict attention to functions P which are hyperbolic, time-orientable and energy-distinguishing reduced homogeneous polynomials in each cotangent fiber. Here, it is bi-hyperbolicity in particular which generalizes the Lorentzian character of metrics to the much more general geometries studied in this work. So, while there is one single geometric structure on cotangent space, there are two very different structures induced by it on the tangent bundle, the duals $P^{\#}$ and P^*. Hence, the general approaches to Finslerian spacetime geometries are correct in describing massive particles by giving a generically non-polynomial structure P^* on tangent space. However, as we know now from our investigations, it is not possible to describe massless particles with the same structure on tangent space as the polynomial structure $P^{\#}$, and the non-polynomial structure P^* will generically not be identical. Hence, a single Finslerian or Lagrangian geometry cannot describe massive and massless particles at the same time. Starting with the cotangent bundle, instead, we have at our disposal the framework developed above.

Furthermore, for a general tensorial spacetime structure, we defined observer frames and established a $3+1$-decomposition of the massless and massive dispersion relations given in terms of P. Since this is the usual form in which generalized dispersion relations appear in canonical quantum gravity and other approaches of quantum gravity, this makes it possible to connect our framework to the considerations of generalized geometries in the literature. However, we saw that the $3+1$ split depends on the particular observer corresponding to this split. Hence, starting from the non-covariant $3+1$-decomposed dispersion relation it seems to be considerably harder to find the covariant dispersion relation. Even worse, there could be no covariant dispersion relation for a given non-covariant dispersion relation. It is therefore conceptually and physically more meaningful to start from the covariant dispersion relation. This is especially true because the conditions we identified for P would be deeply hidden in the non-covariant dispersion relation.

We found that observer transformations connecting frames that correspond to different observers follow from the properties defining observer frames. Additionally, we showed that also freely falling non-rotating observer frames can be defined from P using a non-linear connection. Hence, the cotangent bundle function P or, in other words, the dispersion relation has an effect on spacetime quantities (such as the field-strength tensor, for example) in terms of quantities that are measurable in a laboratory (like the electric and magnetic field, for example). It is important to realize that the choice of a dispersion relation has such far-reaching implications. At the same time, our algebraic restrictions on the cotangent bundle function P - originally required for other reasons - ensure that all required kinematical notions can be constructed.

By considering a process called the vacuum Cherenkov process in which a massive particle radiates off a massless particle, we found that all massive particles with velocities that could not be tangents to the trajectory of an observer are kinematically unstable. In other words, a particle accelerated to a velocity outside of the convex cone of observers will eventually radiate off a massless particle to end up at a velocity inside the cone of observers. Hence, although in our framework super-luminal motion of massive particles is allowed, the vacuum Cherenkov process gives rise to a “soft limit” which is, in particular, a soft energy cutoff. However, all observers are by definition stable particles.

By assuming an experimental situation that includes a source of massive particles and a detector, we identified different energy regimes of the particles emitted by the source with qualitatively different observations by the detector in a $1+1$-dimensional toy model case. This would be a good testing ground for the framework of general tensorial spacetimes. In contrast to the Cherenkov process, the decay of a massless particle into two massive particles was found to be kinematically forbidden.

We also considered non-tensorial field theories that lead to the massive dispersion relation on a tensorial spacetime that possesses a coordinate system in which the coefficients of the polarization tensor to P are constant. We defined the massive scalar field and obtained conditions on the coefficient matrices for a generalization of the Dirac equation on general tensorial spacetimes. We gave examples of generalized Dirac equations for dispersion relations on bi-metric and area metric spacetimes. We also derived charged currents for these field theories, which makes it possible to couple them to the electromagnetic field. This would be the basis of a quantum electrodynamics on tensorial spacetimes.

To establish this quantum electrodynamics and to deal with the massive field theories further, we would like to have a way to define the corresponding free quantum field theory. That would be especially necessary to calculate the times scale for the vacuum Cherenkov process for experimental situations. Furthermore, having such quantum field theories, other results could be derived that would help to test our theory in concrete examples. For example, one could derive corrections to cross sections in particle experiments arising on general tensorial spacetimes and corrections to results in astroparticle physics of massive objects or of the early universe.

In the second part of this thesis, I was mostly dealing with the GBF. This background-independent QFT framework generalizes QFT to general spacetime regions establishing the corresponding Hilbert spaces on the boundaries of these regions. This is of particular interest in non-perturbative quantum gravity where no a-priori $3+1$ space-time split is given since there is no metric telling us which directions are spacelike and which are timelike.

