Design of Multi Objectives Control Systems to Control Nuclear Reactor Power

By

Eng. Magdy Mahmoud Zaky Abdelaal

A Thesis Submitted to the
Faculty of Engineering at Cairo University
In Partial Fulfillment of the
Requirements for the Degree of
DOCTOR OF PHILOSOPHY
In
ELECTRICAL POWER AND MACHINES ENGINEERING

FACULTY OF ENGINEERING, CAIRO UNIVERSITY
GIZA, EGYPT
2013
Design of Multi Objectives Control Systems to Control Nuclear Reactor Power

By

Eng. Magdy Mahmoud Zaky Abdelaal

A Thesis Submitted to the
Faculty of Engineering at Cairo University
In Partial Fulfillment of the
Requirements for the Degree of
DOCTOR OF PHILOSOPHY
In
ELECTRICAL POWER AND MACHINES ENGINEERING

Under the supervision of
Prof. Dr. Adel Abdel Raouf Hanafy Prof. Dr. Hassan Rashad Emara

Prof. Dr. Sayed Elaraby

FACULTY OF ENGINEERING, CAIRO UNIVERSITY
GIZA, EGYPT
2013
Design of Multi Objectives Control Systems to Control Nuclear Reactor Power

By

Eng. Magdy Mahmoud Zaky Abdelaal

A Thesis Submitted to the
Faculty of Engineering at Cairo University
In Partial Fulfillment of the
Requirements for the Degree of
DOCTOR OF PHILOSOPHY
In
ELECTRICAL POWER AND MACHINES ENGINEERING

Approved by the Examining Committee:

Prof. Dr. Adel Abdel Raouf Hanafy
Main Supervisor

Prof. Dr. Hassen Taher Dorrah
Member

Prof. Dr. Hassan Mohammed Rashad Emara
Supervisor

Prof. Dr. Mahmoud Mohammed Fahmy
Member

FACULTY OF ENGINEERING, CAIRO UNIVERSITY
GIZA, EGYPT
2013
Engineer: Magdy Mahmoud Zaky Abdelaal
Date of Birth: 24/3/1965
Nationality: Egyptian
E-mail: zaky_magdy@yahoo.com
Phone: 01000438243
Address: Egypt, ElObour City, ElShabab, 333.
Registration Date: 19/7/2001
Awarding Date: / / 
Degree: DOCTOR OF PHILOSOPHY
Department: Electrical Power and Machines Engineering
Supervisors: Prof. Dr. Adel Abdel Raouf Hanafy
Prof. Dr. Hassan Mohammed Rashad Emara
Prof. Dr. Sayed Mohammed Sayed Elaraby – Atomic Energy Authority
Examiners: Prof. Dr. Adel Abdel Raouf Hanafy
Prof. Dr. Hassen Taher Dorrah
Prof. Dr. Hassan Mohammed Rashad Emara
Prof. Dr. Mahmoud Mohammed Fahmy Tanta University

Title of Thesis: Design of Multi Objectives Control Systems to Control Nuclear Reactor Power

Key Words: Model order reduction, Robust control system, Linear matrix inequality, Multi objective control, Nuclear reactor power control.

Summary: The Egyptian Testing Research Reactor (ETRR-2) nonlinear twelfth order model is linearized and reduced to lower order model. Model order reduction methodologies such as balanced truncation, Schur reduction method, Hankel approximation and Coprime factorization have been used in the reduction process. The reactor actually controlled by PD controller with fixed tuning parameters. LMI state feedback, LMI-pool assignment, $H_{\infty}$ and observer based controllers based third order model are proposed to be used in the reactor power control instead of the PD controller. A comparison of LMI, LMI-Pole placement, $H_{\infty}$ control systems and those of based observer relative to the PD controller has been performed which showed better response and disturbance rejection for the proposed controllers.
Acknowledgments

Many people have played direct and indirect roles in the stage of my life that has lead to this thesis, and I would like to acknowledge them here. First of all, I thank my advisor, Professor Doctor Adel Abdel Raouf Hanafy, Electrical Power and Machines Department, Faculty of Engineering, Cairo University for having the patience to allow me to pursue my own interests, with the occasional push when required to keep me on track.

I especially thank Professor Doctor Hassan Rashad Emara, Electrical Power and Machines Department, Faculty of Engineering, Cairo University for his personal discussions, as well as for his spirit and dedication which are an inspiration to me. It was an invaluable resource to me, for general advice and technical discussions. Dr. Hassan gave me the opportunity to pursue other interesting topics; he broadened my horizons and taught me a lot, and I thank him.

I would like to thank my supervisor Professor Doctor Sayed Elaraby, Nuclear Research Center, Egyptian Atomic Energy Authority, for his encouragement, helpful advice and the time he offered me during his period of supervision.

Finally I am so grateful to my dear father, mother, and wife for their continuous support and encouragement during research period.

Thanks and my love goes to my son Mahmoud.
# List of Contents

Acknowledgments .................................................................................................................. IV  
List of Contents ...................................................................................................................... V  
List of Tables ........................................................................................................................ IX  
List of Figures ........................................................................................................................ X  
List of Symbols and Abbreviations ......................................................................................... XIV  
Abstract ................................................................................................................................ XVII  

Chapter 1: Introduction ........................................................................................................... 1  
1.1 Introduction ..................................................................................................................... 1  
1.2 Reactor Power Control ................................................................................................. 3  
1.3 PID Controller Tuning Methods ................................................................................... 4  
1.4 Thesis Objectives .......................................................................................................... 5  
1.5 Thesis Organization ....................................................................................................... 5  

Chapter 2: Literature Survey .................................................................................................. 6  
2.1 Introduction ..................................................................................................................... 6  
2.2 Automatic Control ......................................................................................................... 6  
2.3 Linear Matrix Inequality: LMI ....................................................................................... 11  
2.4 Model Uncertainty and Its Representation .................................................................. 12  
2.5 Robust Control .............................................................................................................. 13  
2.5.1 Vector Norms and Signal Norms ......................................................................... 14  
2.5.2 Terms Used in $H_{\infty}$ Robust Control Systems Design ........................................ 15  
2.5.3 $l_2$ and $H_2$ Norms .............................................................................................. 16  
2.5.4 $l_\infty$ and $H_\infty$ Norms ........................................................................................ 17  
2.5.5 The $H_2$ and $H_\infty$ Spaces and Norms ................................................................. 19  
2.6 $H_2$ and $H_{\infty}$ Control Systems Design .................................................................. 20  
2.6.1 $H_2$ Control System ............................................................................................... 20  
2.6.2 $H_{\infty}$ Control System .......................................................................................... 20  
2.6.3 $H_2 / H_{\infty}$ Control System .................................................................................. 24  
2.7 Properties of $H_{\infty}$ Robust Control Design ............................................................... 25  
2.7.1 Comparison of $H_{\infty}$ and $H_2$/LQG Controllers ............................................... 26  
2.7.2 Relationships between Classical Design and $H_{\infty}$ Robust Control .................... 27  
2.7.3 $H_2$ and $H_{\infty}$ Design and Relationship to Conventional Control Systems ......... 27
Chapter 3: Reactor Modeling and Classical Control System

3.1 Introduction .................................................................................................................. 30
3.2 Nuclear Reactor Power Control .................................................................................. 30
3.3 ETRR-2 Control Systems ............................................................................................. 32
3.4 Control Mechanism ..................................................................................................... 34
3.5 ETRR-2 System Model ................................................................................................. 38
3.6 Reactor Dynamics and Point Kinetic Model ............................................................... 40
   3.6.1 Model Investigation ................................................................................................. 42
   3.6.2 Feedback Parameters ............................................................................................. 44
      3.6.2.1 Poisons Physical Equations and Model .......................................................... 44
      3.6.2.2 Model Verification and Results ................................................................. 45
   3.6.3 Heat Transfer and Temperature Feedback Model ................................................ 49
3.7 Disturbances .................................................................................................................. 52
3.8 System Stability and Automatic Reactor Power Control ............................................. 53
   3.8.1 Model Response to Transient Situations .............................................................. 56
   3.8.2 Reactor Power Control Model Investigation ....................................................... 56
3.9 Concluding Remarks .................................................................................................... 61

Chapter 4: Robust Control Systems and Linear Matrix Inequality ................................. 62
4.1 Introduction .................................................................................................................... 62
4.2 Model Uncertainty: Representation ............................................................................. 63
4.3 Robust Control Systems ............................................................................................... 64
4.4 Linear Matrix Inequality (LMI) .................................................................................... 65
   4.4.1 Types of LMI Problems ......................................................................................... 67
   4.4.2 Bilinear Matrix Inequality .................................................................................... 69
4.5 Mathematical Techniques of LMI Problems ............................................................... 70
4.6 Robust State Feedback Controller Design Methods .................................................. 72
   4.6.1 Stabilizing Controller Design in LMI ................................................................. 72
   4.6.2 H∞ Norm Formulation in LMI ............................................................................. 73
   4.6.3 H2 performance .................................................................................................... 74
4.7 Pole Placement in Control Systems Design ............................................................... 76
   4.7.1 Pole Placement in LMI Region ............................................................................. 76
   4.7.2 State Feedback with Regional Pole Placement ................................................... 78
6.7 ETRR-2 Observer Based Control Model Analysis ...............................................................129
6.8 Concluding Remarks ...........................................................................................................135
Chapter 7: Conclusion and Recommendation of Future Work ..............................................136
7.1 Conclusion ..........................................................................................................................136
7.2 Recommendation of future work .......................................................................................138
List of References ......................................................................................................................139
<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table 3.1</td>
<td>Delayed neutron data of the U$^{235}$ fuel of (ETRR-2)</td>
<td>41</td>
</tr>
<tr>
<td>Table 3.2</td>
<td>Delayed neutron fraction parameters</td>
<td>42</td>
</tr>
<tr>
<td>Table 3.3</td>
<td>Poisons (Xenon and Iodine) model parameters</td>
<td>45</td>
</tr>
<tr>
<td>Table 3.4</td>
<td>Heat transfer and temperature feedback data</td>
<td>50</td>
</tr>
<tr>
<td>Table 3.5</td>
<td>Power controller data</td>
<td>56</td>
</tr>
<tr>
<td>Table 3.6</td>
<td>Total reactivity at certain critical position</td>
<td>57</td>
</tr>
<tr>
<td>Table 5.1</td>
<td>Modeling error of full order and reduced order models</td>
<td>103</td>
</tr>
<tr>
<td>Table 5.2</td>
<td>Modeling error of tenth order model reduction model</td>
<td>107</td>
</tr>
<tr>
<td>Table 5.3</td>
<td>Modeling error between full order model and reduced order models</td>
<td>111</td>
</tr>
</tbody>
</table>
List of Figures

Figure 2.1: The regulator of Heron of Alexandria ................................................................. 9
Figure 2.2: $H_2$ norm................................................................................................................. 17
Figure 2.3: $H_\infty$ norm............................................................................................................. 19
Figure 2.4: $H_\infty$ control model of a closed loop system ......................................................... 21
Figure 2.5: Disturbance rejection from $w$ to $z$ is obtained by minimizing $H_\infty$ .................. 23
Figure 2.6: Robust stability is guaranteed if $\|\Pi\|_\infty < 1$ ..................................................... 24
Figure 2.7: $H_2 / H_\infty$ controller .............................................................................................. 25
Figure 3.1: Overall sketch of control system of the nuclear reactor ............................................ 32
Figure 3.2: Automatic reactor power control system ............................................................... 34
Figure 3.3: First shut down system of ETRR-2 .......................................................................... 36
Figure 3.4: Generic stepping motor with its main parts ............................................................ 37
Figure 3.5: Perpendicular view of a generic stepping motor ..................................................... 37
Figure 3.6: Top view of the plant model .................................................................................... 38
Figure 3.7: Reactor kinetic 6-groups model .............................................................................. 41
Figure 3.8: Reactor point kinetic model response to step reactivity changes of 100pcm ......... 43
Figure 3.9: Reference reactor point kinetic response to step reactivity changes of 100pcm. 43
Figure 3.10: Xenon and Iodine model ....................................................................................... 45
Figure 3.11: ETRR-2 Xenon and Iodine response to a step flux change of 1E+14nv from steady state conditions at Zero flux .................................................................................. 46
Figure 3.12: Xenon response to shut down for three steady state initial conditions (8.1E+13, 1E+14 and 2.7E+14)nv .............................................................................................................. 47
Figure 3.13: Xenon135 build up after shutdown for several values of the operating flux before shutdown ........................................................................................................................................ 47
Figure 3.14: Xenon and Iodine density for shutdown states .................................................... 48
Figure 3.15: Model of the reactor coolant and fuel temperature feedback ............................... 51
Figure 3.16: Fuel and coolant temperature when the reactor power stepped change from 100% to 50%.......................................................................................................................... 52
Figure 3.17: Closed loop of reactor control system ................................................................. 55
Figure 3.18: Power changes from 100% to 75%, 50% and 25% respectively ........................ 57
Figure 3.19: Rod positions relative power changes from 100% to 75%, 50% and 25% ....... 58
Figure 3.20: Model response to positive reactivity insertion (20pcm, 50pcm and 100pcm) respectively

Figure 3.21: Rod position changes due to positive reactivity insertion (20pcm, 50pcm and 100pcm) respectively

Figure 3.22: Model response to negative reactivity insertion (-20pcm, -50pcm and -100pcm) respectively

Figure 3.23: Rod position changes due to negative reactivity insertion (-20pcm, -50pcm and -100pcm) respectively

Figure 4.1: General uncertain system model

Figure 4.2: Region $\phi_\alpha, r, \theta$

Figure 4.3: Detailed observer block diagram

Figure 4.4: State feedback with full order observer

Figure 5.1: Model and control system order reduction

Figure 5.2: Flowchart of model order reduction for unstable systems

Figure 5.3: Hankel singular values of the fifth order model

Figure 5.4: Bode diagram of the original and reduced models by Schur, BT and Hankel reduction methods

Figure 5.5: Singular values of the fifth order and the reduced third order models by Schur, BT and Hankel approximation methods

Figure 5.6: Hankel singular values of the tenth order model

Figure 5.7: Bode diagram of full order and reduced order models

Figure 5.8: Singular values of full order and reduced order models

Figure 5.9: Hankel singular values of the twelfth order model

Figure 5.10: Bode diagram of full order and reduced order model by Schur method

Figure 5.11: Singular values of full order and reduced order model by Schur method

Figure 6.1: Normalized power of non linear full order model relative to full order and reduced order LMI control system when power step change from 100% to 50%

Figure 6.2: Normalized power of LMI controllers when step change by 250pcm inserted to the core

Figure 6.3: Normalized power of non linear full order model relative to third order LMI state feedback control system when power step change to 50%

Figure 6.4: Normalized power of non linear full order model relative to third order LMI control system when power step change to 50% (zoom in)
Figure 6.5: Normalized power of LMI and PD controllers when step change 100pcm inserted to the core ................................................................. 120

Figure 6.6: Normalized power of LMI and PD controllers when step change -100pcm inserted to the core ................................................................. 120

Figure 6.7: Normalized power of LMI and PD controllers when step change 250pcm inserted to the core ................................................................. 121

Figure 6.8: Normalized power of LMI and PD controllers and when step change -250pcm inserted to the core ................................................................. 121

Figure 6.9: Normalized power of full order nonlinear model with third order LMI pole placement relative to the PD control system when the power step change to 50% .................. 123

Figure 6.10: Normalized power of full order nonlinear model with third order LMI pole placement relative to the PD control system when the power step change with 100pcm .... 123

Figure 6.11: Control rod worth change of LMI pole placement and PD control systems relative to 100pcm reactivity insertion ............................................ 124

Figure 6.12: Model response of PD and H∞ controllers when power step changed from 100% to 50% ........................................................................... 125

Figure 6.13: Model response of PD and H infinity controllers at steady state when 250pcm positive inserted to the core ....................................................... 127

Figure 6.14: Control rod position changed relative to power step changed by 250pcm ..... 127

Figure 6.15: Model response of PD and H-infinity when the reactor power step changed by -250pcm ................................................................................ 128

Figure 6.16: Control rod worth changed when the reactor power step changed by -250pcm ................................................................................ 128

Figure 6.17: PD and H∞ observer based controller response when reactor power stepped from 100% to 50% ................................................................. 131

Figure 6.18: PD and H∞ observer based response when reactor power stepped from 100% to 75% and stepped back to 100% .................................................. 131

Figure 6.19: PD and H∞ observer based response when reactor power stepped by 200pcm ................................................................................ 132

Figure 6.20: Control rod worth change relative to PD and H∞ observer based controllers when reactor power stepped by 200pcm ........................................ 132

Figure 6.21: PD and LMI observer based response when reactor power stepped from 100% to 75% and stepped back to 100% ........................................... 134
**Figure 6.22:** Control rod worth change relative to PD and LMI observer based controllers when reactor power stepped from 100% to 75% and stepped back to 100%.................. 134
## List of Symbols and Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>ARE</td>
<td>Algebraic Riccati Equation</td>
</tr>
<tr>
<td>ARPCS</td>
<td>Automatic Reactor Power Control System</td>
</tr>
<tr>
<td>BIBO</td>
<td>Bounded Input Bounded Output</td>
</tr>
<tr>
<td>BMI</td>
<td>Bilinear Matrix Inequality</td>
</tr>
<tr>
<td>BT</td>
<td>Balanced Truncation</td>
</tr>
<tr>
<td>CIC</td>
<td>Compensated Ionization Chamber</td>
</tr>
<tr>
<td>CR-1</td>
<td>Control rod number 1</td>
</tr>
<tr>
<td>CR-2</td>
<td>Control rod number 2</td>
</tr>
<tr>
<td>CR-3</td>
<td>Control rod number 3</td>
</tr>
<tr>
<td>CR-4</td>
<td>Control rod number 4</td>
</tr>
<tr>
<td>DAE</td>
<td>Differential Algebraic Equation</td>
</tr>
<tr>
<td>ETRR-2</td>
<td>Egyptian Testing Second Research Reactor</td>
</tr>
<tr>
<td>EVP</td>
<td>Eigenvalue Problem</td>
</tr>
<tr>
<td>GEVP</td>
<td>Generalized Eigenvalue Problem</td>
</tr>
<tr>
<td>HNA</td>
<td>Hankel Norm Approximation</td>
</tr>
<tr>
<td>LMI</td>
<td>Linear Matrix Inequality</td>
</tr>
<tr>
<td>LQG</td>
<td>Linear Quadratic Gaussian</td>
</tr>
<tr>
<td>LQR</td>
<td>Linear Quadratic Regulator</td>
</tr>
<tr>
<td>LTI</td>
<td>Linear Time Invariant</td>
</tr>
<tr>
<td>LTI</td>
<td>Linear Time Invariant</td>
</tr>
<tr>
<td>K</td>
<td>Multiplication factor</td>
</tr>
<tr>
<td>MPR:</td>
<td>Multi Purpose Reactor</td>
</tr>
<tr>
<td>ODE</td>
<td>Ordinary Differential Equation</td>
</tr>
<tr>
<td>PCS</td>
<td>Power Control Signal</td>
</tr>
<tr>
<td>PDE</td>
<td>Partial Differential Equation</td>
</tr>
<tr>
<td>PD</td>
<td>Proportional plus Derivative controller</td>
</tr>
<tr>
<td>PID</td>
<td>Proportional plus Integral plus Derivative</td>
</tr>
<tr>
<td>Pcm</td>
<td>Part per million</td>
</tr>
<tr>
<td>PR</td>
<td>Positive Real</td>
</tr>
<tr>
<td>R</td>
<td>Reactivity in dollar</td>
</tr>
<tr>
<td>RS</td>
<td>Root Square</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>RSF</td>
<td>Real Schur Form</td>
</tr>
<tr>
<td>SVD:</td>
<td>Singular Value Decomposition</td>
</tr>
<tr>
<td>$\alpha_c$</td>
<td>Alpha, (coolant temperature feedback coefficient)</td>
</tr>
<tr>
<td>$\alpha_f$</td>
<td>Alpha, (fuel temperature feedback coefficient)</td>
</tr>
<tr>
<td>$\alpha_\chi$</td>
<td>Alpha, (reactivity coefficient)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Beta, (effective delayed neutron fraction)</td>
</tr>
<tr>
<td>$\beta_i$</td>
<td>Beta, (group-$i$th delayed neutron)</td>
</tr>
<tr>
<td>$T_c$</td>
<td>Coolant average temperature</td>
</tr>
<tr>
<td>$T_i$</td>
<td>Coolant inlet temperature</td>
</tr>
<tr>
<td>$M$</td>
<td>Coolant mass flow rate</td>
</tr>
<tr>
<td>$m_c$</td>
<td>Coolant mass</td>
</tr>
<tr>
<td>$P$</td>
<td>Current reactor power</td>
</tr>
<tr>
<td>$c_c$</td>
<td>Coolant specific heat</td>
</tr>
<tr>
<td>$T_c$</td>
<td>Coolant temperature</td>
</tr>
<tr>
<td>$K_r$</td>
<td>Derivative gain</td>
</tr>
<tr>
<td>$\Delta \rho$</td>
<td>Delta raw, (Reactivity defects)</td>
</tr>
<tr>
<td>$\Delta \chi$</td>
<td>Delta, (minimum distance affects the reactivity)</td>
</tr>
<tr>
<td>$T_f$</td>
<td>Fuel average temperature</td>
</tr>
<tr>
<td>$m_f$</td>
<td>Fuel mass</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>Flux rate</td>
</tr>
<tr>
<td>$c_f$</td>
<td>Fuel specific heat</td>
</tr>
<tr>
<td>$T_f$</td>
<td>Fuel temperature</td>
</tr>
<tr>
<td>$\Phi_T$</td>
<td>Flux, (thermal neutron flux)</td>
</tr>
<tr>
<td>$\Gamma_i$</td>
<td>Gamma, (Iodine fission yield)</td>
</tr>
<tr>
<td>$\Gamma_x$</td>
<td>Gamma, (Xenon fission yield)</td>
</tr>
<tr>
<td>$c_i$</td>
<td>Group-$i$th precursor density</td>
</tr>
<tr>
<td>$h$</td>
<td>Heat transfer coefficient</td>
</tr>
<tr>
<td>$T_{eco}$</td>
<td>Initial coolant temperature</td>
</tr>
<tr>
<td>$T_{fo}$</td>
<td>Initial fuel temperature</td>
</tr>
</tbody>
</table>
I  Iodine density (atoms/cm$^3$)
$\lambda_i$  Lambda, ($i^{th}$ precursor decay constant)
$\lambda_x$  Lambda, (Xenon decay constant)
n  Neutron density
N  Normalized power density
S  Neutron source, (used in case of first criticality)
$K_P$  Proportional gain
$\Lambda$  Prompt neutron generation time
$\rho_r$  Row, (control rod reactivity)
$\rho_{ext}$  Row, (external reactivity feedback)
$a_i$  Relative fraction for delayed neutrons of group i
$P_{ref}$  Reference reactor power
$\rho$  Raw, (total reactivity)
$\rho_{Xenon}$  Raw, (Xenon reactivity feedback)
$\Sigma_f$  Sigma, (average thermal macroscopic fission cross section)
$\sigma_{ax}$  Sigma, (thermal microscopic cross section)
A  Total heat transfer area
$p_{th}$  Thermal power
x  Xenon density (atoms/cm$^3$)
Abstract

The dynamics of nuclear reactors are very different according to the power levels and changes as time goes on because of the nonlinear structure of the reactor dynamics. Therefore, it is a challenging task to improve the power control system of the nuclear reactors. The response of the nonlinear systems such as nuclear reactors, generally, can’t be shaped to a desired trajectory using conventional controllers.

In recent years, different control methodologies such as Fuzzy logic control, neural control, and robust control systems have been proposed to control the nuclear reactors after many of the nuclear reactors accidents.

To design a fine control system an important issue is the accurate mathematical model of a system, which essential for the correct development of the control system. If a plant is not modeled correctly then consequently designed controller will exhibit poor performance.

Nuclear reactors are highly complex, nonlinear, time-varying, and constrained systems, so accurate model development and validation is a very important task. Also plant parameters vary with time, for example, the nuclear plant characteristics vary with power levels, ageing effects and changes in nuclear core reactivity with fuel burn up.

The Egyptian Testing Research Reactor (ETRR-2) power level is controlled by Proportional plus Derivative controller (PD), with fixed tuning parameters. The benefits of this study are to analyze the ETRR-2 model response to create new types of controllers and study their performance at different operation conditions of the reactor.

Model reduction is a tool for the approximation of complex high-order systems, leading to reduced-order systems that allow for efficient analysis or facilitate control design and implementation. Since the reduced-order model is used as a substitute for the original high-order model, it is of importance to preserve key system properties during the reduction process. Herein, stability is the most crucial one. However, the preservation of gain properties or passivity is relevant in many control applications as well. Additionally, the availability of an error bound is highly instrumental in determining the quality of the reduction.

In this thesis, balanced truncation, Schur reduction method, Hankel approximation and Coprime factorization techniques are selected for model reduction for ETRR-2 linear model that, firstly, preserve stability, gain or passivity properties and, secondly, allow for the computation of an error bound.
Nuclear reactors power level control is one of the key techniques which provide safe, stable and efficient operation for nuclear reactors.

The physically-based regulation theory is definitely a promising trend of modern control theory, which can provide a control design method that corrects the power level according to the nominal value.

Based on the ETRR-2 model reduction investigation the following low order control systems: Linear Matrix Inequality (LMI), LMI-pole placement, $H_{\infty}$ robust control system and both LMI and $H_{\infty}$ based observer controllers based on the third order model are used to control the ETRR-2 power level instead of the actually used PD control system.

A comparison of LMI, LMI-Pole placement, $H_{\infty}$ control systems and those of based observer relative to the PD controller has been performed which showed better response and disturbance rejection for the selected controllers.
Chapter 1

Introduction

1.1 Introduction

Utilization of nuclear energy began early in world war II especially after the bombs exploded over Hiroshima and Nagasaki, where the development of the nuclear field went on in two sides, one for using of this energy in the peace utilization and the other in field of weapons. Hundreds of the nuclear reactors around the world are used for different objectives such as

- Generation of electrical power
- Conversion of $^{238}U$ to $^{239}Pu$ to used in nuclear weapons
- The propulsion of ships, aircraft, and rockets
- Medical process
- Research
- Radio Isotopes production.
- Material testing
- Material irradiation
- Industrial applications

The nuclear reactors are classified into the following categories

- Nuclear power reactors
- Research reactors

Nuclear research reactors have many peaceful uses such as neutron radiography, material testing, isotopes production and medical neutron therapy, etc. During the operation of the reactor, different changes in its operating conditions occur such as control rods movement, fuel burn-up, Xenon isotope production, and temperature evolution. Therefore, an automatic reactor power control system is required to compensate the reactivity changes produced by such variations in the operating conditions of the reactor [1].

