

A New and Unifying Approach to Spin Dynamics and Beam Polarization in Storage Rings

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Abstract

With this paper we extend our studies [1] on polarized beams by distilling tools from the theory of principal bundles. Four major theorems are presented, one which ties invariant fields with the notion of normal form, one which allows one to compare different invariant fields, and two that relate the existence of invariant fields to the existence of certain invariant sets and relations between them. We then apply the theory to the dynamics of spin-1/2 and spin-1 particles and their density matrices describing statistically the particle-spin content of bunches. Our approach thus unifies the spin-vector dynamics from the T-BMT equation with the spin-tensor dynamics and other dynamics. This unifying aspect of our approach relates the examples elegantly and uncovers relations between the various underlying dynamical systems in a transparent way.

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1 Introduction

We continue our study [1] of spin dynamics in storage rings and extend our tool set by employing a method developed in the 1980s by R. Zimmer, R. Feres and others for Dynamical-Systems theory [2, 3]. In contrast to [1], we now employ a discrete-time treatment (continuous-time would do as well). We present four major theorems, the Normal Form Theorem tying invariant fields with the notion of normal form, the Decomposition Theorem, allowing one to compare different invariant fields, the Invariant Reduction Theorem, giving new insights into the question of the existence of invariant fields and which is supplemented by the Cross Section Theorem. It turns out that the well-established notions of invariant frame field, spin tune, and spin-orbit resonance are generalized by the normal form concept whereas the well-established notions of invariant polarization field and invariant spin field are generalized to invariant (E, l) -fields. Here the notation (E, l) indicates an $SO(3)$ -space which is shorthand for l being a continuous $SO(3)$ -action on a topological space E , i.e., $l : SO(3) \times E \rightarrow E$ is continuous and $l(I; x) = x$ and $l(r_1 r_2; x) = l(r_1; l(r_2; x))$. With the flexibility in the choice of (E, l) , we have a unified way to study the dynamics of spin-1/2 and spin-1 particles and the density matrices of the bunches. Accordingly several (E, l) 's are discussed in some detail. The origins of our formalism, in bundle theory, are pointed out and we also mention the relation to Yang-Mills Theory.

2 General Particle-Spin Dynamics

We assume integrable particle motion so that for fixed action the particle motion is on the d -torus \mathbb{T}^d . The particle one-turn map is given by

$$z \in \mathbb{T}^d \mapsto j(z), \quad j \in \text{Homeo}(\mathbb{T}^d). \quad (1)$$

Here $\text{Homeo}(\mathbb{T}^d)$ denotes the set of homeomorphisms on \mathbb{T}^d . Now we add to each particle an additional E -valued “spin-like” quantity x . The one-turn particle-“spin” map is the function $\mathcal{P}[E, l, j, A] : \mathbb{T}^d \times E \rightarrow \mathbb{T}^d \times E$ defined by

$$\mathcal{P}[E, l, j, A](z, x) = (j(z), l(A(z); x)), \quad (2)$$

where $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$ is the one-turn spin transfer matrix. Here $\mathcal{C}(X, Y)$ denotes the set of continuous functions from X to Y and $SO(3)$ is the group

of real orthogonal 3×3 -matrices of determinant 1. In our formalism, (2) is the most general description of particle-spin dynamics and the choice of (E, l) depends on the situation as we illustrate in the examples.

3 General Field Dynamics

We are primarily interested in the field dynamics induced by the particle-spin dynamics. Let $f : \mathbb{T}^d \rightarrow E$ be an E -valued field on \mathbb{T}^d and set $x = f(z)$ in (2). Then, after one turn, z becomes $j(z)$ and the field value at $j(z)$ becomes $l(A(z); f(z))$. Thus after one turn the field f becomes the field $f' : \mathbb{T}^d \rightarrow E$ where $f'(z) := l(A(j^{-1}(z)); f(j^{-1}(z)))$. Thus we have the one-turn field map

$$f \mapsto f' = l(A \circ j^{-1}; f \circ j^{-1}) , \quad (3)$$

where \circ denotes composition. Iteration of (3) gives the field after n turns resulting in a formula involving the n -turn spin transfer matrix. We call $f \in \mathcal{C}(\mathbb{T}^d, E)$ an “invariant (E, l) -field of (j, A) ”, or just “invariant field”, if it is mapped by (3) into itself, i.e.,

$$f \circ j = l(A; f) . \quad (4)$$

Our main focus is on exploring invariant fields as these describe the spin equilibrium of a bunch [1].