In Chapter 6, we investigated the Unruh effect from the GBF perspective. We applied the general boundary formulation of quantum field theory to quantize a massive scalar field in Minkowski and Rindler spacetimes. By comparing the two quantum theories, we were able to study the Unruh effect from a GBF perspective. Our result is that the expectation value of Weyl observables with compact spacetime support in the interior of the right Rindler wedge, computed in the Minkowski vacuum state, coincides with the one calculated in an appropriately chosen mixed state in Rindler space, as long as the observables are quantized according to the Feynman quantization scheme, one of the quantization schemes for observables in the GBF. This result could be interpreted as the manifestation of the Unruh effect within the GBF. Furthermore, we showed that the coincidence of the expectation values does not hold in the Berezin-Toeplitz quantization, which is the second quantization scheme for observables in the GBF. One possible conclusion would be that there might be something wrong with Berezin-Toeplitz quantization or with the thermal state defined in this quantization prescription.

The work presented in Chapter 6 is of immediate relevance for the GBF program. It represents a concrete application of the quantization of observables and the opportunity to compare the Feynman and Berezin-Toeplitz schemes in a specific context. Moreover, the computation of observable maps involved the use of mixed states for the first time within the GBF.

Recall that the spacetime regions we considered for the derivation of the Unruh effect are bounded by spacelike hyperplanes which represents the usual setup in the derivation of the Unruh effect in the standard formulation of QFT. Certainly, the GBF for a consideration of much more general regions like, for example, regions bounded by timelike hypersurfaces or even compact regions. One possible generalization of the work presented here is to consider a region bounded by one hyperbola of constant Rindler spatial coordinate ρ extending up to spacelike infinity, i.e., $\tilde{x} \to \infty$. Since the origin of Minkowski space does not lie in this region, the Hilbert spaces of Minkowski quantization and Rindler quantization can be compared without the mathematical difficulties explained at the end of section 6.2. An article about these calculations by Daniele Colosi and me, will be finished soon.

A particular example of a generalization to compact regions would be to consider diamond-shaped regions in Minkowski space as was done in [156]. For that purpose we would need to know how to deal with boundaries that consist of hypersurfaces with boundaries in the GBF. There was an article about the GBF with corners for 2-dimensional Yang-Mills theory by Oeckl [58]. However, for the general case it is still not clear which algebraic structure should be associated with a hypersurface with boundaries and how the decomposition of boundaries has to be performed on the algebraic structures of the GBF. The work presented in this section could be the testing ground for solving this question: The hyperplane $t = 0$ in Minkowski spacetime is the union of the two hypersurfaces in the right and left Rindler wedges with boundaries at the origin of Minkowski space. The relation between the Hilbert space corresponding to the hypersurface at $t = 0$ in Minkowski spacetime and the Hilbert spaces corresponding to the two parts of that hypersurface may offer a concrete example of how to deal with such situations. In particular,

it is of interest how the with the boundary condition at the origin of Minkowski space may be dealt with, which has to be imposed on the field to perform the split.
The application of the GBF to black hole radiation, usually denoted as the Hawking effect, is of course another possible generalization. In particular, it would be possible to consider a region bounded by timelike hypersurfaces in a Schwarzschild spacetime modeling a black hole within the framework of the GBF.

In Chapter 7, we used the background-independent geometric quantization formalism that we introduced, originally, for the holomorphic representation of the GBF to quantize a massive scalar field theory on a general tensorial spacetime. In our investigations, we found that to obtain a microcausal theory, solutions have to be included in the quantization process that do not correspond to classical, massive particles. We obtained the following statement: Lorentzian spacetimes are the only tensorial spacetimes on which one can consistently establish a microcausal, unitary quantum scalar field theory fulfilling canonical commutation relations (CCRs) such that only classically interpretable particles exist.
If one wants to go beyond Lorentzian spacetimes and at the same time work with quantum scalar fields, one must either accept the existence of non-classical particles and interpret them or drop the restriction to CCRs or both. Notice, that dropping the CCRs may mean loosing the spin statistics theorem for the scalar particles.
We showed that accepting the existence of the states which are not classically interpretable leads to mathematical problems in the background-independent QFT framework I have chosen: we found that the complex structure does not lead to a positive metric on the space of solutions to the field equations. Furthermore, the non-classical particles resemble the particles of the Klein-Gordon field of imaginary mass. We reviewed briefly the ongoing research concerning the Klein-Gordon field of imaginary mass and found that until now, there has been no quantum theory for the imaginary mass Klein-Gordon field that is at the same time microcausal and Lorentz invariant.
This result suggested that in the case of higher order field theories, different inertial observers may see a different content of non-classical particles in the same state of the quantum field. Such an effect should be in principle experimentally detectable. In particular, it suggests the existence of a preferred frame in which the state of the field is seen as devoid of non-classical particles. This should be a good testing ground for the framework of matter fields on general tensorial spacetimes we developed in Chapter 4.