Egyptian Testing Research Reactor (ETRR-2) is an open pool type reactor: its thermal power is 22 MW, cooled and moderated by light water and reflected by Beryllium. Its operation
conditions change due to the effect of the following feedback parameters: Movement of control rods, fuel burn up, Xenon isotope production, temperature, and experiments. The main reactor pool houses the core matrix, in core irradiation grid, outer irradiation grid, nuclear and conventional instrumentation and some of the reactor irradiation facilities.

There are six control rods, two of them are always used as safety rods, and the other four are used for power compensation in normal operation mode. When these rods are extracted up, positive reactivity is inserted; otherwise negative reactivity is inserted manually or automatically by the automatic reactor power control system. Final actuator element speed is limited to constant step motion to verify the operation safety conditions [2].

The control system of the ETRR-2 is a conventional Proportional plus Derivative (PD) controller, which used to regulate the reactor power surround the operating set point. by moving control rods up and down inside the reactor core [3].

The reactor was designed to cover the following design concepts: Radioisotopes production, Material testing, Neutron radiography, Activation analysis, Material irradiation and Technical training.

- **Motivation**

Most of the experiments to be done in the ETRR-2 require stabilized flux with minimum over or down shoot and with minimum settling time.

However the action of PD controller versus any transients during the normal operation of the reactor tends to stabilize the reactor power associated by over or down shoot depending on the flux rate.

This thesis proposes multi objective controllers to be used instead of the conventional PD controller under the same conditions to control the reactor power. These controllers are designed to handle several design constraints/objectives simultaneously (minimizing the overshoot, handling plant uncertainties…etc.). The use of this type of controller is motivated by their successful applications in many fields such as:

- Robots
- Medical technology
- Industrial Control
- Fault diagnosis
• Helicopters control [4]

Due to the complexity usually involved in the design of multi-objective controllers, reduced order models are preferred to simplify the design process. The thesis surveys several model reduction techniques and concerning on some of them to be used in the reduction of the reactor model.

1.2 Reactor Power Control

The change of the power level and the state of the reactor requires the movement of control rods or their equivalent in such manner that the multiplication factor (k), which denotes to the difference between the new generation of neutrons to the last generation of neutrons in the reactor core. Any deviation in the multiplication factor from the unity will affect the reactor power. In changing power level two conditions must ultimately be satisfied by any control system.

The power level of the reactor must be the demanded power level within a given error, and the multiplication factor must ultimately be one.

A given reactor may contain many control rods. These control rods may be operated individually or in-groups or banks. Large groups of rods are moved quite slowly and usually function in control work as a very sensitive final control element. In some cases a fine tuning rod (regulator rod) is also used. It may move quite quickly, but to ensure safety, the maximum worth of the regulator rod is usually kept below prompt critical in reactivity [5-6].

• Control Loop Response requirements:

The general loop response requirements are:

a) The first requirement is that the loop be unconditionally stable. That is, under no condition of conceivable practical reactor operation are sustained oscillations to be permitted.

b) The second requirement is that the loop shall respond satisfactorily under transient conditions. The transients to be considered are generally of two types:

• Change in the power demand level
• Change in the reactivity of the reactor
1.3 PID Controller Tuning Methods

Today, a number of different controllers are used in industry and in many other fields. In quite general way those controllers can be divided into two main groups:

- conventional controllers
- unconventional controllers

As conventional controllers we can count a controllers known for years now, such as P, PI, PD, PID, Otto-Smith, all their different types and realizations, and other controller types. It is a characteristic of all conventional controllers that one has to know a mathematical model of the process in order to design a controller. Unconventional controllers utilize new approaches to the controller design such as fuzzy controller and neuro or neuro-fuzzy controllers.

Proportional-Integral-Derivative (PID) control is still widely used in industries because of its simplicity. The user just installs a controller and adjusts the three gains to get the best achievable performance. Most PID controllers nowadays are digital.

As mentioned above, the conventional controller used in ETRR-2 is a PD controller (two parts of PID). The controller output $u(t)$ is given by:

$$u(t) = K_p e(t) + K_d \frac{de(t)}{dt}$$

In this equation $K_p$ is the proportional tuning constant, $K_d$ is the derivative tuning constant and the error $e(t)$ is the difference between the set point $r(t)$ and the process variable $c(t)$ at time $t$.

The different techniques for selecting appropriate values for the tuning constants $K_p$ and $K_d$ have been developed over last 50 years. These methods can generally be classified as: [7-8]

Conventional Tuning Methods

- Simple trial and error method
- The analytical approach to the tuning problem
- The ultimate sensitivity method
- The compromise between purely heuristic trial-and-error techniques and the more rigorous analytical techniques.
- Ziegler-Nichols transient response method
**Tuning based on Evolutionary Optimization Methods**

Evolutionary computation techniques such as genetic algorithm and particle swarm optimization are proposed in the literature to tune the parameters of the conventional controllers. These techniques are also used to optimize the performance of the artificial intelligence based techniques such as neural controller and fuzzy control system [9-10].

1.4 Thesis Objectives

The thesis introduces a study of the steady state and transient response of the ETRR-2 nonlinear model. Model order reduction is proposed to be used in the design of low order state feedback control system based on linearized model of the reactor. A reduced order control system is used to control the full order nonlinear model. Multi objective control systems such as LMI and robust control systems will be used instead of the actual used PD controller to improve the system response, stability and disturbance rejection.

Linear Matrix Inequality (LMI) control toolbox offers tools for state-feedback and output feedback design with a combination of robust controllers such as Linear Quadratic Gaussian (LQG), $H_2$, and $H_\infty$ controller…etc. The thesis introduces the applications of LMI; LMI-pole placement and $H_\infty$ controllers based on model reduction to control the full order nonlinear reactor model.

1.5 Thesis Organization

This thesis lies in seven chapters; chapter 1 introduces the thesis structure overview, chapter 2 presents the literature survey and thesis objectives. Chapter 3 presents the reactor modeling with simulink and Matlab software; this chapter illustrates the response investigation of the current automatic reactor power control system at steady state condition and transient states due to variations in the feedbacks parameters. Chapter 4 studies the robust control systems, Linear Matrix Inequality (LMI) and Bilinear Matrix Inequality (BMI) and their control applications. Chapter 5 studies the different methodologies of model and controller order reduction and applications to the (ETRR-2). Chapter 6 introduces the applications of LMI and robust control based on model order reduction, case study: the (ETRR-2). Chapter 7 gives the thesis conclusion.
Chapter 2

Literature Survey

2.1 Introduction
This chapter illustrates the literature survey of the scientific work that had been done by many researchers in the field of our study. The literature survey is divided into the following sections:

- Automatic control: to illustrate the history of the automatic control system and its applications in nuclear field
- Linear Matrix Inequality (LMI): to represent the history of the LMI and its applications in control systems
- Model uncertainty: to show how it affects on the model performance
- Robust control systems: $H_\infty$ robust control systems applications
- Model order reduction: Methods and applications

2.2 Automatic Control
Throughout history mankind has tried to control the world in which he lives. From the earliest days he realized that his puny strength was no match for the creatures around him. He could only survive by using his wits and cunning. His major asset over all other life forms on earth was his superior intelligence. Stone Age man devised tools and weapons from flint, stone and bone and discovered that it was possible to train other animals to do his bidding and so the earliest form of control system was conceived. Before long the horse and ox were deployed to undertake a variety of tasks, including transport. It took a long time before man learned to replace animals with machines [11].

Fundamental to any control system is the ability to measure the output of the system, and to take corrective action if its value deviates from some desired value. This in turn necessitates a sensing device. Man has a number of “in-built” senses which from the beginning of time he has used to control his own actions, the actions of others, and more recently, the actions of machines. In
driving a vehicle for example, the most important sense is sight, but hearing and smell can also contribute to the driver’s actions.

Automatic control systems have been in existence since ancient times. A well-known ancient automatic control system is the regulator of Heron of Alexandria as shown in figure 2.1. This control system was designed to open the doors of a temple automatically when a fire was lit at the altar located outside the temple and to close the doors when the fire was put out.

The first major step in machine design, which in turn heralded the industrial revolution, was the development of the steam engine. A problem that faced engineers at the time was how to control the speed of rotation of the engine without human intervention. Of the various methods attempted, the most successful was the use of a conical pendulum, whose angle of inclination was a function (but not a linear function) of the angular velocity of the shaft. This principle was employed by James Watt in 1769 in his design of a fly ball, or centrifugal speed governor. Thus possibly the first system for the automatic control of a machine was born.

This in turn results in linear motion of the sleeve which adjusts the steam mass flow-rate to the engine by means of a valve. Watt was a practical engineer and did not have much time for theoretical analysis. He did, however, observe that under certain conditions the engine appeared to hunt, where the speed output oscillated about its desired value. The elimination of hunting, or as it is more commonly known, instability, is an important feature in the design of all control systems. Maxwell (1868) developed the differential equations for a governor, linearized about an equilibrium point, and demonstrated that stability of the system depended upon the roots of a characteristic equation having negative real parts.

The problem of identifying stability criteria for linear systems was studied by Hurwitz (1875) and Routh (1905). This was extended to consider the stability of nonlinear systems by a Russian mathematician Lyapunov (1893). The essential mathematical framework for theoretical analysis was developed by Laplace (1749-1827) and Fourier (1758-1830).

Work on feedback amplifier design at Bell Telephone Laboratories in the 1930s was based on the concept of frequency response and backed by the mathematics of complex variables. This was discussed by Nyquist (1932) in his paper “Regeneration Theory”, which described how to determine system stability using frequency domain methods. This was extended by Bode (1945) and Nichols during the next 15 years to give birth to what is still one of the most commonly used control system design methodologies. Another important approach to control system design was
developed by Evans (1948). Based on the work of Maxwell and Routh, Evans, in his Root Locus method, designed rules and techniques that allowed the roots of the characteristic equation to be displayed in a graphical manner. The advent of digital computers in the 1950s gave rise to the state-space formulation of differential equations, which, using vector matrix notation, lends itself readily to machine computation.

The idea of optimum design was first mooted by Wiener (1949). The method of dynamic programming was developed by Bellman (1957), at about the same time as the maximum principle was discussed by Pontryagin (1962). At the first conference of the International Federation of Automatic Control (IFAC), Kalman (1960) introduced the dual concept of controllability and observability. At the same time Kalman demonstrated that when the system dynamic equations are linear and the performance criterion is quadratic (LQ control), then the mathematical problem has an explicit solution which provides an optimal control law. Also Kalman and Bucy (1961) developed the idea of an optimal filter (Kalman filter) which, when combined with an optimal controller, produced linear-quadratic-Gaussian (LQG) control. The 1980s saw great advances in control theory for the robust design of systems with uncertainties in their dynamic characteristics. The work of Athans (1971), Safanov (1980), Chiang (1988), Grimble (1988) and others demonstrated how uncertainty can be modeled and the concept of the $H_\infty$ norm and $\mu$ synthesis theory.

The 1990s has introduced to the control community the concept of intelligent control systems. An intelligent machine according to Rzevski (1995) is one that is able to achieve a goal or sustained behavior under conditions of uncertainty. Intelligent control theory owes much of its roots to ideas laid down in the field of Artificial Intelligence (AI). Artificial Neural Networks (ANNs) are composed of many simple computing elements operating in parallel in an attempt to emulate their biological counterparts. The theory is based on work undertaken by Hebb (1949), Rosenblatt (1961), Kohonen (1987), Widrow-Hoff (1960) and others. The concept of fuzzy logic was introduced by Zadeh (1965). This new logic was developed to allow computers to model human vagueness. Fuzzy logic controllers, whilst lacking the formal rigorous design methodology of other techniques, offer robust control without the need to model the dynamic behavior of the system. Workers in the field include Mamdani (1976), Sugeno (1985) Sutton (1991) and Tong (1978). There is no field in engineering where automatic control is not employed. Even human system is a very complex automatic feedback control system [12].
In parallel with this modernization in automatic control system and computer networks the designers of nuclear power plant try to use the different control systems to guarantee the safety and reliability of the nuclear reactors.

After Three Mile Island and Chernobyl accidents the automatic control system plays an important role to safe the nuclear reactor operation and shutdown. The history of the application of control system in nuclear plant as a very complex engineering system can be concluded as follows:

A survey of various problems in control of nuclear reactors and needs for research and development in that field is given in. One of the needs mentioned by authors of this survey is the use of process computers and control methods based on the synthesis of optimal LQG discrete regulators [13]. The digital computer is generally accepted as the most suitable of the presently available devices to perform such a task. At the same time, nuclear power stations are now operated entirely under the control of digital computers. However, the similarity between what the theoreticians propose and what the practicing engineers employ ends here.
In part, as a result of the stringent safety requirements, the control system designer tends to rely on the well proven classical techniques of controller design in the frequency domain, and use the computer merely as a convenient form of realizing the desired control configuration.

The other and more important reason for the lack of practical applications of optimal control theory is that the research efforts presented to date invariably assume an over-simplified representation of the reactor kinetics, typically the one delayed neutron group model, and no attempts are made to evaluate the response using a more complete description of the reactor dynamics or to include an adequate representation of the controller mechanisms. Furthermore, the solutions advanced for even these simplified, suboptimal problems, suffer from one or more of the following disadvantages: actual response far from the computed optimum, necessary control effort unrealizable, computer time excessive, memory requirements too great [14].

Optimal control of nuclear reactors has received attention in the recent literature utilizes Pontryagin's Maximum Principle to minimize several performance functions and assumes a point model for reactor kinetics. The application of optimal reactor control for CANDU reactors is studied using an adaptive control scheme for linearized low order plant dynamics models. The coefficients of the linear models are updated for new power levels. Linear feedback control is the optimal control form.

Potential improvements in plant efficiency and reliability are often cited as reasons for developing and applying artificial intelligence (AI) techniques, principally expert systems, to the control and operation of nuclear reactors. Nevertheless, there have been few such applications and then mostly at the prototype level. The slow pace at which artificial intelligence is being applied to the real-time operation of nuclear facilities is a reflection of the difficulties inherent in the nature of the control process itself.

Specifically, plant control, as now exercised by humans, is a complex task involving the simultaneous need for both analytic and heuristic resoling. Expert system technologies currently provide only the second of these capabilities and are therefore not sufficient by themselves for achieving effective and safe operation. Also, enormous resources have been expended over the past five decades for the development of automated, analytic controllers. It would be both unrealistic and financially unsound for industry to abandon this investment. Therefore, if AI techniques are to contribute to process control, methods must be identified by which rule-based and analytic approaches can be merged [15].
Especially after the severe Fukushima accident, safe and stable operation of nuclear reactors has become much more crucial than before. Power-level regulation of nuclear reactors is just one of the important techniques that guarantee safe, stable and efficient operation [16]. Fuzzy logic based intelligent system has been recently used in various nuclear power plant applications like use of neuro-fuzzy systems in power plant transient identification and fuzzy inference in nuclear reliability problems. Several researchers have applied combination of various computational intelligence paradigms together for efficient control of the dynamics of nuclear reactors. Among several other approaches, supervisory control, adaptive estimator based dynamic sliding mode control, gain scheduled dynamic sliding mode and robust nonlinear model predictive control [17].

2.3 Linear Matrix Inequality: LMI

The origin of Linear Matrix Inequalities (LMIs) goes back as far as 1890, although they were not called this way at that time, when Lyapunov showed that the stability of a linear system

\[ \dot{x}(t) = Ax(t) \]  

is stable (i.e., all trajectories converge to zero) if and only if there exists a positive definite matrix \( P \) such that

\[ A^T P + PA < 0 \]  

The requirement \( P > 0, \ A^T P + PA < 0 \) is what we now call a Lyapunov inequality on \( P \), which is a special form of an LMI. Lyapunov also showed that this first LMI could be explicitly solved. We can solve the linear equation \( A^T P + PA < 0 \) for the matrix \( P \), which is guaranteed to be positive-definite if the system (2.1) is stable. In summary, the first LMI used to analyze stability of a dynamical system was the Lyapunov inequality (2.2), which can be solved analytically [18]. The term “Linear Matrix Inequality” was coined by Williams in the 1970’s to refer to this specific LMI, in connection with quadratic optimal control.

Due to the lack of good computers as well as of efficient algorithms to solve them, the LMIs did not receive a great deal of consideration from control and system researchers until the late 1980’s, when Nesterov and Nemirovsky developed Interior-point methods that allowed solving elegantly LMI problems. New algorithms appeared then, triggering a renewed interest in this subject.
A summary of key events in the history of LMI’s in control theory is summarized as follows:

- **1890:** First LMI appears; analytic solution of the Lyapunov LMI via Lyapunov equation.
- **1940’s:** Application of Lyapunov’s methods to real control engineering problems. Small LMI’s solved by hand.
- **Early 1960’s:** Positive Real (PR) lemma gives graphical techniques for solving another family of LMI’s.
- **Late 1960’s:** Observation that solving an Algebraic Riccati Equation (ARE) can solve the same family of LMI’s.
- **Early 1980’s:** Recognition that many LMI’s can be solved by computer via convex programming.
- **Late 1980’s:** Development of interior-point algorithms for LMI’s. It is fair to say that Yakubovich is the father of the field, and Lyapunov the grandfather of the field. The new development is the ability to directly solve general LMI’s.
- **Advanced in computer and programming make the LMI applications in control and optimization very useful with many advantages [19-20].**

### 2.4 Model Uncertainty and Its Representation

**Origins of Model Uncertainty**

Uncertainty in control systems may stem from different sources. Model uncertainty is one of main consideration. Other considerations include sensor and actuator failures, physical constraints, changes in control purposes, loop opening and loop closure, etc. [21].

Model uncertainty may have several origins; in particular, it may be caused by:

1. Parameters in a linear model, which are approximately known or are simply in error
2. Parameters, which may vary due to nonlinearities or changes in the operating conditions
3. Neglected time delays and diffusion processes
4. Imperfect measurement devices
5. Reduced (low-order) models of a plant, which are commonly used in practice, instead of very detailed models of higher order
6. Ignorance of the structure and the model order at high frequencies
7. Controller order reduction issues and implementation inaccuracies.
The above sources of model uncertainty may be grouped into three main categories.

- Parametric or Structured Uncertainty
  In this case the structure of the model and its order is known, but some of the parameters are uncertain and vary in a subset of the parameter space.

- Neglected and unmodeled dynamics uncertainty
  In this case the model is in error because of missing dynamics (usually at high frequencies), most likely due to a lack of understanding of the physical process.

- Lumped Uncertainty or Unstructured Uncertainty
  In this case uncertainty represents several sources of parametric and/or un-modeled dynamics uncertainty combined into a single lumped perturbation of prespecified structure. Here, nothing is known about the exact nature of the uncertainties, except that they are bounded [22].

2.5 Robust Control

Robust control refers to the control of uncertain plants with unknown disturbance signals, uncertain dynamics, and imprecisely known parameters making use of fixed controllers. That is, the problem of robust control is to design a fixed controller that guarantees acceptable performance norms in the presence of plant and input uncertainty. The performance specification may include properties such as stability, disturbance attenuation, reference tracking, control energy reduction, etc [23].

An important change in control theory development took place in the 1950s with the emergence of the state-space approach. The high point of this development was reached during the 1960s with the formulation and solution of what is known as the Linear Quadratic Gaussian (LQG) problem. This development was primarily inspired by the important contributions of Kalman. The LQG approach provided a mathematically elegant method for designing feedback controllers of systems working in a noisy environment. However, it was soon realized that one of its principal drawbacks was its inability to guarantee a robust solution [24].

Control engineers found it difficult to incorporate robustness criteria in a quadratic integral performance index used in the LQG problem. For this reason, during the early 1970s, an attempt was made to generalize some of the useful concepts such as gain and phase margins so that they are made applicable to multi input multi output systems. This brings us to the 1980s, where a
variety of ideas of far-reaching significance were developed in the area of robust control. Some of these ideas are the following:

1. Use of singular values as a measure of gain in transformations
2. The factorization approach in controller synthesis
3. Parameterization of stabilizing controllers
4. $H_\infty$ optimization
5. Robust stabilization and sensitivity minimization
6. Computational aspects of $H_\infty$ optimization, such as
   - Interpolation methods based on Nevanlinna-Pick interpolation theory
   - Hankel norm approach
   - Operator-theoretic approach
   - State space approach using separation principle
7. Kharitonov’s theory and related approaches

With all these developments, robust control gained a great momentum, and it is currently one of the most important areas of research in the field of control theory and practice [25].

2.5.1 Vector Norms and Signal Norms

Let the linear space $X$ be $f^m$, where $f = R$ for the field of real numbers, or $f = C$ for complex numbers. For $x = [x_1, x_2, x_3, ..., x_m]^T \in X$ the $p$-norm of the vector $x$ is defined by

1. 1-norm $\|x\|_1 := \sum_{i=1}^{m} |x_i|$, for $p = 1$

2. $p$-norm $\|x\|_p := \left(\sum_{i=1}^{m} |x_i|^p\right)^{1/p}$ for $1 < p < \infty$

3. $\infty$-norm $\|x\|_\infty := \max 1 \leq m \leq |x_i|$, for $P = \infty$

When $x$ is a linear space of continuous or piecewise continuous time scalar valued signals $x(t), t \in R$, the $p$-norm of a signal $x(t)$ is defined by [26]:

- 1-norm $\|x\|_1 := \int_{-\infty}^{\infty} |x(t)|dt$, for $p = 1$

- $p$-norm $\|x\|_p := \left(\int_{-\infty}^{\infty} |x(t)|^p dt\right)^{1/p}$, for $1 < p < \infty$
\[ \|x\|_\infty := \sup_{t \in \mathbb{R}} |x(t)|, \quad \text{for} \quad p = \infty \]

The normed spaces, consisting of signals with finite norm as defined correspondingly are called:
\( L^1(R), L^p(R)\) and \( L^\infty(R)\) respectively.

From a signal point of view,
- The 1-norm, \( \|x\|_1 \) of the signal \( x(t) \) is the integral of its absolute value.
- The square of the 2-norm, \( \|x\|_2^2 \) is often called the energy of the signal \( x(t) \) since that is what it is when \( x(t) \) is the current through a 1 \( \Omega \) resistor [27].
- The \( \infty \)-norm, \( \|x\|_\infty \) is the amplitude or peak value of the signal, and the signal is bounded in magnitude if \( x(t) \in L^\infty(R) \).

### 2.5.2 Terms Used in \( H_\infty \) Robust Control Systems Design

The following summary of terms used in \( H_\infty \) synthesis and design problems may be a useful reference source for the thesis. In most cases, more precise mathematical definitions are presented at the point of use and summarized as follows [28]:

1. \( H_\infty \) norm: A measure of the size of a transfer function that may be estimated by plotting the Bode amplitude diagram and noting the maximum value of the frequency response.
2. Stability robustness: The ability of the system to remain stable, even though the model used for system design is very different from the plant model which exists in practice.
3. Performance robustness: Ability of the system to maintain good performance, measured in terms of its tracking accuracy, given that modeling errors exist when designing the controller.
4. Singular values \( \sigma_i(A) \) of real or complex matrix are nonnegative square roots of \( A^H A \).

The dimension of \( A^H A \) is \( n \times n \). Therefore, there will be \( n \) eigenvalues of \( A^H A \). However, if the rank of the matrix \( A^H A \) is \( r \), \( n - r \) eigenvalues of \( A^H A \) will be zero.
5. Structured singular value: Similar to the singular value, but enables the structure of the uncertainty to be taken into account so that less conservative designs can be achieved.
6. Hardy space: Mathematical setting in which \( H_\infty \) optimization work is posed and is a space of all stable transfer functions.
7. Parametric uncertainty: The uncertainty in the definition of gains or time constants in transfer functions that are otherwise well defined.

8. Un-modeled dynamics: Represents part of the system transfer function or state model, which is neglected or unknown when basing a design on a nominal plant model (usually low order).

9. Unstructured uncertainty: May be an unrealistic representation of uncertainty, since it allows for the presence of modeling errors in all of the elements of a system transfer function, whereas modeling errors may only be possible of a well defined structure.

10. \( \mu \)-Synthesis: This is a design method which uses repeated iterations of \( H_\infty \) design algorithm and invokes the structured singular value to test if the design is robust [29].

### 2.5.3 \( l_2 \) and \( H_2 \) Norms

**The \( H_2 \) norm** has an attractive, physically meaningful interpretation. If we consider \( G(s) \) being the transfer function matrix of a system driven by independent, zero mean, unit intensity white noise, \( u \), then the sum of the variances of the outputs \( y \) is exactly the square of the \( H_2 \) norm of \( G(s) \) [30].

\[
\|G\|_2 = \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}\left[ G^*(j\omega)G(j\omega) \right] d\omega \right\}^{\frac{1}{2}} = \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{i=1}^{r} \sigma_i^2 \left[ G(j\omega) \right] d\omega \right\}^{\frac{1}{2}}
\]

Where: \( \sigma_i^2 \) denotes the square of \( ith \) singular values of the system, \( r \) is the rank of \( G(j\omega) \) and \( G^*(j\omega) \) is the complex conjugate transpose of \( G(j\omega) \). It is easy to see that the \( l_2 \) norm defined previously is finite if and only if the transfer matrix \( G \) is strictly proper; that is \( G(\infty) = 0 \). Hence we will generally assume that the transfer matrix is strictly proper whenever, we refer to the \( l_2 \) norm of \( G \) (the same idea is also applied to \( H_2 \) norm). One straightforward way of computing the \( l_2 \) is to use the contour integral.

Suppose \( G \) is strictly proper, \( D=0 \)

\[
\|G\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}\left[ G^*(j\omega)G(j\omega) \right] d\omega = \frac{1}{2\pi} \int \text{trace}\{G^*(s)G(s)\} ds
\]
The last integral is contour integral along the imaginary axis and around on infinite semicircle in the left-half plane. It can be calculated based on the impulse response \( h(t) \) of a system \( G \), as shown in figure 2.2 [31].

\[
\|G\|_2 = \left( \int_0^{\infty} h^*(t)h(t)dt \right)^{\frac{1}{2}} = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega)H^*(j\omega) \, d\omega \right)^{\frac{1}{2}} \tag{2.4}
\]

where \( \sigma_i \) denotes the \( ith \) singular values of the system, and equal the square roots of the eigenvalues of the system and \( G^*(j\omega) \) is the complex conjugate transpose of \( G(j\omega) \).

![Figure 2.2: \( H_2 \) norm](image)

2.5.4 \( l_\infty \) and \( H_\infty \) Norms

We shall first consider, as in the \( l_2 \) case, how we compute the infinity norm of \( Rl_\infty \) transfer matrix.