We work in the framework of topological dynamics. So A, j, l, f are continuous functions. Frameworks using weaker or stronger conditions than continuity are possible.

4 Examples of $SO(3)$ -spaces (E, l)

4.1 Example 1: $(E, l) = (\mathbb{R}^3, l_v)$

For spin-1/2 particles the spin variable is the spin-vector $x = S \in \mathbb{R}^3$. The T-BMT equation [1] leads us to describe the particle-spin and field dynamics in terms of the $SO(3)$ -space (\mathbb{R}^3, l_v) where $l_v(r, S) := rS$ and $r \in SO(3)$. Thus, by (2), $\mathcal{P}[\mathbb{R}^3, l_v, j, A](z, S) = (j(z), A(z)S)$.

4.2 Example 2: $(E, l) = (E_t, l_t)$

The $SO(3)$ -space (\mathbb{R}^3, l_v) does not suffice for spin-1 particles. One also needs the spin variable $x = M \in E_t$, called the spin-tensor, where $E_t := \{M \in$

$\mathbb{R}^{3 \times 3} : M^t = M, \text{Tr}(M) = 0\}$. The particle-spin tensor dynamics and corresponding field dynamics is described by (E_t, l_t) where $l_t(r, M) := rMr^t$ and $M \in E_t, r \in SO(3)$ and where the superscript t denotes transpose.

4.3 Example 3: $(E, l) = (E^{1/2}, l^{1/2})$

The statistical description of a bunch of spin-1/2 particles, in terms of density matrices, requires the $SO(3)$ -space $(E^{1/2}, l^{1/2})$ where $E^{1/2} := \{R \in \mathbb{C}^{2 \times 2} : R^\dagger = R, \text{Tr}(R) = 1\}$, where \dagger denotes the hermitian conjugate and $l^{1/2}$ is defined as follows. Let $\alpha \in \text{Homeo}(\mathbb{R}^3, E^{1/2})$ be defined by

$$\alpha(S) := \frac{1}{2}(I_{2 \times 2} + \sum_{i=1}^3 S_i \sigma_i), \quad (5)$$

where S_i is the i -th component of S and the σ_i are the well-known Pauli matrices. Then

$$l^{1/2}(r; \alpha(S)) := \alpha(l_v(r; S)), \quad (6)$$

i.e., α is a so-called “ $SO(3)$ -map” from (\mathbb{R}^3, l_v) to $(E^{1/2}, l^{1/2})$.

4.4 Example 4: $(E, l) = (E^1, l^1)$

To complete our treatment of spin-1 particles and their motions, we must combine the vector and spin-tensor motion. The required $SO(3)$ -space is $(E_{v \times t}, l_{v \times t})$ where $E_{v \times t} := \mathbb{R}^3 \times E_t$ and $l_{v \times t}(r, S, M) := (l_v(r, S), l_t(r, M)) = (rS, rMr^t)$.

The statistical description of a bunch of spin-1 particles, in terms of density matrices, requires the $SO(3)$ -space (E^1, l^1) where $E^1 := \{R \in \mathbb{C}^{3 \times 3} : R^\dagger = R, \text{Tr}(R) = 1\}$, and $l = l^1$ is defined as follows. Let $\alpha \in \text{Homeo}(E_{v \times t}, E^1)$ be defined by

$$\alpha(S, M) = \frac{1}{3}(I + \sum_{i=1}^3 S_i \sigma_i + \sqrt{\frac{3}{2}} \sum_{i=1}^3 M_{ij} (\mathfrak{J}_i \mathfrak{J}_j + \mathfrak{J}_j \mathfrak{J}_i))$$

where the hermitian matrices \mathfrak{J}_i are given by their nonzero components $\mathfrak{J}_1(1, 2) = \mathfrak{J}_1(2, 3) = -i/\sqrt{2}$, $\mathfrak{J}_2(1, 1) = -\mathfrak{J}_2(3, 3) = 1$ and $\mathfrak{J}_3(1, 2) = \mathfrak{J}_3(2, 3) = 1/\sqrt{2}$. Then

$$l^1(r; \alpha(S, M)) := \alpha(l_{v \times t}(r; S, M)), \quad (7)$$

i.e., α is an $SO(3)$ -map from (E_t, l_t) to (E^1, l^1) .

