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ON THE DETERMINATION OF GENERAL PLANE STRESS STATES IN ORTHOTROPIC MATERIALS FROM ULTRASONIC VELOCITY DATA IN NONSYMMETRY PLANES

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ABSTRACT

This work reports the progress in the development of a new experimental protocol for plane stress determination in orthotropic materials based on the ultrasonic velocity of bulk waves propagating in nonsymmetry planes with oblique incidence. The presence of stress-induced deformation introduces an acoustic anisotropy in the material in addition to that defined by its texture. Orthotropic materials under general plane stress states become acoustically monoclic and its orthotropic planes orthogonal to the stress plane become nonsymmetry planes. The inverse solution of the generalized Christoffel equation for ultrasonic bulk waves propagating in nonsymmetry planes of anisotropic bodies is known to be numerically unstable. The suggested protocol deals with this numerical instability without recourse to bulk wave propagation in the stress plane as proposed in the literature. Hence, it should be useful for plane stress analysis of thin wall pressure vessels where ultrasonic measurements in the direction of the wall plane are not possible. For the initial validation of the suggested protocol and verification of the stability of the inversion algorithm, computer simulation of stress determination have been performed from synthetic sets of velocity data obtained by the forward solution of the generalized Christoffel equation. Preliminary results for slightly orthotropic aluminium highlight the potential of the suggested protocol.

1. INTRODUCTION

Extensive studies carried out in the last decades on the propagation of ultrasonic waves in solids led to the development of nondestructive techniques for the assessment of the safety and integrity of industrial structures and components. The interest in the application of ultrasound techniques for stress measurement, for instance, comes from the measurable change in the speed of the ultrasonic elastic waves in the presence of a stress field, a phenomenon known as acoustoelastic effect. The study of this phenomenon led to the development of a new branch of engineering mechanics termed acoustoelasticity. In 1953, Hughes and Kelly [1] developed the first acoustoelastic theory considering homogenous isotropic materials. They used the finite elasticity theory of Murnaghan [2] including elastic constants of second and third orders in the constitutive equation to describe the stress dependence of the wave velocities. King and Fortunko [3] and Thompson et al. [4] reported the first successful results for the ultrasonic stress measurement in structural materials with slight anisotropy from fabrication processes. The resulting anisotropy causes shifts in the speed of the ultrasonic waves that are of the same order of magnitude as those due to the presence of stresses. Thompson et al. method [5] suffered critics with regard to its applicability to an anisotropic body because of its assumption on hyperelastic deformation and on the existence of a natural (stress free) material state [6].

This work uses the formulation of Man and Lu [7] to study the propagation of ultrasonic waves in homogenous orthotropic solids under stress. Their formulation, based on the concepts of the theory of linear elasticity with initial stress [8-10], does not postulate a material natural state, deals with complex stress loadings (including plastic deformation) and material anisotropy, and leads to a nonlinear eigenvalue problem described by the generalized Christoffel equation. The nonlinearity characteristic of the problem derives from the interdependence between the stresses and the material effective elastic constants, which are determined from the material second and third order elastic constants, and the stress-induced deformation.

The presence of stress-induced deformation introduces an *acoustic* anisotropy in the material in addition to that defined by preferential orientation of the material grains (texture). Under general plane stress states, the principal stress directions generally do not coincide with the material symmetry axes, the overall symmetry of orthotropic materials becomes *acoustically* monoclic and material orthotropic planes orthogonal to the stress plane become nonsymmetry planes. Hence, four new stress-dependent elastic constants must be added to the existing nine independent elastic constants of the orthotropic symmetry. These new stress-induced elastic constants are much smaller than the previous ones, having values of the order of magnitude of the stress-induced changes in the existing elastic constants. Sixteen unknowns given by the nine elastic constants characteristic of the orthotropic symmetry plus the four stress-induced elastic constants and the three plane stress components characterize the problem of interest.

In 1995, Degtyar and Rokhlin [11] proposed a method to determine plane stresses in orthotropic materials from the angular dependence of the velocities of waves propagating in the material symmetry planes orthogonal to the stress plane with oblique incidence. They reported that when the principal stress directions coincide with the material symmetry axes of an orthotropic material, the generalized Christoffel equation can be decoupled leading to closed-form solutions for the longitudinal and transversal wave velocities. When the principal stress directions do not coincide with the material symmetry axes, however, the two material symmetry planes orthogonal to the stress plane become *acoustically* nonsymmetry planes and the generalized Christoffel equation cannot be decoupled. They also verified that the reconstruction of stresses and stress-dependent elastic constants from nonsymmetry planes are numerically unstable. To overcome this difficulty they considered waves propagating in the stress plane itself since it remained a symmetry plane even when the principal stress directions do not align with the orthotropic axes, and the generalized Christoffel equation could again be decoupled. This procedure, however, limits the application of the technique, as ultrasonic measurements along the wall plane of components are rather unpractical, if not impossible.