Let \( G(s) \in Rl_\infty \) and recall that the \( l_\infty \) norm of a matrix rational transfer function \( G \) is defined as:

\[
\|G\|_\infty = \sup_{\omega} \sigma_{\max} [G(j\omega)]
\]

where “Sup” denotes the supremum or the least upper bound of the function \( \sigma_{\max} [G(j\omega)] \)[32].
In control system engineering, the infinity norm of a scalar function \( G \) can be defined as the distance in the complex plane from the origin to the farthest point on the Nyquist plot of \( G \). It also appears as the peak value on the Bode magnitude plot of \( |G(j\omega)| \). Hence the infinity norm of a transfer function can in principle be obtained graphically. It means that the \( H_\infty \) norm of \( G(s) \) is the maximum value of \( \sigma_{\max}[G(j\omega)] \) overall frequencies \( \omega \). \( H_\infty \) norms also have a physically meaningful interpretation when considering the system \( y(s) = G(s)u(s) \). Recall that when the system is driven with a unit magnitude sinusoidal input at a specific frequency, \( \sigma_{\max}[G(j\omega)] \) is the largest possible output size for the corresponding sinusoidal output. Thus the \( H_\infty \) norm is the largest possible amplification over all frequencies of a unit sinusoidal input. That is, it classifies the greatest increase in energy that can occur between the input and output of a given system. Figure 2.3 shows the \( H_\infty \) norm as a worst gain point [33].

**Graphical Interpretation of the \( H_\infty \) Norm**

The \( H_\infty \) norm of a certain input/output transfer function is a measure of the gain of the system. It can be represented by the maximum value of the ratio of the output and input energy of the respective signals. However, it is easier to understand the physical significance of the \( H_\infty \) norm by looking at the frequency response magnitude of a scalar transfer function.

The maximum amplitude of the frequency response, or maximum gain, is equal to the \( H_\infty \) norm in the scalar case. When robustness is to be improved there is often the need to limit the gain in a particular frequency range and this can be achieved by minimizing some weighted transfer function. The weighting can be achieved by simply multiplying the transfer of interest by a weighting filter of suitable frequency response [34].
2.5.5 The \( H_2 \) and \( H_\infty \) Spaces and Norms

The theory of Hardy (\( H_P \)) spaces has its origins in discoveries made about 90 years ago by mathematicians including: G.H. Hardy, Privalov, F. and M. Riesz, V. Smirnov and G. Szego [35]. An introduction to this class of spaces was given by Duren. Renewed interest in the \( H_P \) spaces occurred in the 1970s with the development of the subject of functional analysis and new applications have arisen in recent years, particularly in systems science. The historical starting point for the theory of \( H_P \) spaces is the 1915 paper by Hardy, dealing with Hardy’s convexity theorem.

The \( H_\infty \) robust design approach to control systems design was motivated and stimulated by the work of Zames and Francis in 1981. Zames argued that the mathematical framework for linear quadratic (LQ) or linear quadratic Gaussian (LQG) optimal control did not enable uncertain systems to be treated rigorously. He proposed a new optimization problem in the \( H_\infty \) space that was more suitable for systems with uncertainty.

The approach was designed for systems with un-modeled dynamics in the plant or in the disturbance models. He was able to show that the families of LQ control laws, which can be related to the solution of problems in the \( H_2 \) space, were not robust to different forms of uncertainty. That is, the traditional least squares optimization problems were equivalent to minimizing a \( H_2 \) norm and in some cases do not provide robust solutions. The \( H_\infty \) space is the
space of transfer functions of causal stable linear time-invariant systems. Such systems have bounded input bounded output (BIBO) stability for square integral ($L^2$) inputs and outputs. In the scalar case, the $H_\infty$ norm is simply the maximum amplitude of the frequency response of the transfer function. Thus, the $H_\infty$ norm represents an important measure of gain, often used in classical design for measuring the peaks of sensitivity or closed-loop transfer functions [36].

2.6 $H_2$ and $H_\infty$ Control Systems Design

2.6.1 $H_2$ Control System

In this section we briefly discuss the motivation, from a control system perspective, for the design of an optimal $H_2$ feedback controller based on minimization of the norm of the map from $w$ to $z$. This technique allows removing the stochastic ingredient of the LQG formulation [37]. In many applications it is difficult to establish the precise stochastic properties of the disturbances and noise signals. In the application of the LQG to control system design the noise intensities act as the role of design parameters rather than that the model reality.

- **The standard $H_2$ Design Problem**

The standard $H_2$ optimization problem is the problem of choosing the controller $K$ that performs the following:

1. Stabilizes the closed loop system
2. Minimizes the $H_2$ norm of the closed loop system from $w$ to $z$ as an input and output respectively. For a function $T \in H_2$, its norm $\|T\|_2$ can be written as:

$$\|T\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace} \left( T^T(j\omega)T(j\omega) \right) d\omega}$$

(2.6)

It has been proved that standard $H_2$ and the LQG control problems are equivalent [38].

2.6.2 $H_\infty$ Control System

The goal of $H_\infty$ control theory is to minimize the $H_\infty$ norm of a certain closed loop transfer function matrix, which is chosen to enforce various desirable closed loop attributes. Roughly speaking, making the $H_\infty$ norm of transfer function small corresponds to making the energy gain,
or equivalently, the maximum gain across all frequency. The design goal of finding a controller $K(s)$ such that the $H_\infty$ norm of a certain closed loop transfer function $T_{zw}(k,G)$ is minimized, or at least lower than a certain bound. [39].

**Figure 2.4: $H_\infty$ control model of a closed loop system**

Figure 2.4 shows the general form of the $H_\infty$ robust control system, where matrix $P(s)$ represents the interconnection structure between, the actual plant $G(s)$, any designed specified weights, $W_i(s), i = \{1,2,\ldots\}$ and the inputs and outputs.

The vector $W$ contains all exogenous inputs, which represent disturbances, uncertainties, etc., some of which may be fictitious but useful for design. The vector $Z$ contains all signals, which we require to be minimized. Standard software, such as Matlab analysis and Synthesis Toolbox, can be used to find a controller $K(S)$ with insures that the infinity norm of the system, $\|T_{zw}\|_\infty$ is less than $\gamma$ as follows: $\|T_{zw}\|_\infty \prec \gamma$ .

**Disturbance rejection and robustness ($H_\infty$ control system criterion)**

1. **Disturbance rejection**

**Definition:** the $H_\infty$ norm of the system shown in figure 2.4. Let the transfer matrix from $w$ to $z$ be $T(s)$. The $H_\infty$ norm of $T(s)$ is defined by $\|T\|_\infty = \sup_{|H|=0} \|z\|_2 = \sup_{w} \sigma(T(j\omega))$ [40].
where $\sigma$ is the maximum singular value. On the other hand, the $l_2$ gain of a linear time invariant (LTI) system is equal to the $H_\infty$ norm of the corresponding transfer function matrix. As we seek a state feedback gain $K$ such that the $l_2$ gain $\sup_{\|w\|_2=0} \|z\|_2 = \sup_{\|\cdot\|_2=1} \frac{\|z\|_2}{\|w\|_2}$.

There are two common $H_\infty$ control problems:

- Optimal $H_\infty$ control problem: in this problem we seek to minimize $\|T\|_\infty$ over all stabilized controller $k(s)$.
- Suboptimal control problem: alternatively, we can specify some maximum value $\gamma$ for the closed loop RMS gain and we are looking for a stabilized controller $k(s)$ that insure that $\|T\|_\infty < \gamma$, which describe the $H_\infty$ performance.

It can be noticed from the definition of the $H_\infty$ norm that $\|z\|_2 \leq \gamma \|w\|_2$ which means that the $H_\infty$ norm of a stable transfer function $T(s)$ is its largest input/output RMS gain or the worst case of $H_\infty$. In other words, the value of the $H_\infty$ norm of $T$ reflects the maximum amplification in the energy of the output signals $Z$ for certain input disturbance with a finite energy. Therefore by minimizing the closed loop root mean square (RMS) gain from $W$ to $Z$ then we are minimizing the effect of the worst case disturbance $W$ on the system output $Z$ (good disturbance rejection from $w$ to $z$) as shown in figure 2.5.
Figure 2.5: Disturbance rejection from \( w \) to \( z \) is obtained by minimizing \( H_\infty \).

2. Robustness point of view

\( H_\infty \) optimization is useful for the design of robust control systems with effect of the unstructured uncertainty. Consider the uncertain system shown in figure 2.6 where the uncertainty \( \Delta(.) \) is an arbitrary, Bounded input bounded output (BIBO) stable system satisfying the norm bound \( \|\Delta\|_\infty < 1 \). From the small gain theorem the controller \( u(s) = k(s) y(s) \) robustly stabilizes the uncertain system if the nominal closed loop transfer function from \( w \) to \( z \) denoted by \( T(s) \) satisfies \( \|T\|_\infty < 1 \). Thus minimizing the value of \( H_\infty \) norm of the nominal closed loop system (\( \|T\|_\infty \prec \gamma \)) results that the maximum norm bound of the uncertainty \( \|\Delta\|_\infty \prec \gamma^{-1} \) for which the system robust stability may be increased. Thus the system is more robust to plant uncertainties.
Figure 2.6: Robust stability is guaranteed if $\|T\|_\infty < 1$

2.6.3 $H_2 / H_\infty$ Control System

Before we discuss $H_2 / H_\infty$ control synthesis, let us mention that the $H_2 / H_\infty$ approach is very different from the optimal control approach. In the optimal control approach, we start with the bounds on uncertainties. We then design a controller based on these bounds. It is noted that, if the controller exists, then it is guaranteed to robustly stabilize the perturbed system. On the other hand, in the $H_2 / H_\infty$ approach, the bounds on the uncertainties are not given in advance. The synthesis will try to achieve the largest tolerance range on uncertainty. However, there is no guarantee that the range is large enough to cover all possible uncertainties.

In other words, the $H_2 / H_\infty$ approach may guarantee the robustness of the resulting controller. Whether the best is good enough depends on the nature of the uncertainties. Figure 2.7 shows the structure of $H_2 / H_\infty$ Controller [41].
2.7 Properties of $H_\infty$ Robust Control Design

The $H_\infty$ design approach is a strong contender to provide a general purpose control design procedure, which can account for uncertainties and is simple to use with computer-aided design tools [42].

The following features and properties distinguish the $H_\infty$ design approach:

- It is a design procedure developed specifically to allow for the modeling errors, which are inevitable and limit high performance control systems design.
- There is a rigorous mathematical basis for the design algorithms, which enables stability and robustness properties to be predicted with some certainty.
- The $H_\infty$ technique can enable classical frequency design intuition to be invoked, since most of the previous applications work has been for linear time invariant systems.
- There are close similarities between state-space versions of $H_\infty$ controllers and the well known Kalman filtering or $H_2$/LQG control structures.
- If the uncertainty lies within the class considered, stability properties can be guaranteed and safe reliable systems can be assured. Note that the design procedures cannot be used blindly, since poor information can still lead to controllers with poor performance properties.
- The tradeoffs between good stability properties and good performance are easier to make in $H_\infty$ context than with many of the competing designs.
The approach can be interpreted in terms of the stochastic nature of the system, but if disturbances and noise are important H₂/LQG may still be the preferred solution.

The $H_\alpha$ design technique is easy to use, since the algorithms are available in commercial software.

2.7.1 Comparison of $H_\alpha$ and $H_2$/LQG Controllers

The similarities and differences between the $H_\alpha$ and the $H_2$/LQG approaches are summarized as follows:

- **Similarities**
  
  (a) Both $H_2$ and $H_\alpha$ optimal controllers are based on the minimization of a cost index.
  
  (b) Some of the closed-loop poles of the LGQ solution will be the same as those of the $H_\alpha$ solution in certain limiting cases.
  
  (c) The dynamic cost weights have a similar effect in both types of cost function, e.g. integral action can be introduced via an integrator in the error weighting term in both cases.
  
  (d) Closed loop stability can be guaranteed, whether the plant be non minimum phase, or unstable (neglecting for the moment uncertainty and assuming controllers are implemented in full) [43].

- **Differences**
  
  (a) The basic conceptual idea behind $H_\alpha$ design involves the minimization of the magnitudes of a transfer function, which is quite different from the $H_2$/LQG requirement to minimize a complex domain integral representing error and control signal power spectra.
  
  (b) The $H_\alpha$ design approach is closer to that of classical frequency response design in that the frequency response shaping of desired transfer functions is attempted.
  
  (c) The calculation of $H_\alpha$ controllers is more complicated than the equivalent H2/LQG controllers, whether this be in the time or frequency domains.
  
  (d) Improvements in the robustness of H2/LQG designs to model inaccuracies must be achieved. The $H_\alpha$ approach is more suited to the design of controllers for systems with uncertainty.
2.7.2 Relationships between Classical Design and $H_\infty$ Robust Control
There are many similarities between classical frequency domain design procedures and $H_\infty$ robust design, which can be exploited, such as:

- Classical gain and phase margins can be related to sensitivity function magnitudes.
- The peak magnitude of the closed-loop frequency response ($H_\infty$ norm of the complementary sensitivity) has often provided an indication of the likely overshoot on the step response.
- Robustness measures have been assessed in frequency domain terms with the distance from the critical (-1) point on a Nyquist diagram, providing a measure of robustness.
- Cost weightings in $H_\infty$ design can be parameterized to provide the same type of tuning in PID or lead/lag compensation.
- As in classical design, stochastic properties of the system must be shaped indirectly, using the target frequency response characteristics.
- Engineers trained in classical frequency domain design methods should be able to use $H_\infty$ design procedures easily (assuming formal design steps are followed) [43].

2.7.3 $H_2$ and $H_\infty$ Design and Relationship to Conventional Control Systems
A technique has been proposed, whereby the proportional, integral, derivative (PID) controller coefficients can be selected to improve $H_\infty$ robustness properties. For practical application of the $H_\infty$ technique the research results can often be translated into a simpler PID design algorithm. The advantage of such a procedure is that it offers the possibility of achieving more robust PID designs. A disadvantage is that a pure PID structure is not obtained, although in practice PID controllers are usually not implemented without additional filtering. The PID controller is valuable in many industrial applications, but robustness is not guaranteed. The use of $H_\infty$ techniques to enable the robustness of PID designs to be improved therefore has some merit.
A more recent approach to designing $H_2$ and $H_\infty$ controllers, with a PID structure, is referred to as restricted structure optimal control [43].
In this approach a controller of restricted structure is optimized to minimize a given $H_\infty$ or $H_2$ cost index, which is very valuable for applications.
The same restricted structure ideas may be applied to feed-forward and tracking control problems. Adaptive versions of such controllers are available, and the dual restricted structure estimation problems have also been considered [43].

2.8 Model Order Reduction
In general the purpose of model order reduction is to replace a full order model of the system by a smaller one, which preserves the properties of the full order model. The smaller (reduced order) model must be an approximation of full order system, in a sense that the input/output behavior of the reduced system is comparable with the full order one within certain accuracy. In certain conditions another properties like stability must be preserved in the reduced order model [44-45].

There exist many different model reduction methods, but only a few are optimal in some sense. We can classify the model order reduction as follows: Relative error model reduction methods and absolute error model reduction methods [46].

The relative error model reduction methods consist of the following model order reduction:

- Hankel matrix approximation
- Dominant eigenvalue approach
- Aggregation model reduction method
- Subspace projection method

The absolute error model reduction methods consist of the following model order reduction:

- Balanced truncation model reduction method
- Schur model reduction method
- Hankel norm model reduction method
- Singular perturbation method
- Optimal reduction method

Another type of model reduction in frequency domain, called model approximation consists of the following model approximation methods:

- Frequency weighted approximation methods
- Singular value decomposition-Krylov (SVD-Krylov) based method
- Reduction based on Routh stability criterion
- Continued fraction expansion (CFE) and truncation method
• Moment matching method
• Matching frequency response
• Simplification using canonical form
• Reduction using orthogonal polynomials
• Pade’ approximation and Krylov methods
• Approximation based on integral least square method
• Coprime factor model approximation

One method of those is the balancing truncation model reduction of controlled systems, which consists of two steps. The first step is to find a transformation that balance the observability and controllability grammians in order to determine which states have the greatest contribution to input output behavior. The second step is to perform Galerkin projection onto the states corresponding to the largest singular value of the balanced grammians [47].

Another method is based on Padé approximation and Krylov subspaces. Motivated by applications in circuit simulation, quite a large number of publications on model reduction of large systems have appeared in the last decade. Reduction method by Hankel Norm Approximation method is based on using the Hankel norm of a stable transfer function model \( \|G\|_\infty \).

It is possible to compute an approximation error by minimizing \( \|A_a\|_\infty = \|G - G_r\|_\infty \) for a given order of the reduced order system. Relative error approximation for unstable and non minimum phase systems is considered to obtain the error of reduction process. Schur method is in exact arithmetic’s mathematically equivalent to these two methods in the sense that the transfer matrices of the reduced realizations are identical.

Schur model reduction method has become quite popular because it generates truncation matrices, which have generally much smaller condition numbers compared to those computed by the square roots [48].
3.1 Introduction
In a reactor of given volume in which fission is caused by neutrons of specified energy, the thermal power is proportional to the neutron flux and macroscopic fission cross-section \([49]\). As the reactor operates, the macroscopic cross section decreases as number of fissile nuclides decreases. However, over an essentially short period of time, the cross section remains constant, and the power is assumed to change only with neutron flux. In most situations varying the neutron flux controls a reactor. Among the general methods available, the insertion and withdrawal of a neutron absorber is most commonly used in power reactors. Materials used as a control absorber have large absorption cross-sections and a long lifetime as an absorber (do not burn out rapidly), like boron, cadmium or hafnium \([49]\).

Strong absorbers in a core compete with fissile material for neutrons. In other words, neutrons, which are absorbed by the controller, are no longer available to induce fission, thus reducing the power. An automatic Reactor Power Control System (ARPCS) is used in ETRR2 to compensate the reactivity changes produced in the reactor core during normal operation of the reactor. ARPCS is in fact a conventional PD controller. It generates a Power Control Signal (PCS), based on the relative error in power (neutron flux) and neutron flux rate to control the reactor at any condition of operation.

3.2 Nuclear Reactor Power Control
Before setting out to design a specific reactor plant, the designer of the control system must have a complete philosophy of operation in mind. This philosophy is usually the one in which the control system as well as all other auxiliary systems must be super safe. The peculiar position of nuclear power plants has been such that, if one were to blow up inadvertently, the resulting damage might severely harm the entire program of nuclear power for several years. The control
system of a nuclear reactor can be considered in three distinct but interconnected regimes. These regimes are defined as follows:

- The startup regime
- The operating at certain power regime
- The reactor shut down regime.

The startup loop receives its primary information from a group of neutron detecting elements. The reactor power level is generally derived from neutron information, with neutron level and power level being synonymous. The neutron detectors give out electrical signals, which can be used to measure either the actual power level of the reactor or the rate of change of this power level [50].

Once the power level of the reactor has been raised to the desired level, a new group of electronic circuits may be employed. It depends upon the neutron detectors for their primary information, and these detectors may be of different design than those used in the startup loop for now considerably more neutrons are available for measurement than were during start up. These measurements present the power of the reactor.

If the output of the reactor is the same as the demanded power output, then the error is equal zero and no signal is given out from the comparison circuit. However, if the two signals are different, manual or automatic control then takes over and ultimately the control rod position will be changed according to their distribution and movement band with fixed speed to modify the power level in such a manner that the error signal is reduced to zero.

The system, which is moved, is sometimes called a regulator rod or compensating rod [51].

Fig 3.1 describes the control system of a nuclear reactor as a general illustration.

In modern research reactors, automatic reactor power control system is used to regulate power on a wide operating range, from about 1% to 100% of full power and reject normal reactor perturbations, such as those originated on irradiation target operation.

It must maintain process variables within operative limits to avoid the actuation of the safety systems and increase reactor availability [52].
3.3 ETRR-2 Control Systems

The total reactivity associated with the point kinetics equations in the reactor core depending on the external reactivity which is introduced through the regulating rod and the reactivity feedback due to changes in the fuel temperature and the xenon concentration can be written as described in equations 3.1 and 3.2.

The change in reactivity caused by control rod motion is referred to as control rod worth. The maximum effect (insertion of the most negative reactivity) of a control rod is at the location in the reactor where the flux has its maximum value. The material used for the control rods varies depending on reactor design. The relation between the reactor core total reactivity and the control rods worth is given by:

\[ \frac{d\rho_r}{dt} = G_r z_r \]  
\[ \rho = \rho_r + \alpha_f (T_f - T_{fo}) + \alpha_c (T_c - T_{co}) + \rho_{xenon} + \rho_{ext} \]  

Figure 3.1: Overall sketch of control system of the nuclear reactor
where:

- $G_r$: total reactivity of the rod
- $z_r$: control input, control rod speed (fraction of core length per second)
- $\rho$: total reactivity
- $\rho_r$: reactivity due to control rod movement
- $\alpha_f$: fuel temperature reactivity feedback coefficient
- $T_f$: fuel temperature
- $\alpha_c$: coolant temperature feedback coefficient
- $T_c$: coolant temperature
- $T_{fo}$: initial equilibrium fuel temperature
- $T_{co}$: initial equilibrium coolant temperature
- $\rho_{Xenon}$: Xenon reactivity feedback
- $\rho_{ext}$: External reactivity feedback

According to criticality position, a control signal controls the motion of the final actuator (stepper motor). The control rod withdrawal is limited by a percentage of the active length of the rod itself.

Let $M$ is the rod withdrawal then the effective percentage range is a function of $M$ as follow [53]:

$$20\% \leq M \leq 80\%$$  \hspace{1cm} (3.3)

With limited velocity to the stepper motor as follows

$$V \leq \pm 4\text{mm/sec}$$  \hspace{1cm} (3.4)

Figure 3.2 shows the structure of the ETRR2 automatic reactor power control system as a closed loop, which use four compensated and control rods.
The reactivity control system of ETRR-2 consists of two in-core elements: guide boxes and neutron absorber plates. Each guide box has the function of housing three neutron absorber plates, ensuring that they will be guided the entire path, guaranteeing the right performance in normal operation and also in case of an accident [54].

3.4 Control Mechanism
ETRR-2-ARPCS as shown in figure 3.3 consists of the following main parts:

- Control rods
- Final actuator elements (stepper motors)
- Passing mechanism

Figure 3.2: Automatic reactor power control system.
• Air tanks and valves
• Coupling and electromagnet and control channel

1. Control Rods

There are six control rods designed to control the reactor in start up, operation regimes and safe shut down. Four control rods are used to control and compensate the reactor power at the desired operating condition. The other two rods are used as safety rods in normal operation. The six control rods control and safe the reactor in case of scram requested to guarantee the reactor safe shutdown. Physically, each control rod absorber plate is an alloy (Ag-In-Cd alloy) and it has a total length of 1000 mm including cladding [55-56].

Control rod motion in both -directions is controlled in two modes:

• Manual mode: this manual mode depends on the operator’s considerations of the reactor kinetic. The operator can give orders to extract or insert the control rod to control and compensate the reactor power according to the reactor criticality. The manual control mode depending on the monitoring of the three independent and isolated power channels through three compensated ionization chamber (CIC)parameters

• Automatic mode: In this mode, the PD controller generates a PCS to control the speed and direction of control rod through a stepper motor, which acts as a final actuator element.

Automatic mode depends on the monitoring of the control channel through a compensated ionization chamber CIC [57-58].
2. **Final Actuator: Stepper Motor**

The second main part of the control mechanism is the stepper motor with fixed magnets and 200 step/turns complete stepping, fed by a continuous current source.

There are a number of characteristics that make a stepper motor is used in reactor control mechanism, including:

- Stepper motors are able to operate with a basic accuracy of ±1 step in an open-loop system. This inherent accuracy removes the requirement for a positional or speed transducer, and it therefore reduces the cost of the overall system.
- Stepper motors can produce high output torques at low angular velocities, including standstill with the hybrid stepper motor.
- A holding torque can be applied to the load solely with direct-current (d.c.) excitation of the stepper motor's windings.
- The operation of stepper motors and their associated drive circuits is effectively digital and permitting a relatively simple interface to a digital controller or to a computer.
- The mechanical construction of stepper motors is both simple and robust, leading to high mechanical reliability.

The motors and controllers are designed so that the motor may be held in any fixed position as well as being rotated one way or the other. In general, most stepping motors can be stepped at audio frequencies, allowing them to spin quite quickly with an appropriate controller.

Figure 3.4 illustrates the generic stepping motor with its main parts, which contains the stator, rotor, axe, bearing, coil and the cover box. Figure 3.5 shows a perpendicular view of the stepping motor [59].

![Generic stepping motor with its main parts](image1)

**Figure 3.4:** Generic stepping motor with its main parts

![Perpendicular view of a generic stepping motor](image2)

**Figure 3.5:** Perpendicular view of a generic stepping motor
3. Control Channel
The control channel consists of the following subsystems:

- out of core nuclear detector: Compensated Ionization Chamber (CIC)
- signal conditioning module
- electronic modules

3.5 ETRR-2 System Model
The ETRR-2 reactor model consists of the following sub models:

- Reactor point kinetic model
- Poisons (xenon and iodine) model
- Temperature feedback model
- Fuel feedback model
- Rod withdrawal model
- Control system model
- External environment model

Figure 3.6 illustrates the top view of the ETRR-2 reactor model.

Figure 3.6: Top view of the plant model.
**ETRR-2 Model Response Analysis**

Operation regimes of the reactor are limited by the following three states

- Critical
- Sub critical
- Super critical

These states are usually defined according to the value of the reactor multiplication factor \( k \), where \( k \) is defined by equation (3.5) and \( \rho \) is the total reactor reactivity of the reactor [60].

\[
\rho = \frac{k - 1}{k} \tag{3.5}
\]

The fission rate varies exponentially with time, and it is convenient to write as equation (3.6),

\[
N(t) = N_0 e^{t/T} \tag{3.6}
\]

Where \( N(t) \) and \( N_0 \) are the neutron flux at time \( t \), the initial neutron flux at time \( t_0 \), respectively, and \( T \) is the reactor period.

The total reactor reactivity affected by the following parameters [61-62]:

- Control rod: the movement of the control rods (insertion-extraction) changes the absorption level of the reactor, and the value of the multiplication factor.
- Fuel burn up: operation of the reactor consumes the fissionable material according to the consuming relation between the power level and the mass of the fissionable material.
- Isotope production: certain numbers of isotopes, known as poisons, are strong neutron absorbers and affect the value of multiplication factor.
- Temperature changes: the reactor is designed based on the negative temperature feedback consideration. Many parameters affected by the variation in the temperature such as the power level, moderator, coolant...etc.
- Environmental changes
- Accidents.
3.6 Reactor Dynamics and Point Kinetic Model

A fission chain reaction produces 24 elements called fission fragments, with various half-life times. These fragments are grouped into 6 groups; each group contains the elements of comparable half-life times.