5 Invariant fields

We now continue our general discussion. Let $E_x := l(SO(3); x) := \{l(r; x) : r \in SO(3)\}$. Then the E_x partition E and each set $\mathbb{T}^d \times E_x$ is invariant under the particle-spin motion of (2) and so we “decompose” $\mathbb{T}^d \times E$. Of central interest are invariant (E, l) -fields in the situation when E is Hausdorff and j is topologically transitive. The latter means that a $z_0 \in \mathbb{T}^d$ exists such that the set $\{j^n(z_0) : n = 0, \pm 1, \pm 2, \dots\}$ is dense in \mathbb{T}^d , i.e., that its closure equals \mathbb{T}^d and we then say for short that j is $TT(z_0)$. Remarkably, in this situation, every invariant (E, l) -field of (j, A) takes values only in one E_x ; the proof [4] uses the compactness of $SO(3)$. Thus our main theorems below are formulated for fields taking values in only one E_x (though these theorems could be generalized).

In Example 1, a field $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ is mapped, according to (3), to f' where $f'(z) = A(j^{-1}(z))f(j^{-1}(z))$. Moreover an invariant (\mathbb{R}^3, l_v) -field is also called an “invariant polarization field (IPF)”. Furthermore if $|f| = 1$, it is called an “invariant spin field (ISF)”. Clearly $E_x = \{S \in \mathbb{R}^3 : |S| = |x|\}$ is a sphere centered at $(0, 0, 0)$. Thus if j is topologically transitive and f is an IPF of (j, A) then $|f(z)|$ is independent of z . The so-called “ISF-conjecture” states: If j is topologically transitive then an ISF exists.

6 Main Theorems

6.1 The Normal Form Theorem (NFT)

Let $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$, $x \in E$ and $A'(z) := T^t(j(z))A(z)T(z)$. Then the NFT states that $f(z) := l(T(z); x)$ is an invariant (E, l) -field of (j, A) iff $l(A'(z); x) = x$ for all $z \in \mathbb{T}^d$. This leads to two important concepts in our work, the normal form and the isotropy group. If $A'(z)$ above belongs to a subgroup H of $SO(3)$ for all z then we call (j, A') an H -normal form of (j, A) . Moreover the subgroup $H_x := \{r \in SO(3) : l(r; x) = x\}$ is called the isotropy group at x (note that for every H one can find (E, l) and x such that $H = H_x$). Thus the NFT is reformulated as: The function f is an invariant (E, l) -field of (j, A) iff (j, A') is an H_x -normal form of (j, A) . We emphasize that the NFT gives us a way to view invariant fields which is distinctively different from (4).

The normal form also impacts the notion of “spin tune”. By definition,

(j, A) has spin tunes iff $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ exists such that $A'(z)$ is independent of z [4]. For $\nu \in [0, 1)$, define $G_\nu := \{R(2\pi n\nu) : n \in \mathbb{Z}\}$ where the matrix $R(\mu)$ is given by its nonzero components $R(\mu)(1, 1) = R(\mu)(2, 2) = \cos \mu$, $R(\mu)(2, 1) = -R(\mu)(1, 2) = \sin \mu$ and $R(\mu)(3, 3) = 1$. Then G_ν is a subgroup of $SO(3)$ and we have proven [4] that (j, A) has spin tunes iff it has a G_ν -normal form for some ν .