This paper reports the progress in the development of a new protocol for general plane stress determination in orthotropic materials using only the velocities of ultrasonic bulk waves propagating in *acoustically* nonsymmetry planes orthogonal to the stress plane with oblique incidence. It explores a result described by Chu et al. [12] related to the determination of the elastic constants of orthotropic materials from velocity data in nonsymmetry planes. Instead of trying to determine all the sixteen unknowns from velocities measurements in one pre-selected nonsymmetry plane only, a numerically unstable procedure, the suggested protocol, exploring the fact that the values of the effective elastic constants and stress components are invariant irrespective of the propagation planes selected, utilizes velocity data from three specially selected nonsymmetry planes. If successfully established, this protocol should be useful for the plane stress analysis of thin wall pressure vessels and piping systems.

2. ULTRASONIC WAVE PROPAGATION IN PRESTRESSED SOLID

2.1. The acoustoelastic theory of Man and Lu

In Man and Lu approach [7], the prestressed configuration is the only reference configuration and the initial residual or applied stresses are included in the material constitutive equation

$$\sigma_{ij} = \sigma_{ij}^{0} + C_{ijkl}\, \varepsilon_{kl} + \mu_{i,k}\, \sigma_{kj}^{o} \, , \tag{1}$$

where σ_{ij} is the first Piola-Kirchhoff stress, σ_{ij}^{0} is the initial static (residual or applied) stress, ε_{kl} is the elastic strain due to wave propagation, $\mu_{i,k}$ is the displacement gradient and C_{ijkl} is the fourth order tensor of effective (stress dependent) elastic constants.

The dynamical equilibrium equation for small elastic deformations superimposed to the prestressed state is given by

$$\sigma_{ij,\,j} = \rho\, \frac{\partial^2 \mu_i}{\partial t^2} \, , \tag{2}$$

where μ_i é is the displacement vector and ρ is the material density.

Introducing Eq. (1) into Eq. (2) and assuming that the material and the local stresses are homogeneous, Eq. (2) can be rewritten as

$$\left(C_{ijkl} + \sigma_{jl}^{0}\, \delta_{ik} \right) \mu_{k.\,jl} = \rho\, \frac{\partial^2 \mu_i}{\partial t^2} \, , \tag{3}$$

where δ_{ij} is the Kronecker tensor.

Eq. (3) can be solved considering a plane wave solution for the displacement vector

$$\mu_k = A P_k \exp\left[i\, K \left(n_s\, x_s - V_p\, t \right) \right] \tag{4}$$

where A is the wave magnitude, P_k the unit displacement vector, K is the wave number, V_p the wave phase velocity, n_s is the unit wave normal vector in the direction of the wave propagation and x_s é o position vector.

Substituting Eq. (4) into equilibrium equation (3) leads to the generalized Christoffel equation,

$$\left[C_{ijkl}\, n_i\, n_l + \left(\sigma_{il}^{0}\, n_i\, n_l - \rho V_p^2 \right) \delta_{jk} \right] P_k \quad = 0 \ \ . \tag{5}$$

For orthotropic materials, Eq. (5) is a polynomial equation of third order in relation to the square of the wave phase velocity V_p. Its solution gives one quasi-longitudinal velocity and two quasi-transversal velocities and their respective polarization directions.

2.2. Degtyar and Rokhlin method

Degtyar and Rokhlin approach [11] uses an optimization technique by minimum squares for the simultaneous reconstruction of the stresses σ_{il} and the stress-dependent elastic constants C_{ijkl} from the angular dependence of the phase velocity V_i of the ultrasonic waves. They propose the minimization of the functional I defined by

$$I \;=\; \min_{c_{ijkl},\,\sigma_{ij}\,\in R^n}\; \frac{1}{2}\sum_{i}^{m}\left(V^e - V^c\right)^2 \tag{6}$$

where n is the number of parameters to be determined, m is the number of velocity data in different propagation directions, V^e is the experimental phase velocity and V^c is the computed phase velocity. The stress-dependent elastic constants and the stresses are the unknown variables in a multidimensional space n. In the inversion process, the effective elastic constants C_{ijkl} and the stresses are assumed as independent variables, although the effective elastic constants C_{ijkl} are indeed stress dependent. Thus is necessary to confirme that the iteration process converges to the correct values.