Figure 3.7 shows the point reactor kinetic of the 6-groups model. These 6 groups point kinetic equations used to study the ETRR-2 dynamics as follows [63] to [66]:

\[
\frac{dn}{dt} = \left( (\rho - \beta) / \Lambda \right) n + \sum_{i=1}^{6} \lambda_i c_i + S \tag{3.7}
\]

\[
\frac{dc_i}{dt} = \left( \beta_i / \Lambda \right) n - \lambda_i c_i \tag{3.8}
\]

where:

- \( \rho \) is the total reactivity
- \( \beta \) is the effective delayed neutron fraction
- \( \beta_i \) is the group-\( i \)th delayed neutron
- \( \Lambda \) is the prompt neutron generation time
- \( \lambda_i \) group-\( i \)th precursor decay constant
- \( n \) neutron density
- \( c_i \) group-\( i \)th precursor density
- \( S \) neutron source (used in case of first criticality)

The normalized power density is rewritten as follows (assuming no neutron source \( S \)):

\[
\frac{dN}{dt} = \left( \beta / \Lambda \right) \left[ \{ R - 1 \} N + \sum \theta_i a_i \right] \tag{3.9}
\]

and the normalized delayed neutron precursor density is given by

\[
\frac{d\theta_i}{dt} = \lambda_i (N - \theta_i) \tag{3.10}
\]

Where:

- \( N : n/n_{\text{o}} \) is the normalized power density
- \( a_i : \beta_i / \beta \) is the relative fraction for delayed neutrons of group \( i \)
- \( R : \rho / \beta \) is the reactivity in dollars and is expressed in part per million (pcm), or \( 10^{-5} \)
Table 3.1 shows the delayed neutron data of the U$^{235}$ fuel of (ETRR-2) and table 3.2 shows the delayed neutron fraction $\beta$ and the prompt neutron generation time $\Lambda$, that are used in ETRR-2 kinetic model. These data is constant only for one core configuration.

![Diagram of Reactor Kinetic 6-Groups Model]

**Figure 3.7:** Reactor kinetic 6-groups model

**Table 3.1: Delayed neutron data of the U$^{235}$ fuel of (ETRR-2)**

<table>
<thead>
<tr>
<th>Group</th>
<th>$\lambda_i$, sec$^{-1}$</th>
<th>$\beta_i/\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0127</td>
<td>0.038</td>
</tr>
<tr>
<td>2</td>
<td>0.0317</td>
<td>0.213</td>
</tr>
<tr>
<td>3</td>
<td>0.1150</td>
<td>0.188</td>
</tr>
<tr>
<td>4</td>
<td>0.3110</td>
<td>0.407</td>
</tr>
<tr>
<td>5</td>
<td>1.4000</td>
<td>0.128</td>
</tr>
<tr>
<td>6</td>
<td>3.8700</td>
<td>0.026</td>
</tr>
</tbody>
</table>
### Table 3.2: Delayed neutron fraction parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.00705</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>7.E-05 sec</td>
</tr>
</tbody>
</table>

3.6.1 Model Investigation

The model is investigated based on the thermal fission of $\text{U}^{235}$ fuel parameters and the results compared with the academic press as follows:

- **Step Response**

Figure 3.8 shows the response of the open-loop reactor point kinetic model to step reactivity changes in the reactor power by 100pcm. The model response is compared to the reference published as described in figure 3.9.

We have to know that each time we test the reactor model, we will use reactivity unit, which defined as follows:

The natural unit of reactivity comes in “dollar”. These considerations show that pcm is a convenient but conventional unit; for physics, and in particular to identify such a threshold not to be approached, the natural unit of reactivity is the proportion $\beta$ of neutrons emitted with a delay.

The Americans suggested calling this unit a “dollar”, and one-hundredth of it a “cent”. The value of a dollar in terms of pcm (real reactivity) depends on the fuel. For example, it is three times lower for a plutonium-based fuel than a uranium-based fuel [67].
Figure 3.8: Reactor point kinetic model response to step reactivity changes of 100pcm.

Figure 3.9: Reference reactor point kinetic response to step reactivity changes of 100pcm.
3.6.2 Feedback Parameters

There are many parameters having positive and negative feedback effect on the power stabilization according to its properties as follows:

3.6.2.1 Poisons Physical Equations and Model

The most effect fission product poison is Xe$^{135}$ due to its high thermal neutron absorption cross section about (2.7E+6 barns). This isotope is produced directly by U$^{235}$ fission and indirectly by I$^{135}$ decay. Because, Xenon is produced in part by Iodine decay, Xenon concentration at any time depends upon the current Iodine concentration at that time. The rate of concentration of Iodine and Xenon are given by the following differential equations [68-69]:

\[
\frac{dI}{dt} = \Gamma_i \Sigma_f \Phi_t - \lambda_i I \tag{3.11}
\]

\[
\frac{dx}{dt} = \Gamma_x \Sigma_f \Phi_t + \lambda_i I - \lambda_x x - \sigma_{as} \Phi_t x \tag{3.12}
\]

where:

- $\Gamma_i$ : Iodine fission yield
- $\Gamma_x$ : Xenon fission yield
- $\lambda_i$ : Iodine decay constant
- $\lambda_x$ : Xenon decay constant
- $\sigma_{as}$ : Thermal microscopic cross section
- $\Sigma_f$ : Average thermal macroscopic fission cross section
- $I$ : Iodine density (atoms/cm$^3$)
- $x$ : Xenon density (atoms/cm$^3$)
- $\Phi_t$ : Thermal neutron flux

Figure 3.10 presents the Xenon and Iodine model, which is used to study the behavior of Xenon and Iodine and their effect on the neutron flux. Table 3.3 gives the Xenon and Iodine model parameters.
Table 3.3: Poisons (Xenon and Iodine) model parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iodine</td>
<td></td>
</tr>
<tr>
<td>$\Gamma_i$</td>
<td>0.061</td>
</tr>
<tr>
<td>$\lambda_i$</td>
<td>2.87e-05 sec$^{-1}$</td>
</tr>
<tr>
<td>Xenon</td>
<td></td>
</tr>
<tr>
<td>$\Gamma_x$</td>
<td>0.003</td>
</tr>
<tr>
<td>$\lambda_x$</td>
<td>2.09e-05 sec$^{-1}$</td>
</tr>
<tr>
<td>$\sigma_{ax}$</td>
<td>2.7e-18 cm$^2$</td>
</tr>
<tr>
<td>ETRR-2</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_F$</td>
<td>0.118 cm$^{-1}$</td>
</tr>
</tbody>
</table>

Figure 3.10: Xenon and Iodine model

3.6.2.2 Model Verification and Results

Figure 3.11 shows the behavior of the poison model to a step change in reactivity and it is verified by comparing the results with their corresponding analytical solutions that are found on the saturation situation.

Figure 3.12 illustrates an important case, the reactor shutdown at three situations as follows: the first case illustrates the reactor shutdown when it is working and the neutron flux equal the
Minimum flux (8.1E+13nv). The second situation is reactor shutdown when the reactor neutron flux equal the neutron flux (1E+14nv) and the last condition is illustrates that the reactor is shutdown when it is working at the maximum flux (2.7E+14nv). The poisons model responses of these conditions are compared with the reference shutdown response to declare the xenon behavior after shutdown as an important factor in reoperation the reactor after shutdown. As presented in figure 3.13.

![Graph showing xenon and iodine density over time](image)

**Figure 3.11:** ETRR-2 Xenon and Iodine response to a step flux change of 1.E+14nv from steady state conditions at Zero flux.
Figure 3.12: Xenon response to shut down for three steady state initial conditions (8.1E+13, 1E+14 and 2.7E+14)\(n_{\nu}\)

Figure 3.13: Xenon\(^{135}\) build up after shutdown for several values of the operating flux before shutdown
Poisons negative feedback has a main role in restart up the reactor after shutdown and power raising or decreasing to different level at the same operation cycle. The variation of the reactor power has a relative variation in Xenon and Iodine build up and their saturation according to equations 3.13 and 3.14.

\[ I(t) = I_0 \exp (-\lambda_i t) \]  
(3.13)

\[ x(t) = (x_0 + \frac{\lambda_0 I_0}{\lambda_i - \lambda_x}) \exp (-\lambda_i t) - \frac{\lambda_0 I_0}{\lambda_i - \lambda_x} \exp (-\lambda_x t) \]  
(3.14)

Figure 3.14 illustrates the relation between the Iodine and Xenon and how the indirect way to Xenon builds up as a result of Iodine decay and power variation.

**Figure 3.14:** Xenon and Iodine density for shutdown states
### 3.6.3 Heat Transfer and Temperature Feedback Model

One product of the fission chain reaction is this heat or thermal energy produced inside the reactor core, transferred to the cooling system and dissipated through the cooling tower to the air and environment without any radioactive release to the public and environment. This part of the reactor model studies the dynamic behavior of the heat generation inside the reactor core according to the following model equations [70] to [72].

\[ m_f c_f \left( \frac{dT_f}{dt} \right) = p_{th} - hA (T_f - T_c) \]  \hfill (3.15)

\[ m_c c_e \left( \frac{dT_c}{dt} \right) = -2M c_e (T_e - T_f) + hA (T_f - T_c) \]  \hfill (3.16)

where:
- \( A \): Total heat transfer area
- \( c_f \): Fuel specific heat
- \( c_e \): Coolant specific heat
- \( M \): Coolant mass flow
- \( h \): Heat transfer coefficient
- \( m_c \): Coolant mass
- \( m_f \): Fuel mass
- \( p_{th} \): Thermal power
- \( T_e \): Coolant average temperature
- \( T_f \): Fuel average temperature
- \( T_i \): Coolant inlet temperature

The negative feedback reactivity due to coolant and fuel temperature is determined by equations 3.17 and 3.18. Table 3.4 summarizes the ETRR-2 heat transfer and temperature feedback data. Generally, the reactor fuel temperature and coolant temperature feedback coefficient \( \alpha_f \) and \( \alpha_e \) respectively are defined as the logarithmic derivative of the effective multiplication factor \( k \) or reactivity \( \rho \) with respect to the reactor temperature [73].

\[ \frac{d\rho}{dT_f} = \alpha_f \]  \hfill (3.17)

\[ \frac{d\rho}{dT_e} = \alpha_e \]  \hfill (3.18)
where:

\( \rho \) : Total reactivity

\( \alpha_f \) : Fuel temperature feedback coefficient

\( \alpha_c \) : Coolant temperature feedback coefficient

\( T_f \) : Fuel average temperature

\( T_c \) : Coolant average temperature

**Table 3.4: Heat transfer and temperature feedback data**

<table>
<thead>
<tr>
<th>Factor</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coolant</td>
<td>( \alpha_c )</td>
<td>-9.7pcm/ ( \dot{\varepsilon} )</td>
</tr>
<tr>
<td></td>
<td>( C_c )</td>
<td>4178.82J/Kg- ( \dot{\varepsilon} )</td>
</tr>
<tr>
<td></td>
<td>( M )</td>
<td>551.297Kg/sec</td>
</tr>
<tr>
<td></td>
<td>( h )</td>
<td>22.43Kw/(m(^2)- ( \dot{\varepsilon} ))</td>
</tr>
<tr>
<td></td>
<td>( m_c )</td>
<td>94.483Kg</td>
</tr>
<tr>
<td></td>
<td>( T_I )</td>
<td>40 ( \dot{\varepsilon} )</td>
</tr>
<tr>
<td>Fuel</td>
<td>( \alpha_f )</td>
<td>-2pcm/ ( \dot{\varepsilon} )</td>
</tr>
<tr>
<td></td>
<td>( A )</td>
<td>28.211m(^2)</td>
</tr>
<tr>
<td></td>
<td>( C_f )</td>
<td>25925.9 J/Kg- ( \dot{\varepsilon} ))</td>
</tr>
<tr>
<td></td>
<td>( m_f )</td>
<td>94.829 Kg</td>
</tr>
</tbody>
</table>
Figure 3.15: Model of the reactor coolant and fuel temperature feedback

Figure 3.15 shows the model of both ETRR-2 fuel and coolant temperature feedbacks. Figure 3.16 illustrates the change in fuel and coolant temperature relative to step changes of the thermal power from 22Mw to 11Mw. Power decreased to 50% at 50 sec till 200 sec; the average temperature of fuel and coolant is 80°C and 45°C respectively to become 60°C and 42°C respectively.
3.7 Disturbances

During nuclear reactors operation there are many parameters may be classified as internal and external and it may affect the operation stability due to its positive or negative effect. The internal and external disturbances associated to the normal operation can be listed as follows:

1. Core inlet temperature change
2. Core temperature difference change
3. Extraction of neutron absorbent control rod
4. Extraction of an irradiation material during normal operation
5. Extraction of non regulating control rod at maximum velocity
6. Noise effect
7. Moreover, abnormalities affects the reactor operations such as:
8. Human error
9. Control rod stucked at any position
10. External disturbance
11. Control loop failure
12. One of the core cooling pumps trip
13. All the design basis accident consequences [74].

### 3.8 System Stability and Automatic Reactor Power Control

Based on the reactor sub models presented in section 3.6, the overall model of the reactor dynamic models can be integrated in a full order nonlinear model contains the reactor point kinetic model, heat transfer model, poisons model and control rod model to investigate the closed loop control system as follows:

\[
\frac{dn}{dt} = [\rho - \beta] / \Lambda \] n + \sum_{i=1}^{6} \lambda_i c_i \tag{3.19}
\]

\[
\frac{dc_i}{dt} = (\beta_i / \Lambda) n - \lambda_i c_i \tag{3.20}
\]

\[
m_j c_j (d T_f / dt) = p_{th} - h A (T_f - T_r) \tag{3.21}
\]

\[
m_c c_c (d T_c / dt) = -2 M c_c (T_c - T_r) + h A (T_f - T_c) \tag{3.22}
\]

\[
\frac{dI}{dt} = \Gamma_i \sum_j \Phi_{rj} - \lambda_i I \tag{3.23}
\]

\[
\frac{dx}{dt} = \Gamma_i \sum_j \Phi_{rj} + \lambda_i I - \lambda_x x - \sigma_{xs} \Phi_{r} x \tag{3.24}
\]

\[
\rho = \rho_r + \alpha_f (T_f - T_fo) + \alpha_c (T_c - T_{co}) + \rho_{xenon} + \rho_{ext} \tag{3.25}
\]

\[
\frac{d\rho_r}{dt} = G_zr \tag{3.26}
\]

A critical reactor (steady state condition at certain reactor power) has an effective multiplication factor \( k_{eff} \) equal to unity. When nuclear reactor deviates from criticality its multiplication factor can be larger or less than unity [75]. In this case, it has an excess multiplication factor, \( k_{ex} = k_{eff} - 1 \), which can be positive or negative. The ratio of excess multiplication factor to the effective multiplication factor is designed as excess reactivity, \( \rho \)

\[
\rho = \frac{k_{ex}}{k_{eff}} = \frac{k_{eff} - 1}{k_{eff}} \tag{3.27}
\]
The change in the neutron population can be written as follows:

\[ \frac{d n(t)}{dt} = n(t) k_{\text{eff}} - n(t) = n(t) (k_{\text{eff}} - 1) \]  

(3.29)

The rate of change in the neutron population would be

\[ \frac{d n(t)}{dt} = n(t) \frac{k_{\text{eff}} - 1}{\tau} \]  

(3.30)

where \( \tau \) is the average neutron lifetime between generations.

From the stability point of view the reactor performance of open loop control system is unstable. In addition there is a non-linearity due to the non-linear behavior of the reactor feedback parameters and its effect on the reactor power as illustrated before [76].

A conventional PD controller used to control the reactor power around an equilibrium operating point as illustrated in figure 3.17. The stability of the system model with PD controller will be investigated at different conditions.

The power control signal (PCS) is generated according to equation 3.31, the power controller parameters are listed in table 3.5.

\[ PCS = K_p \left( \frac{P}{P_{\text{ref}}} - 1 \right) + K_r \Phi \]  

(3.31)

Where

- \( K_p \) : Proportional gain
- \( K_r \) : Derivative gain
- \( P \) : Current reactor power
- \( P_{\text{ref}} \) : Reference reactor power
- \( \Phi \) : Reactor flux rate

To quantify the effect that a variation in parameter (that is, increase in temperature, control rod insertion, increase in neutron poison) will have on the reactivity of the core, reactivity coefficients are used. Reactivity coefficients are the amount that the reactivity will change for a given change in the parameter. For instance, an increase in moderator temperature will cause a decrease in the reactivity of the core. The amount of reactivity change per degree change in the moderator temperature is the moderator temperature coefficient [77].
Reactivity coefficients are generally symbolized by $\alpha x$, where $x$ represents some variable reactor parameters that affects reactivity. The definition of a reactivity coefficient in equation format is shown below

$$\Delta \rho = \alpha x \Delta x$$

Where:
- $\Delta \rho$: Reactivity defects
- $\alpha x$: Reactivity coefficient
- $\Delta x$: Reactor parameter that affects the reactivity

**Figure 3.17:** Closed loop of reactor control system

Table 3.5 summarizes the closed loop PD control system limitation and saturation data such as the fixed gains $k_p$ and $k_i$, dead band, maximum and minimum limit of power.
Table 3.5: Power controller data

<table>
<thead>
<tr>
<th>Term</th>
<th>Parameter</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gains</td>
<td>$K_p$</td>
<td>9.524</td>
</tr>
<tr>
<td></td>
<td>$K_t$</td>
<td>16.667</td>
</tr>
<tr>
<td>Dead band</td>
<td>Minimum</td>
<td>-0.1</td>
</tr>
<tr>
<td></td>
<td>Maximum</td>
<td>0.1</td>
</tr>
<tr>
<td>$P/P_{ref} - 1$</td>
<td>Minimum</td>
<td>-0.07</td>
</tr>
<tr>
<td></td>
<td>Maximum</td>
<td>0.07</td>
</tr>
<tr>
<td>$T$</td>
<td>Minimum</td>
<td>-0.01 sec$^{-1}$</td>
</tr>
<tr>
<td></td>
<td>Maximum</td>
<td>0.02 sec$^{-1}$</td>
</tr>
</tbody>
</table>

3.8.1 Model Response to Transient Situations

The response of the reactor model to different transients’ conditions is driven by the reference power variations due to the feedback parameters and operation condition. Table 3.6 gives the reactor parameters situation at selected operation condition as follows: the distribution of control rods is (62%, 62%, 100%, 62%, 100% and 50%) withdrawal relative to the reactor core with its corresponding reactivity feedback. The other parameters have direct effect in the reactor operation stability are the poisons and temperature feedbacks respectively.

3.8.2 Reactor Power Control Model Investigation

The reactor power control system is investigated in different operation conditions to evaluate the performance in case of sudden change in the reactor power due to any disturbance. In all cases, once achieving the desired power, the compensating rod continues inserted in the reactor core since the reactor is not yet in equilibrium due to non-equilibrium delayed neutrons. Figure 3.18 shows the model response to step change in the reference power from 100% to 75%, 50% and 25% respectively while figure 3.19 shows the withdrawal control rod change relative to the step change in the reactor reference power by 100% to 75%, 50% and 25% respectively to stabilize the reactor at the desired power.
Table 3.6: Total reactivity at certain critical position

<table>
<thead>
<tr>
<th>parameters</th>
<th>Withdrawal %</th>
<th>Reactivity feedback</th>
</tr>
</thead>
<tbody>
<tr>
<td>CR-1</td>
<td>62%</td>
<td>-1120</td>
</tr>
<tr>
<td>CR-2</td>
<td>62%</td>
<td>-1120</td>
</tr>
<tr>
<td>CR-3</td>
<td>100%</td>
<td>0.0</td>
</tr>
<tr>
<td>CR-4</td>
<td>62%</td>
<td>-1120</td>
</tr>
<tr>
<td>CR-5</td>
<td>100%</td>
<td>0.0</td>
</tr>
<tr>
<td>CR-6</td>
<td>50%</td>
<td>-1450</td>
</tr>
<tr>
<td>Xenon</td>
<td>No</td>
<td>-2444</td>
</tr>
<tr>
<td>Coolant/Fuel Temperature</td>
<td>No</td>
<td>-300</td>
</tr>
<tr>
<td>Fuel Burn-Up</td>
<td>No</td>
<td>7500</td>
</tr>
<tr>
<td>Total percentage</td>
<td></td>
<td>0.0</td>
</tr>
</tbody>
</table>

Figure 3.18: Power changes from 100% to 75%, 50% and 25% respectively
Figure 3.19: Rod positions relative power changes from 100% to 75%, 50% and 25%

Figure 3.20: Model response to positive reactivity insertion (20pcm, 50pcm and 100pcm) respectively
Figure 3.20 shows the variation in the reactor normalized power at steady state due to a positive step reactivity insertion with 20pcm, 50pcm and 100pcm respectively. Note that the maximum positive reactivity insertion allowed during the normal operation is limited by 20pcm to save the reactor operation. Figure 3.21 illustrates the regulator rod positions change as a consequence of that different positive reactivity insertion.

Figure 3.22 shows the variation in the reactor normalized power to negative step reactivity insertion with the following values –20pcm, -50pcm and –100pcm respectively. Figure 3.23 illustrates the regulator rod position relative to this variation of negative step reactivity insertion to the reactor core at steady state condition.

**Figure 3.21:** Rod position changes due to positive reactivity insertion (20pcm, 50pcm and 100pcm) respectively
Figure 3.22: Model response to negative reactivity insertion (-20pcm, -50pcm and -100pcm) respectively

Figure 3.23: Rod position changes due to negative reactivity insertion (-20pcm, -50pcm and -100pcm) respectively

60
3.9 Concluding Remarks

This chapter introduces a general study on nuclear reactors theory and operation. The general equations that model the reactor power generation and neutron density is presented. Feedback parameters like poisons, external environment, coolant and fuel temperature and fuel burned are considered to minimize the modeling error. The final actuator element and its practical limitations on the achievable control actions are discussed. The total effect of these parameters is investigated and compared with scientific published references.

From this analysis, it is clear that the PD controller provides a moderate damping for the system response. This response is expected to degrade if the plant drifts from the presented model due to un-modeled dynamics, aging, external disturbances, etc. Hence, it is expected that a multiobjective controller can provide better response for the reactor plant.
Chapter 4

Robust Control Systems and Linear Matrix Inequality

4.1 Introduction

This chapter introduces the robust control system and the linear matrix inequality (LMI) as important tools to design a robust control system. Both methodologies are very attractive for the designers and the researchers for the last two decades because of their advantages in optimization, disturbance rejection and model uncertainty [78-79].

The design of a controller should in general satisfy the following basic requirements:

- Closed Loop Stability
- Robustness: The closed loop controlled system has to remain stable despite uncertainty in the mathematical plant description or disturbances.
- Performance: The controlled system has to have certain dynamical or steady state characteristics such as rise time, overshoot, controller bandwidth or steady state error etc.
- Robust Performance: The controller has to remain well performing and certainly stable, although disturbances and uncertainties affect the plant.

These control design requirements are usually best encoded in the form of an optimization criterion subject to some constraints e.g., terminal constraints or practical constraints on plant variables.

Control design optimization criteria were initially based on the idea of Linear Quadratic Gaussian Control, which was later, generalized to the idea of $H_2$ controller design. For linear systems and suitable optimization criteria, such as $H_2$ and $H_{\infty}$, the solution to the optimization problem is readily found solving Riccati-equations. Many of these optimal control problems can be stated in terms of linear matrix inequalities.
4.2 Model Uncertainty: Representation

- **Origins of Model Uncertainty**

Uncertainty in control systems may stem from different sources. In particular, it may be caused by:

1. Parameters in a linear model, which are approximately known or are simply in error
2. Parameters, which may vary due to nonlinearities or changes in the operating conditions
3. Neglected time delays and diffusion processes
4. Imperfect measurement devices
5. Reduced (low-order) models of a plant, which are commonly used in practice, instead of very detailed models of higher order
6. Ignorance of the structure and the model order at high frequencies
7. Controller order reduction issues and implementation inaccuracies [80].

The above sources of model uncertainty may be grouped into three main categories.

- Neglected and un-modeled dynamics uncertainty

In this case the model is in error because of missing dynamics (usually at high frequencies), most likely due to a lack of understanding of the physical process.

- Lumped uncertainty or unstructured uncertainty

In this case uncertainty represents several sources of parametric and/or unmodeled dynamics uncertainty combined into a single lumped perturbation of prespecified structure. Here, nothing is known about the exact nature of the uncertainties, except that they are bounded [81].

- Parametric uncertainty

In this case the structure of the model and its order is known, but some of the parameters are uncertain and vary in a subset of the parameter space.

**General Uncertain Model**

We can conclude the general uncertain model as shown in figure 4.1 while the general uncertain system model equations are as follows [82]:

63
\[
\begin{align*}
\dot{x} &= A(t)x + B_1(t)w + B_2(t)u \\
z &= C_1(t)x + D_{11}(t)w + D_{12}(t)u \\
y &= C_2(t)x + D_{21}(t)w + D_{22}(t)u
\end{align*}
\]   

(4.1)

Where \( x \in \mathbb{R}^n \) is the state vector, \( u \in \mathbb{R}^m \) is the input vector, \( w \in \mathbb{R}^p \) is the vector of exogenous inputs (inputs that we have no control over such as noises, reference inputs, etc.), \( z \in \mathbb{R}^q \) is vector of outputs of interest (tracking error or control effort \( u \)), \( y \in \mathbb{R}^q \) is the output vector, \( A(t) \in \mathbb{R}^{n \times n}, B_1(t) \in \mathbb{R}^{n \times m}, B_2(t) \in \mathbb{R}^{n \times m'}, C_1(t) \in \mathbb{R}^{q_1 \times n}, \\
C_2(t) \in \mathbb{R}^{q_2 \times n}, D_{11}(t) \in \mathbb{R}^{q \times m}, D_{12}(t) \in \mathbb{R}^{q \times m'}, D_1(t) \in \mathbb{R}^{q_1 \times m}, D_{21}(t) \in \mathbb{R}^{q_1 \times m'}, D_{22}(t) \in \mathbb{R}^{q_2 \times m'} \).

Define \( S(t) = \begin{bmatrix} A(t) & B_1(t) & B_2(t) \\ C(t) & D_{11}(t) & D_{12}(t) \\ C(t) & D_{21}(t) & D_{22}(t) \end{bmatrix} \in \Omega \) (is uncertain set).

4.3 Robust Control Systems

Control engineers are always aware that any design of a controller based on a fixed plant model is very often unrealistic. This is because there is always an expected doubt about the performance specifications if the model on which the design is based deviates from the assumed value over a certain range [83].

Robust control refers to the control of uncertain plants with unknown disturbance signals, uncertain dynamics, and imprecisely known parameters making use of fixed controllers. That is, the problem of robust control is to design a fixed controller that guarantees acceptable performance norms in the presence of plant and input uncertainty. The performance
specifications may include properties such as stability, disturbance attenuation, reference
tracking, control energy reduction, etc.

It is well known that if stability is a minimum requirement in control synthesis, it is not sufficient
in practice and some performance level has to be guaranteed. The modern control systems have
very important properties such as:

- Robustness properties
- Asymptotically stability properties

Linear Quadratic Gaussian, (LQG) has been widely studied since 1960s. \( H_\infty \) control theory has
become a standard design method in the last 20 years, which shows the usefulness of \( H_\infty \) norm
performance index [84].

LMI control toolbox offers tools for state-feedback design with a combination of the following
objectives:

- \( H_2 \) performance (for LQG aspects)
- \( H_\infty \) performance (for tracking, disturbance rejection, or robustness aspects)
- Robust pole placement specifications (to ensure fast and well-damped transient
  responses, reasonable feedback gain, etc.)