6.2 The Decomposition Theorem (DT)

Choose (E, l) and (\hat{E}, \hat{l}) , where E and \hat{E} are Hausdorff and consider, in terms of functions $\beta \in \mathcal{C}(E_\eta, \hat{E}_{\hat{\eta}})$, the relation between fields which take values only in $E_\eta := l(SO(3); \eta)$ and $\hat{E}_{\hat{\eta}} := \hat{l}(SO(3); \hat{\eta})$. Then let $f \in \mathcal{C}(\mathbb{T}^d, E)$ take values only in E_η whence, according to (3), f is mapped to f' which takes values only in E_η too. Of course $g \in \mathcal{C}(\mathbb{T}^d, \hat{E})$, defined by $g(z) := \beta(f(z))$, takes values only in $\hat{E}_{\hat{\eta}}$ whence, according to (3), g is mapped to g' which takes values only in $\hat{E}_{\hat{\eta}}$ too. To relate (E, l) and (\hat{E}, \hat{l}) dynamically, we impose the condition on β that $g'(z) = \beta(f'(z))$ for all f and all (j, A) . It is easy to show that this is equivalent to the condition

$$\hat{l}(r, \beta(x)) = \beta(l(r; x)) , \quad (8)$$

i.e., that β is an $SO(3)$ -map. We can now state the DT [4]. Firstly, assume an $SO(3)$ -map β exists, then $g'(z) = \beta(f'(z))$ for all (j, A) . Thus if f is an invariant (E, l) -field of (j, A) then g is an invariant (\hat{E}, \hat{l}) -field of (j, A) . Secondly if, in addition, β is a homeomorphism then f is an invariant (E, l) -field of (j, A) iff g is an invariant (\hat{E}, \hat{l}) -field of (j, A) ; this allows us to classify invariant fields up to homeomorphisms.

We now address the computation of the β 's. One can show [4] that an $SO(3)$ -map β exists iff H_η (see NFT above) is conjugate to a subgroup of the isotropy group $\hat{H}_{\hat{\eta}} := \{r \in SO(3) : \hat{l}(r; \hat{\eta}) = \hat{\eta}\}$, i.e. an $r_0 \in SO(3)$ exists so that $r_0 H_\eta r_0^t \subset \hat{H}_{\hat{\eta}}$. The proof is constructive by showing that β can be defined by $\beta(l(r; x)) = \hat{l}(r r_0^t; \hat{x})$. Note that every $SO(3)$ -map β is a homeomorphism if H_η and $\hat{H}_{\hat{\eta}}$ are conjugate, i.e., an $r_0 \in SO(3)$ exists so that $r_0 H_\eta r_0^t = \hat{H}_{\hat{\eta}}$. In contrast, if H_η and $\hat{H}_{\hat{\eta}}$ are not conjugate then no $SO(3)$ -map β is a homeomorphism [4].

In summary, the DT enables one to classify and relate invariant fields for the various choices of η and $\hat{\eta}$ and it does so in terms of the functions β and

the subgroups H_η and \hat{H}_η of $SO(3)$. The terminology DT refers to E being “decomposed” into the E_η (similarly for \hat{E}).

6.3 The Invariant Reduction Theorem (IRT) and the Cross Section Theorem (CST)

Throughout this section we consider a fixed $SO(3)$ -space (E, l) and $x \in E$ and $f \in \mathcal{C}(\mathbb{T}^d, E)$ where $f(\mathbb{T}^d) \subset E_x$. Let $\check{\Sigma}_x[f] := \bigcup_{z \in \mathbb{T}^d} \{z\} \times \mathcal{R}_x(f(z))$ where, for $y \in E_x$, we define $\mathcal{R}_x(y) := \{r \in SO(3) : l(r; x) = y\}$. Moreover $\check{\mathcal{P}}[j, A] \in \text{Homeo}(E_d)$ is defined by $\check{\mathcal{P}}[j, A] := \mathcal{P}[E_d, l_d, j, A]$ where $E_d := \mathbb{T}^d \times SO(3)$ and $l_d(r'; z, r) := (z, r'r)$, i.e., $\check{\mathcal{P}}[j, A](z, r) := (j(z), A(z)r)$. Then the IRT [5, 6, 4] states that f is an invariant field iff $\check{\Sigma}_x[f]$ is $\check{\mathcal{P}}[j, A]$ -invariant, i.e., $\check{\mathcal{P}}[j, A](\check{\Sigma}_x[f]) = \check{\Sigma}_x[f]$.

