International Conference on Mathematics and Computational Methods Applied to Nuclear Science and Engineering (M&C 2011)
Rio de Janeiro, RJ, Brazil, May 8-12, 2011, on CD-ROM, Latin American Section (LAS) / American Nuclear Society (ANS)
ISBN 978-85-63688-00-2

GENERALIZED HYBRID MONTE CARLO – CMFD METHODS FOR FISSION SOURCE CONVERGENCE

Emily R. Wolters, Edward W. Larsen, and William R. Martin
Department of Nuclear Engineering & Radiological Sciences
University of Michigan
2355 Bonisteel Boulevard, Ann Arbor, MI 48109
ewolters@umich.edu; edlarsen@umich.edu; wrm@umich.edu

ABSTRACT

In this paper, we generalize the recently published "CMFD-Accelerated Monte Carlo" method [1][2] and present two new methods that reduce the statistical error in CMFD-Accelerated Monte Carlo. The CMFD-Accelerated Monte Carlo method uses Monte Carlo to estimate nonlinear functionals used in low-order CMFD equations for the eigenfunction and eigenvalue. The Monte Carlo fission source is then modified to match the resulting CMFD fission source in a "feedback" procedure. The two proposed methods differ from CMFD-Accelerated Monte Carlo in the definition of the required nonlinear functionals, but they have identical CMFD equations. The proposed methods are compared with CMFD-Accelerated Monte Carlo on a high dominance ratio test problem. All hybrid methods converge the Monte Carlo fission source almost immediately, leading to a large reduction in the number of inactive cycles required. The proposed methods stabilize the fission source more efficiently than CMFD-Accelerated Monte Carlo, leading to a reduction in the number of active cycles required. Finally, as in CMFD-Accelerated Monte Carlo, the apparent variance of the eigenfunction is approximately equal to the real variance, so the real error is well-estimated from a single calculation. This is an advantage over standard Monte Carlo, in which the real error can be underestimated due to inter-cycle correlation.

Key Words: hybrid, Monte Carlo, CMFD, nonlinear functional, source convergence

1. INTRODUCTION

The CMFD (Coarse Mesh Finite Difference) method is a well-known deterministic method for solving the Boltzmann transport equation. CMFD was proposed by K. Smith in 1983 [3] and has been used successfully for several years. Recently, Lee, et al., proposed the adaptation of CMFD to accelerate Monte Carlo fission source convergence [1][2]. CMFD-Accelerated Monte Carlo has been shown to accelerate source convergence and provide a more reliable estimate of the variance because of decreased inter-cycle correlation. In this paper, we show that CMFD-Accelerated Monte Carlo can be improved by using angular moments of the neutron transport equation to reduce statistical error. We present the derivation of two generalized Monte Carlo – CMFD methods and numerical results demonstrating improvement over CMFD-Accelerated Monte Carlo.

2. THEORY

2.1 Derivation of the Generalized Monte Carlo – CMFD Methods

To derive the generalized Monte Carlo – CMFD methods, we consider the following planar geometry k-eigenvalue problem:

$$\mu \frac{\partial}{\partial x}\psi(x,\mu)+\Sigma_t(x)\psi(x,\mu)=\frac{1}{2}\left[\Sigma_s(x)+\frac{\nu\Sigma_f(x)}{k_{eff}}\right]\int_{-1}^{1}\psi(x,\mu)d\mu', \qquad 0<x<X, \tag{1a}$$

$$\psi(0,\mu)=\psi(X,-\mu)=0, \qquad 0<\mu\le 1. \tag{1b}$$

We impose a spatial grid $0=x_{1/2}<x_{3/2}<\cdots<x_{k+1/2}<\cdots<x_{K+1/2}=X$ on the system; this grid is "coarse" because the cross sections can vary within each k^{th} spatial cell (the interval $x_{k-1/2}<x<x_{k+1/2}$, with width $h_j=x_{j+1/2}-x_{j-1/2}$).

To derive Monte Carlo – CMFD methods for Eqs. (1), we define angular flux moments

$$\phi_n(x)=\int_{-1}^{1}\mu^n\psi(x,\mu)d\mu, \qquad n=0,1,2,$$

and we operate on Eq. (1) by $\int_{-1}^{1}\mu^n(\cdot)d\mu$ for $n=0$ and 1. This yields the following two identities:

$$\frac{d}{dx}\phi_1(x)+\Sigma_a(x)\phi_0(x)=\frac{\nu\Sigma_f(x)}{k_{eff}}\phi_0(x), \tag{2}$$

$$\frac{d}{dx}\phi_2(x)+\Sigma_t(x)\phi_1(x)=0, \tag{3}$$

where $\Sigma_a(x)=\Sigma_t(x)-\Sigma_s(x)$. Next, we integrate Eq. (2) over $x_{k-1/2}<x<x_{k+1/2}$. If we define the cell-averaged scalar fluxes, flux-weighted cross sections, and cell-edge currents by

$$\phi_{0,k}=\frac{1}{h_k}\int_{x_{k-1/2}}^{x_{k+1/2}}\phi_0(x)dx, \tag{4}$$

$$\Sigma_{t,k} = \frac{\int_{x_{k-1/2}}^{x_{k+1/2}} \Sigma_t(x)\phi_0(x)dx}{\int_{x_{k-1/2}}^{x_{k+1/2}} \phi_0(x)dx}, \tag{5a}$$

$$\Sigma_{a,k} = \frac{\int_{x_{k-1/2}}^{x_{k+1/2}} \Sigma_a(x)\phi_0(x)dx}{\int_{x_{k-1/2}}^{x_{k+1/2}} \phi_0(x)dx}, \tag{5b}$$

$$\nu\Sigma_{f,k} = \frac{\int_{x_{k-1/2}}^{x_{k+1/2}} \nu\Sigma_f(x)\phi_0(x)dx}{\int_{x_{k-1/2}}^{x_{k+1/2}} \phi_0(x)dx}, \tag{5c}$$

$$\phi_{1,k\pm1/2} = \phi_1(x_{k\pm1/2}), \tag{6}$$

then we obtain, for $1 \le k \le K$, the following neutron balance equations:

$$\phi_{1,k+1/2} - \phi_{1,k-1/2} + \Sigma_{a,k}h_k\phi_{0,k} = \frac{\nu\Sigma_{f,k}h_k}{k_{eff}}\phi_{0,k}. \tag{7}$$