3. THE PROPOSED PROTOCOL

The protocol under development employs quasi-longitudinal and quasi-transverse shear waves propagating in three planes orthogonal to the stress plane with oblique incidence. The protocol comprises three major steps; in each step, a subset of unknowns are solved and taken as fixed values for use in the subsequent one. This reduces the number of unknowns to be solved in each step and provides more stability to the Levenberg-Marquardt [13] least square minimization algorithm employed for the inverse solution of the generalized Christoffel equation. The next section describes the suggested protocol in more detail.

3.1. The protocol steps

Consider an orthogonal Cartesian system (x_1, x_2, x_3) and let the x_1 and x_2 axes define the stress plane 1-2 in a thin solid (idealized plate) and the x_3 axis be along the plate thickness direction. The first two planes orthogonal to the stress plane 1-2 used in the protocol are the planes defined by the axes x_1 and x_3 (plane 1-3) and by the axes x_2 and x_3 (plane 2-3). The third orthogonal plane used in the protocol makes an anticlockwise angle of 45 degrees from the plane 1-3.

The first step of the protocol consists in considering the propagation of quasi-longitudinal and quasi-transversal waves in the plane 1-3. In this plane, the second component of the unit vectors normal to the wave fronts propagated in plane 1-3 is null. Using the Voigt's notation to represent the components of elastic constants order C_{ijkl} and the stress tensor σ_{il}, it is not difficult to verify that only nine stress-dependent elastic constants – C_{11}, C_{13}, C_{16}, C_{33}, C_{36}, C_{44}, C_{45}, C_{55} and C_{66} and one stress component σ_{11} – are retained in the generalized Christoffel equation. The reconstruction process then solves ten out of the sixteen problem unknowns.

The second step of the protocol is similar to the first one. Considering that the first components of each unit vector normal to the wave fronts propagated in plane 2-3 are null, it possible to show that only nine stress-dependent elastic constants – C_{22}, C_{23}, C_{26}, C_{33}, C_{36}, C_{44}, C_{45}, C_{55} and C_{66} and one stress component σ_{22} (using Voigt's notation) – are involved in the generalized Christoffel equation. Only four of these variables (C_{22}, C_{23}, C_{26} and σ_{22}) are indeed unknowns since C_{33}, C_{36}, C_{44}, C_{45}, C_{55} and C_{66} have been determined in the first step. Nonetheless, at this stage of development of the protocol, it has been decided to consider all ten variables related to plane 2-3 as unknowns in the reconstruction process and to verify if the corresponding values of C_{33}, C_{36}, C_{44}, C_{45}, C_{55} and C_{66} computed in these first two steps of the protocol are equal, as expected (the values of the effective elastic constants and of the stress components are invariant irrespective of the propagation planes selected). This comparison serves as an indirect indication of the stability of the proposed protocol.

With fourteen unknowns solved, the third and final step of the protocol determines the two remaining unknowns C_{12} and σ_{12} in the generalized Christoffel equation. The reconstruction process considers the speeds of quasi-longitudinal and quasi-transversal waves propagated in a plane orthogonal to the stress plane 1-2 and rotated 45 degrees anticlockwise around the axis x_3 from the plane 1-3 (hereafter referred to as plane +45º).

4. RESULTS

For initial validation of the suggested protocol and verification of the stability of the inversion algorithm, the stresses acting on a plate element subjected to a general plane stress loading. Experimental velocity data were simulated by synthetic data generated by the forward solution of the generalized Christoffel equation assuming that the material second order and third order elastic constants were known and that the applied stresses resulted from elastic deformation.

The material considered in this application was a textured aluminium with a slight (1%) anisotropy. Its engineering constants are compiled in table 1. Since experimental data for third order elastic constants of orthotropic aluminium are not generally available, data for isotropic aluminium were employed [14]. They are approximately one order of magnitude greater than those of the second order elastic constants. The aluminium second and third order elastic constants are listed in table 2. The material density was taken as 2,700 kg/m^3.