The definitions of $\check{\Sigma}_x[f]$ and the name IRT $\check{\mathcal{P}}[j, A]$ are suggested by bundle theory, as discussed in a separate section. To sketch one-half of the IRT proof, note that if $\check{\mathcal{P}}[j, A](z, r) = (j(z), A(z)r) \in \check{\Sigma}_x[f]$ then $l(A(z)r; x) = f(j(z))$ whence, if $(z, r) \in \check{\Sigma}_x[f]$, $l(A(z); f(z)) = f(j(z))$ so that f is invariant.

The IRT gives new insights into invariant fields. Let $(z_0, y) \in \mathbb{T}^d \times E_x$ and define

$$\hat{\Sigma}_x[j, A, z_0, y] := \bigcup_{n \in \mathbb{Z}} \check{\mathcal{P}}[j, A]^n(\{z_0\} \times \mathcal{R}_x(y)) .$$

Then a corollary to the IRT states: If $\check{\Sigma}_x[f] = Cl(\hat{\Sigma}_x[j, A, z_0, y])$, where Cl indicates closure, then j is $TT(z_0)$ and f is an invariant field [4]. Furthermore if $(z, r) \in Cl(\hat{\Sigma}_x[j, A, z_0, y])$, then $f(z) = l(r; x)$ so that y explicitly determines the invariant field f via the set $Cl(\hat{\Sigma}_x[j, A, z_0, y])$.

Let $p_x[f] : \check{\Sigma}_x[f] \rightarrow \mathbb{T}^d$ be defined by $p_x[f](z, r) := z$. Hence $p_x[f]$ is continuous w.r.t. the subspace topology on $\check{\Sigma}_x[f]$. One calls $\sigma \in \mathcal{C}(\mathbb{T}^d, \check{\Sigma}_x[f])$ a “cross section” of $p_x[f]$ if $p_x[f](\sigma(z)) = z$. Then the CST states that $p_x[f]$ has a cross section iff $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ exists such that $f(z) = l(T(z); x)$ (the proof uses the fact that $\sigma(z) := (z, T(z))$ is a cross section). Note that, under the assumptions of the NFT, $p_x[f]$ has a cross section. The CST will be illustrated in Example 1 and will be tied with bundle theory below.

We now state a partial converse to the above corollary. Let j be $TT(z_0)$, let f be invariant and let $p_x[f]$ have a cross section. Then

$$\check{\Sigma}_x[f] = Cl(\hat{\Sigma}_x[j, A, z_0, f(z_0)]) . \tag{9}$$

Thus, as mentioned after the corollary, $f(z_0)$ explicitly determines the invariant field f via the set $Cl(\hat{\Sigma}_x[j, A, z_0, f(z_0)])$. The fact that f can be determined by the single value $f(z_0)$ is not surprising since j is $TT(z_0)$ and since the iteration $f(j^{n+1}(z_0)) = l(A(j^n(z_0))r; f(j^n(z_0)))$ gives f on a dense subset of \mathbb{T}^d and, by continuity, everywhere. In fact using the “map” version of Weyl’s equidistribution theorem [7] one obtains an explicit form of f from $f(z_0)$ when $(E, l) = (\mathbb{R}^3, l_v)$. In contrast, (9) is an alternative method of obtaining an explicit form of f from $f(z_0)$ and it does so for arbitrary (E, l) . If j is not $TT(z_0)$ then $f(z_0)$ does not necessarily determine f . For example let $j(z) = z$ and $A(z) := I$ then every $f \in \mathcal{C}(\mathbb{T}^d, E)$ is an invariant (E, l) -field and hence is not determined by $f(z_0)$. In contrast if $(E, l) = (\mathbb{R}^3, l_v)$ and if (j, A) are chosen such that only two ISF’s $f, -f$ exist, then both ISF’s are uniquely determined by their value at z_0 . Note finally that for real storage rings there is also a well-known numerical method of constructing f in terms of $f(z_0)$ when $(E, l) = (\mathbb{R}^3, l_v)$ and $(E, l) = (E_t, l_t)$ [8, 9, 10, 11].