Next, we define the dimensionless quantities

$$\tilde{D}_{k+1/2} = \frac{2}{3\left(\Sigma_{t,k+1}h_{k+1} + \Sigma_{t,k}h_k\right)}, \tag{8a}$$

$$\hat{D}_{k+1/2} = \frac{\phi_{1,k+1/2} + \tilde{D}_{k+1/2}\left(\phi_{0,k+1} - \phi_{0,k}\right)}{\phi_{0,k+1} + \phi_{0,k}}. \tag{8b}$$

Then for $1 \le k \le K-1$, we have the identities:

$$\phi_{1,k+1/2} = -\tilde{D}_{k+1/2}\left(\phi_{0,k+1} - \phi_{0,k}\right) + \hat{D}_{k+1/2}\left(\phi_{0,k+1} + \phi_{0,k}\right). \tag{9}$$

At the left edge of the system, we have

$$\begin{aligned}\phi_{1,1/2} &= -\left(\frac{\int_{-1}^{0}|\mu|\psi(0,\mu)d\mu}{\frac{1}{h_1}\int_{x_{1/2}}^{x_{3/2}}\phi_0(x)dx}\right)\phi_{0,1} \\ &= -B_1\phi_{0,1},\end{aligned} \tag{10a}$$

and at the right edge, we have

$$\phi_{1,K+1/2} = \left(\frac{\int_0^1 \mu\psi(X,\mu)d\mu}{\frac{1}{h_K}\int_{x_{K-1/2}}^{x_{K+1/2}} \phi_0(x)dx} \right) \phi_{0,K} \qquad (10b)$$
$$= B_K \phi_{0,K}.$$

Eqs. (9) and (10) enable us to formally eliminate the ϕ_1 terms in Eq. (7). For $2 \le k \le K-1$, we get

$$-\tilde{D}_{k+1/2}\left(\phi_{0,k+1} - \phi_{0,k}\right) + \tilde{D}_{k-1/2}\left(\phi_{0,k} - \phi_{0,k-1}\right) + \Sigma_{a,k} h_k \phi_{0,k}$$
$$= \frac{\nu\Sigma_{f,k} h_k}{k_{eff}} \phi_{0,k} - \hat{D}_{k+1/2}\left(\phi_{0,k+1} + \phi_{0,k}\right) + \hat{D}_{k-1/2}\left(\phi_{0,k} + \phi_{0,k-1}\right), \qquad (11)$$

and similar equations are obtained for $k = 1$ and K.

We note that each of the preceding equations was derived without approximation from Eqs. (1). Thus, if the flux-weighted quantities $\Sigma_{a,k}$, $\nu\Sigma_{f,k}$, and $\hat{D}_{k+1/2}$ are known exactly, then Eqs. (11) exactly determine the coarse-mesh scalar fluxes $\phi_{0,k}$, and the eigenvalue k_{eff}.

The basic Monte Carlo-CMFD method can now be described:

1. A standard Monte Carlo method is used to simulate the random histories of "fission" neutrons born from an estimated fission "bank". After the fission neutrons have been released from the bank, the Monte Carlo process generates their random histories; when this task is completed, a new fission "bank" has been obtained for the next generation.

2. While the standard Monte Carlo process generates the fission neutron histories, it also (using tracklength and surface estimators) generates estimates of the integrals and currents in Eqs. (4) and (6). After all the fission neutrons in a generation have been processed, the quantities $\Sigma_{t,k}$, $\Sigma_{a,k}$, and $\nu\Sigma_{f,k}$ [Eqs. (5)], $\tilde{D}_{k+1/2}$ [Eq. (8a)] and $\hat{D}_{k+1/2}$ [Eq. (8b)] are calculated.

3. Using the estimated quantities from step 2, the coarse-grid Eqs. (11) are solved for $\phi_{0,k}$ and k_{eff}.

4. The Monte Carlo fission source "bank" obtained in step 1 is modified to become consistent with the normalized fission source $\frac{\nu\Sigma_{f,k}\phi_{0,k}h_k}{\sum_{k'=1}^{K}\nu\Sigma_{f,k'}\phi_{0,k'}h_{k'}}$ obtained from the solution of the CMFD equations in step 3. This step is known as "feedback" and is performed by randomly duplicating or deleting fission sites in each coarse cell, making the spatial distribution of the fission bank consistent with the CMFD fission source distribution.

5. Return to step 1 and repeat for as many inactive and active cycles as are deemed necessary.

In this paper, we focus on the statistical properties of the coarse-grid cell-averaged fluxes $\phi_{0,k}$ and eigenvalue k_{eff}, obtained by (i) the Monte Carlo-CMFD method described above, and (ii) the standard Monte Carlo method, in which steps 2-4 above are deleted. We also consider two new generalized Monte Carlo-CMFD methods that improve the statistical properties of the above Monte Carlo-CMFD method.