Table 1: Aluminium engineering properties (in GPa)

Properties	x_1 axis	x_2 axis	x_3 axis
Young modulus	70.00	69.88	69.80
Poisson ratio	0.350	0.350	0.350
Shear modulus	25.92(6)	25.92(6)	25.92(6)

Table 2: Second and third order elastic constants

Second order elastic constants [a]	Value (in GPa)
$C_{11}^{(0)}$	112.170
$C_{12}^{(0)}$	60.2680
$C_{13}^{(0)}$	60.2169
$C_{22}^{(0)}$	112.004
$C_{23}^{(0)}$	60.1901
$C_{33}^{(0)}$	111.858
$C_{44}^{(0)} = C_{55}^{(0)} = C_{66}^{(0)}$	25.9259
Third order elastic constants	
$C_{111} = C_{222} = C_{333}$	-1,333.5
$C_{112} = C_{113} = C_{122} = C_{133} = C_{223} = C_{233}$	-242.71
C_{123}	-54.110
$C_{144} = C_{255} = C_{366}$	-94.300
$C_{155} = C_{166} = C_{244} = C_{266} = C_{344} = C_{355}$	-272.70
C_{456}	-89.200

a. The superscript $^{(0)}$ indicates second order constants for a stress free material.

Table 3 lists the effective elastic constants for the free stress and the general biaxial stress cases. The free stress case helps to study the stability of the inversion procedure due to material anisotropy only. For the general plane stress case, the existence of a shear stress component reduces the acoustic symmetry of the material from orthotropic to monoclic. These constants were calculated during the forward solution of the generalized Christoffel equation for generation of the synthetic velocity data. The simulation used 61 quasi-longitudinal waves with refraction angles in the range 20º to 80º, 20 fast quasi-transversal waves with refraction angles in the range 20º to 39º and 10 slow quasi-transversal waves with refraction angles range 10º to 29º. These refraction ranges were selected based on the results of Chu et al. [12].

Table 3: Effective elastic constants and applied stresses

Constant (in GPa)	Unloaded (in MPa) $\sigma_{11} = \sigma_{22} = \sigma_{12} = 0.0$	Biaxial stresses (in MPa) $\sigma_{11} = 100.0$; $\sigma_{22} = 50.0$; $\sigma_{12} = 5.0$
C_{11}	112.170	111.185
C_{12}	60.2680	60.0995
C_{13}	60.2169	60.1141
C_{16}	0.00000	-0.0094
C_{22}	112.004	111.637
C_{23}	60.1901	60.1530
C_{26}	0.00000	-0.00937
C_{33}	111.858	112.113
C_{36}	0.00000	0.00503
C_{44}	25.9259	25.9162
C_{45}	0.00000	-0.0072
C_{55}	25.9259	25.7944
C_{66}	25.9259	25.6721

Tables 4 and 5 compare the reconstructed values of the effective stress-dependent elastic constants and the stress components with those used for generation of the synthetic velocity data for the loading cases considered.

Table 4: Effective elastic constant reconstruction (stress free)

Con-stants	Expected values	Initial guesses: +10% deviation	SINGLE INVERSION		PROPOSED PROTOCOL (three inversion planes)				
			Plane +45º	error (%)	Plane 1-3	Plane 2-3	Plane +45º	Final values	error (%)
C_{11}	112.170	123.387	86.1253	-23.2	112.170	---	---	112.170	0.0
C_{12}	60.2680	66.2948	68.8332	14.2	---	---	60.2680	60.2680	0.0
C_{13}	60.2169	66.2386	28.3544	-52.9	60.2169	---	---	60.2169	0.0
C_{16}	0.0000	0.0000	0.0000	0.00	0.0000	0.0000	0.0000	0.0000	0.0
C_{22}	112.004	123.205	189.956	69.6	---	112.004	---	112.004	0.0
C_{23}	60.1901	66.2091	83.1163	38.1	---	60.1901	---	60.1901	0.0
C_{26}	0.0000	0.0000	0.0000	0.00	0.0000	0.0000	0.0000	0.0000	0.0
C_{33}	111.858	123.044	111.858	0.00	111.858	111.858	---	111.858	0.0
C_{36}	0.0000	0.0000	0.0000	0.00	0.0000	0.0000	0.0000	0.0000	0.0
C_{44}	25.9259	28.5185	25.9259	0.00	25.9259	25.9259	---	25.9259	0.0
C_{45}	0.0000	0.0000	0.0000	0.00	0.0000	0.0000	0.0000	0.0000	0.0
C_{55}	25.9259	28.5185	25.9259	0.00	25.9259	25.9259	---	25.9259	0.0
C_{66}	25.9259	28.5185	-0.0275	-100.	25.9259	25.9259	---	25.9259	0.0