7 Application to the Examples

7.1 Example 1: $(E, l) = (\mathbb{R}^3, l_v)$

We begin with the NFT. So let $x = (0, 0, 1)$ and $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ be of the form $f(z) = l_v(T(z); x) = T(z)(0, 0, 1)$ where $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ whence f is the third column of T . Then the isotropy group is $H_x = \{R(2\pi\nu) : \nu \in [0, 1)\} =: SO(2)$ and the NFT states that $T^t(j(z))A(z)T(z) \in SO(2)$ for all $z \in \mathbb{T}^d$ iff the third column of T is an ISF of (j, A) . Note that as in [1, 4] T is called an “invariant frame field” (IFF) iff its third column is an ISF. Thus our introduction of normal forms gives a new view on IFF’s and generalizes it from the group $SO(2)$ to an arbitrary subgroup of $SO(3)$. We emphasize that the notions of IFF and ISF are tied to the group $SO(2)$.

To apply the DT we recall that the E_x are the $l_v(SO(3); \lambda(0, 0, 1))$ which are spheres of radius λ centered at $(0, 0, 0)$ where $\lambda \in [0, \infty)$. Thus, for topologically transitive j , IPF’s can be classified by applying the DT for the case where $(E, l) = (\hat{E}, \hat{l}) = (\mathbb{R}^3, l_v)$ and $\eta = \lambda(0, 0, 1), \hat{\eta} = \hat{\lambda}(0, 0, 1)$ with $\lambda, \hat{\lambda} \in [0, \infty)$. One can show [4] that an $SO(3)$ -map β which is a homeomorphism only exists if either $\lambda, \hat{\lambda} > 0$ or $\lambda = \hat{\lambda} = 0$. Thus, for topologically transitive j , there are only two classes of IPF’s namely the class containing the ISF’s and the class consisting only of the zero field.

If $x = (0, 0, 1)$ and $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ with $|f| = 1$ one observes, by the CST, that $p_x[f]$ has a cross section iff a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ exists whose third column is f . Thus the CST gives another view on IFFs. Using simple arguments from Homotopy Theory one can also show [5] that, if $d = 1$, $p_x[f]$ has a cross section while for $d \geq 2$, $p_x[f]$ does not always have a cross section. Thus, in light of (9), the case $d = 1$ of the ISF conjecture is exceptional.

7.2 Example 2: $(E, l) = (E_t, l_t)$

For brevity we only address the DT. We call the fields $f \in \mathcal{C}(\mathbb{T}^d, E_t)$ polarization tensor fields and so invariant (E_t, l_t) -fields are invariant polarization tensor fields (IPTF's). Since E_t is Hausdorff one can show [4], for topologically transitive j , that IPTF's can be classified by applying the DT for the case where $(E, l) = (\hat{E}, \hat{l}) = (E_t, l_t)$ and where η and $\hat{\eta}$ are the diagonal matrices $\eta = \text{diag}(y_1, y_2, -y_1 - y_2)$ and $\hat{\eta} = \text{diag}(\hat{y}_1, \hat{y}_2, -\hat{y}_1 - \hat{y}_2)$ with $y_1, y_2, \hat{y}_1, \hat{y}_2 \in \mathbb{R}$. One can also show [4] that an $SO(3)$ -map β which is a homeomorphism only exists if η and $\hat{\eta}$ have the same number of eigenvalues. Thus, for topologically transitive j , there are only three classes of IPTF's: the class containing the invariant fields which have values in E_η where either $\eta = \text{diag}(0, 1, -1)$, $\text{diag}(1, 1, -2)$, or $\text{diag}(0, 0, 0)$. Note that computing the underlying isotropy groups is easily done by Linear Algebra since $l_t(r; M)$ is linear in M (same for l_v).