2.2 Definition of the HCMFD-I, HCMFD-II, and HCMFD-III Methods

To define the new generalized CMFD methods, we use the fact that the exact solution of Eqs. (1) satisfies Eq. (3) as well as Eq. (2). Multiplying Eq. (3) by a to-be-specified function $f_{k+1/2}(x)$, integrating over $x_{k-1/2} < x < x_{k+3/2}$, and then integrating by parts, we obtain

$$f_{k+1/2}(x)\phi_2(x)\Big|_{x_{k-1/2}}^{x_{k+3/2}} - \int_{x_{k-1/2}}^{x_{k+3/2}} \frac{df_{k+1/2}}{dx}\phi_2(x)dx + \int_{x_{k-1/2}}^{x_{k+3/2}} f_{k+1/2}(x)\Sigma_t(x)\phi_1(x)dx = 0.$$

This directly gives the following identity:

$$0 = F_{k+1/2} \equiv \frac{\int_{x_{k-1/2}}^{x_{k+3/2}} f_{k+1/2}(x)\Sigma_t(x)\phi_1(x)dx}{\int_{x_{k-1/2}}^{x_{k+3/2}} f_{k+1/2}(x)\Sigma_t(x)dx} + \frac{f_{k+1/2}(x)\phi_2(x)\Big|_{x_{k-1/2}}^{x_{k+3/2}} - \int_{x_{k-1/2}}^{x_{k+3/2}} f'_{k+1/2}(x)\phi_2(x)dx}{\int_{x_{k-1/2}}^{x_{k+3/2}} f_{k+1/2}(x)\Sigma_t(x)dx}, \quad (12)$$

which is satisfied by the solution of Eqs. (1) for any choice of $f_{k+1/2}(x)$. However, $F_{k+1/2}$ is not likely to be zero if $\psi(x,\mu)$ does not exactly satisfy Eqs. (1). In particular, $F_{k+1/2}$ can be calculated at the end of each fission generation using estimates of the quantities in its numerator obtained during the Monte Carlo tracking process. A generalized Monte Carlo – CMFD method can then be defined by subtracting $F_{k+1/2}$ from the numerator of Eq. 8b):

$$\hat{D}_{k+1/2} = \frac{\phi_{1,k+1/2} - F_{k+1/2} + \tilde{D}_{k+1/2}\left(\phi_{0,k+1} - \phi_{0,k}\right)}{\phi_{0,k+1} + \phi_{0,k}}. \quad (13)$$

The rationale for doing this is that for cells that are not optically thick, the terms involving $\phi_1(x)$ in Eqs. (8b) and (12) will – to some degree – cancel in Eq. (13). Experimentally, we have found that this reduces the variance in $\hat{D}_{k+1/2}$, and hence the variance in the solution of Eqs. (11).

In our numerical simulations, we have considered three definitions of $\hat{D}_{k+1/2}$:

1. $\hat{D}_{k+1/2} = \hat{D}^{(I)}_{k+1/2}$ is defined by Eq. (8b). This is the original Monte Carlo – CMFD method proposed by Lee, et al. [1]. We call this the "HCMFD-I" method. (The "H" denotes "hybrid".)

2. $\hat{D}_{k+1/2} = \hat{D}^{(II)}_{k+1/2}$ is defined by Eq. (13), where $F_{k+1/2} = F^{(II)}_{k+1/2}$ is defined by Eq. (12) with $f_{k+1/2}(x) = f^{(II)}_{k+1/2}(x) = 1.$

$$F^{(II)}_{k+1/2} = \frac{\int_{x_{k-1/2}}^{x_{k+3/2}} \Sigma_t(x)\phi_1(x)dx}{\int_{x_{k-1/2}}^{x_{k+3/2}} \Sigma_t(x)dx} + \frac{\phi_2(x_{k+3/2}) - \phi_2(x_{k-1/2})}{\int_{x_{k-1/2}}^{x_{k+3/2}} \Sigma_t(x)dx}. \tag{14}$$

   The use of this modified functional definition comprises the "HCMFD-II" method.

3. $\hat{D}_{k+1/2} = \hat{D}^{(III)}_{k+1/2}$ is defined by Eq. 13), where $F_{k+1/2} = F^{(III)}_{k+1/2}$ is defined by Eq. 12) with $f_{k+1/2}(x) = f^{(III)}_{k+1/2}(x)$ defined as the tent function

$$f_{k+1/2}(x) = \begin{cases} \dfrac{1}{h_{k+1}}\left(x_{k+3/2} - x\right) & x_{k+1/2} \le x \le x_{k+3/2} \\ \dfrac{1}{h_k}\left(x - x_{k-1/2}\right) & x_{k-1/2} \le x \le x_{k+1/2}. \end{cases} \tag{15}$$

   In this case,

$$F^{(III)}_{k+1/2} \equiv \frac{\int_{x_{k-1/2}}^{x_{k+3/2}} f^{(III)}_{k+1/2}(x)\Sigma_t(x)\phi_1(x)dx}{\int_{x_{k-1/2}}^{x_{k+3/2}} f^{(III)}_{k+1/2}(x)\Sigma_t(x)dx} + \frac{\phi_{2,k+1} - \phi_{2,k}}{\int_{x_{k-1/2}}^{x_{k+3/2}} f^{(III)}_{k+1/2}\Sigma_t(x)dx}. \tag{16}$$

   The use of this modified functional definition comprises the "HCMFD-III" method.

Other choices of $f_{k+1/2}(x)$, $F_{k+1/2}(x)$, and $\tilde{D}_{k+1/2}$ are possible but will not be considered here.

3. NUMERICAL RESULTS

3.1 Test Problem Definition

In this paper, we compare the standard Monte Carlo method and Monte Carlo with HCMFD-I, -II, and -III feedback for the following planar geometry criticality test problem.

Table I. Material specifications of test problem.

Location [cm]	Σ_t [cm^{-1}]	Σ_s [cm^{-1}]	Σ_f [cm^{-1}]	ν
$0 < x < 70$	1.0	0.5	0.2	2.4

Using diffusion theory, the dominance ratio (DR) of this problem was calculated to be 0.996. In the following figures, the Monte Carlo Shannon entropy [4], eigenfunction, and eigenfunction standard deviation are reported. In addition, the Monte Carlo *and* HCMFD eigenvalues are reported; we show that the HCMFD eigenvalues are more accurate. In addition, a benchmark S_{16} solution was computed with a 0.01 cm grid.