Finally, let me remark that it was possible to tackle to some extent the problem of finding equations for the dynamics of the tensorial spacetime geometries itself. Using the framework we developed above, the authors of [52] derived a set of linear partial differential equations that must be fulfilled by the constraints defining the dynamics of a given tensorial spacetime geometry via its coupling to matter fields. In particular, for the case of Lorentzian spacetimes, the resulting dynamics are those given by the Einstein-Hilbert action.

Dynamical equations for the geometrical background would be necessary to obtain generalizations of particular tensorial spacetimes from known Lorentzian spacetimes. In particular, it would be interesting to obtain the generalization of the Schwarzschild- and the Friedmann-Lemiatre-Robertson-Walker-Spacetime in the context of tensorial spacetimes. That would put us in the position to also give quantitative results besides the qualitative ones derived in this thesis. This would help in designing experiments to test the whole framework of tensorial spacetimes.

Appendix A

An identity

Here we prove the identity

$$\sum_{k=0}^{\infty} \frac{(k+n)!}{k!n!} e^{-2\pi kp/a} = \frac{1}{(1-e^{-2\pi p/a})^{n+1}}\,, \tag{A.1}$$

We start by defining the function

$$f(n) := \sum_{s=0}^{\infty} \frac{(s+n)!}{s!n!} e^{-2\pi sp/a} \tag{A.2}$$

for which we obtain the recursion relation

$$f(n+1) = \left(1 - \frac{1}{n+1}\frac{a}{2\pi}\frac{d}{dp}\right) f(n)\,. \tag{A.3}$$

For $n = 0$ we find

$$f(0) = \frac{1}{1-e^{-2\pi p/a}}\,. \tag{A.4}$$

So we start the induction step with the ansatz

$$f(n) = \frac{1}{(1-e^{-2\pi p/a})^{n+1}} \tag{A.5}$$

and find

$$f(n+1) = \left(1 - \frac{1}{n+1}\frac{a}{2\pi}\frac{d}{dp}\right) \frac{1}{(1-e^{-2\pi p/a})^{n+1}} \tag{A.6}$$

$$= \frac{1}{(1-e^{-2\pi p/a})^{n+1}} + \frac{e^{-2\pi p/a}}{(1-e^{-2\pi p/a})^{n+2}} = \frac{1}{(1-e^{-2\pi p/a})^{n+2}} \tag{A.7}$$

which proves that the ansatz was correct. □

Appendix B

Well posedness

In this section we show that field theories with the dispersion relation (4.10) have a well-posed Cauchy problem. We start with the basic definitions of hyperbolic differential operators and some theorems from [157] and [158]. We are interested in differential operators $Q(-i\partial)$ with constant coefficients that are $d \times d$ matrices. We say that $Q(-i\partial)$ is hyperbolic with respect to $\vartheta \in \mathbb{R}^n$ if there exists a distribution $E(Q, \vartheta, x)$ (a fundamental solution) with

$$Q(-i\partial_x)E(Q, \vartheta, x) = \delta(x) \tag{B.1}$$

and support in some closed cone K with its vertex at the origin and $x.\vartheta > 0$ for all $x \in K - \{0\}$. When $Q(-i\partial_x)$ is hyperbolic w.r.t. ϑ the distribution $E(Q, \vartheta, x)$ is unique (Lemma 3.1. of [158]). Theorem 3.5. of [158] tells us that $Q(-i\partial_x)$ is hyperbolic w.r.t. ϑ if and only if its characteristic polynomial $P(\xi) := \det Q(\xi)$ is hyperbolic w.r.t. ϑ which is defined as follows:

Definition. *Let P be a polynomial of degree r and P_r the homogeneous part of P of degree r (the principal part of P). Then P is called hyperbolic w.r.t. the real vector ϑ if $P_r(\vartheta) \neq 0$ and there exists $\tau_0 \in \mathbb{R}$ s.t. $P(\xi + i\tau\vartheta) \neq 0$ if $\xi \in \mathbb{R}^n$ and $\tau < \tau_0$.*

In the case of a homogeneous polynomial the above definition of hyperbolicity coincides with the definition of hyperbolicity given above. The characteristic polynomial P telling us whether $Q(-i\partial)$ has a well-posed Cauchy problem is of course exactly the polynomial solvability condition in (4.10) which we identified as the dispersion relation. In the following we show that the polynomial $P = P_r - m^{\deg P}$ is hyperbolic with respect to ϑ if the corresponding principal polynomial P_r is hyperbolic w.r.t. ϑ. For that purpose, we define a sort of norm of P and introduce some notions to compare different polynomials:

$$\tilde{P}(\xi, t) := \left(\sum_{\alpha} |P^{(\alpha)}(\xi)|^2 t^{2|\alpha|} \right)^{1/2} \tag{B.2}$$

and $\tilde{P}(\xi) := P(\xi, 1)$ where

$$P^{(\alpha)}(\eta) := \frac{\partial^{|\alpha|} P(\eta)}{\partial \eta_1^{\alpha_1} \cdots \eta_n^{\alpha_n}} \tag{B.3}$$