4.4 Linear Matrix Inequality (LMI)

The theory of linear matrix inequalities (LMI) has been attracting the attention of research
communities for a decade especially from researchers in the control systems community. The
concept of LMI and its applications are based on the fact that LMIs can be reduced to linear
programming problems which can easily be solved by computers. In this section, the concepts of
LMI will be presented first, followed by the LMI problems and its applications in control. The
LMI problem solution becomes easier with MATLAB using the Robust Control Toolbox, as well
as a free YALMIP Toolbox.

Matrix Inequality Definiteness:

Inequalities in the case of matrices are in the sense of positive and negative (semi) definiteness.
A matrix \( Q \) is defined to be positive definite (denoted as \( Q > 0 \)) if

\[
x^T Q x > 0 \quad \forall x \neq 0
\]  

Likewise, \( Q \) is said to be positive semi-definite (denoted as \( Q \geq 0 \)) if
\( x^T Q x \geq 0 \quad \forall x \) \hspace{1cm} (4.3)

The negative (semi) definiteness can be defined as follows: \( P < 0 \) \( (P \leq 0) \) [85].

- **Lyapunov Inequality**

In the first Lyapunov stability problem, Lyapunov theory states that for a given positive definite matrix \( Q \), if the Lyapunov equation \( A^T X + XA = -Q \) has positive definite solution \( x \). The matrix \( A \) is stable, i.e., all the eigenvalues of the matrix are located in the left hand side of the complex plane. The previous equation can also be converted into a Lyapunov inequality, \( A^T X + XA < 0 \)

- **LMI forms:**

  1. **Basic LMI form**

The basic form of an LMI is given by:

\[
F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \succ 0 \tag{4.4}
\]

where \( x \in \mathbb{R}^m \) is a variable and \( F_0, F_i \) are given constant symmetric matrices

The basic LMI problem—the feasibility problem—is to find \( x \) such that inequality holds. Note that \( f(x) > 0 \) describes an affined relationship in terms of the matrix \( x \).

  2. **General LMI form**

Normally the variable \( x \), which we are interested in, is composed of one or many matrices whose columns have been ‘stacked’ as a vector [86].

That is,

\[
F(x) = F(X_1, X_2, X_3, \ldots, X_n) \tag{4.5}
\]

Where \( X_i \in \mathbb{R}^{q_i \times p_i} \) is a matrix, \( \sum_{i=1}^{n} q_i p_i = m \) and the columns of all the matrix variables are stacked up to form a single vector variable. Hence, the general form also covers the form

\[
F(X_1, X_2, X_3, \ldots, X_n) = F_0 + G_1 X_1 H_1 + G_2 X_2 H_2 + \ldots + F_0 + \sum G_i X_i H_i \succ 0 \tag{4.6}
\]

where \( F_0, G_i, H_i \) are given matrices and \( X_i \) are the matrix variables which we seek.
• **LMI Systems**

In general, we are frequently faced with LMI constraints of the form

\[ F_i(X_1, \ldots, X_n) \succ 0 \]

\[ \ldots \ldots \succ 0 \]

\[ F_p(X_1, \ldots, X_n) \succ 0 \]

Where: \( F_i(X_1, \ldots, X_n) = F_{0j} + \sum_{i=1}^{n} G_{ij} X_i H_{ij} \)

By defining \( \overline{F}_0, \overline{G}_i, \overline{H}_i, \overline{X}_i \) as

\[ \overline{F}_0 = \text{diag} (F_{01}, \ldots, F_{0p}) \]

\[ \overline{G}_i = \text{diag} (G_{i1}, \ldots, G_{ip}) \]

\[ \overline{H}_i = \text{diag} (H_{i1}, \ldots, H_{ip}) \]

\[ \overline{X}_i = \text{diag} (X_1, \ldots, X_i) \]

We actually have the following inequality

\[ F_{big}(X_1, \ldots, X_n) = \sum_{i=1}^{n} G_{ij} \overline{X}_i \overline{H}_i \succ 0 \]

That is, we can represent a (big) system of LMI’s as a single LMI. Therefore, we do not distinguish a single LMI from a system of LMI’s; they are the same mathematical entity.

**4.4.1 Types of LMI Problems**

The most important LMI problems that can be classified into three typical problems, i.e., the feasibility solution problems, linear objective functions minimization problems and the generalized eigenvalue problems [87]

• **LMI Feasibility Problems**

The LMI feasibility problems (LMIPs) are simply problems for which we seek a feasible solution \( (x_1, \ldots, x_n) \) such that

\[ f(x_1, \ldots, xn) \succ 0 \]

In this case, the solver is not trying to optimize any function to select a solution from the solution set.

The typical application in control is the stability verification of a linear system in the form:
\[ \dot{x} = Ax \] (4.12)

Then the Lyapunov LMI problem for proving asymptotic stability of this system is to find a matrix \( P > 0 \) such that

\[ A^T P + PA < 0 \] (4.13)

This is obviously an LMI feasibility problem in \( P > 0 \). Note that, given any \( P \succ 0 \), which satisfies this LMI, \( \alpha P \) also will be a solution for any positive \( \alpha \). The solver just picks a solution from the solution set.

- **Linear Objective Minimization Problems**

These problems are also called eigenvalue problems. They involve the minimization (or maximization) of some linear scalar function, \( \alpha() \) of the matrix variables, subject to LMI constraints:

Minimize \( \alpha(X_1,\ldots,X_n) \) subject to

\[ F(X_1,\ldots,X_n) \succ 0 \]

In this case, we are therefore trying to optimize some quantity whilst ensuring some LMI constraints are satisfied.

- **Eigenvalue Problems**

The Eigenvalue Problem (EVP) is to minimize the maximum Eigenvalue of \((n \times n)\) matrix \( A(x) \) that depends affinely on a variable, subject to an LMI symmetric constraint \( B(x) \succ 0 \) (or determine that the constraint is infeasible) [88].

\[ \lambda_{\text{max}}(A(x)) \rightarrow \min_{x \in \mathcal{X}} \]

\[ B(x) \succ 0 \]

or

\[ \text{Minimize } \lambda \text{ Subject to } (\lambda I - A)(x) \succ 0, \ B(x) \succ 0 \]

This problem can be equivalently represented as follows:

\[ \lambda \rightarrow \min_{\lambda, x \in \mathcal{X}} \]
\[
\begin{bmatrix}
\lambda I_{n \times n} - A(x) & 0 \\
0 & B(x)
\end{bmatrix} \succ 0
\]

Where \( A \) and \( B \) are symmetric matrices that depend affinely on the optimization variable \( x \). EVP’s can appear in the equivalent form of minimizing a linear function subject to an LMI. The following relation explains the eigenvalue problem

\[
\text{Minimize } c^T x \text{ Subject to } F(x) \succ 0 \text{ with } F \text{ an affine function of } x
\]

In the special case when the matrices \( F_i \) are all diagonal, this problem reduces to the general linear programming problem: minimizing the linear function \( C^T x \) subject to a set of linear inequalities on \( x \).

Another equivalent form for the EVP is:

\[
\text{Minimize } \lambda \text{ Subject to } A(x, \lambda) \succ 0
\]

where \( A \) is affine in \( (x, \lambda) \)

The later formulation is called positive definite programming (PDP) or if the inequality is non strict, semi definite programming (SDP). As an application of PDP we consider the bounded real lemma, which determines the \( H_\infty \) norm of a system \( G(s) = c(sI - A)^{-1}B + D \) by minimizing

\[
\gamma \text{ with respect to } P \succ 0 \begin{bmatrix}
PA + A^T P & PB & C^T \\
B^T P & -\gamma I & D^T \\
C & D & -\gamma I
\end{bmatrix} \prec 0
\]

4.4.2 Bilinear Matrix Inequality

Interest in BMI originated in control system applications mid 1990, by Safonov’s team. Typical BMI: static output feedback of the system.

- **Definition:**

A bilinear matrix inequality (BMI) is a matrix inequality of the form

\[
F(x, y) := F_0 + \sum_i F_i x_i + \sum_j G_i y_j + \sum_i \sum_j x_i y_j H_{ij} \succ 0
\] (4.14)
Where $G_i$ and $H_j$ are symmetric matrices of the same dimension of $F_i$, a BMI is an LMI in $x$ for fixed $y$ and an LMI in $y$ for fixed $x$ and so is convex in $x$ and convex in $y$. The bilinear term make the set not jointly convex in $x$ and $y$ [89].

4.5 Mathematical Techniques of LMI Problems

Although many control problems can be cast as LMI problems, a substantial number of these need to be manipulated before they are in a suitable LMI problem format. Fortunately, there are a number of common tools that can be used to transform many problems into suitable LMI forms [90].

- **Congruence Transformation**

  For a given positive definite matrix $Q \in \mathbb{R}^{n,n}$ it can be easily shown that for another real matrix $W \in \mathbb{R}^{m,n}$ such that rank $(W) = n$, the following inequality holds $WQW^T > 0$

  In other words, definiteness of a matrix is invariant under pre and post-multiplication by a full rank real matrix, and its transpose, respectively. The process of transforming $Q > 0$ into equation using a real full rank matrix is called a ‘congruence transformation’. It is very useful for ‘removing’ bilinear terms in matrix inequalities and is often used, in conjunction with a change of variables (as implemented in the state feedback design above) [91].

- **Change of Variables**

  Many control problems can be posed in the form of a set of nonlinear matrix inequalities; that is, the inequalities are nonlinear in the matrix variables we seek. However by defining new variables it is sometimes possible to be ‘linearized’ the nonlinear inequalities, hence making them solvable by LMI methods.

  In state feedback control synthesis problem: Consider the problem of finding a matrix $F \in \mathbb{R}^{m,n}$ such that the matrix $A + BF \in \mathbb{R}^{n,n}$ has all of its eigenvalues in the open left-half complex plane. By the theory of Lyapunov equations, this is equivalent to finding a matrix $F$ and a positive definite matrix $P \in \mathbb{R}^{n,n}$ such that the following inequality holds

  \[ (A + BF)^T P + P (A + BF) < 0 \]  

  \[ A^T P + PA + F^T B^T P + PBF < 0 \]

  (4.15)  

  (4.16)
This problem is not in LMI form due to the terms, which contain products of \( F \), and \( P \) these terms are ‘nonlinear’ and as there are products of two variables they are said to be ‘bilinear. If we multiply on either side of equation by \( Q = P^{-1} \) which does not change the definiteness of the expression since \( \text{rank}(P) = \text{rank}(Q) = n \) we obtain

\[
QA^T + AQ + QF^TB^T + BFQ < 0 \tag{4.17}
\]

This is a new matrix inequality in the variables \( Q > 0 \) and \( F \), but it is still nonlinear. To rectify this, we simply define a second new variable \( L = FQ \) giving

\[
QA^T + AQ + L^TB^T + BL < 0 \tag{4.18}
\]

We now have an LMI feasibility problem in the new variables \( Q > 0 \) and \( L \in \mathbb{R}^{m \times n} \). Once this LMI has been solved we can recover the resulting state-feedback matrix [92].

- **Schur Complement**

The main use of the Schur complement is to transform quadratic matrix inequalities into linear matrix inequalities, or at least as a step in this direction. Schur’s formula indicates that the following statements are equivalent [93]:

\[
Q = \begin{bmatrix}
Q_{11} & Q_{12} \\
Q_{12} & Q_{22}
\end{bmatrix} < 0 \tag{4.19}
\]

\[
\begin{cases}
Q_{22} < 0 \\
Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^T < 0
\end{cases} \tag{4.20}
\]

As a common application of Schur complement, consider the LQR type matrix inequality (Riccati inequality) as follows [94]:

\[
A^TP + PA - PB^{-1}B^TP + Q < 0 \tag{4.21}
\]

Using a congruence transformation for \( S = P^{-1} \) we get:

\[
SA^T + AS - BR^{-1}B^T + SQS < 0 \tag{4.22}
\]

We can define

\[
\begin{align*}
Q_{11} &:= SA^T + AS - BR^{-1}B^T \\
Q_{12} &:= S \\
Q_{22} &:= -Q^{-1}
\end{align*} \tag{4.23}
\]

and use the Schur complement identities. Thus, we can transform the Riccati inequality into
\[
\begin{bmatrix}
S + A^T S A - B R^{-1} B^T & S \\
* & -Q^{-1}
\end{bmatrix} \prec 0
\] (4.24)

Hence the original quadratic inequality has been transformed into a linear matrix inequality.

4.6 Robust State Feedback Controller Design Methods

In feedback control system problems, we need to design a controller, \( u = kx \) with a feedback gain to guarantee the following requirements:

- Good disturbance rejection (criterion); the exogenous signal that enters the system has a minimum effect on the tracking errors and the control effort.
- Command tracking as well as reduction of measurement noise on the plant output
- Robust stability: the closed loop system is stable under various external disturbance and model uncertainties
- Noise insensitivity: the system should perform well under noisy measurements.

4.6.1 Stabilizing Controller Design in LMI

A continuous linear time invariant (LTI) system

\[
x = Ax + Bu
\]

\[
y = Cx
\]

Applying the Lyapunov LMI stability condition

The closed loop matrix becomes \( A_{cl} = (A + Bk) \)

\[(A + Bk)^T P + P (A + Bk) \prec 0\]

\[P \succ 0\]

This inequality is bilinear matrix inequality (BMI) and nonconvex. To solve this problem as an LMI we need to convert BMI to LMI as follows:

Let \( P^{-1} \succ 0 \)

Multiplied left and right of the equation (congruence)

\[P^{-1}[(A + Bk)^T P + P (A + Bk)] P^{-1} \prec 0\]

Now we can apply the change of variables: \( Q = P^{-1}, Y = k P^{-1} \)

\[AQ + QA^T + BY + YB \prec 0\]
\[ Q > 0 \]

Hence, solving for \( Q \) and \( Y \), we can calculate the controller \( k = YQ^{-1} \), which guarantee the system stability.

### 4.6.2 \( H_\infty \) Norm Formulation in LMI

The \( H_\infty \) norm of the transfer functions from \( w \) to \( z \) denoted by \( \|T_w\|_\infty \) of the following system:

\[
\begin{align*}
\dot{x}_{cl} &= A_{cl} x_{cl} + B_{cl} w \\
z &= C_{cl} x_{cl} + D_{cl} w
\end{align*}
\]  

(4.25)

The value of \( \|T_w\|_\infty \) does not exceed \( \gamma \) if and only if there exists a symmetric matrix \( P > 0 \) such that:

\[
\begin{bmatrix}
A_{cl}^T P + PA_{cl} & PB_{cl} & C_{cl}^T \\
\ast & -\gamma I & D_{cl}^T \\
\ast & \ast & -\gamma I
\end{bmatrix} < 0
\]

(4.26)

**Proof:**

Suppose that there exist a quadratic Lyapunov function:

\[ \nu(x) := x_{cl}^T P x_{cl}, \ldots, P > 0, \gamma > 0, \] such that:

\[ \dot{\nu}(x_{cl}) + \gamma^{-1} z^T z - \gamma w^T w < 0 \]

(4.27)

for all \( x_{cl}, w \) satisfying equation (4.25).

Integration the inequality (4.27) results:

\[ \nu(x_{cl}(T)) + \int_0^T \gamma^{-1} z^T z - \gamma w^T w dt \leq 0 \Rightarrow \int_0^T \gamma^{-1} z^T z - \gamma w^T w dt \leq 0 \]

(4.28)

Since \( \nu(x_{cl}(T)) > 0 \)

\[ \Rightarrow \frac{\|z\|_2}{\|w\|_2} < \gamma \]

The condition 4.27 is equivalent to
\[
\begin{bmatrix}
 x_{cl}^T \\
 w
\end{bmatrix} = \begin{bmatrix}
 A_{cl}^T P + PA_{cl} + \gamma^{-1} C_{cl}^T C_{cl} & PB_{cl} + \gamma^{-1} C_{cl}^T D_{cl} \\
 * & \gamma^{-1} D_{cl}^T D_{cl} - \gamma
\end{bmatrix} \begin{bmatrix}
 x_{cl} \\
 w
\end{bmatrix} < 0
\] (4.29)

The inequality (3) could be written in the following form

\[
\begin{bmatrix}
 A_{cl}^T P + PA_{cl} & PB_{cl} \\
 * & -\gamma I
\end{bmatrix} + \begin{bmatrix}
 \gamma^{-1} C_{cl}^T C_{cl} & \gamma^{-1} C_{cl}^T D_{cl} \\
 * & \gamma^{-1} D_{cl}^T D_{cl}
\end{bmatrix} = \begin{bmatrix}
 A_{cl}^T P + PA_{cl} & PB_{cl} \\
 * & -\gamma I
\end{bmatrix} + \begin{bmatrix}
 C_{cl}^T \\
 D_{cl}
\end{bmatrix} \gamma^{-1} \begin{bmatrix}
 C_{cl} & D_{cl}
\end{bmatrix} < 0
\]

Then we apply the Schur complement the inequality (4.26) is obtained [94].

4.6.3 H\textsubscript{2} performance:

By applying, the H\textsubscript{2} norm of the transfer function T from w to z can be defined as follows:

\[
\|H\|_2 := \sqrt{\int_{-\alpha}^{\alpha} \text{trace}(H^T(jw)H(jw))dw} = \|h(t)\|_2
\] (4.30)

The standard H\textsubscript{2} optimization problem is to choose the compensator k that

- Stabilize the closed loop system
- Minimize the H\textsubscript{2} norm of the closed loop system

\[
\|T\|_2 := \sqrt{\int_{-\alpha}^{\alpha} \text{trace}(T^T(jw)T(jw))dw}
\] (4.31)

The H\textsubscript{2} norm of the transfer function from w to z in the system

\[
\begin{align*}
x_{cl} &= A_{cl} x_{cl} + B_{cl} w \\
z &= C_{cl} x_{cl} + D_{cl} w
\end{align*}
\] (4.32)

Denoted by \(\|T\|_2\) does not exceed \(\beta\) if and only if there exist two symmetric matrices P and Q such that:

\[
\begin{bmatrix}
 A_{cl}^T P + PA_{cl} & PB_{cl} \\
 * & -I
\end{bmatrix} < 0
\]

\[
\begin{bmatrix}
 Q & C_{cl} \\
 * & P
\end{bmatrix}
\]

\[
T_{r}(Q) < \beta^2
\]

\[
D_{cl} = 0
\] (4.33)
• **Proof:**

The $H_2$ norm of $T$ may be written as follows:

$$
\|T\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}[T^T(j\omega)T^*(j\omega)]d\omega = \text{trace}\left\{ \int_{0}^{\infty} t(\tau)t^T(\tau)d\tau \right\} = \text{trace}\left\{ \int_{0}^{\infty} B^T(e^{A\tau})^T e^{A\tau} B d\tau \right\} 
$$

(4.34)

where: $t(\tau)$ is the impulse response.

It is well known that the $H_2$ norm can be calculated as follows [95]:

$$
\|T\|_2^2 = \text{trace}(C_{cl} S_0 C_{cl}^T), \text{ where } S_0 \text{ solves the Lyapunov equation } A_{cl} S_0 + S_0 A_{cl}^T + B_{cl} B_{cl}^T = 0
$$

To transform this equality constraint to an inequality constraint, we introduce

$S > S_0$, where $S - S_0 = W > 0$

Substituting $S_0$ by $S$

$$
A_{cl} S + SA_{cl}^T + B_{cl} B_{cl}^T = A_{cl} S_0 + S_0 A_{cl}^T + B_{cl} B_{cl}^T + A_{cl} W + WA_{cl}^T = A_{cl} W + WA_{cl}^T < 0
$$

(4.35)

$A_{cl}$, is stable

$$
A_{cl} S + SA_{cl}^T + B_{cl} B_{cl}^T < 0.
$$

It is readily verified that $\|T\|_2^2 = \text{trace}(C_{cl} S_0 C_{cl}^T) < \text{trace}(C_{cl} S C_{cl}^T)$ by adding an auxiliary parameter $Q = C_{cl} S C_{cl}^T$. The $H_2$ norm of $T$ is upper bounded by $\text{trace}(Q)$.

$$
\|T\|_2^2 = \text{trace}(C_{cl} S_0 C_{cl}^T) < \text{trace}(C_{cl} S C_{cl}^T) < \text{trace}(Q)
$$

(4.36)

And $\|T\|_2^2 < \beta^2$ if

$$
A_{cl} S + SA_{cl}^T + B_{cl} B_{cl}^T < 0
$$

$$
C_{cl} S C_{cl}^T < Q
$$

$\text{trace}(Q) < \beta^2$

Using Schur complement to the second inequality and follow the congruence transformation with $S^{-1}$ to the first inequality then we can obtain those inequalities

$S^{-1} A_{cl} + A_{cl}^T S^{-1} + S^{-1} B_{cl} B_{cl}^T S^{-1} < 0$

$$
\begin{bmatrix}
Q \\
C_{cl}^T S^{-1}
\end{bmatrix} > 0
$$

$\text{trace}(Q) < \beta^2$
By making the change of variables \( P = S^{-1} \) and taking the Schur complement to the first inequality, the inequality (4.33) are obtained. [94-95].

4.7 Pole Placement in Control Systems Design

It is well known that the dynamic behavior of a controlled system is influenced by its closed loop poles. If the system is fully controllable, the poles of the closed loop system can be arbitrarily shifted or placed in any prespecified positions. This is referred to as the pole placement controller design method. Pole placement can be classified into conventional and robust or LMI pole placement region. The conventional pole placement may come in some of the following algorithms [96].

- The Bass-Gura Algorithm
- Ackerman’s Algorithm

Robust pole placement algorithm and design concepts may come in LMI region.

4.7.1 Pole Placement in LMI Region

In most control system design stability is a minimum requirement for control system. In most practical situations, however, a good controller should also deliver sufficiently fast and well damped time response. An important way to guarantee satisfactory transients is to place the closed-loop poles in a suitable region of the complex plane.

For example: fast decay, good damping and enhanced controller dynamics can be imposed by confining the poles in the intersection of a shifted half-plane. More practically LMI regions include relevant region such as sectors and disks, conic, strips, etc., as well as any intersection of the above most common LMI region.

- **General Definition**

Pole placement in convex regions in the left-half plane or in the unit disk can be expressed as LMI constraints, either applied to Lyapunov matrix P involved in such a single constraint or added to other constraints, e.g. in the case of an LQC synthesis with regional pole constraint. LMI region can be defined as follows [97]:

An LMI region is any region \( R \) of the complex plane that can be defined as
\[ R = \left\{ Z \in C : L_{\text{reg}} + ZM_{\text{reg}} + \bar{Z}M_{\text{reg}}^T \right\} < 0 \]  

(4.37)

where: \( L_{\text{reg}} = L_{\text{reg}}^T \), \( M_{\text{reg}} \) are fixed real matrices and \( R \) is a convex region in the complex plane, symmetric about the horizontal axis.

The Lyapunov theory, which applies for the open left-half plane \( \{ Z : Z + \bar{Z} < 0 \} \) can be generalized to any LMI region. The eigenvalues of a given matrix \( A \) are all contained in the LMI region \( R \) if and only if some LMI involving \( A \) is visible. The matrix \( A \) has all its eigenvalues in the LMI region if and only if there exists a symmetric matrix \( P \) such that [98].

\[ L_{\text{reg}} \otimes P + M_{\text{reg}} \otimes A^T P + M_{\text{reg}}^T \otimes PA < 0 \]  

(4.38)

\[ P > 0 \]  

(4.39)

where: \( \otimes \) means Kronecker products and this type of product defined as an important tool for the subsequent analysis. Recall that the Kronecker product of two matrices \( A \) and \( B \) is a block matrix \( C \) with generic block entry the following condition:

\[ C_{ij} = A_{ij} B_{ij} \text{ That is } A \otimes B = \left[ A_{ij} B_{ij} \right] \]  

(4.40)

The following properties of the Kronecker product can be easily established [99].

\[ I \otimes A = A \]  

(4.41)

\[ (A + B) \otimes C = A \otimes C + B \otimes C \]  

(4.42)

\[ (A \otimes B)(C \otimes D) = AC \otimes BD \]  

(4.43)

\[ (A \otimes B)^T = A^T \otimes B^T \]  

(4.44)

\[ (A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \]  

(4.45)

Generally, the eigenvalues of \( A \otimes B \) are the wise pair products \( \lambda_i(A) \lambda_j(B) \) of the eigenvalues of the two matrices \( A \) and \( B \) [100].
By constraining $\lambda$ to be lying in a specific region, specific bounds can be put on these quantities to ensure certain parameters in the transient response. Regions include $\alpha$ mean stability region and the real part of the eigenvalues governs by $R_x \leq -\alpha$.

Another important region for the control purposes is the set $S(\alpha, r, \theta)$ of the complex number $x + jy$ such that $x < -\alpha < 0$, $|x + jy| < r$, $\tan \theta < -|y|$ as shown in figure 4.2. If we can transfer the closed loop poles to this region we will improve the following system parameters:

Minimum decay rate $\alpha$, minimum damping ratio, $\xi = \cos \theta$ and maximum undamped natural frequency, $\omega_d$, $\omega_d = r \sin \theta$. These parameters have direct effect on the maximum overshoot, the rise time and settling time [101].

4.7.2 State Feedback with Regional Pole Placement

For a given system matrix $A \in \mathbb{R}^{n \times n}$ and $P = P^T \in \mathbb{R}^{n \times n}$, we can define a function, $M_D(A, P) = L \otimes P + M \otimes (AP) + M^T \otimes (PA^T)$ (4.46)

Therefore the eigenvalues $\lambda_i(A) \in D$, there exists $P = P^T > 0$ and $M_D(A, P) < 0$
In case of state feedback, the closed loop matrix \((A - BK)\) concerned with assignment of poles in the LMI regions, from this equation substituting in equation (4.46) then the inequality becomes inequality in decision variables \(L\) and \(P\) but it is not an LMI because of the product of \(LP\). The inverse of \(P\) exists. We multiply the left and right of the inequality 4.46 by \(P^{-1}\) and we have compute \(Y = P^{-1}\) and \(W = LY\).

The following LMI is obtained

\[
L_{\text{reg}} \otimes Y + M_{\text{reg}} \otimes (AY - BW)^T + M_{\text{reg}}^T \otimes (AY - BW) < 0
\]

\((4.47)\)

\(Y > 0\)

The LMI can be used to find the state feedback gain matrix of the system as follows:

According to the Lyapunov inequality of this system

\[
A^T P + PA < 0
\]

\((4.48)\)

\(P > 0\)

With the closed loop state feedback matrix the inequality becomes

\[
(A - BK)^T P + P(A - BK) < 0
\]

\((4.49)\)

\(P > 0\)

The inequality \((4.49)\) is nonlinear because of the \(PK\) product

\(Y = PK\), which is linear in \(P\) and \(Y\)

\(k = yP^{-1}\) The inequality becomes linear inequality as follows

\[
PA^T + AP - Y^T B - BY < 0
\]

\((4.50)\)

\(P > 0\)

This LMI can be used to find the state feedback gain matrix. It is used to place the closed loop poles inside the \(\alpha\) region.