3.2 Effect of Feedback on Source Convergence

Lee, et al. showed that applying HCMFD-I feedback converges and stabilizes the fission source in high dominance ratio problems [1][2]. In this section, we show that performing Monte Carlo with HCMFD-II or HCMFD -III feedback also converges and stabilizes the Monte Carlo fission source. When HCMFD feedback is turned "off", the Monte Carlo source destabilizes, so the HCMFD feedback should be kept on for all cycles. The test problem was simulated using no feedback ("MC"), feedback during inactive cycles ("MC-FB-x (Inactive Only)"), and feedback during all cycles ("MC-FB-x"), where "x" is the numeral I, II or III corresponding to the HCMFD-I, -II, or –III method, respectively. The numerical parameters are listed in the following table.

Table II. Numerical parameters for source convergence test.

Histories/Cycle (N)	Inactive Cycles (NI)	Active Cycles (NA)	Fine Grid [cm]	CMFD Grid [cm]
100,000	200	200	0.5	0.5

Figure 1 shows that the standard Monte Carlo source converges around cycle 200. Applying HCMFD-I, -II, or -III feedback at the beginning of the calculation converges the fission source immediately, and keeps the source converged as long as the feedback is applied. When the

feedback is turned off during active cycles, the fission source immediately destabilizes, and the Shannon entropy wobbles and diverges from the asymptote.

These source convergence effects can also be seen in the mean eigenfunction (averaged over active cycles) in Figure 2. The mean eigenfunction is accurate when feedback is applied for all cycles. However, when feedback is turned off during active cycles (or not used at all), the eigenfunction has significant errors.

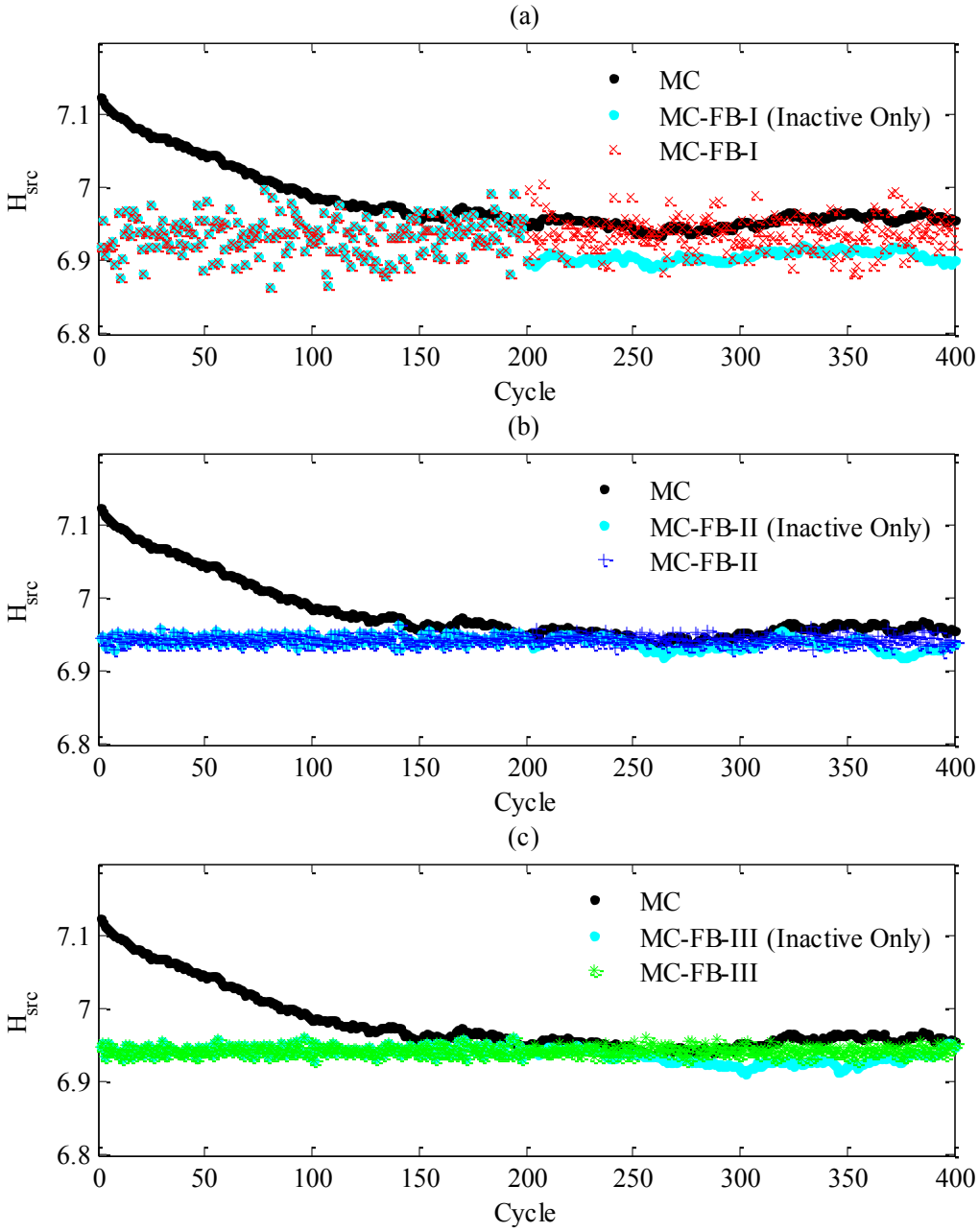

Figure 1. Entropy behavior of Monte Carlo with and without HCMFD feedback.

