Superintegrability of d-dimensional Conformal Blocks

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We observe that conformal blocks of scalar 4-point functions in a d-dimensional conformal field theory can mapped to eigenfunctions of a 2-particle hyperbolic Calogero-Sutherland Hamiltonian. The latter describes two coupled Pöschl-Teller particles. Their interaction, whose strength depends smoothly on the dimension d, is known to be superintegrable. Our observation enables us to exploit the rich mathematical literature on Calogero-Sutherland models in deriving various results for conformal field theory. These include an explicit construction of conformal blocks in terms of Heckman-Opdam hypergeometric functions and a remarkable duality that relates the blocks of theories in different dimensions.

1. INTRODUCTION

Conformal quantum field theories (CFTs) play an important role for modern theoretical physics. In statistical physics they describe the universal behavior of second order phase transitions. At the same time, CFTs also provide a window into interacting and strongly coupled quantum field theories which are very difficult to access otherwise. In $d = 2$ dimensions, the global conformal algebra is extended to an infinite-dimensional symmetry. This was exploited to construct many such models, paving the way for numerous applications in diverse areas of physics and mathematics.

While the symmetry enhancement of 2-dimensional CFT is certainly helpful, it may not be decisive. In fact, CFTs in any dimension d are very strongly constrained by global conformal symmetry. Within the so-called conformal bootstrap programme, the solution of CFTs can be reduced to certain integral equations, the *crossing symmetry* constraints [1–3]. These provide a system of equations for the dynamical coefficients in the operator product expansion involving only the kinematically determined crossing kernel, i.e. group theoretic data. And indeed, recent numerical studies of the crossing symmetry equations, in particular for the conformal Ising model in $d = 3$ dimensions, have provided ample new precision data on this model, see [4] and references therein.

Analytical progress is lagging behind partly because it is restricted to certain limits in which there exists sufficient control of the kinematical input. This is what our work addresses. We will focus on the group theoretic building blocks of scalar 4-point functions, the so-called *conformal blocks* that underly the entire bootstrap programme. Partial waves can be characterized through a second order differential equation [5]. So far, a construction of solutions of conformal Casimir equations in terms of hypergeometric functions is only known in even integer dimensions, where they can be obtained from Gauss hypergeometric functions.

Our main observation is that the Casimir equation for conformal blocks in d dimensions may be transformed into the eigenvalue problem for a Calogero-Sutherland (CS) Hamiltonian, whose eigenfunctions are given by Heckman-Opdam (HO) hypergeometric functions [6]. Thereby we connect the poorly developed theory of conformal blocks to integrability and the modern theory of special functions. The relevant CS Hamiltonian turns out to be superintegrable, i.e. it possesses an additional Runge-Lenz-like integral of motion [7]. The latter is part of the (degenerate) double affine Hecke algebra (DAHA) [8] which provides an extremely powerful algebraic underpinning, introduces a distinguished q-deformation and bridges to the dual Ruijsenaars-Schneider (RS) model [9]. This leads to a wealth of interesting relations for conformal blocks some of which we shall touch upon below.

The plan of this paper is as follows. In the next section we will briefly review the characterization of conformal blocks through the conformal Casimir equation. For pedagogical reasons we shall then explore our general theme in $d = 2$ where the relevant CS model decouples into two Pöschl-Teller systems. These are known to be solvable through hypergeometric functions. Then we turn to the d-dimensional problem in section 4 and explain how the two Pöschl-Teller systems are coupled in order to describe conformal blocks in d-dimensional conformal field theory. The known eigenfunctions of the resulting CS Hamiltonian are used in section 5 to construct conformal blocks from a q-deformed version of HO hypergeometric functions. We conclude by highlighting a few applications of known mathematical results, many of them quite recent, to the conformal bootstrap programme.