If we return back to the

\[
PA^T + AP + 2\alpha P < 0
\]

\((4.51)\)

\(P > 0\)

The system matrix \(A\) is replaced by the closed loop matrix \(A_c\) where \(A_c = (A - BK)\) this leads to the following equation

\[
P(A - BK^T) + (A - BK)P + 2\alpha P < 0
\]

\((4.52)\)
This inequality (4.52) is nonlinear in the variables as mentioned above and can be converted to linear inequality by using parameter change as follows: \( Y = PK \) to obtain

\[
PA_C^T + A_C^T P - Y^T B^T - BY + 2\alpha P < 0
\]

(4.53)

\( P > 0 \)

This inequality (4.53) is linear and can be solved with YALMIP solvers. Once the solution \((P,Y)\) is available, the state feedback controller that places the closed loop poles in the \( \alpha \) region is computed as follows \( K = YP^{-1} \).

The eigenvalues of the closed loop system matrix are inside the \( \alpha \) region, which means that \( (\lambda_i < \alpha) \).

### 4.8 Observer Based Control

In several cases, the system states are not easily measurable. Hence, to apply state feedback, we have to estimate the state of the system from its inputs and outputs, as illustrated in Figure 4.3. In some situations we will assume that there is only one measured signal, i.e., that the signal \( y \) is a scalar and that \( C \) is a (row) vector. We write \( \hat{x} \) for the state estimation given by the observer.

The problem of observability is one that has many important applications, even outside feedback systems. If a system is observable, then there are no “hidden” dynamics inside it; we can understand everything that is going on through observation (over time) of the inputs and outputs. As we shall see, the problem of observability is of significant practical interest because it will determine if a set of sensors is sufficient for controlling a system. Sensors combined with a mathematical model can also be viewed as a virtual sensor that gives information about variables that are not measured directly.
In practice, there two observation techniques are commonly used according to the system states measurement.

- Full order state observer
- Minimum (reduced) order state observer

The full order state observer observes all state variables of the system regardless of whether some state variables are available for direct measurement. The minimum order state observer is an observer that estimates fewer than \( n \) state variables, where \( n \) is the dimension of the state vector, is called a minimum (reduced) order observer. [102].

Design and implementation of the observer base control now comes in two ways as follow:

1. High-gain observer based control
2. Slide-mode observer based control

### 4.8.1 High Gain Observer

High gain observer based control has been suggested and investigated in many scientific articles. The advantages of this observer are its excellent robustness properties. By choosing the observer gain \( k \) large enough therefore the name (“high-gain“), the observer error can be made arbitrarily small. The difficulty in practical applications is, however the determination of an appropriate value for the observer gains. For values too low, the desired bounds on the observer error can’t be achieved. For values unnecessarily high, the sensitivity to noise increases, thus limiting the practical use [103-104].
4.8.2 Sliding Mode Observer

Sliding mode observer (SMO) are robust observers, which estimate the state of nonlinear uncertain system. SMO are well suited for nonlinear uncertain systems with partial state feedback. The main advantages of SMO, over a linear observer such as a Luenberger observer, are that SMO can be made considerably more robust to parametric uncertainty, external disturbances and noisy measurements. The main drawback of SMO is that extensive efforts in the design procedure must be taken to guarantee robustness for bounded modeling errors [105].

4.8.3 Combining Observers and State Feedback

In many of control system design problems the combination between observer and state feedback is used. Let the system described by the following equations:

\[
\dot{x}(t) = Ax(t) + Bu(t) \quad (4.54)
\]
\[
y(t) = Cx(t) \quad (4.55)
\]

We have to check the system controllability and system observability as a first step in the observer design as follows:

The third order model is completely state controllable because the \( n \times n \) controllability matrix has rank \( n \), “The rank of \( V_c = [B, AB, \ldots, A^{n-1}B] \) equal 3”. The third order model is completely state observable because the \( n \times n \) observability matrix has rank \( n \), “The rank of \( V_o = [C, CA, \ldots, CA^{n-1}] \) equal 3”.

\[
\hat{x}(t) = A \hat{x}(t) + Bu(t) + L(y(t) - y(t)) \quad (4.56)
\]
\[
\hat{y}(t) = C \hat{x}(t) \quad (4.57)
\]

To establish full state feedback using the state estimates, the feedback law is calculated as follows:

\[
u(t) = -k \hat{x}(t) + r(t) \quad (4.58)
\]

This immediately leads to the observer state equation

\[
\dot{x} = (A - Bk - LC) \hat{x} + Ly + Br \quad (4.59)
\]
While the estimated error is

\[ e_e = x - \hat{x} \Rightarrow \dot{x} = x - e_e \]  \hspace{1cm} (4.60)

\[ x = (A - BK) x + B e_e + Br \]  \hspace{1cm} (4.61)

The estimated error calculated by the following equation

\[ e_e = (A - LC)e_e \]  \hspace{1cm} (4.62)

Equations (4.61) and (4.62) can be combined to the 2n dimensional state equation as follows:

\[
\begin{bmatrix}
\dot{x} \\
e
\end{bmatrix} = \begin{bmatrix}
A - Bk & Bk \\
0 & A - LC
\end{bmatrix}
\begin{bmatrix}
x \\
e_e
\end{bmatrix} + \begin{bmatrix}
B \\
0
\end{bmatrix} r
\]  \hspace{1cm} (4.63)

Where the characteristic equation of the overall system is calculated by the determinant of the following matrix

\[
\det \begin{bmatrix}
\lambda I - A + Bk & -Bk \\
0 & \lambda I - A + LC
\end{bmatrix}
\]  \hspace{1cm} (4.64)

\[
\det (\lambda I - A + Bk) \cdot \det (\lambda I - A + LC) = 0
\]  \hspace{1cm} (4.65)

We can find the estimated eigenvalues by using pole placement methods. The eigenvalues of the overall system (4.65) consist of the union of the eigenvalues of the controller and the eigenvalues of the observer. In other words, the eigenvalues of the controller and those of the observer are independent of each other. If the controller as well as the observer separately has been given eigenvalues in the left half plane, then one can be sure that the overall system will be asymptotically stable. Figure 4.4 shows the overall system with observer and state feedback control system.
4.9 Concluding Remarks

This chapter illustrates the basic concepts of the robust control system and its application to guarantee robust stability and robust system performance with uncertainty. System norms like $H_2$, $H_\infty$ are explained to deal with the norm meaning relative to the robust control system design methodologies. The robust $H_2$ and $H_\infty$ controllers’ mathematics is explained, with their applications. The chapter gives an introduction to the fundamentals of LMI’s and BMI’s with their problems in optimization and control applications. The rest of this chapter covers the observer based design techniques and the combination between state feedback and observer.
Chapter 5

Model Reduction Methods and Applications to Nuclear Reactor

5.1 Introduction

An important tool in the design of complex high-tech systems is the numerical simulation of predictive models. However, these dynamical models are typically of high order, i.e. they are described by a large number of ordinary differential equations. This results from either the inherent complexity of the system or the discretization of partial differential equations. Model reduction can be used to find a low-order model that approximates the behavior of the original high-order model, where this low-order approximation facilitates both the computationally efficient analysis and controller design for the system to induce desired behavior.

The earliest methods for model reduction belong to the field of structural dynamics, where the dynamic analysis of structures is of interest. Typical objectives are the identification of frequency response functions. Besides the mode displacement reduction method and extensions thereof many important techniques are given by component mode synthesis techniques, which started to emerge in the 1960s. The model reduction problem has also been studied in the systems and control community, where the analysis of dynamic systems and the design of feedback controllers are of interest.

Some of the most important contributions were made in the 1980s by the development of balanced truncation and optimal Hankel norm approximation. Finally, numerically efficient methods for model reduction have been developed in the field of numerical mathematics in the 1990s. Important techniques are asymptotic waveform evaluation, Padé-via-Lanczos and rational interpolation [106].

These methods are often applied in the design and analysis of large electronic circuits. Despite the fact that the above techniques essentially deal with the same problem of model reduction, the results in the fields of structural dynamics, systems and control and numerical mathematics have largely been developed independently.
The thesis aims at providing a thorough comparison between the model reduction techniques from the application in systems and control field, facilitating the choice of a suitable reduction procedure for a given reduction problem. The most popular methods from the field of systems and control will be used. Then, the properties of these techniques will be compared, where both theoretical and numerical aspects will be discussed. In addition, these differences and commonalities will be illustrated by means of application of the model reduction techniques to case studies [107].

5.2 Model Linearization
To use the linear model order reduction methods, the nonlinear model must be transformed into a linear model based on the linearization of model equations. Linearization can be a helpful tool for the following reasons:

1. Linear systems are more easier to be reduced to a lower order than the nonlinear models
2. Linear systems can be analyzed by using available powerful mathematical tools
3. System behavior can be analyzed because relationships between process inputs and process outputs are expressed in terms of process variables and/or operating conditions
4. It is relatively easy to investigate the stability of a linear system [108].

Actually, linearization is carried out about some operating points of the system/model.

5.3 Model Simplification and Reduction
In many engineering applications, processes are described by increasingly complex models that are difficult to analyze and difficult to control. Reduction of the order of the model may overcome some of these difficulties, but it is quite possible that model reduction incurs a significant loss of accuracy. Therefore, the system has to be analyzed in a manner that is useful for the application purpose. Simplification of the model based on this analysis usually results in a model of lower complexity which is easier to handle, and in a corresponding simplification of synthesis procedures for control and filtering problems [109].

5.3.1 Model Order Reduction
Reducing the complexity of the functional expressions in the model equations is not always enough to achieve the model reduction. In many cases, the number of the model equations needs
to be reduced also. In the model-order reduction process, two main groups of methods can be distinguished: linear and nonlinear model-order reduction, depending on the type of the full order model. The models of dynamical systems contain a large number of equations, both differential and algebraic, and, quite often, these equations contain complex functional expressions. Such models can be written in the following, general form:

\[
\frac{dx(t)}{dt} = f[x(t), u(t), \alpha] \tag{5.1}
\]

\[
y(t) = g[x(t), u(t), \alpha] \tag{5.2}
\]

Here \( t \) is the time variable, \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R}^m \) the input vector, \( y(t) \in \mathbb{R}^p \) the output variable vector, \( \alpha \) is the parameters vector, and \( n \) is the state space dimension. The dimension of the input vector \( m \) and the dimension of the output variable vector \( p \), are much smaller than \( n \), and usually \( m \geq p \).

In many engineering applications, processes are described by increasingly complex models that are difficult to analyze and difficult to control. Reduction of the order of the model may overcome some of these difficulties, but it is quite possible that model reduction incurs a significant loss of accuracy. Therefore, the system has to be analyzed in a manner that is useful for the application purpose. Simplification of the model based on this analysis usually results in a model of lower complexity which is easier to handle, and in a corresponding simplification of synthesis procedures for control problems. Furthermore, the simplification decreases the computational effort. Every application has its own demands, and different model reduction methods have different properties [110].

**Linear systems model reduction**

The main idea of model reduction is to construct a reduced order model from the original model. The closeness of the reduction process is normally measured by the absolute error of \( \|G(s) - G_r(s)\| \) where \( G(s) \) and \( G_r(s) \) are respectively the transfer function matrices of the original (full order) and the reduced order models. The aim of the following methods is to find a reduced-order system \( G_r(s) \) such that the absolute error is minimized.
**Nonlinear systems model reduction**

Most nonlinear model order reduction methods are extensions of linear methods for nonlinear systems. The simplest approach for generating reduced order models for nonlinear systems is based on linearization of systems nonlinearity and subsequent application of linear model order reduction methods. The drawback of this approach is that the obtained reduced model is valid only locally, around the initial operating point of the nonlinear system. Hence, these techniques do not work for cases when the models have a strongly nonlinear behavior [111].

### 5.3.2 Model Order Reduction and Control System Design

Since small order controllers are appealing from the point of view of real applications, the directions to obtain a lower-order controller for a relatively high-order system are illustrated in figure 5.1. These techniques can be summarized as:

1. System model reduction followed by controller design, this technique mainly used to reduce the reactor model into third order model followed by a low order controller based design as illustrated in the next chapter.
2. Controller design followed by controller-order reduction; and,
3. Direct design of low-order controllers.

Techniques (1) and (2) are widely used and can be used together. When a controller is designed using a robust design method, technique (1) would usually produce a stable closed loop, though the reduction of the model order is likely to be limited.

In technique (2), there is freedom in choosing the final order of the controller, but the stability of the closed-loop system should always be verified. The third technique usually would heavily depend on some properties of the plant, and require numerous computations. It is found that both models and controllers order reduction follow the same considerations and may reduced by the same methods [112].
5.4 State Space Model Reduction

Methods for reducing the order of models by truncation of state variable realizing are usually based on some transformation of the full order model. The formula for the additive error associated with a truncated model is given by the difference between both full order model and reduced model. The transfer function of the full order system and truncated approximation match at $\omega = \infty$ [113].

Several methods are commonly used for state space model reduction, namely as follows:

- Balanced truncation method
- Schur model reduction method
- Hankel norm model reduction method

Balanced truncation can be used in the case of systems, which are so large that numerical standard for controller design or simulation of the system can’t be applied due to their extensive numerical costs. The error bound in this case is computed based on values of absolute error $\|G - G_r\|_\infty$ of the full order model transfer function $G(s)$ and reduced order model $G(s)$ respectively [114].

---

**Figure 5.1:** Model and control system order reduction
For a stable model; these Hankel singular values indicate the respective state energy of the model. Hence reduced order can be directly determined by examining the model Hankel singular values $\sigma_i$. This method guarantees an error $\|G - G_r\|_\infty$ for well conditioned reduced problems; this error is guaranteed to satisfy twice-the sum of Hankel singular values of the original model, 

$$
\|G - G_r\|_\infty \leq 2 \sum_{i=r+1}^{n} \sigma_i \quad [115].
$$

### 5.4.1 Balanced Realization Algorithm

Suppose that the $n$-th order system $G(s)$ is stable and minimal. Thus it is both controllable and observable. Therefore, the controllability and observability gramians $C_G$ and $O_G$ respectively are symmetric and positive definite and hence admit the Cholesky factorizations [116].

Let: $C_G = L_cL_c^T$ and $O_G = L_oL_o^T$ be the respective Cholesky factorizations.

Let: $L_o^TL_c^T = U\sum V^T$ be the singular value decomposition (SVD) of $L_o^TL_c^T$

Define now: $T = L_cV\sum^{-1/2}$

Where: $\sum^{1/2}$ denotes the square roots of $\sum$.

The $T$ is nonsingular, and furthermore using definitions of $C_G$ and $T$, we see that the transformed controllability gramian is

$$
\tilde{C}_G = T^{-1}C_GT^{-T} = \sum^{1/2}V^TL_c^{-1}L_cL_c^TV\sum^{1/2} = \sum
$$

Similarly, using the definitions of $O_G, L_o^TL_c^T$ and $T$, we see that the transformed observability gramian $O_G$ is

$$
\tilde{O}_G = T^TO_GT = \sum^{-1/2}V^TL_c^TL_oL_cV\sum^{-1/2} = \sum^{-1/2}V^TV\sum U^TU\sum V^TV\sum^{-1/2} = \sum^{1/2}, \sum^{1/2} = \sum
$$

Thus, the particular choice of

$$
T = L_cV\sum^{-1/2}
$$

Reduce both the controllability and observability Gramians to the same diagonal matrix $\sum$, the same system ($\tilde{A}, \tilde{B}, \tilde{C}$), where the system matrices are defined by
\[ \tilde{A} = T^{-1}AT, \quad \tilde{B} = T^{-1}B \text{ and } \tilde{C} = CT \] is then balanced realization of the system \((A, B, C)\). The decreasing positive numbers \(\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n\) in \(\sum = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)\), are the Hankel singular values.

We can summarize the above explanation to the following algorithm:

**Algorithm for internal balancing of a continuous time model reduction**

**System input**

- **A**: \(n \times n\) State matrix
- **B**: \(n \times m\) input matrix
- **C**: \(r \times n\) output matrix

**System output:**

- **T**: \(n \times n\) Nonsingular balancing transforming matrix.

\(\tilde{A}, \tilde{B}, \tilde{C}\): the matrices of internally balanced realization

\[ \tilde{A} = T^{-1}AT, \quad \tilde{B} = T^{-1}B \quad C = CT \] with assumptions of \((A, B)\) is controllable and \((A, C)\) is observable, and \(A\) is stable.

\[ T^{-1}C_GT^{-T} = TTO_GT \sum, \] a diagonal matrix with positive diagonal entries

**The algorithm in steps:**

**Step 1:** compute the controllability and observability grammians matrices \(C_G\) and \(O_G\) by solving the Lyapunov equations \([117]\):

\[ AC_G + C_GA^T + BB^T = 0 \]  \hspace{1cm} (5.6)

\[ A^TO_G + O_GA + C^TC = 0 \]  \hspace{1cm} (5.7)

**Step 2:** find the Cholesky factors \(L_c\) and \(L_o\) of \(C_G\) and \(O_G\):

\[ C_G = L_cL_c^T \text{ and } O_G = L_oL_o^T \]

**Step 3:** find the SVD of the matrix \(L_c^TL_c\):

\[ L_c^TL_c = U \sum V^T \]

**Step 4:** compute \(\sum^{-1/2} = \text{diag}(\frac{1}{\sqrt{\sigma_1}}, \frac{1}{\sqrt{\sigma_2}}, \ldots, \frac{1}{\sqrt{\sigma_n}})\) where \(\sum = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)\)

**Step 5:** form \(T = L_cV\sum^{-1/2}\)

**Step 6:** compute the matrices of the balanced realization: \(\tilde{A} = T^{-1}AT, \quad \tilde{B} = T^{-1}B\) and \(\tilde{C} = CT\).
The most important property of a balanced system (or a balanced realization of a system) is that each state is as controllable as it is observable, and the measure of state’s observability and controllability is given by its associated Hankel singular value. This property becomes interesting for the model reduction techniques, which remove the states with little effect on the system’s input-output behavior [118].

The balanced realization can be easily obtained by simple state similarity transformations, and routines for doing this are available in simple software, like MATLAB.

In the balanced realization method, a new state space description is obtained in such a way that the controllability and observability Gramians are equal and diagonal [119].

5.4.2 Model Reduction via Balanced Truncation Method

The main idea behind model reduction via balanced truncation is to obtain a reduced-order model by deleting those states that are least controllable and observable (as measured by the size of Hankel singular values).

We now assume that the state-space model of the original system $G(s)$, $[A,B,C,D]$, is already in the balanced realization form.

Assume $\Sigma = \text{diag}(\Sigma_1, \Sigma_2)$, $\Sigma_1 = \text{diag}(\sigma_1, \sigma_2, ..., \sigma_r)$ where $\sigma_r > \sigma_{r+1}$. The matrices $A$, $B$ and $C$ can be partitioned compatibly as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

The balanced realization implies that the states corresponding to the Hankel singular values $\sigma_{d+1}, ..., \sigma_N$ are less controllable and less observable than those corresponding to $\sigma_1, ..., \sigma_d$. Thus, the reduced-order model obtained by eliminating these less controllable and less observable states are likely to retain some desirable information about the original system [120].

Then, a reduced-order system $G_r(s)$ can be defined as follows: $G_r = \begin{bmatrix} A_{11} & B_1 \\ C_1 & D \end{bmatrix}$

It can be shown that $G_r(s)$ is stable, in the balanced realization form, and $\|G(S) - G_r(s)\|_\infty \leq 2\text{tr}(\Sigma_2)$ [121].
Where $\text{trace}(\Sigma_2)$ denotes the trace of the matrix $(\Sigma_2)$, i.e. $\text{trace}(\Sigma_2) = \sigma_{r+1} + \ldots + \sigma_n$ the sum of the last $(n - r)$ Hankel singular values. In most applications, to reduce the full order model into an $r$th-order one there should be a large gap between $\sigma_r$ and $\sigma_{r+1}$, it means that $\sigma_r \gg \sigma_{r+1}$.

### 5.4.3 Schur Model Reduction

The Schur reduction method has been designed to avoid possible ill-conditioning, because of the requirement of explicitly computing the product of the controllability and observability Gramians [122]. This problem can be overcome if orthogonal matrices are used to transform the system model to another equivalent model from which the reduced order model is extracted.

The Schur reduction method, described below, constructs such matrices using the Real Schur Form (RSF) of the matrix $C_GO_G$. The RSF of a matrix $A$ can be obtained using orthogonal similarity transformations.

The Schur method for model reduction does not give balanced realization but the essential properties of the original model are preserved in the reduced order model [123].

The following part summarizes Schur reduction algorithm.

**Schur reduction -Algorithm outlines**

**Inputs:** $A$: the $n \times n$ state matrix, $B$: the $n \times m$ control matrix, $C$: the $r \times n$ output matrix and $Q$: the dimension of the desired reduced order model  

**Outputs:** $A_r$: the $q \times q$ reduced state matrix, $B_r$: the $q \times M$ reduced control matrix, $C_r$: the $r \times q$ reduced output matrix

**Steps of Schur Model Reduction Algorithm**

- Step 1: compute both controllability and observability Grammians $C_G$ and $O_G$ respectively by solving the following Lyapunov equations

  \[ AC_G + C_GA^T + BB^T = 0 \quad \text{and} \quad AO_G + O_GA^T + CC^T = 0 \]  

- Step 2: transform the matrix $C_GO_G$ to RSF of $y$ to find the eigenvalues of a matrix $y$ and find an orthogonal matrix $\chi$; such that $y = \chi^TC_GO_G\chi$. It is obtained by using the well-known QR iteration method.
• Step 3: reorder the eigenvalues of \( y \) in ascending and descending order to find orthogonal matrices \( U \) and \( V \) such that

\[
u^T y u = u^T \chi C_g O_g \chi u = u_s^T C_g O_g u_s = \begin{bmatrix} \lambda_1 & * \\ \vdots & \ddots \\ 0 & \lambda_n \end{bmatrix}
\]

\[
u^T y v = v^T \chi C_g O_g \chi v = v_s^T C_g O_g v_s = \begin{bmatrix} \lambda_n & * \\ \vdots & \ddots \\ 0 & \lambda_1 \end{bmatrix}
\] (5.9) (5.10)

Where \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \) and \( \lambda_i = \sigma_i^2, \lambda_{i-1} = \sigma_{i-1}^2, \ldots \), and so on, where \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \) are the Hankel singular values.

• Step 4: partition the matrices \( u_s \) and \( v_s \), as follows: \( u_s = (u_{1s}, u_{2s}), \ v_s = (v_{1s}, v_{2s}) \)

• Step 5: find the SVD of \( u_{2s}^T v_{1s} : Q \sum R^T = u_{2s}^T v_{1s} \)

• Step 6: compute the transforming matrices: \( s_1 = u_{2s}^T Q \sum^{-\frac{1}{2}}, s_2 = v_{1s}^T R \sum^{-\frac{1}{2}} \)

• Step 7: compute the reduced order model matrices as follows:

\( A_R = S_1^T A S_2, B_R = S_1^T B, \text{and} \ C_R = C S_2 \)

**Definition of QR Factorization:**
The QR factorization recall that the square matrix \( Q \) is said to be an orthogonal matrix if \( Q O^T = O^T Q = I \). Given an \( m \times n \) matrix \( A \), there exist an \( m \times m \) orthogonal matrix \( Q \) and an \( m \times n \) upper triangular matrix \( R \) such that \( A = QR \). Such a factorization of \( A \) is called QR factorization.

If \( m \geq n \), and if the matrix \( Q \) is portioned as \( Q = [Q_1, Q_2] \), where \( Q_1 \) is the matrix of the first \( n \) columns of \( Q \) and if \( R_1 \) is defined by \( R_1 = \begin{pmatrix} R_{11} \\ 0 \end{pmatrix} \) where \( R_{11} \) is \( n \times n \) upper triangular, then \( A = Q_1 R_1 \).

This QR factorization is called “economy size of the “thin” QR factorization of \( A \). There are several ways to compute QR factorization of a matrix such as

• House-holder’s and Givans methods can be used to compute both types of QR factorization.
• On the other hand, the classical Gram-Schmidt (MGS) compute \( Q \in R^{m \times n} \) and \( R \in R^{m \times n} \) such that \( A = QR \). The MGS has better numerical properties than CGS [124].

**Definition of Real Schur Form (RSF)**

In this section, we describe how to obtain the RSF of a matrix, the RSF of a matrix \( A \) displays the eigenvalues of \( A \). It is obtained by using the well known QR iteration method. This method is nowadays a standard method for computing the eigenvalues of a matrix. In the Schur triangularization theorem, let \( A \) be an \( n \times n \) complex matrix, then there exists an \( n \times n \) unitary matrix \( U \) such that \( U^*AU = T \), where \( T \) is an \( n \times n \) upper triangular matrix and the diagonal entries of \( T \) are the eigenvalues of \( A \) [125].

**5.4.4 Hankel Norm Approximation Algorithms**

It is important to define the Hankel norm as the largest Hankel singular value of a stable system \( G(s) \). This system is called balanced if the solution of \( P \) and \( Q \) to the following Lyapunov equation \( AP + PA^T + BB^T = 0 \) and \( A^T Q + QA + C^T C = 0 \) are such that \( P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n) := \sum \), with \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0 \), \( P \) and \( Q \) are called the controllability and observability grammians respectively. When the system is balanced, both grammians are diagonal and equal. \( \sigma_i, \, i = 1,2,\ldots,n \), is the \( i-th \) Hankel singular value of the system [126]. The largest Hankel singular value \( \sigma_1 \) is defined as the Hankel norm of \( G(s) \) and the Hankel norm denotes the largest possible \( L_2 \) gain. In some model order reduction cases, minimization of the Hankel norm of the error system is more appropriate and thus required.

**5.4.4.1 Approximation Algorithm 1**

Let a stable and square system with a state space model \([A,B,C,D]\) of a minimal and balanced realization. The gramian \( P = Q = \text{diag}(\sum_i \sigma I_L) \) where \( \sigma \) is the smallest Hankel singular value with multiplicity \( L \) and every diagonal element of \( \sum_i \) is larger than \( \sigma \). An \( (n-1)th \)–order system \( G_h(s) \) can be computed as follows:

\[
\bar{A} = \Gamma^{-1}(\sigma^2 A_{11}^T + \sum_i A_{11} \sum_i - \sigma C_1^T U B_1^T)
\]  

(5.11)
\[ \tilde{B} = \Gamma^{-1} \left( \sum_i B_i + \sigma C_i^T U \right) \]  
(5.12)

\[ \tilde{C} = C_1 \sum_i + \sigma UB_i^T \]  
(5.13)

\[ \tilde{D} = D - \sigma U \]  
(5.14)

Where \( U \) is an orthonormal matrix and satisfies

\[ B_2 = -C_2^T U, \quad \text{and} \quad \Gamma = \sum_i -\sigma^2 I. \]  
(5.15)

The reduced order system

\[ G_h(s), G_{h_1}(s) = \tilde{C} (sI - \tilde{A})^{-1} \tilde{B} + \tilde{D} \]  
(5.16)

\((n-1)\text{th}-\text{order}, G_h(s)\) is stable and is an optimal approximation of \( G(s) \) satisfying

\[ \left\| G(s) - G_h(s) \right\|_H = \sigma. \]  
It is also true that \( G(s) - G_h(s) \) is all-pass with the infinity norm \( \left\| G(s) - G_h(s) \right\|_\infty = \sigma \) [127].