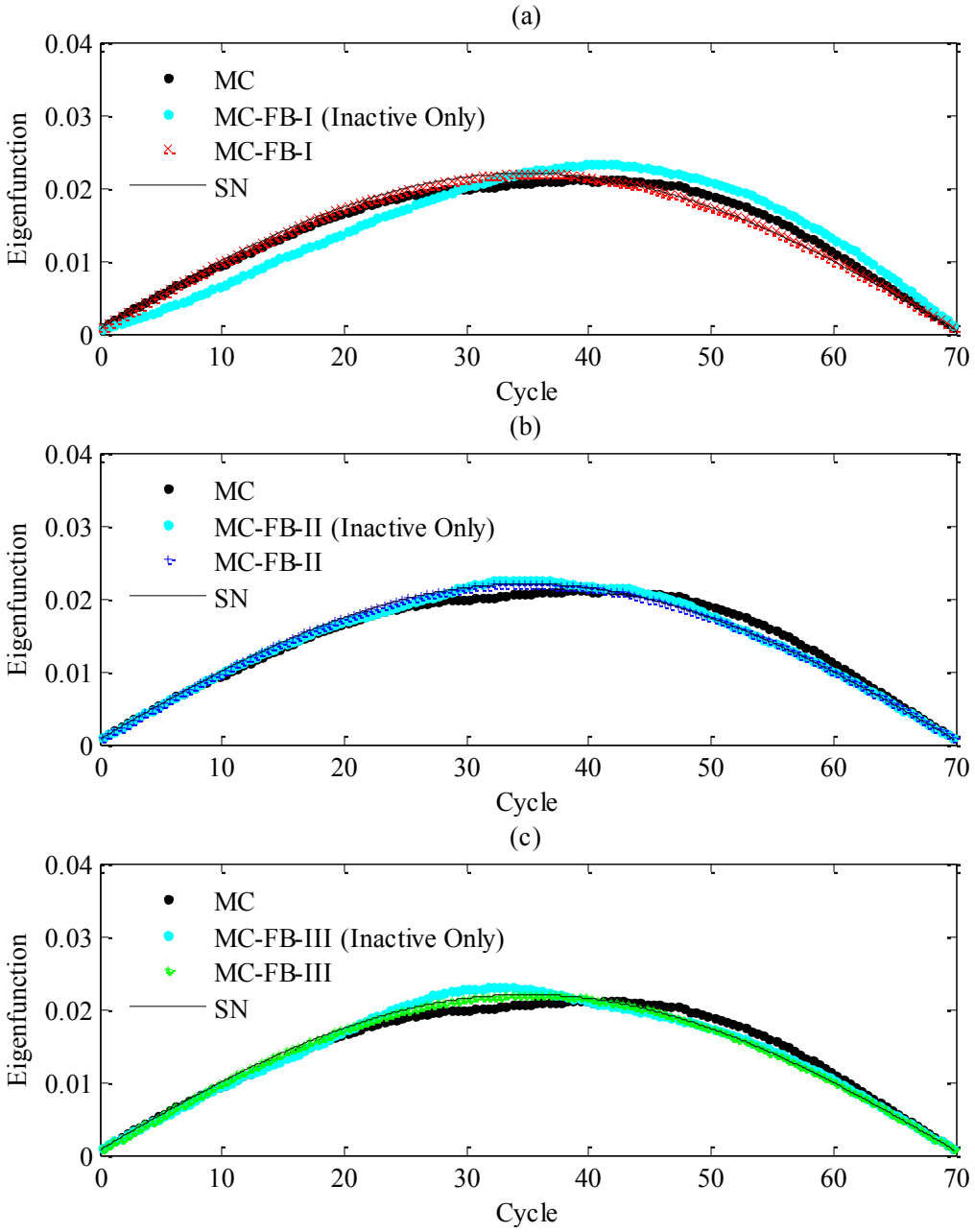

Figure 2. Monte Carlo eigenfunctions averaged over active cycles, with and without HCMFD feedback.

The Monte Carlo eigenvalue is less sensitive to the dominance ratio than the eigenfunction. Applying feedback during active cycles improves the accuracy of the Monte Carlo eigenvalue slightly but does not change its apparent variance. This is expected because the Monte Carlo eigenvalue is computed by *standard Monte Carlo transport* within each cycle whether or not feedback is applied. The HCMFD eigenvalues (computed at the end of each Monte Carlo cycle from the CMFD equations) are more accurate and have lower variance than the Monte Carlo eigenvalues because they are computed from a well-converged fission source.

The benchmark S_N eigenvalue for this problem is $k_{eff} = 0.958764$. Table III lists the Monte Carlo eigenvalues and standard deviations for Monte Carlo with and without feedback. In addition, the HCMFD eigenvalues (computed from either the HCMFD-I, -II, or -III equations after each cycle depending on the feedback type) are listed.

In Section 3.4, we show that the apparent variance of each HCMFD solution is approximately equal to the real variance, while apparent variance of the standard Monte Carlo solution underestimates the real variance. Therefore, the real standard deviations for standard Monte Carlo are higher than the apparent standard deviations in Table III.

Table III. Eigenvalue means and apparent standard deviations for Monte Carlo with and without feedback.

Method	Monte Carlo Eigenvalue	Monte Carlo Apparent S.D.	HCMFD Eigenvalue	HCMFD Apparent S.D.
MC	0.958328	0.000290	-	-
MC-FB-I (Inactive)	0.958468	0.000267	0.958747	0.000009
MC-FB-II (Inactive)	0.958823	0.000273	0.958766	0.000002
MC-FB-III (Inactive)	0.958184	0.000288	0.958765	0.000002
MC-FB-I	0.958991	0.000302	0.958755	0.000008
MC-FB-II	0.958322	0.000284	0.958766	0.000002
MC-FB-III	0.958747	0.000292	0.958765	0.000002

The results of this section show that applying HCMFD-I, HCMFD-II or HCMFD-III feedback to Monte Carlo converges and stabilizes the Monte Carlo fission source, thereby lowering the number of inactive and active cycles required for accurate eigenfunction estimation. In addition, the eigenvalues computed in the HCMFD calculations are much more accurate, and have less variance, than the Monte Carlo eigenvalues. Therefore, we recommend that feedback be turned on for all cycles, and that the HCMFD eigenvalue be reported rather than the mean Monte Carlo eigenvalue.