Definition (Def. 3.2.1. of [157]). *If $P(D)$ and $Q(D)$ are differential operators such that $\tilde{Q}(\xi)/\tilde{P}(\xi) < C$ for all $\xi \in \mathbb{R}^n$ we shall say that Q is weaker than P and write $Q \prec P$, or* *that P is stronger than Q and write $P \succ Q$. If $P \prec Q \prec P$, the operators are called equally strong.*

DEFINITION (Def. 3.3.1. of [157]). *We shall say that P dominates Q and write $P \succcurlyeq Q$ if*

$$\sup_{\xi} \tilde{Q}(\xi,t)/\tilde{P}(\xi,t) \to 0, \quad t \to \infty \tag{B.4}$$

where ξ and t denote real variables.

With these definitions we can give the useful theorems:

THEOREM B.0.1 (Thm. 5.5.7. of [157]). *If the principal part P_r of P is hyperbolic w.r.t. ϑ and P is weaker than P_r, then P is also hyperbolic w.r.t. ϑ and P and P_r are equally strong.*

and

THEOREM B.0.2 (Thm. 3.3.4. of [157]). *In order that $P \prec P + aQ \prec P$ for every complex number a it is necessary and sufficient that $P \succcurlyeq Q$.*

Now we can give our main result of this section:

PROPOSITION B.0.3. *Let P be a homogeneous hyperbolic polynomial then $P_a = P + a$ is hyperbolic for all complex numbers a.*

Proof: By setting $Q = 1$ we obtain that $\tilde{Q}(\xi,t) = 1$. With

$$\tilde{P}(\xi,t) = \left(\sum_{|\alpha|=\deg P} |P^{(\alpha)}|^2 t^{2|\alpha|} + \sum_{|\alpha|<\deg P} |P^{(\alpha)}(\xi)|^2 t^{2|\alpha|} \right)^{1/2} \tag{B.5}$$

where the factor $\sum_{|\alpha|=\deg P} |P^{(\alpha)}|^2$ is positive and independent of ξ. It follows then that for fixed positive t

$$\inf_{\xi} \tilde{P}(\xi,t) = \left(\sum_{|\alpha|=\deg P} |P^{(\alpha)}|^2 \right)^{1/2} t^{|\alpha|} \tag{B.6}$$

and eventually

$$\sup_{\xi} \frac{\tilde{Q}(\xi,t)}{\tilde{P}(\xi,t)} = \left(\sum_{|\alpha|=\deg P} |P^{(\alpha)}|^2 \right)^{-1/2} t^{-|\alpha|} \to 0 \tag{B.7}$$

for $t \to \infty$ which proves that P dominates Q as defined in B. With theorem B.0.2 we obtain that P_a is hyperbolic which proves the proposition. □

We have that for field theories with dispersion relation $P_r(p) - m^{\deg P} = 0$ and P_r a hyperbolic, time and energy orientable polynomial the Cauchy problem is well-posed. Furthermore, the above theorems lead directly to two corollaries concerning the support of the fundamental solutions as (7.26) of massive field theories on HPSTs:

COROLLARY B.0.4. *If the principal part P_r of P is hyperbolic w.r.t. ϑ and P is weaker than P_r, then $C(P_r,\vartheta) \subseteq C(P,N)$ and $C^{\perp}(P,\vartheta) \subseteq C^{\perp}(P_r,\vartheta)$, i.e. the support of the fundamental solutions of P is contained in that of P_r.*

Proof: If P_r is hyperbolic w.r.t. ξ and P is weaker it follows from theorem B.0.1 that P is hyperbolic w.r.t. ξ. This is true for all $\xi \in C(P_r, \vartheta)$ and hence it follows that $C(P_r, \vartheta) \subseteq C(P, \vartheta)$. That $C^{\perp}(P, \vartheta) \subseteq C^{\perp}(P_r, \vartheta)$ follows from the definition of the dual cone. $\square$
which gives

Corollary B.0.5. *The support of the fundamental solutions of a field theory with dispersion relation $P = P_r - m^{\deg P}$ is contained in that of the massless field theory with dispersion relation P_r, i.e. $C(P_r, \vartheta) \subseteq C(P, \vartheta)$ and $C^{\perp}(P, \vartheta) \subseteq C^{\perp}(P_r, \vartheta)$.*

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