### 5.4.4.2 Approximation Algorithm 2

Let the Hankel singular values of \( G(s) \) be \( \sigma_1 \succ \sigma_2 \succ \ldots \succ \sigma_r \) with multiplicities \( m_i, i = 1, \ldots, r \), i.e. \( m_1 + m_2 + \ldots + m_r = n \) by repeatedly applying equations (5.11)-(5.16), we may have:

\[ G(s) = D_0 + \sigma_1 E_1(s) + \sigma_2 E_2(s) + \ldots + \sigma_r E(s) \]  
(5.17)

Where \( D_0 \) is a constant matrix and \( E_i(s), i = 1, 2, \ldots, r \), are stable, norm 1, all-pass transfer function matrices. \( E_i(s) \) are the differences at each approximation and the reduced order models for \( k = 1, \ldots, r-1 \).

\[ \hat{G}_k(s) = D_0 + \sum_{i=1}^k \sigma_i E_i(s) \]  
(5.18)

Such a \( \hat{G}_k(s) \) is stable with the order, \( m_1 + \ldots + m_k \), and satisfies

\[ \left\| G(s) - \hat{G}_k(s) \right\|_\infty \leq (\sigma_{k+1} + \ldots + \sigma_r) \]  
(5.19)

However, \( \hat{G}_k(s) \) is not an optimal Hankel approximation, for \( k \prec r-1 \). The method to obtain an optimal Hankel approximation with general order is given in the next algorithm.
5.4.4.3 Approximation Algorithm 3

Let the Hankel singular values of $G(s)$ be.

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k \geq \sigma_{k+1} = \ldots = \sigma_{k+l} \geq \ldots \geq \sigma_n$$  \hspace{1cm} (5.20)

Apply appropriate state similarity transformations to make the Gramians of $G(s)$ be arranged as follows.

$$\sum = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k, \sigma_{k+1}, \ldots, \sigma_n, \sigma_{k+1}, \ldots, \sigma_{k+l})$$  \hspace{1cm} (5.21)

Define the last $l$ Hankel singular values to be $\sigma$. Following the formulae (5.11) to (5.16), define an $(n-1)$th-order Hankel optimal approximation of $G(s)$. This $\tilde{G}(s)$ is not stable but has exactly $k$ stable poles.

The $k$th-order stable part $G_{h,k}(s)$ of $\tilde{G}(s)$, obtained by using model decompositions, is an $k$th-order Hankel optimal approximation of $G(s)$ and satisfies

$$\|G(s) - G_{h,k}(s)\|_H = \sigma$$ \hspace{1cm} (5.22)

5.5 Model Reduction of Unstable Systems

Model reduction of unstable systems can be performed by two methods. The first method consists of finding only the reduced-order model of the stable part and then including the unstable part in the resulting reduced model. The second method is based on computing a stable rational coprime factorization of the transfer function matrix and then reducing the stable system.

The model order reduction of unstable systems can be summarized as follows [128]:

- Step 1: decompose the transfer function matrix $G(\lambda)$ additively as:

  $$G(\lambda) = G_s(\lambda) + G_u(\lambda)$$ \hspace{1cm} (5.23)

Where: $G_s(\lambda)$ is stable part and $G_u(\lambda)$ is unstable part of $G(\lambda)$.

- Step 2: find a reduced-order model $G_{Rs}(\lambda)$ of the stable part $G_s(\lambda)$

- Step 3: compute the final reduced order model by $G_{Rs}(\lambda) = G_{Rs}(\lambda) + G_{ru}(\lambda)$.

Figure 5.2 shows the flow chart of model order reduction of unstable system [129].
Start

Input (A, B, C, D) of G(s)

minimal realization of A, B, C and D

Controllability and observability of A, B, C and D

G(s) = Gs(s) + Gu(s)

Check the system stability

G(s) is Gs(s)

Apply suitable method to Reduce the stable part Gs, to find Grs(s)

Find the final reduced order model
Gr = Gs (reduced of stable part) + Gu (unstable part)
Gr = Gs + Gu

Reduced order model Ar, Br, Cr and Dr

Check the error with frequency response

Yes

No

Stable part, Gs

Figure 5.2: Flowchart of model order reduction for unstable systems
5.6 ETRR-2 Model Reduction Applications

5.6.1 ETRR-2 Fifth Order Model Reduction by Schur, Hankel and BT Methods

We have noticed that the majority of the publication relative to the analytical solution of point kinetic equation used the fifth order model or at least third order model to make it easier in mathematical study. The reactor nonlinear model equations are grouped from equation (5.24) to equation (5.28) and linearized into state space model with system matrices $A_5$, $B_5$ and $C_5$. The linearized fifth order reactor model parameters $A_5$, $B_5$, $C_5$ and $D_5$ consists of one point kinetic, fuel and coolant temperature and control rod equations parameters [130]. The reactor linearized fifth order model will be reduced to a third order model by Schur reduction, balanced truncation and Hankel norm approximation methods as follows:

$$\frac{dn}{dt} = [(\rho - \beta) / \Lambda]n + \sum_{i=1}^{6} \lambda_i c_i$$  \hspace{1cm} (5.24)

$$\frac{dc_i}{dt} = (\beta_i / \Lambda) n - \lambda_i c_i$$  \hspace{1cm} (5.25)

$$m_j c_j (dT_j / dt) = p_{th} - hA(T_f - T_c)$$  \hspace{1cm} (5.26)

$$m_c c_c (dT_c / dt) = -2M_c (T_c - T_f) + hA(T_f - T_c)$$  \hspace{1cm} (5.27)

$$\frac{d\rho_r}{dt} = r z_r$$  \hspace{1cm} (5.28)

The linearized state-space model is described by the following equations [131]:

$$\dot{x} = Ax + Bu$$  \hspace{1cm} (5.29)

$$y = Cx + Du$$  \hspace{1cm} (5.30)

Where A is the system states matrix, B is the control matrix, C is the output matrix and D is zero because the system is strictly proper.
\[ A_5 = \begin{bmatrix}
-\beta & \beta & \alpha_f n_r & \alpha_c n_r & n_r \\
\Lambda & \Lambda & \Lambda & \Lambda & \Lambda \\
\lambda & -\lambda & 0 & 0 & 0 \\
1 & 0 & -hA & hA & 0 \\
mf & 0 & mf & mf & 0 \\
0 & 0 & hA & -hA - 2MC & 0 \\
-G_c \beta / 100 & 0 & 0 & mcC & 0 \\
\end{bmatrix} \]

\[ B_5 = [0 \ 0 \ 0 \ 0 \ G_c / 100]^T, \quad C_5 = [1 \ 0 \ 0 \ 0 \ 0] \] and the state vector is defined as follows:
\[ x = (\delta n_r, \sigma_c r, \sigma_f T, \sigma T_c, \sigma n_r) \]

Where: \( n_r = n/n_0 \) “the neutron density relative to equilibrium density”.

The first step is to calculate the transfer function of all models to present if there is a pole zero cancelation in the reduction process through the three reduction method and to compare the poles locations and determine what is the eliminated ones. The fifth order model and the reduced third order models by Schur reduction method, balanced truncation method and Hankel norm approximation method have the following transfer functions respectively \( G_s, G_b, G_b, G_h \).

\[ G_s(s) = \frac{1000 \ (s + 55.55) \ (s + 10.85) \ (s + 0.0127)}{(s + 99.12) \ (s + 59.09) \ (s + 12.45) \ (s + 2.881) \ (s + 0.01275)} \] (5.31)

\[ G_b(s) = \frac{-0.35716 \ (s - 2.843e4) \ (s + 11.87)}{(s + 105.5) \ (s + 13.78) \ (s + 2.89)} \] (5.32)

\[ G_b(s) = \frac{-0.35716 \ (s - 2.843e4) \ (s + 11.87)}{(s + 105.5) \ (s + 13.78) \ (s + 2.89)} \] (5.33)

\[ G_h(s) = \frac{22.6473 \ (s + 306.5) \ (s + 20.66)}{(s + 59.81) \ (s + 28.5) \ (s + 2.934)} \] (5.34)

The second step is applying the reduction algorithm and calculates the modeling error. The error equals the difference between the infinity norm of full order model and reduced model for each case respectively. It is controlled by the relation between the approximation error and modeling
error bounds as follows: \[ \|G - G_r\|_\infty = 2 \sum_{i=r+1}^{n} \sigma_i \] \( \sigma \) are the Hankel singular values, \( G \) and \( G_r \) describes the full order and reduced models respectively.

Figure 5.3 illustrates the Hankel singular values of the full order (fifth order) model, which equal [14.4720, 0.3490, 0.2143, 0.0551, 0.0035]. Clearly a large number of the Hankel singular values are close to zero and we should be able to eliminate or truncate them from the model without introducing pronounced modeling errors.

Figure 5.4 presents the bode diagram of full order, which is denoted by M1 and reduced models by Schur, balanced truncation and Hankel norm approximation, which are denoted by M2, M3 and M4 respectively.

It is noted that the third order model reduced by Schur and balanced truncation comes closest to the full order model with lower error than the reduced order model of Hankel norm approximation.

Figure 5.5 shows the singular values plot of the full order model relative to the reduced third order model reduced by Schur, balanced truncation and Hankel norm approximation over a wide range of frequencies. The reduced order model exhibited good match with the full order model up to frequency of about \( 10^3 \) in case of Schur and balanced truncation methods and \( 10^2 \) rad/sec in case of Hankel norm approximation method.

The infinity error bound between the full order model M1 and reduced models by Schur and balanced truncation methods M2 and M3 becomes equal 0.1428 and the error bound between M1 and the reduced model by Hankel norm approximation method M4 becomes equal 0.1498.

Table 5.1 gives the modeling error values of reduced models relative to the full order model, which shows that the Schur and BT methods give the lower bounded error. The four models come in table 5.1 are defined as follows:

- M1: full order model (fifth order)
- M2: third order model reduced by Schur reduction method
- M3: third order model reduced by balanced truncation method
- M4: third order model reduced by Hankel norm approximation
Hankel Singular Values (State Contributions)

![Hankel Singular Values Graph]

**Figure 5.3:** Hankel singular values of the fifth order model

![Bode Diagram Graph]

**Figure 5.4:** Bode diagram of the original and reduced models by Schur, BT and Hankel reduction methods
**Figure 5.5:** Singular values of the fifth order and the reduced third order models by Schur, BT and Hankel approximation methods

**Table 5.1:** Modeling error of full order and reduced order models

<table>
<thead>
<tr>
<th>Model reduction method</th>
<th>Models error</th>
<th>$|G - G_r|_{\infty}$</th>
<th>$|G - G_r|_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full order (M1)</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Schur reduction (M2)</td>
<td>M1-M2</td>
<td>0.1428</td>
<td>0.0377</td>
</tr>
<tr>
<td>Balanced truncation (M3)</td>
<td>M1-M3</td>
<td>0.1428</td>
<td>0.0377</td>
</tr>
<tr>
<td>Hankel norm approximation (M4)</td>
<td>M1-M4</td>
<td>0.1498</td>
<td>1.4895</td>
</tr>
</tbody>
</table>
5.6.2 ETRR-2 Tenth Order Model Reduction

The second case study is the reduction of the ETRR-2 tenth order model, which described by equations (5.23) to (5.27) with the six group point kinetic equations, fuel and temperature feedbacks and control rod feedback. As we illustrated before the nonlinear full order model has to be linearized and reduced into the third order model.

The linearized tenth order model is defined by state space linearized system parameters $A_{10}$, $B_{10}$ and $C_{10}$, which reduced into third order model using the Schur reduction method, Hankel norm approximation method, balanced truncation method and normalized coprime factors method. We use the coprime factors method as a frequency domain approximation method that also suitable for unstable model reduction to be sure that the system has no restriction for reduction process.

The corresponding full order model matrices are given by:

$$A_{10} = \begin{bmatrix} -\beta & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \alpha_jn_r & \alpha_in_r & n_r \\ \Lambda & \Lambda & \Lambda & \Lambda & \Lambda & \Lambda & \Lambda & \Lambda & \Lambda & \Lambda \\ \lambda_1 & -\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_2 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_3 & 0 & 0 & -\lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_4 & 0 & 0 & 0 & -\lambda_4 & 0 & 0 & 0 & 0 & 0 \\ \lambda_5 & 0 & 0 & 0 & 0 & -\lambda_5 & 0 & 0 & 0 & 0 \\ \lambda_6 & 0 & 0 & 0 & 0 & 0 & -\lambda_6 & 0 & 0 & 0 \\ P_0 & 0 & 0 & 0 & 0 & 0 & 0 & -hA & hA & 0 \\ m_f c_f & 0 & 0 & 0 & 0 & 0 & 0 & m_f c_f & m_f c_f & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & hA & -hA - 2M_e C_e & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_e C_c & m_e C_c & 0 \\ -Gc\beta / 100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_e / 100 \end{bmatrix}^T$$

$$C_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The ETRR-2 10th order model state vector is defined as follows:

$$X = (\delta n_r, \sigma c_{r1}, \sigma c_{r2}, \sigma c_{r3}, \sigma c_{r4}, \sigma c_{r5}, \sigma c_{r6}, \sigma T_f, \sigma T_e, \sigma \rho_r)$$

(5.35)

Where: $c_r = c / c_0$, precursor density relative to initial equilibrium precursor density.
Figure 5.6 shows the computed Hankel singular values of the tenth order model, it equals \[15.3871 0.6469 0.5308 0.1830 0.0155 0.0038 0.0029 0.0013 0.0001\]. Clearly a large number of the Hankel singular values are close to zero and we can truncate them from the model without introducing pronounced modeling errors.

Figure 5.7 shows the bode diagram of the full order model, and reduced models, which are denoted by model1 (full order), model2 (reduced model by Schur reduction), model3 (reduced model by Hankel norm approximation), model4 (reduced model by balanced truncation) and model5 (reduced model by normalized coprime factors) respectively.

The dc gain of all models is nearly the same and equal to that of the full order model. The reduced order models exhibited good match with the full order model up to frequency of about \(10^4\) rad/sec except for the Hankel norm approximation method (model3) for which the resulting model deviates from the original model at \(10^3\) rad/sec. The error between full order model and reduced models are calculated by the infinity error bound and listed in table 5.2.

Figure 5.8 illustrates the singular values plot over a wide range of frequencies for the full order model compared with the reduced third order model by each reduction method.

\[\text{Figure 5.6: Hankel singular values of the tenth order model}\]
Figure 5.7: Bode diagram of full order and reduced order models

Figure 5.8: Singular values of full order and reduced order models
### Table 5.2: Modeling error of tenth order model reduction model

<table>
<thead>
<tr>
<th>Model reduction method</th>
<th>Models error</th>
<th>$|G - G_r|_\infty$</th>
<th>$|G - G_r|_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full order model (M1)</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Schur reduction method (M2)</td>
<td>M1-M2</td>
<td>0.3178</td>
<td>0.9610</td>
</tr>
<tr>
<td>Hankel norm approximation (M3)</td>
<td>M1-M3</td>
<td>0.3656</td>
<td>4.7957</td>
</tr>
<tr>
<td>Balanced truncation (M4)</td>
<td>M1-M4</td>
<td>0.3178</td>
<td>0.9610</td>
</tr>
<tr>
<td>Normalized coprime factors (M5)</td>
<td>M1-M5</td>
<td>1.0801</td>
<td>0.7218</td>
</tr>
</tbody>
</table>

The five models come in table 5.2 are defined as follows:

- M1: full order model (fifth order)
- M2: third order model reduced by Schur reduction method
- M3: third order model reduced by Hankel norm approximation
- M4: third order model reduced by balanced truncation method
- M5: third order model reduced by coprime factors

The value of the infinite norm identified by $\|G - G_r\|_\infty$, equal the error between the full order model $G$ and reduced model $G_r$ for each method.

### 5.6.3 Case Study: ETRR-2 Twelfth Order Model Reduction to Design LMI and $H_\infty$ Control Systems

The third case study is the reduction of the twelfth order model, which described by equations (5.24) to (5.28) with respect to the poisons equations (3.11) and (3.12). The twelfth order model, which defined by parameters $A_{12}$, $B_{12}$ and $C_{12}$, is reduced to ninth, sixth and third order models respectively using the Schur reduction method.

In a reactor of given volume in which fission is caused by neutrons of specified energy, the thermal power is proportional to the neutron flux and macroscopic fission cross-section. As the reactor operates, the macroscopic cross section decreases as number of fissile nuclides decreases. However, over an essentially short period of time, the cross section remains constant, and the power is assumed to change only with neutron flux. In most situations varying the neutron flux...
controls a reactor. Among the general methods available, the insertion and withdrawal of a neutron absorber is most commonly used in power reactors [132-133].

As discussed in chapter 3 and 5, the reactor model equations are integrated in the following full order state space model to be used in LMI and robust control systems applications.

\[ x = Ax + Bu \]  
\[ y = CX \]

(5.36)  
(5.37)

Where \( A, B \) and \( C \) are the linearized model parameters and \( D=0 \) because the system is strictly proper and the states of the twelfth order model can be represented as follows:

\[ X = (\delta n_r, \sigma c_{r1}, \sigma c_{r2}, \sigma c_{r3}, \sigma c_{r4}, \sigma c_{r5}, \sigma c_{r6}, \sigma T_f, \sigma T_c, \sigma I, \sigma x, \sigma \rho_r) \]

(5.38)

\[
A_{12} = \begin{bmatrix}
-\beta & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \alpha f n_r & \alpha c n_r & \alpha a n_r & \alpha p n_r & n_r \\
\lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
\lambda_i & -\lambda_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda_2 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda_3 & 0 & 0 & -\lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda_4 & 0 & 0 & 0 & -\lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda_5 & 0 & 0 & 0 & 0 & -\lambda_5 & 0 & 0 & 0 & 0 & 0 \\
\lambda_6 & 0 & 0 & 0 & 0 & 0 & -\lambda_6 & 0 & 0 & 0 & 0 \\
\frac{\rho_o}{m_c f} & 0 & 0 & 0 & 0 & 0 & 0 & -hA & hA & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -hA & -2*Mc & 0 & 0 \\
\Gamma_i \sum_j y & 0 & 0 & 0 & 0 & 0 & 0 & m_c f & m_c f & \lambda_i & 0 \\
\Gamma_i \sum_j (v-\sigma ax_r^v) y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_i & \lambda_i + \sigma_x \Phi_o & 0 \\
-G_c \beta /100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
B_{12} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(5.39)

\[
C_{12} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(5.40)

Schur reduction method can be used to reduce the 12th order model into ninth, sixth and third order models respectively in order to simplify controller design.

Figure 5.9 shows the computed Hankel singular values of full order model, it equals [15.4899 0.8192 0.3898 0.1453 0.1169 0.0212 0.0118 0.0117 0.0034 0.0028 0.0016 0.0002].
Clearly a large number of the Hankel singular values are close to zero and can be truncated without introducing pronounced modeling errors.

Figure 5.10 shows the Bode diagram of the full order and reduced order models. The difference in phase shift between full order and reduced models is noticed; the amplitude ratio is only slightly different for high frequencies. It can be seen that the full order model and the reduced models have the same frequency response for low to moderate frequencies, for moderate to high frequencies the two frequency response curves start to deviate depending on the reduction degree.

Figure 5.11 illustrates the singular values plot over a wide range of frequencies for the original state space model compared with the reduced order state space models. The resulting reduced order models exhibited good match with the full order model up to frequency of about $10^4$ rad/sec. According to the above results, it is clear that the third order model can be used to design a low order controller as shown in the following sections.

![Hankel Singular Values](image)

**Figure 5.9:** Hankel singular values of the twelfth order model
**Figure 5.10**: Bode diagram of full order and reduced order model by Schur method

**Figure 5.11**: Singular values of full order and reduced order model by Schur method
Table 5.3 indicates the maximum modeling error between full order model and reduced models respectively. The error bound is computed based on Hankel singular values $\sigma_i$ of the full order model transfer function $G(s)$. The table summarizes the error between M1, M2, M3 and M4 respectively. The four models come in table 5.3 are defined as follows: M1 refers to the full order model, M2 refers to the ninth order model with the first step of full order model reduction by Schur method, M3 refers to the sixth order model as a second step of full order model reduction, and M4 refers to the third order model as the last step of full order model reduction.

<table>
<thead>
<tr>
<th>Model reduction method</th>
<th>Models</th>
<th>$|G - G_r|_\infty$</th>
<th>$2 \sum_{i=r+1}^n \sigma_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>full order model (M1)</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>ninth order model (M2)</td>
<td>M1-M2</td>
<td>0.0027</td>
<td>34.7228</td>
</tr>
<tr>
<td>sixth order model (M3)</td>
<td>M1-M3</td>
<td>0.0076</td>
<td>34.7140</td>
</tr>
<tr>
<td>third order model (M4)</td>
<td>M1-M4</td>
<td>0.3758</td>
<td>34.4822</td>
</tr>
</tbody>
</table>

The error associated with the third order model is acceptable according to the reduction error calculation condition as we illustrated in table 5.3, it is less than the twice sum of the tail of the Hankel singular values of the system. The third order model will be used to design a low order robust control system to control the ETRR-2 power as shown in the next chapter.

5.7 Concluding Remarks

This chapter introduces the model order reduction methodologies and their applications to the linearized model of the ETRR-2. Several methods are presented such as balanced realization, which is not suitable to be used in our application due to the reduction order limitation. On the other hand the model reduction methodologies such as balanced truncation method, Schur reduction method and Hankel norm approximation method are widely used in model and control reduction applications. These methods are successfully applied to reduce the ETRR-2 models into a third order model as follows:
- The ETRR-2 fifth order model reduced to third order model by using Schur reduction method, balanced truncation method and Hankel approximation method.

- The tenth order model reduced to third order model by using Schur reduction method, balanced truncation method, Hankel approximation method and coprime factors. The bounded error between the full order model and reduced models is calculated, which indicate that the third order is acceptable and the Schur method is more applicable and give the minimum error relative to the other method except the balanced reduction method.

- The twelfth order model finally reduced to ninth, sixth and third order models respectively by using Schur reduction method. The reduced third order model will be used to design a low order robust control system to control the ETRR-2 power as shown in the next chapter.
Chapter 6

LMI and Robust Control Systems Applications to Nuclear Reactor Power Control

6.1 Introduction

The power control system is a key control system for a nuclear reactor, which directly concerns the safe operation and safe shutdown of a nuclear reactor. Much effort is paid to improve the power control system performance. Generally, the power control system should operate safely and reliably, should maintain maximum power output of nuclear reactor with less static error and should possess certain stability margin, suitable overshoot and transient time [134].

The reactor power level can be changed manually or automatically by using the control rods according to the situation of the safety systems and operation condition. In the manual control mode, positions of the control rods are adjusted by an operator. In the automatic control mode only the regulating rod position is arranged by an automatic controller with electro-mechanical subsystems. The regulating rod or the compensating rod only is handled to automatically control the power level of the reactor.

The reactivity is inserted into the reactor core by movement of the regulating or compensating rod, so the power changes are provided by this way. The production of neutrons rises with withdrawing the control rod and decreases with inserting the control rod into the reactor core. Thus, control of the reactor power level is achieved.

The regulating or compensating rod position is adjusted by a stepper motor through a complete control channel as illustrated in chapter three. The available PD control system is used to control the control rod (regulating rod) motion to compensate the reactor power and adjust the flux shape inside the reactor core.

To improve the reactor control system performance, a robust control and LMI state feedback control systems are proposed to control the reactor model to guarantee the system stability and robust performance. The selected controllers are in fact low order controllers; they are designed based on the third order model has been investigated before. The designed low order controller
will be used to control the full order nonlinear model and compared with the PD control system Model [135].

### 6.2 Control Systems Design Specifications

In designing control systems using algebraic techniques, we usually apply linear state feedback or output feedback control systems as well as design of reactor power control system according to the given data, operation situations and users requirements. Feedback control techniques may reduce the influence generated by uncertainties and achieve desirable performance. However, an inadequate feedback controller may lead to an unstable closed loop system though the original open-loop system is stable. In this chapter, we used another control systems based on the reduction of the ETRR-2 twelfth order model by Schur reduction method. The designed control systems are robust and essential controllers, which guarantee the system robust performance and stability.

### 6.3 LMI Control System Application to ETRR-2

Referring to the full order nonlinear model and the linearized state space model given in equation (5.23) to (5.30) and its transfer function, $G(s)$, we can compute a state feedback controller $K$ such that the closed loop system is stable $\dot{x} = Ax + Bu$, $y = Cx + Du$ and $u = -Kx$

For that linear system to be stable there is a Lyapunov function

$$V(x) = x^T px$$

Where, $p$ is symmetric positive definite matrix such that $\dot{V}(x)$ is negative $\forall x \neq 0$

$$\dot{V}(x) = x^T px + x^T p\dot{x}, \forall x \neq 0$$

So

$$(A - BK)^T p + p(A - BK) \prec 0, p > 0$$

The system designer has to find both $p$ and $K$. Hence (6.3) is bilinear. Using congruence transformation (by multiplying both sides by $p^{-1}$) and putting $W = p^{-1}$ then we get the following inequality [90]:

$$W(A - BK)^T + (A - BK)W \prec 0, x > 0$$

Which $L = KW$, we derive the LMI as follows:

$$WA^T + AW - L^T B^T - BL \prec 0, W > 0$$

and $K = LW^{-1}$. (6.5)
Equation (6.5) is used to design full order and low order LMI state feedback control system based on the different models of ETRR-2 such as twelfth order model and the reduced ninth order, sixth order and third order models respectively that are derived from the 12th order linearized model by Schur reduction method in chapter 5.

The designed controllers have a number of parameters equal to the model orders as illustrated before.

Figure 6.1 illustrates the normalized power of the full order nonlinear reactor model governed by the following designed LMI state feedback controllers for a power set point change by 50%.

- Full order LMI control system (LMI12) controls the full order nonlinear reactor model
- Ninth order LMI control system (LMI9) controls the full order nonlinear model
- Sixth order LMI control system (LMI6) controls the full order nonlinear model
- Third order LMI control system (LMI3) controls the full order nonlinear model

Figure 6.1 shows that all controllers are able to control the nonlinear full order model with good performance.

**Figure 6.1:** Normalized power of non linear full order model relative to full order and reduced order LMI control system when power step change from 100% to 50%
Figure 6.2: Normalized power of LMI controllers when step change by 250pcm inserted to the core

Figure 6.2 illustrates the normalized power of the full order nonlinear model response to a disturbance of 250pcm positive reactivity inserted into the reactor core at steady state.

It illustrates that the controllers’ responses is oscillatory and it has a steady state error for some controllers. The oscillation that appears is due to the fact that the LMI based controllers are satisfying only sufficient conditions for stability without any specific requirements on the error norm or the response damping.