3.3 Performance for Small Number of Active Cycles

In high dominance ratio criticality problems, Monte Carlo estimates of the eigenfunction may need to be averaged over many active cycles in order to reduce statistical errors due to the slow "wobbling" of the fission source. In this section, we show that applying feedback increases the accuracy of the Monte Carlo eigenfunction for individual cycles, and consequently fewer active cycles are needed. The test problem was repeated with NA=1 and NA=10 active cycles. The number of inactive cycles for the feedback methods was reduced to NI=20.

The mean eigenfunctions obtained with and without feedback are compared in Figure 3.

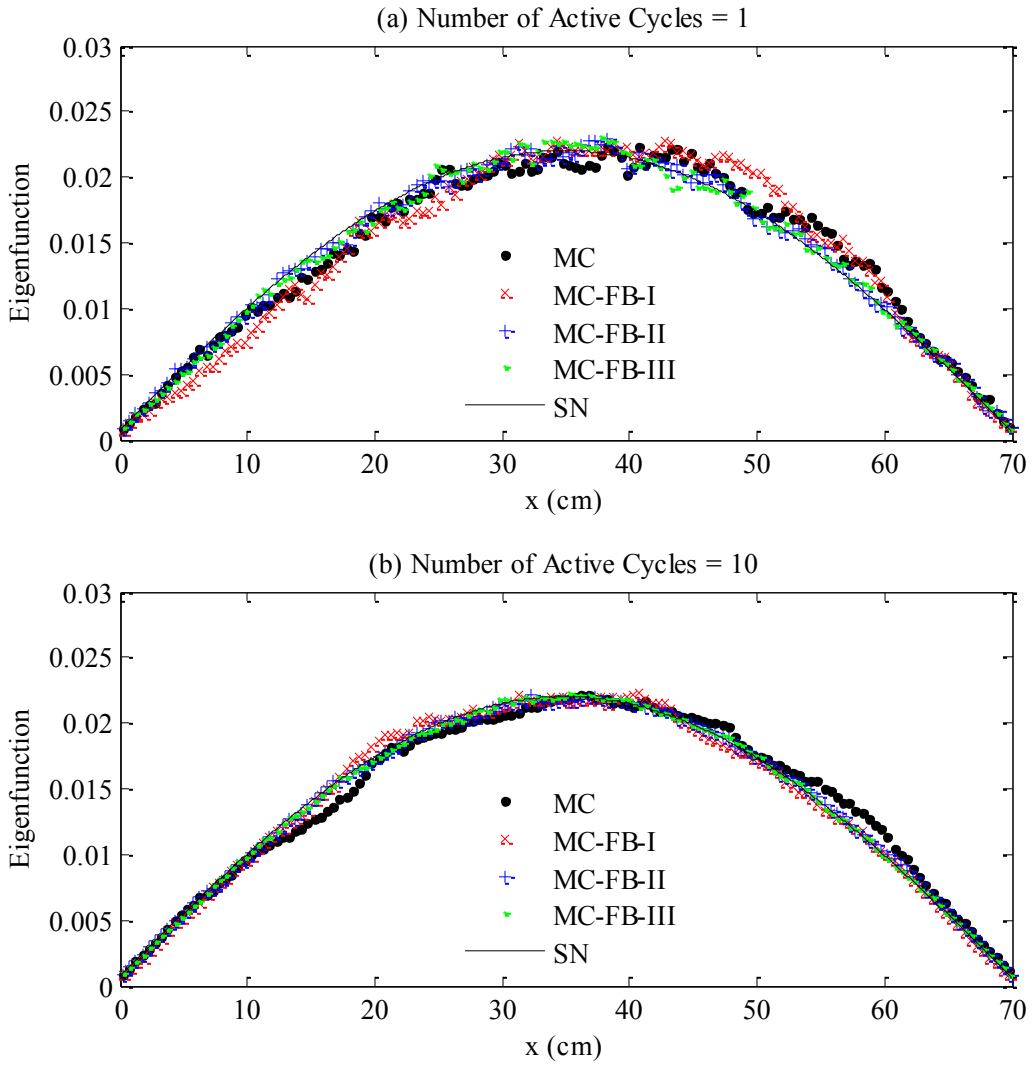

Figure 3. Eigenfunctions averaged over one and ten active cycles.

The Monte Carlo eigenfunctions with HCMFD-II and HCMFD-III feedback are more accurate than the Monte Carlo eigenfunction with HCMFD-I feedback and without feedback for both NA=1 and NA=10 active cycles. For the individual cycle comparison (NA=1), the HCMFD-III feedback resulted in significantly better accuracy than the HCMFD-II feedback. When the number of active cycles increased to 10 (NA=10), the mean HCMFD-II and HCMFD-III eigenfunctions behaved similarly.

The mean Monte Carlo and HCMFD eigenvalues are compared to the benchmark S_N eigenvalue, $k_{eff} = 0.958764$, in Table IV. The error is listed in per cent mille $(1\ \text{pcm} = 10^{-5}\frac{\Delta k}{k})$.

Table IV. Eigenvalues averaged over various numbers of active cycles.

Method	Eigenvalue (Error in pcm)		HCMFD Eigenvalue (Error in pcm)	
	NA=1	NA=10	NA=1	NA=10
MC	0.955099 (-382)	0.955341 (-357)	-	-
MC-FB-I	0.964376 (585)	0.959102 (35)	0.958920 (16)	0.958810 (5)
MC-FB-II	0.958500 (-28)	0.959263 (52)	0.958801 (4)	0.958770 (1)
MC-FB-III	0.963589 (503)	0.959334 (59)	0.958814 (5)	0.958763 (0)

Turning on feedback did not improve the accuracy of the Monte Carlo eigenvalue. However, the HCMFD eigenvalues were much more accurate than the Monte Carlo eigenvalues in all cases (NA=1 and NA=10). In fact, for a single cycle, the HCMFD-I, -II, and –III eigenvalues had errors of only 16, 4, and 5 pcm (compared to 382 pcm in standard Monte Carlo). This is a large improvement in accuracy.