2. CONFORMAL PARTIAL WAVES

In this section want to set up the problem by briefly reviewing some material from [5]. The correlation function of four scalar conformal primary fields of weight $\Delta_i, i = 1, \ldots, 4$ in a d-dimensional conformal field theory can be decomposed as

$$\begin{aligned}&\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle\\ &= \frac{1}{x_{12}^{\frac{1}{2}(\Delta_1+\Delta_2)}x_{34}^{\frac{1}{2}(\Delta_3+\Delta_4)}}\left(\frac{x_{14}}{x_{24}}\right)^a\left(\frac{x_{14}}{x_{13}}\right)^b G(z,\bar z)\end{aligned} \quad (1)$$

with $x_{ij} = x_i - x_j$ and $2a = \Delta_2 - \Delta_1, 2b = \Delta_3 - \Delta_4$. The conformal invariants $z, \bar z$ were introduced to parameterize the more familiar cross ratios as

$$\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar z \quad (2)$$

$$\frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar z)\ . \quad (3)$$

For a Euclidean theory, $z, \bar z$ are complex variables. The function G receives contributions from all the primary fields that can appear in the operator product expansion of the field ϕ_1 and ϕ_2

$$G(z,\bar z) = \sum_{\Delta,l}\lambda_l^{12}(\Delta)\lambda_l^{34}(\Delta)G_{\Delta,l}(z,\bar z)\ . \quad (4)$$

This expansion separates the dynamically determined coefficients λ of the operator product from the kinematic *conformal blocks* $G_{\Delta,l}$. The latter are eigenfunctions of the conformal Laplacian D_ϵ^2,

$$D_\epsilon^2 G(z,\bar z) = \frac{1}{2}C_{\Delta,l}G(z,\bar z) \quad (5)$$

with eigenvalues

$$C_{\Delta,l} = \Delta(\Delta-d) + l(l+d-2) \quad (6)$$

and subject to an additional boundary condition that selects a unique (up to normalization) combination of solutions. The form of the conformal Laplacian can be worked out easily, see e.g. [5],

$$D_\epsilon^2 := D^2 + \overline{D}^2 + \epsilon\left[\frac{z\overline{z}}{\overline{z}-z}\left(\overline{\partial}-\partial\right) + (z^2\partial - \overline{z}^2\overline{\partial})\right] \quad (7)$$

where $\epsilon = d-2$ and

$$D^2 = z^2(1-z)\partial^2 - (a+b+1)z^2\partial\ - abz. \quad (8)$$

$\overline{D}^2$ is defined similarly in terms of $\bar z$. In $d = 2$ dimensions the Hamiltonian splits into sum of two independent pieces and the corresponding eigenvalue equations are straightforwardly related to hypergeometric differential equations. Our main goal in this work is to solve the eigenvalue problem for the conformal Laplacian in terms of some known special functions.

3. PÖSCHL-TELLER POTENTIAL

In order to get a bit more insight into the structure of the eigenvalue problem for the conformal Laplacian we will temporarily set $d = 2$. The Laplacian then decomposes into a sum of operators acting on z and $\bar z$ only and we shall focus on the eigenvalue problem for D^2. This problem leads to the following second order differential equation

$$D^2G(z) = h(h-1)G(z)$$

Now let us now define a new function which is related to G by a 'gauge transformation' of the form

$$\psi(x) := \frac{(z-1)^{\frac{a+b}{2}+\frac{1}{4}}}{\sqrt{z}}\ G(z) \quad (9)$$

where the coordinates z and x are related by

$$z = -\sinh^{-2}\frac{x}{2}\ . \quad (10)$$

Note that this relation maps the complex z-plane to a semi-infinite strip with $Re\, x \geq 0$ and $Im\, x \in [0,\pi]$. Inserting these relations it is easy to see that the function ψ is an eigenfunction of the Pöschl-Teller Hamiltonian with potential

$$V_{\rm PT}^{(a,b)}(x) = \frac{(a+b)^2 - \frac{1}{4}}{\sinh^2 x} - \frac{ab}{\sinh^2(x/2)} \quad (11)$$

for the eigenvalue $\varepsilon := 2mE/\hbar^2 = -(2h-1)^2/4$. The original Schrödinger problem studied by Pöschl and Teller in [10] was a trigonometric version of eq. (11). After such rotation to $y = ix$ the associated Schrödinger problem describes a particle that is confined to the interval $y \in [0,\pi]$. The Hamilton operator possesses a discrete spectrum with eigenfunctions given by ordinary Jacobi polynomials.