We now apply the DT in the case where $(E, l) = (\mathbb{R}^3, l_v)$ and $(\hat{E}, \hat{l}) = (E_t, l_t)$ and where $\eta = (0, 0, 1)$ and $\hat{\eta} = \text{diag}(y, y, -2y)$ where $y \in \mathbb{R}$. We know from Example 1 that $H_\eta = SO(2)$ and one easily computes $\hat{H}_{\hat{\eta}}$ and finds that $H_\eta \subset \hat{H}_{\hat{\eta}}$ whence $\beta \in \mathcal{C}(E_\eta, \hat{E}_{\hat{\eta}})$ defined by $\beta(l_v(r; \eta)) := l_t(r; \hat{\eta})$ is an $SO(3)$ -map. Easy computation shows that $\beta(S) = yI_{3 \times 3} - 3ySS^t$ where $S \in \mathbb{R}^3$ with $|S| = 1$ so that if f is an ISF of (j, A) then, by the DT, the function $g \in \mathcal{C}(\mathbb{T}^d, E_t)$, defined by

$$g(z) := \beta(f(z)) = yI_{3 \times 3} - 3yf(z)f^t(z), \quad (10)$$

is an IPTF of (j, A) . This confirms the observation [12] obtained by a different method.

7.3 Example 3: $(E, l) = (E^{1/2}, l^{1/2})$

We call every $\rho \in \mathcal{C}(\mathbb{T}^d, E^{1/2})$ a spin-1/2 density matrix and since α is a homeomorphism we can write $\rho = \alpha \circ P$ which defines the polarization field

$P \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$. With (4) it follows that ρ is an invariant field iff P is an IPF. Thus the theorems about IPF's (e.g., from the four main theorems) can be applied to invariant ρ 's.

The spin-1/2 density matrix is the key to the statistical description of a bunch of spin-1/2 particles. Assume for simplicity that the particle bunch is in equilibrium, so that a stationary bunch density $\rho_{\text{eq}}(J)$ exists where J is the action-variable. Define $\rho_{\text{tot}}(n; z, J) := \rho_{\text{eq}}(J)\rho_{\text{spin}}(n; z, J) = \rho_{\text{eq}}(J)\alpha(P(n; z, J))$ where each of the $\rho_{\text{spin}}(n; \cdot, J)$ evolves under a (j_J, A_J) as a spin-1/2 density matrix. Then $P(n; \cdot, J)$ moves as a polarization field on the torus from turn n to $n+1$. Note that $|P| \leq 1$. Every ‘‘physical observable’’ $\mathcal{O} \in \mathcal{C}(\mathbb{T}^d \times \Lambda, \mathbb{C}^{2 \times 2})$ can be written as $\mathcal{O}(z, J) = h_0(z, J) + \sum_{i=1}^3 h_i(z, J)\sigma_i$, and its expectation value $\langle \mathcal{O} \rangle(n)$ at turn n is defined by

$$\langle \mathcal{O} \rangle(n) := \int_{\mathbb{T}^d \times \Lambda} dz dJ \text{Tr}(\rho_{\text{tot}} \mathcal{O}) , \quad (11)$$

where $\Lambda \subset \mathbb{R}^d$ is the domain of J . For example, in case of the spin observable where $\mathcal{O}(z, J) := \sigma_i$, the expectation value of \mathcal{O} is the i -th component of the polarization vector of the bunch at time n . The choice $(E^{1/2}, l^{1/2})$ and the above theory of ρ_{tot} follows from the semiclassical treatment of Dirac's equation in terms of Wigner functions where the particle-variables z and J are purely classical (see [8] and the references therein).

7.4 Example 4: $(E, l) = (E^1, l^1)$

We call the fields $\rho \in \mathcal{C}(\mathbb{T}^d, E^1)$ spin-1 density matrices and since α is a homeomorphism we can write $\rho(z) = \alpha(P(z), m(z))$ where the polarization field P and the polarization tensor field m are uniquely determined by ρ . With (4) one can show that ρ is an invariant field iff P is an IPF and m an IPTF of (j, A) . Thus the theorems about IPF's and IPTF's can be applied to invariant ρ 's.