These results show that the individual cycle estimates of the Monte Carlo eigenfunction and the HCMFD eigenvalues obtained with feedback are very accurate. Therefore, fewer active cycles are necessary to obtain accurate solutions for high dominance ratio problems when feedback is used.

3.4 Real vs. Apparent Variance

In problems with high dominance ratios, the standard Monte Carlo source converges very slowly, causing the *apparent variance* of the eigenfunction to underestimate the *real variance*. We demonstrate that when CMFD feedback is used to modify the Monte Carlo fission source, the apparent variance of the resulting Monte Carlo eigenfunction is much closer to the real variance.

The test problem was simulated with N=100,000 histories per cycle and NA=25 active cycles. The simulation was performed K=25 independent times with and without feedback, generating 25 independent estimates of the mean eigenfunction and apparent variance: $\hat{\phi}_{NA,k}$ and $\sigma_k^2\left(\hat{\phi}_{NA,k}\right)$. The *average apparent variance* and *real variance* were calculated from Eqs. (17) and (18):

$$\sigma_A^2 = \frac{1}{K}\sum_{k=1}^{K}\sigma_k^2\left(\hat{\phi}_{NA,k}\right), \tag{17}$$

$$\sigma_R^2 = \frac{K}{K-1}\left[\frac{1}{K}\sum_{k=1}^{K}\hat{\phi}_{NA,k}^2 - \left(\frac{1}{K}\sum_{k=1}^{K}\hat{\phi}_{NA,k}\right)^2\right]. \quad (18)$$

The apparent and real relative standard deviations (RSD) are plotted in Figure 4.

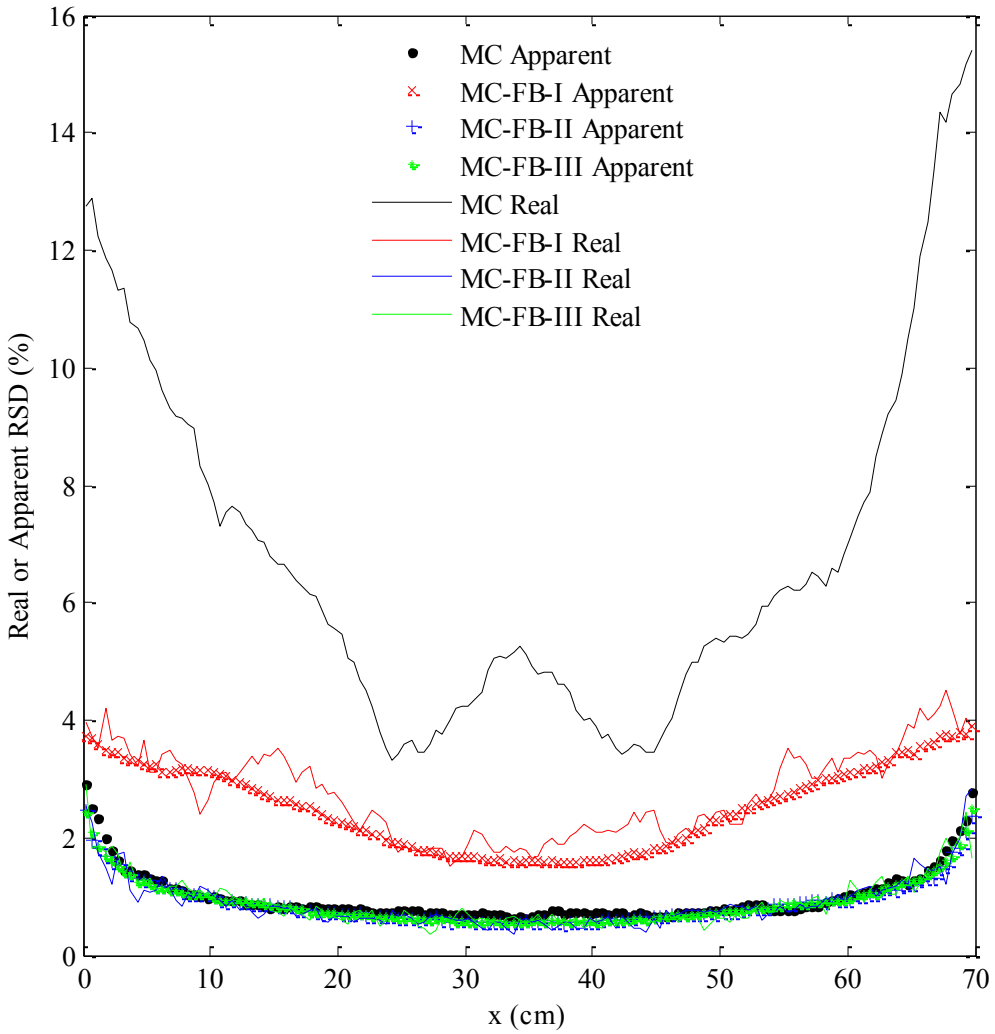

Figure 4. Real and apparent relative standard deviations of the mean eigenfunction.

The real RSDs of the Monte Carlo eigenfunctions with HCMFD-II and -III feedback (MC-FB-II and MC-FB-III) range between 1-3% for this test problem. The real RSD of MC-FB-I is 2-4%, and the real RSD of the standard Monte Carlo eigenfunction is 4-15%. Therefore, use of HCMFD-II and –III feedback results in a significant reduction of *real* error in standard Monte Carlo. Applying HCMFD-II and –III feedback to the Monte Carlo simulation results in real variances which are 3-5% of the real standard Monte Carlo variance. Therefore, Monte Carlo calculations with HCMFD-II or -III feedback are 20 to 35 times faster than standard Monte Carlo calculations. This *significant* computational savings refers to the savings during active cycles.

The actual savings is higher when feedback is applied because fewer inactive cycles are required for source convergence.

The *ratio of real to apparent standard deviation,* $r_{R/A} \equiv \frac{\sigma_R}{\sigma_A}$, is plotted in Figure 5.