The hyperbolic version we are dealing with here is also referred to as Pöschl-Teller Hamiltonian of second kind. It describes a particle on the half-line $x \geq 0$. Since the potential falls off to zero for large x, the Hamiltonian has a continuous part in its spectrum. The eigenfunctions are given by

$$\psi_h(x) \sim z^{h-\frac{1}{2}}(z-1)^{\frac{a+b}{2}+\frac{1}{4}}\ {}_2F_1\left(\begin{matrix}h+a, h+b\\ 2h\end{matrix}; z\right)\ .$$

Before we move on we stress that the Pöschl-Teller problem is related to some classical theory of special functions. Let us describe this for the trigonometric case in which eigenfunctions are classical Jacobi polynomials. Like all other hypergeometric orthogonal polynomials in a single variable, Jacobi polynomials are obtained from a degeneration of the so-called Askey-Wilson polynomials. The latter may be constructed from the q-deformed version ${}_4\Phi_3$ of the hypergeometric function ${}_4F_3$ by specializing its parameters, see e.g. [11]. Of course, all these relations can be lifted to the hyperbolic theory, i.e. from polynomials to functions.

4. CALOGERO-SUTHERLAND POTENTIAL

Historically, the Schrödinger problem for the Pöschl-Teller potential was solved through the relation with the hypergeometric differential equation. But today it is much more interesting to look at the relation in the opposite direction. Following work of Calogero, Moser and Sutherland in the early 1970s, see [12–14], the solvable Pöschl-Teller problem has been generalized in several directions. In particular it was understood that the Pöschl-Teller potential is just the simplest example of a large family of superintegrable Schrödinger problems involving multiple particles. The relevant potentials are associated with reflection groups and give rise to so-called (trigonometric or hyperbolic) Calogero-(Moser)-Sutherland models [15].

In the last section we recalled that the Casimir equation for blocks in 2-dimensional chiral conformal field theory is equivalent to the Pöschl-Teller problem. Our main claim is that this extends to the full Casimir equation for conformal blocks in d dimensions. In complete analogy to the discussion above, it turns out that the Casimir equation is equivalent to the hyperbolic CS model for reflection group BC_2. Its potential is given by

$$V_{\mathrm{CS}}^{(a,b,\epsilon)}(x_1,x_2) \;=\; V_{\mathrm{PT}}^{(a,b)}(x_1)+V_{\mathrm{PT}}^{(a,b)}(x_2)+ \\ +\frac{\epsilon(\epsilon-2)}{8\sinh^2\frac{x_1-x_2}{2}}+\frac{\epsilon(\epsilon-2)}{8\sinh^2\frac{x_1+x_2}{2}}\;. \tag{12}$$

It is built from two Pöschl-Teller systems with an interaction term whose coupling explicitly depends on the dimension d. The six terms of this potential reflect the six positive roots of the BC_2 root system. To relate the associated Schrödinger problem on the BC_2 Weyl chamber with the eigenvalue equation (5) for the conformal Laplacian we generalize the gauge transformation (9) to become

$$\psi(x_1,x_2) := \prod_i \frac{(z_i-1)^{\frac{a+b}{2}+\frac14}}{z_i^{\frac12+\frac{\epsilon}{2}}}|z_1-z_2|^{\frac{\epsilon}{2}}G(z_1,z_2) \tag{13}$$

where $z_1=z$ and $z_2=\bar z$. It is not difficult to verify that this gauge transformation, along with the relation

$$z_i = -\sinh^{-2}\frac{x_i}{2}\;. \tag{14}$$

between the coordinates z_i and x_i, turns the conformal Laplacian into the CS Hamiltonian for the potential (12), with the eigenvalue $\varepsilon = -d(d-2)/4-(C_{\Delta,l}+1)/2$. The appearance of the BC_2 root lattice possesses a natural explanation in the corresponding harmonic analysis formulation of the problem [16], where it enters as a projection of the root lattice of the conformal algebra $\mathfrak{so}(1,d+1), d\geq 5$, to the 2-dimensional plane spanned by the Cartan generators of an embedded $\mathfrak{so}(1,3)$.