This oscillation comes from the effect of the system nonlinearity. To improve the system response, the following sections introduce some modifications in control system design to improve the model response.
Figure 6.3 compares the response of the low order LMI control system and the PD controller due to a disturbance in the reactor power by 50% step down. It shows that the low order LMI has a better performance than the performance of the PD controller. The low order LMI control system illustrates a faster settling time with smaller down shoot than the PD controller. The PD control system has certain steady state error in the normalized output power response and more down shoot. Now, it is clear that, the LMI shows a relatively better response compared with PD controller.

Figure 6.4 shows the response of the low order LMI control system and the PD controller zooming in 50%, which illustrates the oscillation of the model response around the set point. It shows that the PD control system has certain steady state error in the normalized output power response and more down shot.

Figure 6.3: Normalized power of non linear full order model relative to third order LMI state feedback control system when power step change to 50%
Figure 6.4: Normalized power of non linear full order model relative to third order LMI control system when power step change to 50% (zoom in)

Figure 6.5 illustrates the difference in the response of the low order LMI control system and the PD controller due to a disturbance in the reactor power by 100pcm as a positive reactivity insertion into the reactor core at steady state condition. It shows that the low order LMI controller has a better performance than the performance of the PD controller. The LMI control system gives a faster settling time with minimum over shoot, while the PD controller has big over shoot and certain steady state error in the normalized output power response. Now, it is clear that, the low order LMI controller shows a relatively better response compared with PD control system.

Figure 6.6 illustrates the difference in the response of the low order LMI control system and the PD controller due to a disturbance in the reactor power by -100pcm as a fast negative reactivity insertion into the reactor core at steady state condition. It illustrates that the low order LMI controller has a better performance than the performance of the PD controller.
The LMI control system gives a faster settling time with minimum down shoot, while the PD controller has big down shoot and certain steady state error in the normalized output power response. Now, it is clear that, the low order LMI controller shows a relatively better response compared with PD control system.

Figure 6.7 compares the response of the low order LMI control system and the PD controller due to a disturbance in the reactor power by 250pcm as a positive reactivity insertion into the reactor core at normal operation condition. It presents that the low order LMI controller has a better performance than the performance of the PD controller. The LMI control system gives a faster settling time with minimum over shoot, while the PD controller has unaccepted over shoot and certain steady state error in the normalized output power response. Now, it is clear that, the low order LMI controller shows a relatively better response compared with PD control system.

Figure 6.8 shows the difference in the response of the low order LMI control system and the PD controller due to a disturbance in the reactor power by -250pcm as a a large and negative reactivity insertion into the reactor core at steady state condition. It illustrates that the low order LMI controller has a better performance than the performance of the PD controller. The LMI control system gives a faster settling time with minimum down shoot, while the PD controller has big down shoot and certain steady state error in the normalized output power response. Now, it is clear that, the low order LMI controller shows a relatively better response compared with PD control system.
Figure 6.5: Normalized power of LMI and PD controllers when step change 100pcm inserted to the core

Figure 6.6: Normalized power of LMI and PD controllers when step change -100pcm inserted to the core
Figure 6.7: Normalized power of LMI and PD controllers when step change 250pcm inserted to the core

Figure 6.8: Normalized power of LMI and PD controllers and when step change -250pcm inserted to the core
6.4 LMI Pole Placement Control System Application to ETRR-2

Referring to section 4.7 and subsections 4.7.1 and 4.7.2, we assume that the $\alpha = -1$, the half plane with real part of $x < \alpha$, circle: $|x| < r$ with sector of $\text{Re}(x) \tan \theta < -|\text{Im}(x)|$ in the LMI region, which used to design a control system to control the ETRR-2 power in different operation conditions.

The LMI region is determined by $(\alpha, r, \theta)$, where $(\alpha = -1, r \simeq 2.0242, \theta \simeq 55.89)$ respectively.

Figure 6.9 compares the response of the low order LMI pole placement control system and the PD controller due to a disturbance in the reactor power such as step changed to 50% of the full power.

It shows that the low order LMI pole placement has a better performance than the performance of the PD controller.

The LMI pole placement control system shows a faster settling time with no over shoot but the PD controller has downshoot and small steady state error in the normalized output power response. Now, it is clear that, the LMI pole placement shows a relatively better response compared with PD controller.

Figure 6.10 illustrates the difference in the response of the low order LMI pole placement control system and the PD controller due to a disturbance in the reactor power by 100pcm as a positive reactivity insertion to the reactor core. It shows that the low order LMI pole placement has a better performance than the performance of the PD controller.

The LMI pole placement control system shows a faster settling time with minimum over shoot but the PD controller has certain steady state error in the normalized output power response. It is clear that, the LMI pole placement shows a relatively better response compared with PD control system.

Figure 6.11 shows the change in the control rod worth relative to the reactor core due to the step change in the reactor power by 100pcm as positive reactivity insertion into the reactor core at steady state operation condition.

It illustrates that the variation in the reactor power due to any disturbance will change the control rod position to compensate the error in the reactor criticality.

The LMI pole placement controller presents small change in the control rod position relative to the reactor power change compared with the PD control system.
**Figure 6.9:** Normalized power of full order nonlinear model with third order LMI pole placement relative to the PD control system when the power step change to 50%

**Figure 6.10:** Normalized power of full order nonlinear model with third order LMI pole placement relative to the PD control system when the power step change with 100pcm
6.5 $H_\infty$ Robust Control System Application to ETRR-2

Referring to state space model equations (5.29) and (5.30) with that strictly proper LTI system model, the Bounded real lemma can be used to compute $H_\infty$ controller gain such that the infinity norm of the closed loop system is minimized.

By using bounded real lemma, we can design a robust state feedback control system as follows:

\[
(A - BK)^T p + p(A - BK) + \gamma^{-2} pBB^T p + C^T C < 0
\]  
(6.6)

Multiply equation from left and right by $x = p^{-1}$,

\[
x(A - BK)^T + (A - BK)x + \gamma^{-2} BB^T + xC^T Cx < 0
\]  
(6.7)

Denoting \( y = Kx \)

\[
xA^T + Ax - y^T B^T - By + \gamma^{-2} BB^T + xC^T Cx < 0
\]  
(6.8)

This inequality can be converted into an LMI by using Schur lemma as follows:

Multiplying the inequality by $\gamma$ and then we take $X = \gamma x$ and $Y = \gamma y$ then we can rewrite the inequality in the following form [136-137]:
\[ XA^T + AX - Y^T B^T - BY + \gamma^{-1}BB^T + \gamma^{-1}XCCX < 0 \]  \hfill (6.9)

By applying the Schur lemma and we obtain
\[
\begin{bmatrix}
XA^T + AX - Y^T B^T - BY & X \gamma T & B \\
CX & -\gamma I & 0 \\
B^T & 0 & -\gamma I
\end{bmatrix} < 0, \quad X > 0
\]  \hfill (6.10)

Minimizing \( \gamma \) subject to the LMI constraints to compute the controller \( K = YX^{-1} \)

The robust control system gain can be used to compute the control system to control the full order nonlinear model as follows:

Figure 6.12 compares the response of the low order robust \( H_\infty \) control system and the PD controller due to a disturbance in the reactor power as a step changed to 50\% of the full power. It shows that the low order robust \( H_\infty \) has a good performance than the performance of the PD controller. The \( H_\infty \) control system shows a faster settling time with down shoot but the PD controller has also down shoot. Now, it is clear that, the \( H_\infty \) shows a relatively better response compared with PD controller.

**Figure 6.12**: Model response of PD and \( H_\infty \) controllers when power step changed from 100\% to 50\%
Figure 6.13 compares the response of the low order $H_\infty$ control system and the PD controller due to a disturbance in the reactor power a step change the reactor criticality by 250pcm and the relative disturbance in the reactor power. It shows that the low order $H_\infty$ has a better performance than the performance of the PD controller.

The low order $H_\infty$ control system shows a faster settling time with smaller over shoot than the PD controller. The PD control system has certain steady state error in the normalized output power response. It is clear that, the $H_\infty$ shows a relatively better response compared with PD controller.

Figure 6.14 shows the change in the control rod worth relative to the reactor core due to the step change in the reactor power by 250pcm as positive reactivity insertion into the reactor core at steady state operation condition.

It illustrates that the variation in the reactor power due to any disturbance will change the control rod position to compensate the error in the reactor criticality.

The $H_\infty$ controller presents small change in the control rod position relative to the reactor power change compared with the PD control system.

Figure 6.15 compares the response of the low order $H_\infty$ control system and the PD controller due to a disturbance in the reactor power a step change the reactor criticality by -250pcm and the relative disturbance in the reactor power.

It shows that the low order $H_\infty$ has a better performance than the performance of the PD controller. The low order $H_\infty$ control system shows a faster settling time with smaller over shoot and down shoot than the PD controller. The PD control system has certain steady state error in the normalized output power response. Now, it is clear that, the $H_\infty$ shows a relatively better response compared with PD controller.

Figure 6.16 illustrates the change in the control rod worth relative to the reactor core due to the step change in the reactor power by -250pcm as negative reactivity insertion into the reactor core at steady state operation condition.

It illustrates that the variation in the reactor power due to any disturbance will change the control rod position to compensate the error in the reactor criticality.

The $H_\infty$ controller presents small change in the control rod position relative to the reactor power change compared with the PD control system.
**Figure 6.13:** Model response of PD and H infinity controllers at steady state when 250pcm positive inserted to the core

**Figure 6.14:** Control rod position changed relative to power step changed by 250pcm
Figure 6.15: Model response of PD and H-infinity when the reactor power step changed by -250pcm

Figure 6.16: Control rod worth changed when the reactor power step changed by -250pcm
6.6 Observer based on LMI Applications to ETRR-2

Referring to equations 4.54, 4.55, 4.63, 4.64 and 4.65 with a state feedback controller \( u = -kx \), we now can consider the design of an observer \( L \), which solves the problem of the closed loop system 5.29 and 5.30 for a given controller \( k \). The observer design is requested here because we need to observe the third order model that characterized the full order model.

If we consider that
\[
P\succ 0
\]
\[
A^\top P + PA < 0, \tag{6.11}
\]
\[
A = \begin{bmatrix}
A - BK & BK \\
0 & A - LC
\end{bmatrix} \tag{6.12}
\]

With \( P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \) we can separate the bilinear terms of equation 6.12 and obtain
\[
\begin{bmatrix}
A - BK & BK \\
0 & A
\end{bmatrix}^\top P + P \begin{bmatrix}
A - BK & BK \\
0 & A
\end{bmatrix} - \begin{bmatrix}
0 & P_{12} LC \\
C^T L^T P_{12} & C^T L^T P_{22} LC
\end{bmatrix} < 0 \tag{6.13}
\]

This equation can’t be written as an LMI because of the product of \( P \) s and \( L \) at this moment we solve this problem in two steps

- Step 1 we calculate the \( H_\infty \) controller gain \( K \) as we illustrated before
- Step 2 we calculate the observer gain \( L \) in another LMI of \( (A - LC) \) [138].

6.7 ETRR-2 Observer Based Control Model Analysis

Referring to the solution of observer based controller design problems, the \( H_\infty \) controller and observer gain values are as follows:
\[
k_{H_\infty} = \begin{bmatrix} -2.2717 & 9.0338 & -1.6974 \end{bmatrix}, l_{H_\infty} = \begin{bmatrix} 25.7 & 7 & 14.04 \end{bmatrix}
\]

and the LMI controller and observer gain are as follows:
\[
k_{LMI} = \begin{bmatrix} -111.8 & -506.86 & -112.88 \end{bmatrix}, l_{LMI} = \begin{bmatrix} -5.9975 & -1.5536 & -3.4998 \end{bmatrix}
\]
Figure 6.17 compares the response of the low order $H_\infty$ observer based control system and the PD controller when the reactor power disturbed by a step changed to 50% of the full power.

It shows that the low order robust $H_\infty$ observer base has better performance than the performance of the PD controller. The low order $H_\infty$ observer based control system illustrates a faster settling time with minimum down shoot relative to the PD controller. Now, it is clear that, the $H_\infty$ observer based presents a relatively better response compared with PD controller.

Figure 6.18 illustrates the response of the low order $H_\infty$ observer based control system and the PD controller when the reactor power disturbed by a step changed to 75% of the full power and stepped back to 100% as two step change.

It shows that the low order robust $H_\infty$ observer base has better performance than the performance of the PD controller.

The low order $H_\infty$ observer based control system illustrates a faster settling time with minimum down shoot relative to the PD controller.

Now, it is clear that, the $H_\infty$ observer based presents a relatively better response compared with PD controller.

Figure 6.19 compares the response of the low order $H_\infty$ based observer control system and the PD controller due to a disturbance in the reactor power a step change the reactor criticality by 200pcm and the relative disturbance in the reactor normalized power.

It shows that the low order $H_\infty$ based observer has a better performance than the performance of the PD controller. The low order $H_\infty$ based observer control system shows a faster settling time with smaller over shoot than the PD controller. The PD control system has certain steady state error in the normalized output power response. Now, it is clear that, the $H_\infty$ based observer shows a relatively better response compared with PD controller.
Figure 6.17: PD and $H_\infty$ observer based controller response when reactor power stepped from 100% to 50%.

Figure 6.18: PD and $H_\infty$ observer based response when reactor power stepped from 100% to 75% and stepped back to 100%.
Figure 6.19: PD and $H_\infty$ observer based response when reactor power stepped by 200pcm

Figure 6.20: Control rod worth change relative to PD and $H_\infty$ observer based controllers when reactor power stepped by 200pcm
Figure 6.20 indicates the change in the control rod worth relative to the reactor core due to the step change in the reactor power by 200pcm as positive reactivity insertion into the reactor core at steady state operation condition.

It illustrates that the variation in the reactor power due to any disturbance will change the control rod position to compensate the error in the reactor criticality.

The $H_\infty$ based observer controller presents small change in the control rod position relative to the reactor power change compared with the PD control system.

Figure 6.21 compares the response of the low order LMI observer based control system and the PD controller when the reactor power disturbed by a step changed to 75% of the full power and stepped back to 100% as two step change.

It shows that the low order LMI observer based control system has better performance than the performance of the PD controller.

The low order LMI observer based control system illustrates a faster settling time with minimum down shoot relative to the PD controller.

Now, it is clear that, the LMI observer based shows a relatively better and fast response compared with PD controller.

Figure 6.22 compares the change in the control rod worth and position relative to the reactor core due to two step change in the reactor power from 100% to 75% and stepped back to 100%.

It shows that the variation in the reactor power will make the control system changes the worth of control rod to compensate the error in the reactor criticality.

The LMI based observer controller presents fast change in the control rod position relative to the reactor power change compared with the PD control system.
Figure 6.21: PD and LMI observer based response when reactor power stepped from 100% to 75% and stepped back to 100%

Figure 6.22: Control rod worth change relative to PD and LMI observer based controllers when reactor power stepped from 100% to 75% and stepped back to 100%
6.8 Concluding Remarks
This chapter introduces the applications of linear matrix inequality and robust control systems based on model reduction. ETRR-2 twelfth order model is reduced to third order model by Schur reduction method with certain and accepted absolute error according to the model reduction error condition.

We design the following low order state feedback control systems based on the third order ETRR-2 model:

- LMI state feedback control system
- LMI-pole placement control system
- H∞ robust control system
- LMI observer based control system
- H∞ observer based control system

In fact these low order control systems used to control the full order nonlinear ETRR-2 model at different transient conditions.

ETRR-2 nonlinear closed loop model is investigated to show the difference between the PD controller and these control systems. From the above mentioned simulations it can concluded that the low order LMI, LMI-pole placement, H∞ and LMI and H∞ based observer controllers show a relatively better response compared with PD controller.
Chapter 7

Conclusion and Recommendation of Future Work

7.1 Conclusion

Nuclear reactors are becoming increasingly popular as a supplement to thermal power plants in providing electrical power. Otherwise the research reactors play an important role in the life for medicine, agriculture, industrial…etc. applications. As such nuclear reactor control is of prime essence due to the safety issues and command following operations. The point kinetic dynamical model of a nuclear reactor is inherently governed by few sets of nonlinear differential equations. Its linearized transfer function models are marginally stable with the magnitudes of the eigen-values differing widely.

It is well known that few time constants in an open loop model of a nuclear reactor are very small due to the effect of the prompt neutron jump and few are quite large due to the delayed neutron jump [139]. Hence at different time instants, different time constants are predominant, which makes the control of such kind of process very difficult.

Robust control system is used because of the nuclear plants are highly complex, nonlinear, time varying and constrained systems and their characteristics vary with operating power levels. Changes in nuclear core reactivity with fuel and coolant temperature, poisons buildup and fuel burn up generally degrade systems performance.

Dynamic model is very important to design the control system because it explains the interactions between the input and output variables and also the nature of the basic dynamic relationships.

The ETRR-2 operation and control basically based on its dynamics and the total reactivity feedbacks. The total core reactivity is calculated as a sum of the feedback reactivity of fuel temperature, coolant temperature, poisons and external reactivity relative to the fuel consumption. The reactor nonlinear full order model is linearized at certain operating point and this linearized model is reduced to a lower order model by using different reduction methods.
In this thesis the different methodologies of model order reduction such as Schur reduction, Hankel approximation and balanced truncation methods are surveyed with several engineering applications. A specific study of ETRR-2 linearized fifth, tenth and twelfth order model are reduced to different orders with different methodologies to study the effectiveness of model order reduction and to deal with the best model order reduction methods.

According to the results of the model order reduction applications, we introduced a new idea to design low order controllers to control the ETRR-2 nonlinear full order model. By using Schur reduction method, the ETRR-2 twelfth order model is reduced finally to third order model. This third order model is adapted to design the following control systems: LMI, LMI-pole placement and observer based on LMI controllers to control the reactor power at different condition.

The same reduced third order model used to design low order robust $H_{\infty}$ and $H_{\infty}$ based observer control systems to control the ETRR-2 power in different operation conditions.

The low order $H_{\infty}$ and $H_{\infty}$ controllers design was carried out by Schur reduction method, linearization process and LMI optimization technique.

A comparison of both $H_{\infty}$ and $H_{\infty}$ based observer proposed control systems relative to the PD controller has been performed which showed better response and disturbance rejection for the proposed controllers.

Simulation results indicate that the low order controllers outperform the PD controller actually used in ETRR-2.

The application of model reduction results in control system design that is computationally simple to implement, and facilitates the controller design as the resulting LMI problem are less prone to numerical problems. We can conclude the design requirements of multi objective control systems such as LMI, LMI-Pole placement, $H_{\infty}$ robust control and observer based control systems as follows:

- Stability robustness.
- Rejection of the disturbances.
- Close tracking of the reference or set-point.
- Performance robustness to bounded additive modeling errors.
- Measurement noise rejection.
Recommendation of future work

Finally the thesis recommended the following issues for future study:

- A comparison of model reduction techniques from structural dynamics, numerical mathematics and systems and control.
- Robust nonlinear model predictive control for a nuclear power plant based model reduction.
- Nonlinear stability analysis of a reduced order model of nuclear reactors.
- Model reduction for physical parameters of nuclear reactor
List of References


الفصل السادس يتعرض بالتفصيل لبناء متحكمات من الطرق غير التقليدية المبنية على استخدام
اختزال النموذج الرياضي للمفاعل كاستخدام نظم التحكم المتينة والمتبانات الخطية واستخدامها
في التحكم في النموذج اللاخطي للمفاعل مع تحليل أداء هذا المتحكمات ومقارنة أداؤها بأداء
المتحكم التقليدي المستخدم بالفعل.
الفصل السابع يقدم ملخص للبحث والنقاط التي تم استنباطها وكذا الدراسات المستقبلية
والتطبيقات المختلفة لنقطة البحث.
الملخص العربي

يرتبط تشغيل المفاعلات النووية بمجموعة من المتغيرات التي تؤثر على ثبات قدرة هذه المفاعلات عند حدود ومتطلبات التشغيل الثابتة والمتغيرة.

ولذا يتطلب الأمر إضافة نظام تحكم آلية لضمان ثبات هذه القدرة في ظل وجود العديد من المتغيرات وضمان التشغيل الأمان والمستمر طبقاً لحدود الأمان النووي. ومن هذه المتغيرات التي تؤثر سلباً وإيجاباً على قدرة المفاعلات مايلي:

- حركة قضاءن التحكم - احتراق الوقود - ارتفاع درجات الحرارة. تولد العديد من النظائر مثل نظير الزينون (الزئبق 135) ونظام التحكم الألي لمفاعل مصر البحثي الثاني كا هذه المفاعلات يحتوي على متحكم (تفاضل - تناسب).

وقد تم استبداله في هذا البحث بعد من المتحكمات المبنية على استخدام التطبيقات الحديثة للتحكم الألي وكذلك المتحكمات التي لها القدرة على مواقع التغيرات التي تحدث في المعاملات المتعلقة بثبات القدرة والتي تم مقارنتها مع المتحكم المستخدم بالفعل للتحكم في قدرة المفاعل.

والفصل الأول يشتغل على مقدمة عامة ومدخل للبحث. الفصل الثاني يحتوي على دراسة استرجاعية شاملة للموضوعات التي شملها البحث مثل نظام التحكم الألي واستخدام أنظمة التحكم الألي بصورة عامة. كما ويشتمل هذا الفصل على التعرف بنظام التحكم المتين والمتعددة ودراسة عدم الدقة في النماذج الرياضية وكذلك المتباينات الخطية واللاخطية.

الفصل الثالث يعرض بالدراسة والتحليل لبناء محاكى رياضى للمفاعل والمتحكم التقليدي المستخدم بالفعل ودراسة أداء كل منهما عند ظروف التشغيل المختلف من قبل التطبيق المماثل.

الفصل الرابع يقدم دراسة شاملة للمتغيرات الخطية واللاخطية وتطبيقاتها المختلفة في أنظمة التحكم كما يقدم هذا الفصل دراسة تحليلية لنظام التحكم المتين وكذلك عدم الدقة المصاحبة للنماذج الرياضية الخطية واللاخطية للمتغيرات المختلفة.

الفصل الخامس يقدم دراسة شاملة لأساليب اختزال درجات النماذج الرياضية ذات الرتبة الأعلى إلى رتب أدنى ودراسة الطرق المختلفة وتطبيقاتها على النموذج الرياضي الخطى للمفاعل وفي حالات مختلفة. فقد تم بناء نموذج رياضى خطي للنموذج اللاخطى للمفاعل من الرتبة الخامسة والعشرينية والثانية مصدقاً لانجاز محدد عند القدرة الفصوى ثم استخدام طرق التقريب المختلفة مثل Schur and Hankel norm approximation وحساب نسبة الخطأ بينهما.
مهنـدـس:

مجدى محمود زكى عبدالعال

تاريخ الميلاد: 3/24/1965

المشيـرة:

الجنسية: مصرى

تاريخ التسجيل: 7/6/2006

القسم:

الدرجة:

المنحة:

المشرفون:

أ.د عادل عبدالرؤف حنفى

أ.د حسن طاهر درة

أ.د حسن محمد رشاد عماره

أ.د سيد محمد سيد العربى

أمتحنـون:

أ.د عادل عبدالرؤف حنفى

أ.د حسن طاهر درة

أ.د حسن محمد رشاد عماره

كلية الهندسة جامعة طنطا

عنوان الرسالة: تصمـيم نظـم تحـكم متتعدد الأغراض للتحكم في قدرة مفاعل نووى

الكلمات الدالة: - اختزال النموذج الرياضي - نظم التحكم المتتيبة الخطية - نظم التحكم المتعددة - التحكم في قدرة مفاعل نووى

ملخص البحث:

يرتبط تشغيل المفاعلات النووية بمجموعة من المتغيرات التي تؤثر على ثبات قدرة هذه المفاعلات عند حدود ومتطلبات التشغيل الثابتة والمتحركة. وإذا يتطلب الأمر إضافة نظم تـحكم آلية لضمان ثبات هذه القدرة في ظل وجود العديد من المتغيرات وضمان التشغيل الأمان والمستمر طبقًا لحدود الأمان النووي. ومن هذه المتغيرات التي تؤثر سلبا

إيجابا على قدرة المفاعلات مايلي:

حركة قضاة التشغيل - احترس الوقود - درجات الحرارة - تولد العديد من التظاهرات المؤثرة في الفيض البيئي و معدل التبريد... الخ. وغير ذلك من التحكم الالتي لتفاعل بنية على متغيرات الضمان ذاتى. حيث هذه التضمنات بحثها على مصلحة (تضايقي - تناسبي) وقد تم استخدامه في هذا البحث بنظام تـحكم عشوائي على هـدف التشغيل الذاتي الخطأ للمفاعل ثم تصمـيم نظم تحكم ذو دuccوسية متوازنة مع هذا النموذج الرياضي المتصل للتحكم في النموذج الالخلي الأصلى للمفاعل. وقد تم تصمـيم نظم تحكم باستخدام المثابنة الخطية وكذلك نظم التحكم التي لها القدرة على تـ(rotation) الظاهرات التي تحدث في المعاملات المتعلقة بثبات القدرة وبعـمان التشغيل والتي تم مقايرتها مع المحاكاة المستخدم بالفعل للتحكم في قدرة المفاعل. وقد أسـتيت نتائج أفضل عند حالات تشغيل متغيرة.
تصميم نظم تحكم متعددة الأغراض للتحكم في قدرة مفاعل نووي

إعداد
مهندس/ مجدى محمود زكي عبد العال

رسالة مقدمة إلى كلية الهندسة، جامعة القاهرة كجزء من متطلبات الحصول على درجة الدكتوراه في هندسة القوى والآلات الكهربائية

يعتمد من لجنة الممتحنين:

peater

عضو لجنة التحكيم

عضو لجنة التحكيم

كلية الهندسة، جامعة القاهرة الجيزة، جمهورية مصر العربية
2013
تصميم نظم تحكم متعددة الأغراض للتحكم في قدرة مفاعل نووي

إعداد

مهندس/ مجرى محمود زكي عبد العال

رسالة مقدمة إلى كلية الهندسة، جامعة القاهرة

كجزء من متطلبات الحصول على درجة الدكتوراه

في

هندسة القوى والآلات الكهربائية

تحت إشراف

د. عادل عبد الرؤف حنفى
كلية الهندسة جامعة القاهرة

د. حسن محمد رشاد
كلية الهندسة جامعة القاهرة

د/ سيد محمد سيد العربي
هيئة الطاقة الذرية

كلية الهندسة، جامعة القاهرة
الجيزة، جمهورية مصر العربية

2013
تصميم نظم تحكم متعددة الأغراض للتحكم في قدرة مفاعل نووي

إعداد
مهندس/ مجدى محمود زكى عبد العال

رسالة مقدمة إلى كلية الهندسة، جامعة القاهرة كجزء من متطابقات الحصول على درجة الدكتوراه في هندسة القوى والآلات الكهربية

كلية الهندسة، جامعة القاهرة
الجيزة، جمهورية مصر العربية
2013