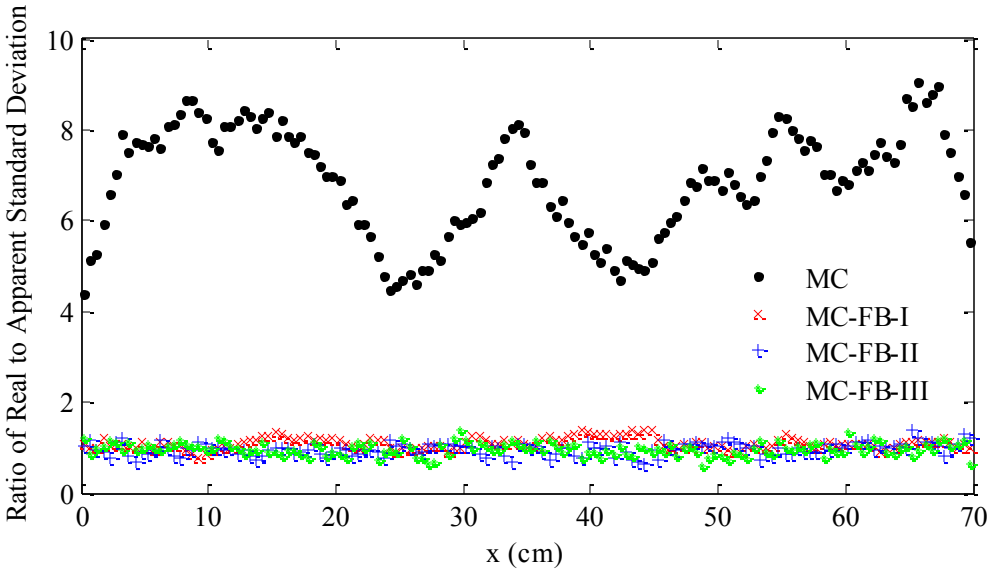

Figure 5. Ratio of real to apparent standard deviation when feedback is applied.

For the specified problem, the Monte Carlo apparent standard deviation underestimates the actual standard deviation by a large factor and should not be trusted. However, the real standard deviation in the HCMFD methods is *approximately equal* to the apparent standard deviation. Therefore, performing feedback is advantageous for two reasons: it reduces the real error in the Monte Carlo estimate of the eigenfunction, and this reduced real error is well-estimated by the apparent error obtained from a single calculation. These results confirm work by Lee, et al. for HCMFD-I [1][2], and show that the close relationship between real and apparent variance also holds for HCMFD-II and HCMFD-III.

4. CONCLUSIONS

In this paper, we have derived generalized hybrid Monte Carlo-CMFD methods that can be used to accelerate Monte Carlo fission source convergence. We considered the previously published CMFD-Accelerated Monte Carlo method (called "HCMFD-I" in this paper), and proposed two specific new hybrid Monte Carlo – CMFD methods called "HCMFD-II" and "HCMFD-III". The three methods involve the stochastic calculation of different nonlinear functionals, which are used in the CMFD equations to solve for the eigenfunction and eigenvalue. The HCMFD-II and HCMFD-III nonlinear functionals were defined by making transport-consistent modifications to

the HCMFD-I functional in order to reduce its statistical error. (Details on the reasoning behind this procedure are found in Reference [5].)

The HCMFD fission source is used to stabilize the Monte Carlo fission source during a Monte Carlo criticality calculation in a procedure known as "feedback". This "feedback" procedure accelerates fission source convergence, resulting in fewer inactive cycles required. It also stabilizes the fission source, preventing "wobbling" in high dominance ratio problems. When feedback is turned "off", the Monte Carlo fission source destabilizes, so we recommend that feedback be applied for all inactive and active cycles. HCMFD-II and HCMFD-III stabilize the source better than HCMFD-I, but all three converge the source equally quickly.

Applying feedback during active cycles alters the Monte Carlo transport procedure by modifying the fission source. Therefore, the solutions obtained using feedback during active cycles are *not* "pure" Monte Carlo solutions. However, the modification is performed by using source biasing factors from an exact system of low-order equations that has no truncation error. The parameters in this system of equations are nonlinear functionals determined by the same Monte Carlo simulation. Moreover, we have presented numerical results that the solutions obtained using feedback are more accurate than standard Monte Carlo solutions because feedback "pushes" the standard Monte Carlo solution in the correct direction for difficult problems. In addition, fewer active cycles are required when feedback is applied because each individual cycle eigenfunction estimate is more accurate (each cycle begins with a converged fission source). Finally, fewer histories per cycle are required because the nonlinear functionals in the HCMFD methods are not as sensitive to statistical errors as direct Monte Carlo estimates of the eigenfunction.

The Monte Carlo eigenfunction obtained using feedback is much more accurate than the standard Monte Carlo eigenfunction. The Monte Carlo eigenvalue obtained using feedback is also more accurate than the standard Monte Carlo eigenvalue, but the HCMFD eigenvalue calculated after each cycle is much more accurate than both of this quantities. We recommend that the HCMFD eigenvalue be reported rather than the Monte Carlo eigenvalue. The HCMFD-II and HCMFD-III solutions were consistently more accurate, and had less variance, than the HCMFD-I and standard Monte Carlo solutions.

Application of feedback reduces inter-cycle correlation in the Monte Carlo calculation, and makes the apparent standard deviation of the eigenfunction obtained with feedback approximately equal to the real standard deviation. Therefore, performing feedback is advantageous for two reasons: it reduces the real error in the Monte Carlo estimate of the eigenfunction, and this reduced real error is well-estimated from a single calculation.

These methods are successfully demonstrated on more realistic problems (e.g., a highly heterogeneous one-group reactor core) in the recently published doctoral dissertation by E. Wolters [5].

ACKNOWLEDGMENTS

This work was funded by a University of Michigan Rackham Predoctoral Fellowship and a University of Michigan Rackham Presidential Fellowship. The authors would like to acknowledge Han Joo and Kord Smith for their helpful discussions relating to this work.

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