Just as the 1-dimensional Pöschl-Teller problem is exactly solvable, so are the higher-dimensional CS extensions and hence, by the relation (13), the eigenvalue problem for the conformal Laplacian. Let us note that the coupling constant in front of the interaction term is $\varepsilon = d-2$. In $d=2$ dimensions, we are just dealing with two independent integrable Pöschl-Teller systems. Going away from $d=2$ introduces a new coupling in the potential. It is quite remarkable that this coupling is also integrable. One may notice that in $d=4$ dimension the interaction terms vanishes once again. This implies that 2d and 4d conformal blocks are simply related by a gauge transformation. The latter switches between bosonic/fermionic statistics of the wave function. For conformal blocks, the simple relation between $d=2$ and $d=4$ dimensions is indeed consistent with the standard expressions [5].

5. SOME APPLICATIONS

What makes the observed relation between conformal blocks and the CS Hamiltonian interesting are the connections of the latter with integrability and the modern theory of special functions. The integrability of the CS model can be established using so-called *Dunkl operators*, i.e. a special set of linear first order operators that involve reflections. From powers of these Dunkl operators one can construct sufficiently many commuting operators to render the problem integrable, in fact even superintegrable (in rational/hyperbolic cases). Along with the multiplication by coordinates and Weyl-reflections, Dunkl operators generate a (trigonometric) degeneration of the so-called *double affine Hecke algebra*. The latter involves an additional deformation parameter q that is sent to $q=1$ when dealing with the (undeformed) CS model. In order to understand the origin of the parameter q one needs to turn to the rational Ruijsenaars-Schneider model which is related to our hyperbolic CS model by a bi-spectral duality [17]. Within the dual theory, q controls the deformation from the rational to the hyperbolic version. Many more details on these topics can be found e.g. in [8, 18].

All this structure is an integral part of the modern theory of special functions. In the context of the (trigonometric) Pöschl-Teller problem we briefly sketched the relation between classical Jacobi and q-deformed Askey-Wilson polynomials. The latter possess well developed multi-variable extensions which are known as q-deformed HO or (Macdonald)-Koornwinder polynomials K. Just as the trigonometric Pöschl-Teller problem can be solved through a degenerate limit of Askey-Wilson polynomials, eigenfunctions of the trigonometric CS Hamiltonian may be obtained from Koornwinder polynomials in the limit $q\to 1$. This web of interrelations may be lifted from polynomials to functions, i.e. from the trigonometric to the hyperbolic theory. The lift turns Koornwinder or q-deformed HO polynomials into what Rains refers to as *virtual* Koornwinder polynomials $\hat K$, see [19]. We can also think of them as q-deformed HO hypergeometric functions, up to normalization issues.

Before we can spell out a concrete formula we need to split the data Δ, l that characterize the internal field into a partition (l_1, l_2) and a real parameter χ. Upon imposing usual unitarity bounds, this is done as follows

$$l_2 := \lfloor \frac{1}{2}(\Delta - l) \rfloor \tag{15}$$

$$l_1 := l_2 + l \quad , \quad \chi := \frac{1}{2}(\Delta - l) - l_2 \ . \tag{16}$$

The virtual Koornwinder polynomials $\hat{K}^{(2)}_{l_1,l_2}$ for the root system BC_2 are functions of two variables $u_i, i = 1, 2$ that are associated to two-row partitions (l_1, l_2). By appropriate choice of their seven parameters, we can obtain conformal blocks as

$$(-4)^{-\Delta} \left(\frac{z\bar{z}}{16}\right)^a G_{\Delta,l}(z_i) = (u_1 u_2)^{\chi+a} \ \times \tag{17}$$
$$\lim_{q\to 1^-} \hat{K}^{(2)}_{\lambda_1,\lambda_2}\left(u_i; q, q^{\epsilon/2}, q^{-\chi-a}; q^{a-b+1}, -q^{a+b+1}, 1, -1\right)$$

where $\epsilon = d - 2$ and

$$u_i = -\frac{z_i}{(1+\sqrt{1-z_i})^2} \tag{18}$$

for $i = 1, 2$ for $z_1 = z, z_2 = \bar{z}$ are obtained by inverting the relations (14) with $u_i = \exp x_i$. Note that the arguments u_i agree with the radial coordinates of [20] up to a sign.