Table 5: Effective elastic constants (in GPa) and stress reconstruction (in MPa)

Con-stants	Expected values	Initial guesses +10% deviation	SINGLE INVERSION		PROPOSED PROTOCOL (three inversion planes)				
			Plane +45º	error (%)	Plane 1-3	Plane 2-3	Plane +45º	Final values	error (%)
C_{11}	111.185	122.303	111.939	-0.68	111.185	---	---	111.185	0.0
C_{12}	60.0995	66.1095	60.2823	0.30	---	---	60.0995	60.0995	0.0
C_{13}	60.1141	66.1255	60.3986	0.47	60.1141	---	---	60.1141	0.0
C_{16}	-0.0093	-0.0103	-0.0090	-3.89	-0.0082	---	---	-0.0082	-12.6
C_{22}	111.637	122.801	111.251	-0.35	---	111.637	---	111.637	0.0
C_{23}	60.1530	66.1683	59.8656	-0.48	---	60.1530	---	60.1530	0.0
C_{26}	-0.0094	-0.0103	-0.0088	-5.96	---	-0.0090	---	0.00000	-3.95
C_{33}	112.113	123.324	112.113	0.0	112.113	112.113	---	112.113	0.0
C_{36}	0.00503	0.00553	0.00585	16.2	0.00622	0.00541	---	0.00622	23.6
C_{44}	25.9162	28.5079	25.9163	0.0	25.9162	25.9162	---	25.9162	0.0
C_{45}	-0.0072	-0.0079	-0.0068	-5.63	-0.0072	-0.0072	---	-0.0072	0.0
C_{55}	25.7944	28.3738	25.7943	0.0	25.7944	25.7944	---	25.7944	0.0
C_{66}	25.6721	28.2393	25.4880	-0.72	25.6721	25.6721	---	25.6721	0.0
σ_{11}	100.000	110.000	90.134	-9.9	100.00	---	---	100.00	0.0
σ_{22}	50.000	55.000	56.069	12.1	---	50.000	---	50.000	0.0
σ_{12}	5.000	5.500	6.493	29.9	---	---	5.020	5.020	0.39

Table 4 indicates that the relative errors in the reconstruction of all effective elastic constants from the plane +45 were different from zero for several constants reaching, for example, -23.2 % for C_{11}, 69.6 % for C_{22} up to -100% for C_{66}. The reconstruction of the effective elastic constants using the suggested protocol were exact for all the nine effective elastic constants typical of orthotropic symmetry.

Table 5 in turn shows that the relative errors in the reconstruction of the effective elastic constants from plane +45 were different from zero for all, except three (C_{33}, C_{44} and C_{55}) elastic constants. They were, however, smaller in magnitude (maximum of 16.2 % for C_{36}) than for the stress free case. The relative errors in the reconstruction of the stresses from plane +45 were significant for all stress components, reaching about 30 % for σ_{12}. The reconstruction of the effective elastic constants using the proposed protocol were exact for all effective elastic constants typical of orthotropic symmetry, but different from zero for the stress-induced elastic constants reaching 23.6 % for C_{36}. These latter non-null values are of the same order of magnitude than those encountered from the reconstruction using plane +45 only. The reconstruction of the stresses using the proposed protocol were exact for the normal stress components σ_{11} and σ_{22} and the relative error in the reconstruction of the tangential stress component σ_{12} was small (less than 0.5 %).

It can also be verified in tables 4 and 5 that, except for C_{36} in the biaxial stress case, the corresponding values of C_{33}, C_{36}, C_{44}, C_{45}, C_{55} and C_{66} computed in these first two steps of the protocol are equal as expected. This result provides further indication of the stability of the proposed protocol.

5. CONCLUSIONS

This work reported the progress in the development of a new experimental protocol for plane stress determination in orthotropic materials using velocity data of bulk waves propagating in nonsymmetry planes with oblique incidence.

The preliminary results obtained in the reconstruction of the acting stresses and of the effective stress-dependent elastic constants of a textured aluminium subjected to a biaxial stress state highlighted the potential of the proposed protocol. Particularly good results have been found for the reconstruction of the applied stresses.

Further developments of the proposed protocol require testing for

- greater deviations of the initial guesses from the original data;
- the scatter in the experimental velocity data;
- different number of measurements (refraction angles ranges);
- different degrees of material orthotrophy; and,
- additional plane stress states.

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