We note that given ρ one may construct in complete analogy to the spin-1/2 case a ρ_{tot} . The observables are now continuous hermitian 3×3 matrix functions and one obtains the analogy to (11) by building ρ_{tot} out of ρ_{eq} and polarization fields and polarization tensor fields. In particular one can prepare an equilibrium bunch where the IPF's $P(0, \cdot, J)$ of (j_J, A_J) are built up from the ISF's $p(0, \cdot, J)$ and where the IPTF's $m(0, \cdot, J)$ are built up from the $p(0, \cdot, J)$ via (10), and then ρ_{tot} is completely determined by ρ_{eq} and the

$p(0, \cdot, J)$. The choice (E^1, l^1) and the theory of ρ_{tot} follows, as in the spin-1/2 case, from the semiclassical treatment of Wigner functions.

8 Underlying Bundle Theory

While bundle-theoretic aspects play no role in the above outline of our results we put it into that context here by following [2, 3, 13]. See also [4, 5].

The “unreduced” principal bundle underlying our formalism is a product principal $SO(3)$ -bundle with base space \mathbb{T}^d , i.e., it can be written as the 4-tuple $(E_d, p_d, \mathbb{T}^d, L_d)$ where E_d is the bundle space, $p_d \in \mathcal{C}(E_d, \mathbb{T}^d)$ the bundle projection, i.e., $p_d(z, r) := z$, and (E_d, L_d) the underlying $SO(3)$ -space. For every (j, A) bundle theory gives us a natural particle-spin map on E_d which turns out to be $\check{\mathcal{P}}[j, A]$. The reductions are those principal H -bundles which are subbundles of the unreduced bundle such that their bundle space is a closed subset of E_d and such that H is closed. By the well-known Reduction Theorem [2, Chapter 6], [13, Chapter 6], every $(\check{\Sigma}_x[f], p_x[f], \mathbb{T}^d, L)$, for which E is Hausdorff, is a reduction where L is the restriction of L_d to $H_x \times \check{\Sigma}_x[f]$ and conversely, every reduction is of this form. By bundle theory, the natural particle-spin map on $\check{\Sigma}_x[f]$ for a given (j, A) is that bijection on $\check{\Sigma}_x[f]$ which is a restriction of $\check{\mathcal{P}}[j, A]$. Clearly this function is a bijection iff $\check{\Sigma}_x[f]$ is $\check{\mathcal{P}}[j, A]$ -invariant and then the reduction is called “invariant under (j, A) ”. Thus indeed the IRT deals with invariant reductions and it states that a reduction is invariant under (j, A) iff f is an invariant field.

The bundle-theoretic aspect of the CST follows from the simple fact that the cross sections of $p_x[f]$ are the bundle-theoretic cross sections of the reduction. Thus, by bundle theory, $p_x[f]$ has a cross section iff the principal bundle $(\check{\Sigma}_x[f], p_x[f], \mathbb{T}^d, L)$ is trivial, i.e., is isomorphic to a product principal bundle (this isomorphism is used in proving (9)).

Every (E, l) in the formalism uniquely determines an “associated bundle” (relative to the unreduced bundle) which, up to bundle isomorphism, is of the form $(\mathbb{T}^d \times E, p, \mathbb{T}^d)$ where $p(z, x) := z$. Moreover, since $\check{\mathcal{P}}[j, A]$ is the natural particle-spin map on E_d , the association with (E, l) leads to a natural map on $\mathbb{T}^d \times E$ which turns out to be the particle-spin map $\mathcal{P}[E, l, j, A]$ of (2). In comparison, the matter fields in gauge field theories enter via associated bundles.

As a side aspect, the above mentioned reductions reveal a relation to Yang-Mills Theory via the principal connections. For example, in the pres-

ence of an IFF (see Example 1) we have an invariant $SO(2)$ reduction which has a cross section and describes planar spin motion. Since this reduction is a smooth principal bundle, it has a well-defined class of principal connections leading via path lifting to parallel transport motions which, remarkably, reproduce the form of the well-known T-BMT equation, and thus in discrete time give us $\mathcal{P}[\mathbb{R}^3, l_v, j, A]$. These aspects will be extended to nonplanar spin motion [14].

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