Virtual Koornwinder polynomials possess a binomial expansion in terms of Okounkov's BC_n-type interpolation Macdonald polynomials, see [19] section 7. These should be considered as generalizations of the usual series expansion of hypergeometric functions $F(u)$ in terms of monomials u^k. In the case the BC_2 root system, the interpolation Macdonald reduce to Gegenbauer polynomials upon taking $q \to 1$ and the combinatorial prefactors may be expressed through the hypergeometric functions ${}_4F_3$. The resulting expansion reproduces a formula for conformal blocks that was found by Dolan and Osborn in [5].

In order to demonstrate the powerful consequences of the relation between conformal field theory and CS models we want to sketch a few features of the blocks that seemingly were not observed before. The first one concerns an interesting strong-weak coupling duality of the BC_n CS model that was noted by Serban in [21] and can be neatly derived once the theory has been q-deformed, using so-called Cauchy identities, see [19]. The duality relates wave functions of the model with parameters (a, b, ϵ) and

$$(\tilde{a}, \tilde{b}, \tilde{\epsilon}) = \left(\frac{2(a+1)}{\epsilon} - 1, \frac{2b}{\epsilon}, \frac{4}{\epsilon}\right)$$

This duality is non-perturbative in the integrable coupling ϵ that describes the deformation away from two decoupled Pöschl-Teller systems. Note that $\epsilon = 2$ is the self-dual point at which the duality acts trivially. In the context of conformal blocks, the parameter ϵ is related to the dimension d through $\epsilon = d - 2$ and the self-dual point corresponds to 4-dimensional conformal field theories. Away from this special point, the duality allows to write the blocks of a theory in dimension d as an integral over blocks of another theory in dimension $d' = 2d/(d-2)$. An explicit formula for the integral kernel of this transformation may be inferred from [19, 21].

As we mentioned before, the q-deformation that proved useful at least for the duality we discussed in the previous paragraph, originates from the duality between the hyperbolic CS and the rational RS model. Within the context of the RS model, one can also obtain Gauss-like recurrence relations that describe the behavior of HO hypergeometric functions under finite shifts of their parameters, see e.g. [22]. There also exist growth estimates, see [23] for some recent work, and an interesting relation with (quantum) affine Knizhnik-Zamolodchikov equations for (q-deformed) blocks [24, 25]. We will detail all these features of blocks in a forthcoming longer paper.

6. CONCLUSION AND OUTLOOK

The main observation of this work that the Casimir equation for conformal blocks is equivalent to the Schrödinger equation for the BC_2 CS model, embeds the central objects in the bootstrap programme of d-dimensional conformal field theory into the rich world of superintegrable quantum systems. The deep connections to the modern theory of special functions have powerful implications for conformal blocks of which we have seen just two examples in the previous section.

There are a number of obvious extensions of our work that merit further investigation. The first one concerns the extension to superconformal field theory. In fact, the Casimir equations for conformal blocks in superconformal field theories have been discussed previously, but in most cases explicit formulas are only known for a restricted set of external scalar fields, see e.g. [26] for some interesting recent developments and many references to the earlier literature.

Another interesting direction concerns the so-called crossing kernel of d-dimensional conformal field theory. In the numerical bootstrap program, the crossing symmetry is usually written in terms of conformal blocks, with one side of the equation involving blocks in the so-called s-channel while the other side is expressed in terms of t-channel waves. The blocks in the two different channels are related by the crossing kernel, so that crossing symmetry may be expressed in terms of operator product coefficients $\lambda_l(\Delta)$ and the crossing kernel, stripping off the $(z, \bar{z})$-dependent conformal blocks. With the analytic control of conformal blocks we have described above it is possible to obtain new and more explicit formulas for the crossing kernels.

Let us finally stress again that in our entire discussion, the dimension d enters as a continuous parameter which is interpreted as a coupling constant of the CS model. There exist many conformal field theories for $d = 2$ dimensions that can be solved through their higher spin symmetries. It should be possible to combine the results we outlined above with the ideas that were put forward recently in [27] to study the spectrum of conformal field theories in $2 + \epsilon$ dimensions, at least for small ϵ. We will return to these interesting problems in future work.

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